TUCS

Irina Georgescu

Rational Choice and Revealed Preference: A Fuzzy Approach

Turku Centre *for* Computer Science

TUCS Dissertations
No 60, April 2005

Rational Choice and Revealed Preference: A Fuzzy Approach

Irina Georgescu

ACADEMIC DISSERTATION

To be presented, with the permission of the Faculty of Economics and Social Sciences of Åbo Akademi University, for public criticism in Auditorium 3102 at DataCity on the 25th of May 2005, at 12 o'clock noon.

Åbo Akademi University
Turku Centre for Computer Science
Institute for Advanced Management Systems Research
Lemminkäinengatan 14B FIN-20520 Åbo
Finland
2005

Supervisors

Professor Christer Carlsson
Department of Information Systems
Institute for Advanced Management Systems Research,
Åbo Akademi University
Åbo, Finland

Reviewers

Professor Hannu Nurmi
Department of Political Science,
Turku University
Turku, Finland

Professor Erich Peter Klement
Department of Knowledge-Based Mathematical Systems,
Fuzzy Logic Laboratorium Linz-Hagenberg,
Johannes Kepler University
Linz, Austria

Opponent

Professor Bernard De Baets
Department of Applied Mathematics, Biometrics and Process Control,
Gent University
Gent, Belgium

ISBN 952-12-1528-3
ISSN 1239-1883

Painosalama Oy, 2005
Turku, Finland

Abstract

Rational choice and revealed preference are important issues in social choice theory. A choice act is said to be rational if it is based on the optimization of some preference relation. The revealed preference theory of consumers was created by Samuelson (1938) and Houthakker (1950). Uzawa (1956), Arrow (1959) and Sen (1969, 1971) developed an axiomatic theory of revealed preference in an abstract framework independent of budget sets and demand functions. The results of Uzawa, Arrow, Sen and their followers were proved under the hypothesis that all non-empty finite sets of alternatives are included in the domain of a choice function. Further on, Richter (1966), Hansson (1968) and Suzumura (1976) elaborated a generalized theory of revealed preference with no restriction on the domain of the choice function.

This thesis is concerned with rational choice and revealed preference of a large class of fuzzy choice functions. Our concept of fuzzy choice function includes that of Banerjee. In Banerjee's approach, the range of the fuzzy choice function consists of fuzzy sets and the domain consists of crisp sets, in our approach both the domain and the range of the fuzzy choice function consist of fuzzy sets.

Our contributions can be grouped in five main themes:

–(1) Revealed preference and congruence axioms for fuzzy choice functions;

–(2) Rationality and normality of fuzzy choice functions;

–(3) Consistency conditions for fuzzy choice functions;

–(4) Degree of dominance for fuzzy choice functions;

–(5) Applications.

Our revealed preference results are developed in two directions:

-one generalizes the Uzawa-Arrow-Sen theory

-the second extends the Richter-Hansson-Suzmura theory.

The first direction starts from two hypotheses $(H1)$ and $(H2)$ that extend to a fuzzy context Uzawa-Arrow-Sen theory. In this framework connections between weak and strong congruence axioms $WFCA$, $SFCA$, weak and strong revealed preference axioms $WAFRP$, $SAFRP$ and other properties of rationality and normality are established. The main result is a generalization of the Arrow-Sen theorem. Further consistency conditions $F\alpha$, $F\beta$, $F\delta$

which are fuzzy versions of Sen's conditions α, β, δ are studied. We prove that a fuzzy choice function satisfies $F\alpha$ and $F\beta$ if and only if $WFCA$ holds. Also, $F\delta$ holds if and only if the revealed preference relation R (canonically associated to the fuzzy choice function) is quasi-transitive. Other consistency conditions ($F\alpha 2$, $F\gamma 2$, $F\beta(+)$, path independence) are also discussed.

In the second direction rationality and revealed preference theory for fuzzy choice functions with arbitrary domains are investigated. New axioms of revealed preference $WAFRP^{\circ}$, $SAFRP^{\circ}$, $HAFRP$ are introduced and relations between these axioms and the previous ones are established.

We obtain two main theorems: (1) Axioms $WFCA$ and $WAFRP^{\circ}$ are equivalent. (2) Axioms $SFCA$ and $HAFRP$ are equivalent. We analyze two concepts of rationality: G-rationality and M-rationality. Another result generalizes a part of a Richter theorem.

We define a notion of the degree of dominance of an alternative with respect to an available fuzzy set of alternatives and we introduce new axioms of congruence for fuzzy choice functions.

If we interpret an available set as a criterion, then we can obtain a ranking of alternatives (for each criterion) with respect to the act of choice. This ranking is obtained by using fuzzy choice problems and the instrument by which it is established is the degree of dominance associated with a fuzzy choice function. In defining this fuzzy choice function the revealed preference theory is applied.

Keywords

revealed preference, fuzzy choice function, rationality, normality, consistency, degree of dominance, decision-making

Acknowledgements

I would like to thank my supervisor Professor Christer Carlsson for his persistence, guidance and support that he has given me during these four years of doctoral program. Without his continuous encouragement, I would have never had the resources to write this thesis. He has always shown a large and consistent interest in my research and our discussions have been valuable and constructive. His most contribution to my development as a researcher was to direct me towards independent research work, which I will always be grateful for.

I want also to thank the reviewers of my thesis, Professor Hannu Nurmi from Turku University and Professor Erich Peter Klement from Johannes Kepler University for the useful comments and suggestions that improved the quality of the manuscript. I am grateful to Professor Bernard De Baets from Gent University for accepting the invitation of being my opponent.

My professional knowledge has been founded by the professors from the Department of Economic Cybernetics, Academy of Economic Studies, Bucharest, where I was an undegraduate student. In particular my thanks go to my first mentor and supervisor, Professor Dumitru Marin for his constructive teaching during my undergraduate studies.

Special thanks are due to Turku Centre for Computer Science (TUCS), one of the most powerful research centers, for the excellent working conditions and financial support provided during my doctoral studies. I also have benefited from the friendly and stimulating environment that I have had in the Institute for Advanced Management Systems Research (IAMSR), the place where I have studied and done my research.

I am thankful to the friends that I have here and in my country for their moral support and the nice times that we spent together.

Last but not least, I would like to especially thank my family for the huge patience and endless understanding that they had with me during my doctoral studies.

Turku, April 26, 2005

Contents

Chapter 1

Introduction

The rationality of a consumer is a main subject in classical consumer theory. By Uzawa [61], "the rationality of a consumer may be described by postulating that a consumer has a definite preference over all conceivable commodity bundles and that he chooses those commodity bundles that are optimal with respect to his preference subject to budgetary constraints".

Samuelson's theory of revealed preference expresses the rationality of a consumer in terms of some preference relation associated with a demand function. The foundation of this theory is built on The Weak Axiom of Consumer Behavior [44] and on The Strong Axiom of Consumer Behavior [28]. The second axiom assures that the demand function can be reconstructed from a revealed preference relation.

Following a suggestion of Georgescu-Roegen [23], Uzawa [60] and Arrow [2] have developed a revealed preference theory in an abstract context, with no economic interpretation. The work of Uzawa and Arrow was continued by Sen with his fundamental results concentrated in [49, 48, 50] (see also [51]). In this approach there is no demand function and the basic elements are an arbitrary set X of alternatives and a choice function C defined on a family $\mathcal{B}$ of available sets of alternatives ($\mathcal{B} \subseteq \mathcal{P}(X)$). By the rationality of C we mean to find a preference relation R on X such that for any available S the choice set $C(S)$ coincides with the set of R-greatest elements of S. To a choice function C there are associated more preference relations, each of them leading to a different notion of rationality.

In [2] The Weak Axiom of Revealed Preference $WARP$ and The Strong Axiom of Revealed Preference $SARP$ are formulated. $WARP$ and $SARP$ are abstract versions of The Weak Axiom of Consumer Behavior and of The Strong Axiom of Consumer Behavior. The results obtained by Uzawa, Arrow, Sen and others are based on the hypothesis that the domain of the choice function C should contain all finite subsets of the universe X; in fact it suffices to assume that the domain of C contains the pairs and the triples of alternatives.

An important contribution to the development of choice function theory belongs to Richter [41] by introducing the Weak Congruence Axiom WCA and the Strong Congruence Axiom SCA and by proving the equivalence between rational and congruous choice functions. For the choice functions whose domain includes the finite sets of alternatives, Arrow-Sen theorem [2, 49] establishes the equivalence between the congruence axioms WCA, SCA, the revealed preference axioms $WARP$, $SARP$ and other four conditions of rationality. Richter's result does not use the above–mentioned hypothesis. Richter's line was followed by Hansson [27] and Suzumura [54] that created a revealed preference theory for arbitrary choice functions. They introduced a new axiom of revealed preference $HARP$ and the equivalence of SCA and $HARP$ was established.

The first contribution on fuzzy preference relations belongs to Orlovsky [38] and a vast literature has been dedicated to this subject (for an exhaustive overview see the monograph [16] as well as [43, 33]). Some authors build their results on the thesis that social choice is governed by fuzzy preferences (hence modelled through fuzzy binary relations) but the act of choice is exact (hence choice functions are crisp). They study choice functions generated by fuzzy preference relations [5, 6, 7].

Some papers discuss the topic of fuzzy choice functions. Usually fuzzy choice functions are associated to fuzzy preference relations [58, 43, 33]. In this case the fuzzy choices are the consequence of the fuzzy preferences.

In [4] Banerjee studies fuzzy choice functions that are no longer defined by a fuzzy preference relation. In this way the act of choice is primordial and the preferences are defined by choices. The domain of a Banerjee fuzzy choice function C is the family of all non-empty subsets of a universal set X of alternatives and the range of C is a family of non-zero fuzzy subsets of X. Then any non-empty subset S of X is an available set of alternatives. We have no information about the alternatives in S except that they can be chosen.

This thesis targets the study of fuzzy choice functions in a very general form. In our approach the domain of a choice function will be a family $\mathcal{B}$ of non-zero fuzzy subsets of X; if $S \in \mathcal{B}$ and $x \in X$ then $S(x)$ can be viewed as the availability degree of the alternative x. In this way the alternatives are singled out by their degree of availability. As in [4], the range of a fuzzy choice function contains fuzzy subsets of X. Of course the class of fuzzy choice functions we consider includes that of Banerjee. Banerjee fuzzifies only the range of a choice function; we use a fuzzification of both the domain and the range of a choice function.

Chapter 2 works out the methodological basis by introducing the concept of paradigm as the theoretical framework within which a theory is created. In particular, this chapter focuses on the concept of revealed preference introduced by Samuelson [44] as one of the dominant paradigms of social choice theory.

The preliminaries of Chapter 3 introduces some notions and basic facts on continuous t-norms [11, 26, 31] as well as some notions and results on classical choice functions [2, 41, 42, 49, 48, 50, 27, 54, 55, 60, 61]. In the last section of the chapter a fuzzy version of the Szpilrajn theorem is proved.

Chapter 4 has two sections. The first introduces the fuzzy choice functions, the preference relations associated to them and formulates the axioms of revealed preference $WAFRP$, $SAFRP$ and congruence $WFCA$, $SFCA$. $WAFRP$ and $SAFRP$ (resp. $WFCA$ and $SFCA$) are fuzzy versions of the classical axioms $WARP$ and $SARP$ (resp. WCA and SCA). Following [27], [54] we also consider the axioms $WAFRP^{\circ}$ and $HAFRP$ (a fuzzy version of $HARP$). The fuzzy versions of classical axioms establish relationships between the fuzzy choice functions and various preference relations associated with them. These relationships reflect the way in which the fuzzy choices determine vague preferences and how in their turn these preference relations influence the act of choice. The second section of the chapter deals with M-rational, G-rational, M-normal and G-normal fuzzy choice functions, extending some results obtained by Suzumura in [54] for crisp choice functions.

Chapter 5 is devoted to a fuzzy revealed preference theory following the line introduced by Uzawa, Arrow and Sen. The class of fuzzy choice functions studied in this chapter is subject to hypotheses $(H1)$ and $(H2)$; they come naturally from the assumption that the domain of choice functions includes all finite sets of alternatives. In Section 5.1 we study the relations between the axioms $WAFRP$, $SAFRP$, $WFCA$, $SFCA$ and four other rationality conditions. The main result (Theorem 5.1) establishes equivalences or implications between these properties; some are true for an arbitrary continuous t-norm and other for Gödel or Lukasiewicz t-norms. Section 5.2 is concerned with consistency conditions $F\alpha$ and $F\beta$, which are fuzzy forms of Sen's properties α and β [49, 48, 50]. We prove that a fuzzy choice function satisfies $F\alpha$ and $F\beta$ if and only if $WFCA$ holds. Section 5.3 deals with condition $F\delta$, a fuzzy form of Sen's condition δ. The main result (Theorem 5.3) asserts that for a normal fuzzy choice function, the fuzzy preference relation R is quasi-transitive if and only if $F\delta$ holds. Other consistency conditions ($F\alpha 2$, $F\gamma 2$, $F\beta(+)$, path independence) are discussed in the last section of the chapter.

Chapter 6 studies the revealed preference properties for arbitrary fuzzy choice functions ignoring the hypotheses $(H1)$ and $(H2)$. The investigation follows the trend of the Richter-Hansson-Suzumura theory [41, 27, 54]. In Section 6.1 we prove two main theorems: (1) the axioms $WFCA$ and $WAFRP^{\circ}$ are equivalent; (2) the axioms $SFCA$ and $HAFRP$ are equivalent. Section 6.2 is devoted to the analysis of a particular class of fuzzy choice functions for which the equivalence between $WAFRP^{\circ}$, G-normality, M-normality and two other algebraic conditions holds. The last section of the chapter investigates how the Richter theorem can be extended to a fuzzy choice function theory.

In Chapter 7 a notion of degree of dominance of an alternative with respect to a fuzzy subset is introduced, extending Banerjee's notion of dominance. In the literature several notions of dominance have been studied, but with respect to a fuzzy preference relation [33]. The degree of dominance proposed here refers to the act of choice, and not to a preference relation. In Section 7.1 we mention briefly the setting in which Banerjee formulated his concept of dominance. In Section 7.2 a concept of dominance for the type of fuzzy choice functions studied in this thesis is introduced. We prove that the degree of dominance of an alternative $x \in X$ with respect to a fuzzy set can be expressed by means of a degree of dominance of an alternative with respect to fuzzy subsets of type $[x, y]$, $y \in X$. In Section 7.3, new congruence axioms FC^*1, FC^*2, FC^*3 are formulated based on the degree of dominance introduced in the previous section and we prove that FC^*1 implies FC^*3 and FC^*2 implies FC^*3, generalizing the results of Banerjee [4] and Wang [62]. We introduce a new revealed preference axiom $WAFRP_D$ and we prove the equivalence $WAFRP_D \Leftrightarrow FC^*3$.

Chapter 8 is devoted to the possible applications of fuzzy choice functions and the fuzzy revealed preferences associated to them. Three distinct applications are discussed, showing the importance of the notions introduced in the previous chapters for decision making processes, by ranking the alternatives according to multiple criteria. The applications try to model economic situations in which partial information or human subjectivity generate vague choices and vague preferences. The mathematical modelling of these situations is done by fuzzy choices examples where criteria are represented by available fuzzy sets of alternatives. For each case one builds a fuzzy choice space, determines the fuzzy choice function and computes the degree of dominance of alternatives for each fuzzy subset of the universe of alternatives. This leads to a ranking of alternatives with respect to each criterion. The decision-maker will rely on the information obtained in such way.

Chapter 9 contains a summary and the main conclusions of the thesis and poses some open problems for continued and future research.

This thesis is based on the following original papers:

I I. Georgescu. Rational and congruous fuzzy consumers. In *Proceedings of the International Conference on Fuzzy Information Processing (FIP'03)*, Beijing, China, Springer Verlag, pages 133–137, 2003.

II I. Georgescu. A fuzzy analysis of a Richter theorem in fuzzy consumers. In *Proceedings of the 3rd International Conference in Fuzzy Logic and Technology (EUSFLAT 2003)*, Zittau, Germany, pages 423–428, 2003.

III I. Georgescu. On the axioms of revealed preference in fuzzy consumer theory. In *Journal of Systems Science and Systems Engineering*, volume 13, pages 279–296, 2004.

IV I. Georgescu. Consistency conditions in fuzzy consumers theory. In *Fundamenta Informaticae*, volume 61, pages 223–245, 2004.

V I. Georgescu. Revealed preference, congruence and rationality: a fuzzy approach. In *Fundamenta Informaticae*, volume 65, pages 307–328, 2005.

VI I. Georgescu. Degree of dominance and congruence axioms for fuzzy choice functions. In *Fuzzy Sets and Systems*, forthcoming.

Chapter 2

Research Methodology

2.1 Normal-scientific research and paradigms

In his book *The Structure of Scientific Revolutions* the physicist and philosopher Thomas S. Kuhn [32] calls "normal science" that scientific research based on achievements that a community of scientists acknowledges and that in time forms a basis for further research. In order to be called a scientific achievement, the research results should be innovative and sufficiently powerful to attract several supporters and to raise new topics to be explored. In economic science, such scientific achievements include von Neumann and Morgenstern's *Theory of Games and Economic Behaviour*, Samuelson's *Foundations of Economic Analysis*, Dantzig's simplex algorithm, Arrow's contribution to the theory of social choice, etc.

Normal-scientific research does not aim at producing "scientific revolutions" or major discoveries. Its intention is neither to open new territories nor to test old beliefs. On the contrary, it is directed towards extending the knowledge that a major discovery reveals.

Kuhn connects the concept of "paradigm" to "normal science". In its dictionary meaning, a paradigm is a conceptual framework within which scientific theories are constructed. In science the appearance of a paradigm produces a conversion from old beliefs to new ones. This process is called "a paradigm shift". The common view is that a paradigm is shared by the members of a scientific community and conversely, a scientific community consists of people who share the same paradigm. As a result of a paradigm shift, [32], p. 7: "a scientist's world is qualitatively transformed as well as quantitatively enriched by fundamental novelties of either fact or theory". Kuhn believes that a paradigm is rather produced by young people who are not so conservative or by people who are new in a scientific field.

Normal-scientific research is based on paradigms and develops the facts that arise from them. Normal-scientific research is cumulative and it progresses relatively fast, compared to paradigms that appear as a result of scientific revolutions.

The discovery that brings a paradigm is not a usual process. Not all theories bring paradigms. Many theories pretend that they lead to paradigms, but only those theories that assimilate experiments are proclaimed paradigmatic. After a theory has been universally accepted, it will explain a larger scope of phenomena and a lot of scientific issues will be solved.

According to Kuhn [32], scientists involved in normal science deal with three types of problems:

a) determination of theoretical facts on which the paradigm is based;

b) determination of real situations that can be set in the theoretical framework of the paradigm;

c) matching empirical data with theoretical results, in order to clear out unsolved or ambiguous aspects of the paradigm, i.e. the "articulation" of a theory.

Once a scientific theory is built, there will be cases when empirical observations cannot fit very well in that theory. Empirical observations do not intend to shatter a theory, but to correct its imperfections; the theory should be consistent with the empirical evidence. Revisions of the theory are important because discrepancies between theoretical and empirical evidence can be found in different parts of that theory, such as the basic assumptions, or the auxiliary parts.

Kuhn [32] emphasizes the feature of "puzzle-solver" of the normal science, as an act of creation that involves commitment and skills. In general the scientists challenged with problems dedicate their time and use their creativity to find solutions, assuming that they exist, while paradigms are taken for granted. Normal science-based researchers use already existing techniques for finding solutions to their research problems [32], p. 96: "The man who is striving to solve a problem defined by existing knowledge and technique is not however, just looking around. He knows what he wants to achieve, and he designs his instruments and directs his thoughts accordingly". Normal science has an expected, anticipated character, unlike paradigms that emerge when the expectations regarding the nature of the problem and the tools for solving it prove to be wrong.

2.2 Mathematical methods in economics

The mathematization of a science is the process by which that science has established its structural and formal aspects.

Mathematics plays an important role in economics. One can say that economics became a science when it started to be significantly expressed by mathematical models. In the recent years, economic theory has been built on solid mathematical ground. A mathematical formulation of economic concepts confers rigor and clarity of thought, offers generalizations and opens new directions for research. Usually in pure economics the mathematical formalism is greater than in applied economics. The axiomatic approach that uses abstract mathematics is beneficial for economic theory because it gives straight answers free of interpretations. At the same time by its rigor it leads to strong conclusions that eliminate confusion. The interdisciplinary potential of mathematics lies not only in the generality of its concepts, but also in the mode of reasoning that it proposes.

In mathematical modelling there is the tendency to use the mathematical structures that already exist, but also to ask for new mathematical structures. This tendency is natural and favorable, since it enriches the logical consequences of the modelled theory. There might be the risk that the mathematical structures that are in use cannot give an accurate description of the observed reality, and parts of its essential features might be overlooked.

The language of mathematics is meant to translate non-mathematical concepts into symbols. It has the advantage of simplicity and generality. The economic models based on mathematics have the characteristic of generality, in the sense that they comprise a multitude of possible situations. Sometimes the prose of an economic narration can be very lengthy and imprecise, therefore a mathematical approach would make it intelligible and would simplify its understanding. For the non-mathematical economists, clear explanations of the mathematical conclusions and their possible connections to the real world would be necessary. The difference between the mathematical economist and the non-mathematical economist is that the former is likely to look for similarities with other modelled processes, while the latter is inclined to regard modelled processes as singular and attached to a context.

A comparison between economics as a social science and natural sciences is debatable too. Both social sciences and natural sciences build on similar logical systems with axioms as premises. Natural sciences (especially physics and chemistry) have the premises (=axioms) strongly based on tests and observations and for that reason, those premises are taken as laws of nature. In economics, it is more difficult to formulate the axioms starting only from empirical evidence. Thus in social sciences the adequacy of the axioms to the real phenomena is more difficult to measure than in natural sciences.

There are some opinions that new knowledge springs only from pure research and requires a higher degree of intellectual effort than applied science, that tries to implement already-known results. This idea is erroneous and superficial, since applied science also gives rise to new knowledge and deals with several disciplines. The economist Paul A. Samuelson [46] acknowledges the depth of pure mathematics and rejects the idea that the applied mathematics is inferior in nature; its power is given by its applicability to real situations. In Samuelson's opinion, valuable mathematical economics should have some empirical relevance. Mathematical formulations should offer systematic explanations for a set of observable data. Samuelson also stresses out the responsibility of the mathematical economist towards the reader, to explain the assumptions and the conclusions in a clear language.

Over time, mathematical theory has contributed to the development of applied economics. Without a mathematical framework it would not be possible to test or estimate economic models. In support of this assertion comes the fact that an important part of the Nobel laureates in economics are applied mathematicians. We can show how mathematical ideas have significantly influenced economics. In business, Danzig's simplex algorithm changed the system of industrial planning by a better allocation of resources. In order to derive a consumer's preferences by observing his choices, Samuelson introduced the revealed preference theory. Akerlof showed how asymmetric information of buyers and sellers about product quality can cause an "adverse selection" of low-quality products. Mathematical methods in economics include: linear programming, fixed-point theorems, control theory, calculus, game theory or even Arrow's theory of rational choice.

2.3 The construction of theorems and the building of valid theories

In its incipient stage, each science targets to accumulate particular data, to analyze and classify them and to formulate some hypotheses on the connections between these data. Without a systematization and logical unity, the progress of the science is not satisfactory. Therefore, in its mature stage, the science gets organized in coherent systems of ideas called scientific theories.

Generally speaking, a scientific theory is a body of knowledge coherently organized, which describes the real world phenomena. According to the philosopher and physicist Mario Bunge [10],p. 381: "Such systems, characterized by the relation of deducibility holding among some of its formulas, are called hypothetico–deductive systems, models, or simply theories".

The same author claims that the fundamental objectives of a scientific theory are [10]:

(i) to systematize the knowledge by establishing logical relations between previously non-connected elements;

(ii) to explain facts by means of systems of hypotheses that consist of propositions expressing the facts;

(iii) to enlarge the knowledge by deriving new propositions;

(iv) to increase the testability of hypotheses.

Scientific theories progress in four stages: the direct observation, the inductive stage, the deductive stage and the formalization of the theory.

The *direct observation* is concerned with searching for singular facts on phenomena. These facts are collected and grouped; by their analysis hypotheses on the laws that govern these phenomena are derived. The collection of data and the investigation of the relationships among the hypotheses give rise to new hypotheses and generalizations, configured in systems of hypotheses [10]. The goals of the research are figured out, the observed properties are verified and experiments leading to new knowledge are conceived. The *inductive stage* begins and also elements that belong to deduction appear.

The inductive reasoning moves from specific to general and characterizes a theory that is derived from empirical observations or by experience. The deductive reasoning moves from general to specific and is characteristic for those theories which are derived from assumptions or axioms. The inductive reasoning is more exploratory, while the deductive reasoning narrows down the study to hypothesis testing. Although distinctive, the two types of reasoning intertwine in a scientific theory.

After an inductive itinerary, a scientific theory aspires to a form of *deductive organization*. This confers to the science an intelligible and unifying individuality. The results obtained by deductive reasoning can be applied to a multitude of concrete situations. The deductive reasoning assures the certitude of the proofs; by standardization it produces economy in the effort of thinking.

The organization of science in deductive theories has been influenced by the evolution of mathematics and its expansion of its methods in other sciences.

The structure of deductive theories is connected to the notions of axiom and proof. An axiomatic system should coherently provide a body of knowledge. The justification of choosing axioms is different from the construction of the theory. The confrontation of the axiomatized theory and the body of knowledge concretely obtained does not say anything about what happens inside the theory, but validates more or less the choice of axioms.

The axiomatization of a deductive theory assumes the choice of some *primitive concepts* on whose basis all the notions of the theory are defined; they are called *derived concepts*.

The manner in which the derived concepts are obtained from the primitive concepts is specified by well established *rules of construction*. The *set of propositions* is consequently obtained.

A set of propositions called *axioms* are chosen and the *rules of deduction* are specified. By rules of deduction from axioms the valid propositions or the *theorems* of that theory are derived.

The *formalization* is the last stage in the evolution of the theory. To the propositions of the theory formulated in natural language are assigned expressions in a formal language. One starts with an alphabet composed of a list of primitive symbols. The finite strings of symbols are called *words*. By rules, symbols from the set of the words are combined to form the expressions of the language. They will be the formalized expression of the propositions of the theory.

When the language has been built, its deductive structure is defined.

A list of axioms and a list of deduction rules are given. The strings of expressions obtained from axioms and by applying the deduction rules are called formal proofs. The expressions situated at the end of a formal proof are called formal theorems.

Formal proofs and formal theorems are fundamental elements in the construction of a formal system.

The theory building process described in this section is the one that I have used in my research.

2.4 Revealed preference theory: a paradigm

One of the dominant paradigms of social choice is the revealed preference theory. The classical economic theory has as a basic assumption the rationality of the consumer behaviour, as an optimizing behavior subject to some budgetary constraints. Revealed preference is a concept introduced by Samuelson in 1938, in the attempt to postulate the rationality of a consumer's behaviour in terms of a preference relation associated to a demand function.

The observable variables regarding the behaviour of a consumer are prices, quantities purchased and income, while the unobservable variables are his preferences. In revealed preference theory first choices are given, then preferences are defined by choices. The consumer reveals by choices his/her preferences, hence the term *revealed preferences* [45]:

"By comparing the costs of different combinations of goods at different relative price situations, we can infer whether a given batch of goods is preferred to another batch; the individual guinea-pig, by his market behaviour, reveals his preference pattern - if there is such a consistent pattern".

Unlike the results of his predecessors Marshall and Hicks, who proposed a parametric analysis of the demand functions, Samuelson's theory of revealed preference was meant to construct a theory of demand functions based only on observable variables. Samuelson's non-parametric approach of the theory of demand functions was less subject to errors than the Hicksian and Marshallian parametric approaches to the same theory.

The paradigm of revealed preference theory is emphasized by Suzumura [57], p. 22:

"If the choice behaviour of an agent is guided systematically by some underlying preferences, that fact will infallibly reveal itself in his actual choices, so that by observing his choices under alternative specifications of environmental conditions, we may possibly reconstruct his underlying preferences. This was indeed the original insight of Samuelson that propelled him to open the door to the splendid edifice of revealed preference theory for a competitive consumer."

Samuelson introduced the revealed preference relation R as follows:

In terms of vectors of prices and chosen bundles (p^t, x^t), we say that commodity bundle x^0 is *revealed preferred* to commodity bundle x^1 and write x^0Rx^1 if x^0 is chosen when x^1 is available.

We can explain this definition by the fact that if $p^0x^0 \geq p^0x^1$, the consumer can buy commodity bundle x^1 at price p^0 but he has chosen commodity bundle x^0 instead. If expenditures $p^0x^0 < p^0x^1$ then we say that x^0 is not revealed preferred to x^1.

Samuelson's theory is based on the Weak Axiom of Consumer Behaviour:

If x^0 is revealed preferred to x^1 then x^1 is not revealed preferred to x^0.

In terms of relation R, the Weak Axiom of Consumer Behaviour is briefly written:

If x^0Rx^1 then not x^1Rx^0.

Samuelson's results in revealed preference theory could be applied for two commodity bundles. Houthakker formulated a stronger form, called the Strong Axiom of Consumer Behaviour, managing to prove Samuelson's results for an arbitrary number of commodity bundles. In terms of quantities and prices, the Strong Axiom of Consumer Behaviour can be formulated

If $p^0x^0 \geq p^0x^1$, $p^1x^1 \geq p^1x^2$, $\ldots$, $p^{n-1}x^{n-1} \geq p^{n-1}x^n$ then $p^nx^0 > p^nx^n$.

In terms of commodity bundles, it can be formulated:

If commodity bundle x^0 is revealed preferred to commodity bundle x^1, commodity bundle x^1 is revealed preferred to commodity bundle x^2, $\ldots$, and commodity bundle x^{n-1} is revealed preferred to commodity bundle x^n then x^n is not revealed preferred to x^0, i.e.

If x^0Rx^1, x^1Rx^2,..., $x^{n-1}Rx^n$ then not x^nRx^0.

The Strong Axiom of Consumer Behaviour implies the Weak Axiom of Consumer Behaviour. The main difference between strong and weak axioms of consumer behaviour is that the strong axiom not only makes impossible the existence of two choices that are both revealed preferred to one another, but also makes impossible the existence of sequences of choices that finally lead to two choices that are each revealed preferred to one another.

From these results, the axiomatization of consumer theory using a set-theoretic approach began.

Uzawa and Arrow enlarged Samuelson's theory of revealed preference, introducing in an abstract setting the concept of choice function. Their axiomatic approach started from the assumption that the domain of the choice function contains all finite subsets of a universal set of alternatives [2]: "It is the suggestion [...] that the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets. Indeed, as Georgescu–Roegen has remarked, the intuitive justification of such assumptions as the Weak Axiom of Revealed Preference has no relation to the special form of the budget constraint sets but is based rather on implicit consideration of two–element sets". Their approach goes beyond the study of demand function and lacks any economic interpretation. The underlying structure is the choice space, a pair $\langle X, \mathcal{B}\rangle$ made of a non-empty set X (whose elements are interpreted as alternatives) and a non-empty family of non-empty subsets of X (available sets). This is a level at which the revealed preference theory is developed as a deductive system: from basic concepts and axioms, by deductions, the whole theory is built.

Uzawa and Arrow introduced the weak and strong axioms of revealed preference $WARP$, $SARP$, abstract versions of weak and strong axioms of consumer behaviour. Sen noticed that Arrow's results can be reduced to the study of the choice functions based on two-element and three-element sets of alternatives. An important step in the study of the theory of rational choice was taken by Richter, Hansson and Suzumura. Richter was the first to formulate the strong congruence axiom SCA and proved the equivalence between a rational and congruous consumer. This direction assumed that the domain of a choice function consists of an arbitrary family of non-empty subsets of a universe of alternatives. Later Sen introduced a weaker version of SCA, the weak congruence axiom WCA and proved the equivalence between the axioms of revealed preference and congruence and other four conditions of rationality. The current viewpoint is that the theory of general domains, as the Richter-Hansson-Suzumura approach is called, is the most relevant for most choice situations. Suzumura maintains ([57], p. 27) that under the assumption that the domain of the choice functions is a non-empty collection of non-empty subsets of a universe of alternatives X, "the theory of rational choice functions developed on this minimal domain condition applies to whatever choice situations we may care to specify: Choice of consumers in a competitive or noncompetitive market, of government bureaucracies, of voters, and of whatsoever".

Besides rationality, consistency conditions are another way of expressing the coherency of the act of choice. They describe the choice behaviour when the available set of alternatives expands or contracts. Consistency conditions are related to the lattice operations of feasible sets (union and intersection). We distinguish contraction consistency conditions and expansion consistency conditions. The first ones give "information on what elements are chosen from subsets from information on what elements are chosen from supersets." The second ones give information on "information on what elements are chosen from supersets from information on what elements are chosen from subsets". ([29], pp 30-31).

Human preferences are many times ambiguous and have different degrees of intensity. An explanation for this would be insufficient information or human subjectivity. For this reason instead of saying that an alternative x is better than an alternative y, it is better to evaluate the degree of preference of x to y; this will always be a number in $[0, 1]$. The idea of mathematical modelling of vague preferences is obvious: given a set X of alternatives, the preference will be represented by a binary fuzzy relation on X.

Even if the preference is ambiguous, the choice can be either exact or vague. When the choice is exact, it will be mathematically described by a crisp choice function on X: from any available set $S \subseteq X$ a non-empty crisp subset is selected. This viewpoint has been adopted by several authors (see [33] for a detailed discussion). Various crisp choice functions have been proposed based on a fuzzy preference relation. The most studied is Orlovsky choice function.

There are cases (negotiations on electronic marketplaces) when the decision maker cannot make a definitive choice. In this process of decision making, the choice is potential [4]:

"For instance, a decision-maker, faced with the problem of deciding whether not to choose an alternative x from a set of alternatives A, may feel that he/she is inclined to the extent 0.8 (on, say, a scale from 0 to 1) toward choosing it. Moreover, this fuzziness of choice is, at least potentially, *observable*. For instance, the decision-maker in the example will be able to tell an interviewer the degree of his/her inclinations, or demonstrate these inclinations to an observer by the degree of eagerness or enthusiasm which he/she displays. Hence, while there may be problems of estimation, fuzzy choice functions are, in theory, observable."

In this stage for an available set S, one cannot say if $x \in S$ is chosen, but one can evaluate by the real number $C(S)(x) \in [0, 1]$ "the degree to which x can be chosen". The choice function then assigns a fuzzy subset of X to any available set S. This approach belongs to Banerjee [4] (see also [64], [62]).

In this thesis we enlarge Banerjee's framework introducing a new concept: the degree of availability of an alternative. Here the available sets are fuzzy subsets of the universe of alternatives X. If S is such an available set and $x \in X$ then $S(x)$ is the availability degree of x. In interpretation the available sets correspond to some criteria or attributes of the alternatives. To an available set S (which is a fuzzy subset of X) the choice function will assign a non-zero fuzzy subset $C(S)$ of X such that $C(S)(x) \leq S(x)$ for any $x \in X$. If we identify the crisp subsets of X with their characteristic functions we will see that our framework extends Banerjee's. The pair made by the set of alternatives and the family of available sets is called *fuzzy choice space* and represents the mathematical structure in which this thesis develops the theory of fuzzy choice functions.

As in classic economic theory, the relationship between fuzzy preferences and fuzzy choices is studied twofold. To a fuzzy choice function several fuzzy preference relations are attached and, conversely, there are several ways to define a fuzzy choice function from a fuzzy preference relation. The fundamental concepts of classic economic theory (rationality, normality, revealed preference, consistency etc) appear exactly in the context of this twofold relation.

Fuzzy choice functions reflect vague choices: their behaviour ought to be subject to some rationality conditions. The rationality of fuzzy choice functions is a larger concept than the rationality of crisp choice functions. Similar to the crisp case, the rationality of fuzzy choice functions expresses a principle of optimality through fuzzy preference relations. One can define two cases of fuzzy rationality: G-rationality (the choice of the greatest elements) and M-rationality (the choice of the maximal elements). Related to the properties of the fuzzy preference relation, there exist transitive-rationality, reflexive-rationality, etc.

The fuzzy revealed preference represents the study of the rationality of fuzzy choice functions through certain preference relations canonically associated to them. The axioms of revealed preference reflect a rational behaviour. $WAFRP$ and $SAFRP$ are fuzzy versions of $WARP$ and $SARP$. Congruence axioms $WFCA$, $SFCA$ (fuzzy versions of axioms WCA, SCA) are other conditions that adjust the fuzzy choices.

The connections between rationality, normality, revealed preference and congruence axioms are the main objective of this thesis. In relation with them various consistency conditions are also studied.

Several authors have studied the problem of selecting the alternatives based on a fuzzy preference relation ([5, 6, 16, 33, 38, 43] etc). In this matter they analyzed various concepts of dominance expressed in terms of the fuzzy preference relation (for an exhaustive overview on this topic see [16, 33, 43]). In [4, 62] a concept of dominance directly related to a (Banerjee) fuzzy choice function and *not* to a fuzzy preference relation is investigated. This type of dominance allows for the formulation of new axioms of congruence used in the development of fuzzy revealed preference theory in [4, 62].

In this thesis we have defined the degree of dominance of an alternative x with respect to an available set S. This notion refines the dominance in [4, 62]. If we interpret the available set S as a criterion in decision making, the degree of dominance is important for obtaining a hierarchy of alternatives with respect to that criterion. Banerjee's notion of dominance determines only the dominant alternatives. The degree of dominance is useful for the ranking of all alternatives according to each criterion of choice. This type of multicriterial ranking of alternatives provides more complete information for the decision–makers. In this way the alternatives are ordered by choices (at least by potential choices) and not by preferences.

The transition from crisp revealed preference to fuzzy revealed preference is not a simple translation. Many times classic concepts and properties lead to several fuzzy versions (e. g. axioms $WAFRP$, $WAFRP^{\circ}$ are distinct fuzzy versions of the classic axiom $WAFRP$). From these versions we must choose those that reflect more faithfully this topic (vague preferences and choices) and that allow for well-founded mathematical constructions.

In formulating the notions, axioms and theorems we used the framework of the residuated lattice structure of $[0, 1]$ (induced by an arbitrary continuous t-norm). The proofs of the results are based on the properties of the residuum associated to a t-norm. By this, at the level of formulation and argument, the use of fuzzy logic is more transparent.

Chapter 3

Preliminaries

In the first section of this chapter we present notions and basic facts on continuous t-norms and fuzzy relations. As is known, a continuous t-norm defines a residuum and a negation on $[0, 1]$. The structure of residuated lattice of $[0, 1]$ allows the development of a fuzzy set theory for each continuous t-norm. The main continuous t-norms are Lukasiewicz t-norm, Gödel t-norm and product t-norm [26, 31]. Some results of this thesis will be established for a continuous t-norm and others for one of the three t-norms from above, especially for Gödel t-norm.

Section 3.2 outlines some notions in classical choice function theory. We recall several preference relations associated to a classical choice function and we state the congruence axioms WCA, SCA [41], revealed preference axioms $WARP$, $SARP$ [2, 49, 48, 50] in the form imposed by Arrow [2] as well as consistency conditions α, β, γ, δ [49]. We sum up classical results among which Arrow-Sen theorem [2, 49] and Richter theorem [41]. We assert the equivalent forms of axioms $WARP$, $SARP$ in terms of C-connected sequences and Hansson's revealed preference axiom $HARP$ [27]. The section ends with two theorems belonging to Hansson and Suzumura [27, 54].

In Section 3.3 we formulate and prove a fuzzy version of Szpilrajn theorem that asserts that for a strict $*$-order R on a set X there exists a total strict $*$-order R^* on X that includes R. This theorem will be later used for a fuzzy analysis of Richter theorem [41] in consumer theory.

The results of Section 3.3 are based on our paper [17].

3.1 Continuous t-norms and fuzzy relations

The notion of a triagular norm , briefly a t-norm, was introduced by Menger [36] as a generalization of the triangular inequality of a metric. The current notion of the t-norm belongs to Schweizer and Sklar [47]. The t-norm models the intersection of two fuzzy sets and the conjunction in fuzzy logic [16, 26, 11]. There is a vast literature dedicated to t-norms and to their applications to fuzzy set theory [11, 16, 26, 31].

In this paragraph we shall recall some notions and properties of t-norms. The basic reference for fuzzy preference relations is the monograph [16]. We shall present next some notions and results of fuzzy relations theory with regard to a continuous t-norm.

Let $[0,1]$ be the unit interval. For any $a, b \in [0,1]$ we shall denote $a \vee b = \max\,(a,b)$ and $a \wedge b = \min\,(a,b)$. More generally, for any $\{a_i\}_{i\in I} \subseteq [0,1]$ we denote

(3.1.1) $\bigvee\limits_{i\in I} a_i = \sup\,\{a_i | i \in I\}$; $\bigwedge\limits_{i\in I} a_i = \inf\,\{a_i | i \in I\}$.

Then $([0,1], \vee, \wedge, 0, 1)$ becomes a bounded distributive lattice. Furthermore, $[0,1]$ is a distributive complete lattice.

A *triangular norm* (=t-norm) is a binary operation $*$ on $[0,1]$ such that for any $a, b, c \in [0,1]$ the following axioms are satisfied:

$(T1)$ $a * b = b * a$;

$(T2)$ $a * (b * c) = (a * b) * c$;

$(T3)$ If $a \leq b$ then $a * c \leq b * c$;

$(T4)$ $a * 1 = a$.

An immediate consequence is that $a * 0 = 0$ for any $a \in [0,1]$.

A t-norm $* : [0,1]^2 \to [0,1]$ is said to be *continuous* if it is continuous as a function on the unit interval. With any continuous t-norm $*$ we associate a new binary operation $\to$ on $[0,1]$:

(3.1.2) $a \to b = \bigvee\{c \in [0,1] | a * c \leq b\}$.

The operation $\to$ is called the *residuum* or the implication associated with $*$.

We list here the most well-known continuous t-norms and their corresponding residua:

Lukasiewicz t-norm

$a *_L b = \max\,(0, a + b - 1)$; $a \to_L b = \min\,(1, 1 - a + b)$

Gödel t-norm

$a *_G b = \min\,(a,b)$; $a \to_G b = \begin{cases} 1 & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$

Product t-norm

$a *_P b = ab$; $a \to_P b = \begin{cases} 1 & \text{if} \quad a \leq b \\ b/a & \text{if} \quad a > b \end{cases}$

Let $*$ be a continuous t-norm.

The properties mentioned in the following three lemmas reflect the main connections between the t-norm $*$ and its residuum $\to$.

Lemma 3.1 *[11, 26] For any $a, b, c \in [0,1]$ the following properties hold:*

(1) $a * b \leq c \Leftrightarrow a \leq b \to c$;

(2) $a * (a \to b) = a \wedge b$;

(3) $a * b \leq a$; $a * b \leq b$;

(4) $b \leq a \to b$*;*

(5) $a \leq b \Leftrightarrow a \to b = 1$*;*

(6) $a = 1 \to a$*;*

(7) $1 = a \to a$*;*

(8) $1 = a \to 1$*;*

(9) $a * (b \vee c) = (a * b) \vee (a * c)$*;*

(10) $a \leq b$ *implies* $b \to c \leq a \to c$ *and* $c \to a \leq c \to b$*;*

(11) $a \to (b \to c) = b \to (a \to c) = (a * b) \to c$*;*

(12) $(a \to b) * (b \to c) \leq a \to c$*.*

In accordance with Lemma 3.1 (1), $([0,1], \vee, \wedge, *, \to, 0, 1)$ is a residuated lattice (see [11, 26]).

Lemma 3.2 *[11, 26] For any* $\{a_i\}_{i\in I}, \{b_i\}_{i\in I} \subseteq [0,1]$ *and* $a \in [0,1]$ *the following properties hold:*

(1) $a \to (\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I}(a \to a_i)$*;*

(2) $(\bigvee_{i\in I} a_i) \to a = \bigwedge_{i\in I}(a_i \to a)$*;*

(3) $\bigvee_{i\in I}(a_i \to a) \leq (\bigwedge_{i\in I} a_i) \to a$*;*

(4) $\bigvee_{i\in I}(a \to a_i) \leq a \to (\bigvee_{i\in I} a_i)$*;*

(5) $(\bigvee_{i\in I} a_i) * a = \bigvee_{i\in I}(a_i * a)$*;*

(6) $(\bigvee_{i\in I} a_i) * (\bigvee_{j\in I} b_j) = \bigvee_{i,j\in I}(a_i * b_j)$*;*

(7) $(\bigwedge_{i\in I} a_i) * (\bigwedge_{j\in I} b_j) \leq \bigwedge_{i,j\in I}(a_i * b_j)$*.*

The *negation operation* $\neg$ associated with a continuous t-norm $*$ is defined by

(3.1.3) $\neg a = a \to 0 = \bigvee\{c \in [0,1] | a * c = 0\}$.

Recall the negations associated with Lukasiewicz, Gödel and product t-norms ([26] p. 31):

Lukasiewicz t-norm

$\neg a = 1 - a$

Gödel and product t-norms

$\neg a = \begin{cases} 1 & \text{if} \quad a = 0 \\ 0 & \text{if} \quad a > 0 \end{cases}$

Lemma 3.3 *[11, 26] For any $a, b, c \in [0, 1]$ the following properties hold:*

(1) $a \leq \neg b \Leftrightarrow a * b = 0$;

(2) $a * \neg a = 0$;

(3) $a \leq \neg\neg a$;

(4) $\neg 0 = 1$, $\neg 1 = 0$;

(5) $\neg a = \neg\neg\neg a$;

(6) $a \rightarrow b \leq \neg b \rightarrow \neg a$;

(7) $a \leq b \Rightarrow \neg b \leq \neg a$;

(8) $\neg(a \vee b) = \neg a \wedge \neg b$; $\neg(a \wedge b) = \neg a \vee \neg b$.

The *biresiduum* $\leftrightarrow$ of a continuous t-norm $*$ is defined by

(3.1.4) $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$

for any $a, b \in [0, 1]$.

Let X be a non-empty set and $\mathcal{P}(X)$ the power set of X. For any $A \subseteq X$ the characteristic function χ_A of A is defined by

(3.1.5) $\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A. \end{cases}$

Remark 3.1 *The assignment $A \mapsto \chi_A$ defines a bijection between $\mathcal{P}(X)$ and $\{0,1\}^X$; in fact this bijection is a Boolean isomorphism.*

A *fuzzy subset* of X is a function $A : X \rightarrow [0, 1]$. If $x \in X$ then $A(x)$ is called the degree of membership of x in A. Let us denote by $\mathcal{F}(X)$ the set of fuzzy subsets of X. By identifying a subset A of X with its characteristic function χ_A (cf. Remark 3.1), $\mathcal{P}(X)$ can be considered a subset of $\mathcal{F}(X)$.

For any $x_1, \ldots, x_n \in X$, we shall denote by $[x_1, \ldots, x_n]$ the characteristic function of the set $\{x_1, \ldots, x_n\}$:

$$[x_1, \ldots, x_n](y) = \begin{cases} 1 & \text{if} \quad y \in \{x_1, \ldots, x_n\} \\ 0 & \text{otherwise.} \end{cases}$$

If $A, B \in \mathcal{F}(X)$ we denote $A \subseteq B$ if $A(x) \leq B(x)$ for each $x \in X$. For any $A, B \in \mathcal{F}(X)$ we define the fuzzy subsets $A \cup B$, $A \cap B$, $A * B$ and $A \rightarrow B$ by

(3.1.6) $(A \cup B)(x) = A(x) \vee B(x)$; $(A \cap B)(x) = A(x) \wedge B(x)$;

(3.1.7) $(A * B)(x) = A(x) * B(x)$; $(A \to B)(x) = A(x) \to B(x)$

for each $x \in X$. Let $\{A_i\}_{i \in I} \subseteq \mathcal{F}(X)$; we define $(\bigcup_{i \in I} A_i(x))$, $(\bigcap_{i \in I} A_i(x))$ by

(3.1.8) $(\bigcup_{i \in I} A_i(x))(x) = \bigvee_{i \in I} A_i(x)$; $(\bigcap_{i \in I} A_i(x))(x) = \bigwedge_{i \in I} A_i(x)$

for any $x \in X$.

Remark 3.2 *By (3.1.6) and (3.1.7) we obtain four operations $\cup$, $\cap$, $*$ and $\to$ on $\mathcal{F}(X)$. Denote by $\mathbf{0}$ and $\mathbf{1}$ the constant functions $\mathbf{0}(x) = x$, $\mathbf{1}(x) = 1$ for each $x \in X$. Then $(\mathcal{F}(X), \bigcup, \bigcap, *, \to, \mathbf{0}, \mathbf{1})$ is a complete residuated lattice (see [11], Theorem 3.6, p. 80).*

A fuzzy subset A of X is *non-zero* if $A(x) \neq 0$ for some $x \in X$; A is *normal* if $A(x) = 1$ for some $x \in X$. The support of $A \in \mathcal{F}(X)$ is defined by *supp* $A = \{x \in X | A(x) > 0\}$.

If $A, B \in \mathcal{F}(X)$ then we denote

(3.1.9) $I(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$;

(3.1.10) $E(A, B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x))$.

$I(A, B)$ is called the *subsethood degree* of A in B and $E(A, B)$ the *degree of equality* of A and B. Intuitively $I(A, B)$ expresses the truth value of the statement "A is included in B." and $E(A, B)$ expresses the truth value of the statement "A and B contain the same elements."(see [11], p. 82 and p. 85).

Example 3.1 *([11], p. 83) For Lukasiewicz, Gödel and product t-norms we have*

$I(A, B) = \bigwedge\{1 - A(x) + B(x) | x \in X, A(x) > B(x)\}$ *(Lukasiewicz)*

$I(A, B) = \bigwedge\{B(x) | x \in X, A(x) > B(x)\}$ *(Gödel)*

$I(A, B) = \bigwedge\{B(x)/A(x) | x \in X, A(x) > B(x)\}$ *(product)*

Example 3.2 *([11], p. 85) For Lukasiewicz, Gödel and product t-norms we have*

$E(A, B) = \bigwedge\{1 - |A(x) - B(x)| | x \in X\}$ *(Lukasiewicz)*

$E(A, B) = \bigwedge\{A(x) \wedge B(x) | x \in X, A(x) \neq B(x)\}$ *(Gödel)*

$E(A, B) = \bigwedge\{A(x)/B(x) \wedge B(x)/A(x) | x \in X\}$ *(product)*

Lemma 3.4 *([11], p. 84-85) For any $A, B, C \in \mathcal{F}(X)$ we have*

(i) $I(A, A) = 1$; $E(A, A) = 1$;

(ii) $I(A, B) = 1$ iff $A \subseteq B$;

(iii) $E(A, B) = 1$ iff $A = B$;

(iv) $E(A,B) = E(B,A)$;

(v) $I(A,B) * I(B,C) \leq I(A,C)$;

(vi) $E(A,B) * E(B,C) \leq E(A,C)$.

Besides $I(.,.)$ there exist plenty of indicators expressing the inclusion of one set into another. A large class of such indicators was axiomatically developed by Sinha and Dougherty [53, 13].

Now we fix a continuous t-norm $*$.

A *fuzzy relation* on X is a fuzzy subset of X^2, i.e. a function $R : X^2 \to [0,1]$.

Fuzzy relations model vague preferences: if $x, y \in X$ are two alternatives then the real number $R(x,y)$ shows the degree to which x is preferred to y. In other words, $R(x,y)$ is the degree to which x is "at least as good as" alternative y.

The *asymmetric part* of R is the fuzzy relation P_R on X defined by $P_R(x,y) = R(x,y) * \neg R(y,x)$ for all $x, y \in X$ (here $\neg$ is the negation associated to t-norm $*$). The number $P_R(x,y)$ shows the degree to which alternative x is *strictly preferred* to alternative y.

If R, Q are two fuzzy relations on X then Q is an *extension* of R if $R \subseteq Q$.

The fuzzy relation R on X is said to be:

reflexive if $R(x,x) = 1$ for any $x \in X$;

irreflexive if $R(x,x) = 0$ for each $x \in X$;

$*$-antisymmetric if $R(x,y) * R(y,x) = 0$ for all distinct $x, y \in X$;

$*$-transitive if $R(x,y) * R(y,z) \leq R(x,z)$ for all $x, y, z \in X$;

total if $R(x,y) > 0$ or $R(y,x) > 0$ for all distinct $x, y \in X$;

strongly total if $R(x,y) = 1$ or $R(y,x) = 1$ for all distinct $x, y \in X$;

strongly complete if it is strongly total and reflexive.

If R, Q are two fuzzy relations on X then the product (composition) $R \circ Q$ is the fuzzy relation defined by

(3.1.11) $(R \circ Q)(x,y) = \bigvee_{z \in X} (R(x,z) * Q(z,y))$

for all $x, y \in X$. We shall use the notation $R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n-\text{times}}$.

The product $\circ$ is associative but not commutative.

Obviously R is $*$-transitive iff $R^2 \subseteq R$. Any intersection of $*$-transitive fuzzy relations is $*$-transitive. Thus the *$*$-transitive closure* $T(R)$ of a fuzzy relation R is defined to be the intersection of all $*$-transitive fuzzy relations on X including R. Of course R is $*$-transitive iff $T(R) = R$. Throughout this thesis "transitive" (resp. "transitive closure") means "$\wedge$-transitive" (resp. "$\wedge$-transitive closure").

The following lemma is well-known (see e.g. [16, 3]):

Lemma 3.5 *If R is a fuzzy relation on X then for all $x, y \in X$*

$$T(R)(x,y) = R(x,y) \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n \in X} (R(x,t_1) * \ldots * R(t_n,y)).$$

Remark 3.3 *In this section we have seen that any continuous t-norm defines a structure of complete residuated lattice on the set $\mathcal{F}(X)$ of fuzzy subsets of X. If we change the t-norm $*$ we obtain other operations on $\mathcal{F}(X)$, hence another structure of residuated lattice. In conclusion we can say that for any continuous t-norm there exists a "theory of fuzzy sets" in which we can reflect (more or less adequately) vague phenomena.*

3.2 Crisp choice functions

A choice problem resides in:

a) a universe X of alternatives under the control of an agent;

b) a non-empty family $\mathcal{B}$ of non-empty subsets of X; a member S of $\mathcal{B}$ represents "the set of available states that could possible be presented to the agent under the appropriate specification of the environmental conditions" [57], p. 20;

c) a mechanism by which the agent may choose from an available set S a subset $C(S)$; it is necessary to impose the condition that at least an element of S has to be chosen, therefore $C(S)$ will be always non-empty.

More often, the mechanism of choice has an underlying preference relation on X.

Let X be a non-empty universe of alternatives and $\mathcal{B}$ a non-empty subset of $\mathcal{P}(X) \setminus \{\emptyset\}$. A set $S \in \mathcal{B}$ can be taken as an *available* set of alternatives. The pair $\langle X, \mathcal{B} \rangle$ will be called a *choice space*. In the terminology of consumer theory the pair $\langle X, \mathcal{B} \rangle$ is called a *budget space*; the elements of X are called *bundles* and the sets of $\mathcal{B}$ are called *budgets*.

In this framework we can formulate a choice problem: given $S \in \mathcal{B}$ choose one or more elements in S; we can relate it to the notion of *choice function* or *consumer*.

A *choice function* or a *consumer* on $\langle X, \mathcal{B} \rangle$ is a function $C : \mathcal{B} \to \mathcal{P}(X)$ which to any $S \in \mathcal{B}$ assigns a non-empty subset $C(S)$ of S; $C(S)$ can be taken as the set of bundles or alternatives chosen subject to S; it will be called the *choice set* of S.

A significant part of choice function theory [2, 49, 48, 50] has been developed under the following hypothesis:

(H) $\mathcal{B}$ contains all non-empty finite subsets of X.

A *preference relation* Q on X is a binary relation Q on X; for $x, y \in X$, if $(x,y) \in Q$ then we say that the alternative x is preferred to the alternative y. A *regular preference* relation on X (or ordering in Sen's terminology [49]) is a preference relation on X which is reflexive, transitive and total.

To a preference relation Q we assign two relations P_Q and I_Q:

(3.2.1) $P_Q = \{(x, y) \in X^2 | (x, y) \in Q \text{ and } (y, x) \notin Q\}$

(3.2.2) $I_Q = \{(x, y) \in X^2 | (x, y) \in Q \text{ and } (y, x) \in Q\}$

P_Q (resp. I_Q) is called the *strict preference* relation (resp. the *indifference* relation) associated with Q. If $(x, y) \in P_Q$ then x is preferred to y and y is not preferred to x. In this case we say that alternative x is strictly preferred to alternative y. If $(x, y) \in I_Q$ then alternatives x and y are equally preferred.

The act of choice is connected to a preference relation. The choice functions are related to preference relations in two ways.

In the first case to a preference relation on the set of alternatives a choice function is assigned. The properties of the preference relation will influence the behaviour of the choice function.

The second case follows a converse route. To a choice function a preference relation (or more) is assigned. One investigates how this preference relation reflects the act of choice.

In the first case there exists a correspondence from preference relations to choice functions, while in the second case the correspondence is inverse.

A significant part of the theory of choice functions is devoted to those correspondences. The concept of rational choice function appears as a result of the first construction. A choice function is rational if it is derived from a preference relation Q and if the chosen alternatives verify a criterion of Q-optimality. [54] considers two expressions of Q-optimality: Q-maximality and Q-greatestness. The criteria of optimality give a first classification of rationality (in this thesis G-rationality and M-rationality are considered). The second way to classify rationality of the choice functions is according to the properties of the preference relation (transitivity, reflexivity etc.)

A special case of rationality is normality. A choice function C is normal if it is rationalizable by a preference relation canonically associated to C. Normality reconstructs the choice function from the associated preference relation.

If $S \in \mathcal{B}$ and Q is a preference relation on X then we define

(3.2.3) $M(S, Q) = \{x \in S | (y, x) \notin P_Q \text{ for all } y \in S\}$

(3.2.4) $G(S, Q) = \{x \in S | (x, y) \in Q \text{ for all } y \in S\}$

$M(S, Q)$ is the set of Q-*maximal* elements in S and $G(S, Q)$ is the set of Q-*greatest* elements in S.

For the interpretation of the notions of Q-maximality and Q-greatestness see e.g. [54]: An alternative x in an available set S is "said to be Q-maximal in S if there exists no y in S which is strictly preferred to x in terms of Q". An alternative x in an available set S is "said to be Q-greatest in S if, for all y in S, x is at least as preferable as y in terms of Q".

For a fixed preference relation Q we consider the functions

$M_Q : \mathcal{B} \to \mathcal{P}(X)$; $G_Q : \mathcal{B} \to \mathcal{P}(X)$

defined by $M_Q(S) = M(S, Q)$ and $G_Q(S) = G(S, Q)$ for any $S \in \mathcal{B}$. In general M_Q and G_Q are not choice functions; this is assured by natural conditions on Q.

Let $C : \mathcal{B} \to \mathcal{P}(X)$ be a choice function. Following [49] we shall present some preference relations generated by C:

(3.2.5) $R = \{(x,y) | x \in C(S) \text{ and } y \in S \text{ for some } S \in \mathcal{B}\}$

(3.2.6) $P = \{(x,y) | (x,y) \in R \text{ and } (y,x) \notin R\}$

(3.2.7) $I = I_R = \{(x,y) | (x,y) \in R \text{ and } (y,x) \in R\}$

(3.2.8) $\tilde{P} = \{(x,y) | x \in C(S) \text{ and } y \in S - C(S) \text{ for some } S \in \mathcal{B}\}$

(3.2.9) $\tilde{R} = \{(x,y) | (y,x) \notin \tilde{P}\}$

(3.2.10) $\tilde{I} = \{(x,y) | (x,y) \in \tilde{R} \text{ and } (y,x) \in \tilde{R}\}$

Assuming that hypothesis (H) holds we also define

(3.2.11) $\bar{R} = \{(x,y) | x \in C(\{x,y\})\}$

(3.2.12) $\bar{P} = P_{\bar{R}} = \{(x,y) | (x,y) \in \bar{R} \text{ and } (y,x) \notin \bar{R}\}$

(3.2.13) $\bar{I} = \{(x,y) | (x,y) \in \bar{R} \text{ and } (y,x) \in \bar{R}\}$

We shall denote by W (resp. P^*) the transitive closure of R (resp. $\tilde{P}$).

Relation R (called *revealed preference* in [52]) has been introduced by Samuelson in 1938 (see [44]); he associates R to a demand function. Samuelson's theory of revealed preference studies the rationality of a consumer in terms of the preference relation R. In an axiomatic context it has been resumed by Uzawa [60] and Arrow [2]. Its intuitive significance is obvious: x is weakly revealed preferred to y if x is chosen and y is available.

Relation $\tilde{P}$ (called strong revealed preference in [52]) has been defined by Arrow in [2]. It has the meaning: x is strongly revealed preferred to y if there exists an available set from which x is chosen and y is rejected.

Base relation $\bar{R}$ appears in [60] and [2] and corresponds to a concept of binary choice: x is preferred to y if x is chosen from the available set $\{x, y\}$.

In general revealed preference theory concentrates on how relations R and $\tilde{P}$ (P, I, $\tilde{R}$, $\tilde{I}$ as well) contribute to the description of the choice function. They are used in the formulation of the axioms of revealed preference. In addition, base relation $\bar{R}$ will have a crucial role in the results of Chapter 5.

The axioms of revealed preference $WARP$ and $SARP$ were introduced by Samuelson [44] and Houthakker [28], respectively. The form presented here is given by Arrow [2]:

$WARP$ (Weak Axiom of Revealed Preference)

If $(x,y) \in \tilde{P}$ then $(y,x) \notin R$.

$SARP$ (Strong Axiom of Revealed Preference)

If $(x,y) \in P^*$ then $(y,x) \notin R$.

The *strong congruence axiom* was introduced by Richter [41] and its weaker form by Sen [49]:

WCA (Weak Congruence Axiom)

For any $x, y \in X$ and $S \in \mathcal{B}$, if $x \in S, y \in C(S)$ and $(x,y) \in R$ then $x \in C(S)$.

SCA (Strong Congruence Axiom)

For any $x, y \in X$ and $S \in \mathcal{B}$, if $x \in S, y \in C(S)$ and $(x,y) \in W$ then $x \in C(S)$.

Following [54] a choice function $C : \mathcal{B} \to \mathcal{P}(X)$ is said to be *G-rational* (resp. *M-rational*) if there is a preference relation Q such that $C = G_Q$ (resp. $C = M_Q$); in this case we say that Q is a *G-rationalization* (resp. an *M-rationalization*) of C. C is *G-normal* (resp. *M-normal*) if $C = G_R$ (resp. $C = M_R$) where R is the preference relation defined by (3.2.5).

Sometimes we will denote $\hat{C} = G_R$ and we will say that $\hat{C}$ is "the image" of C.

Now we shall recall three results from [49].

Proposition 3.1 *Assume hypothesis (H) holds. Then WCA implies that R is a regular preference relation; if C is G-normal then the converse is also true.*

Proposition 3.2 *If hypothesis (H) holds then WCA is equivalent to $R = \tilde{R}$.*

The following Arrow-Sen theorem establishes the equivalence between WCA, SCA, $WARP$, $SARP$ and other conditions of rationality.

Theorem 3.1 *[49] (Arrow-Sen) Assume hypothesis (H) holds. For a choice function $C : \mathcal{B} \to \mathcal{P}(X)$ the following assertions are equivalent:*

(i) R is a regular preference and C is normal;

(ii) $\bar{R}$ is a regular preference and C is normal;

(iii) C satisfies WCA;

(iv) C satisfies SCA;

(v) C satisfies $WARP$;

(vi) C satisfies $SARP$;

(vii) $R = \tilde{R}$;

(viii) $\bar{R} = \tilde{R}$ and C is normal.

The previous theorem is stated in the form given in [49]; some implications or equivalences have been established by Arrow in [2].

Besides rationality and normality, a choice function can be related to consistency. Consistency conditions determine the choice by varying between subsets and supersets of available sets.

Now we shall recall the consistency conditions $\alpha, \beta, \gamma, \delta$ introduced by Sen in [49, 48, 50].

Let $C : \mathcal{B} \to \mathcal{P}(X)$ be a choice function.

Condition α . For any pair of sets $S, T \in \mathcal{B}$ and for any $x \in S$, if $x \in C(T)$ and $S \subseteq T$ then $x \in C(S)$.

That is, "if x is best in a set it is best in all subsets of it to which x belongs" [49].

Condition β . For any pair $S, T \in \mathcal{B}$ and for any $x, y \in C(S)$, if $S \subseteq T$ then $x \in C(T)$ iff $y \in C(T)$.

That is, "if x and y are both best in S, a subset of T, then x is best in T if and only if y is best in T" [49].

Condition γ . Let $\mathcal{M} \subseteq \mathcal{B}$, V the union of all sets in $\mathcal{M}$ ($V = \bigcup \mathcal{M}$) and $x \in X$. If $x \in C(S)$ for all $S \in \mathcal{M}$ then $x \in C(V)$.

That is, "if x is best in each set in a class of sets such that their union is V, then x must be best in V" [49].

Condition δ . For any pair of finite sets $S, T \in \mathcal{B}$ and for any (distinct) elements $x, y \in C(S)$, if $S \subseteq T$ then $C(T) \neq \{x\}$.

That is, "if x and y are both best in S, a subset of T, then neither of them can be uniquely best in T" [49].

Besides these properties, in [50] other consistency conditions are studied:

Condition $\alpha 2$. For any $S \in \mathcal{B}$ and for any $x \in S$, if $x \in C(S)$ then $x \in C([x, y])$ for all $y \in S$.

Condition $\gamma 2$. For any $S \in \mathcal{B}$ and for any $x \in S$, if $x \in C([x, y])$ for all $y \in S$ then $x \in C(S)$.

Condition $\beta(+)$. For any pair of sets $S, T \in \mathcal{B}$ such that $S \subseteq T$ and for any $x \in C(S)$ and $y \in S$, if $y \in C(T)$ then $x \in C(T)$.

Path Independence (PI) . For any pair of sets $S, T \in \mathcal{B}$, $C(S \cup T) = C(C(S) \cup C(T))$.

We recall from [49, 48, 50] some propositions on consistency conditions. Later they will be analyzed in the fuzzy framework.

Proposition 3.3 *If hypothesis (H) holds then a G-normal choice function satisfies condition* α.

Proposition 3.4 *Assume hypothesis (H) holds. Then a choice function satisfies WCA iff it satisfies conditions* α *and* β.

A preference relation Q is *quasi-transitive* iff the strict preference relation P_Q is transitive.

Proposition 3.5 *Assume hypothesis (H) holds. For a G-normal choice function the associated preference relation R is quasi-transitive iff condition δ is satisfied.*

We have seen that all the results above hold in the presence of hypothesis (H). In fact it suffices to assume that $\mathcal{B}$ contains the pairs and the triples of alternatives.

Investigation of properties of rationality, congruence and revealed preference axioms in the general case, without hypothesis (H) is a direction started by Richter [41] and continued by Hansson [27], Suzumura [54] and others.

We will recall first an important theorem of Richter [41].

Theorem 3.2 *Let $C : \mathcal{B} \to \mathcal{P}(X)$ be an arbitrary choice function. Then the following are equivalent:*

(i) C satisfies SCA;

(ii) There exists a regular preference relation Q on X such that Q is the G-rationalization of C.

A sequence $(S_1, \ldots, S_n)$ in $\mathcal{B}$ is called *C-connected* if $S_k \cap C(S_{k+1}) \neq \emptyset$ for all $k \in \{1, 2, \ldots, n\}$ where $S_{n+1} = S_1$.

In [27], Hansson proposed the following equivalent form of revealed preference axioms (denoted also $WARP$ and $SARP$):

$WARP$ For any C-connected pair (S_1, S_2) in $\mathcal{B}$, $S_1 \cap C(S_2) = C(S_1) \cap S_2$ holds.

$SARP$ For any C-connected sequence $(S_1, \ldots, S_n)$ in $\mathcal{B}$ we have

$S_k \cap C(S_{k+1}) = C(S_k) \cap S_{k+1}$ for some $k = 1, \ldots, n-1$.

Also in [27, 54] another revealed preference axiom appears:

$HARP$ (Hansson's Axiom of Revealed Preference) For any C-connected sequence $(S_1, \ldots, S_n)$ in $\mathcal{B}$

$S_k \cap C(S_{k+1}) = C(S_k) \cap S_{k+1}$ for all $k = 1, \ldots, n-1$.

These three axioms allowed to obtain some results in the general case.

Following [54] we shall consider the following revealed preference relation:

(3.2.14) $R_* = \{(x, y) | x \notin S \text{ or } x \in C(S) \text{ or } y \notin C(S) \text{ for all } S \in \mathcal{B}\}$.

In interpretation, "x is said to be R_*-revealed preferred to y if there exists no choice situation in which y is chosen and x is available but rejected"[54].

Theorem 3.3 *[54] Conditions $WARP$, WCA and $R \subseteq R_*$ are mutually equivalent.*[1]

Theorem 3.4 *[54] Conditions $HARP$ and SCA are equivalent.*

[1]In [54] R is denoted by R^*.

3.3 An extension theorem

A classical Szpilrajn theorem [59] asserts that any strict partial order is a subrelation of a strict total order. The first result in formulating a fuzzy version of Szpilrajn theorem was established by Zadeh in [66]. A fuzzy version of Szpilrajn theorem was also obtained by Gottwald in [24] for t-norms without zero divisors. Paper [8] contains an analysis of Szpilrajn theorem in fuzzy orderings context with respect to a t-norm $*$ and a $*$-equivalence E.

In particular Szpilrajn theorem is the main tool for proving a well-known theorem of Richter [41] that establishes the equivalence between rational and congruous consumers.

In this section we will formulate and prove a fuzzy version of Szpilrajn theorem, that will be later used in Section 6.3 for a fuzzy analysis of Richter theorem in consumer theory.

Let X be a non-empty set. A *preorder* on X is a reflexive and transitive relation on X. A relation R is *complete* if $R \cup R^{-1} = X \times X$, i.e. $(x, y) \in R$ or $(y, x) \in R$ for all $x, y \in X$. A *weak order* is a complete and transitive relation on X. Of course a weak order is a preorder. For any relation R, $T(R)$ will denote its transitive closure, i.e. the intersection of all transitive relations Q including R. The *asymmetric part* of R is the relation $P_R = R \setminus R^{-1} = \{(x, y)|(x, y) \in R \text{ and } (y, x) \notin R\}$. A relation R is *total* if for any $x \neq y$ we have $(x, y) \in R$ or $(y, x) \in R$. A relation R is complete if and only if it is reflexive and total.

A relation R on X is said to be a partial order if it is reflexive, anti-symmetric and transitive. R is a strict partial order if it is irreflexive and transitive.

Theorem 3.5 *(Szpilrajn [59]) Any strict partial order can be embedded in a strict total order on X.*

An important problem is to see to what extent a fuzzy version of Szpilrajn theorem holds true for an arbitrary continuous t-norm. Our extension theorem will be later used for the fuzzy analysis of Richter theorem in consumer theory.

Let R, Q be two fuzzy relations on X such that $R \subseteq Q$. We say that the extension Q of R *preserves the irreflexivity* of R if $Q(x, x) = 0$ for each $x \in X$ such that $R(x, x) = 0$. A fuzzy relation R on X is a strict partial $*$-order if it is irreflexive and $*$-transitive; a fuzzy relation is a total strict $*$-order if it is total, irreflexive and $*$-transitive.

In this section we shall prove that any $*$-transitive fuzzy relation R on X can be extended to a total $*$-transitive fuzzy relation Q on X preserving the irreflexivity of R. From this one infers that any strict partial $*$-order on X can be extended to a total strict $*$-order on X. This result can be viewed as a fuzzy version of Szpilrajn theorem [59].

Let $*$ be a continuous t-norm. If R is a fuzzy relation on X then $\bar{R}$ will be the fuzzy relation defined by

$$\bar{R}(a,b) = \begin{cases} 1 & \text{if} \quad a = b \\ R(a,b) & \text{if} \quad a \neq b \end{cases}$$

for any $a, b \in X$. It is obvious that $R \subseteq \bar{R}$.

Lemma 3.6 *If R is $*$-transitive then $\bar{R}$ is also $*$-transitive.*

Proof. Let $a, b, c \in X$. We shall prove that $\bar{R}(a,b) * \bar{R}(b,c) \leq \bar{R}(a,c)$. If $a = c$ then $\bar{R}(a,c) = 1$ and the inequality is trivially verified. Assume $a \neq c$. If $a \neq b, b \neq c$ then $\bar{R}(a,b) * \bar{R}(b,c) = R(a,b) * R(b,c) \leq R(a,c) = \bar{R}(a,c)$. Supposing $a = b, b \neq c$ we have $\bar{R}(a,b) * \bar{R}(b,c) = 1 * R(b,c) = R(b,c) = R(a,c) = \bar{R}(a,c)$; the case $a \neq b, b = c$ follows similarly. If $a = b = c$ then the inequality is obvious. ∎

Lemma 3.7 *Let R be a $*$-transitive fuzzy relation on X and p, q two distinct elements of X such that $R(p,q) = R(q,p) = 0$. Then there exists a $*$-transitive fuzzy relation R' on X fulfilling the following conditions: (i) $R \subseteq R'$; (ii) $R'(p,q) = 1$; (iii) R' preserves the irreflexivity of R.*

Proof. The fuzzy relation R' is defined by

$R'(a,b) = R(a,b) \vee [\bar{R}(a,p) * \bar{R}(q,b)]$

for all $a, b \in X$. It is obvious that $R \subseteq R'$ and $R'(p,q) = R(p,q) \vee [\bar{R}(p,p) * \bar{R}(q,q)] = R(p,q) \vee 1 = 1$.

In order to prove the $*$-transitivity of R' one remembers that

$R'(x,y) * R'(y,z) = [R(x,y) \vee (\bar{R}(x,p) * \bar{R}(q,y))] * [R(y,z) \vee (\bar{R}(y,p) * \bar{R}(q,z))] = [R(x,y) * R(y,z)] \vee [R(x,y) * \bar{R}(y,p) * \bar{R}(q,z)] \vee [\bar{R}(x,p) * \bar{R}(q,y) * \bar{R}(y,z)] \vee [\bar{R}(x,p) * \bar{R}(q,y) * \bar{R}(y,p) * \bar{R}(q,z)]$ and

$R'(x,z) = R(x,z) \vee [\bar{R}(x,p) * \bar{R}(q,z)]$.

The following inequalities follow from hypothesis and by Lemma 3.6:

$R(x,y) * R(y,z) \leq R(x,z) \leq R'(x,z)$;

$R(x,y) * \bar{R}(y,p) * \bar{R}(q,z) \leq \bar{R}(x,y) * \bar{R}(y,p) * \bar{R}(q,z) \leq \bar{R}(x,p) * \bar{R}(q,z) \leq R'(x,z)$;

$\bar{R}(x,p) * \bar{R}(q,y) * \bar{R}(y,z) \leq \bar{R}(x,p) * \bar{R}(q,z) \leq R'(x,z)$;

$\bar{R}(x,p) * \bar{R}(q,y) * \bar{R}(y,p) * \bar{R}(q,z) \leq \bar{R}(x,p) * \bar{R}(p,q) * \bar{R}(q,z) = 0$

since $\bar{R}(p,q) = R(p,q) = 0$.

Therefore $R'(x,y) * R'(y,z) \leq R'(x,z)$ hence R' is $*$-transitive.

Assume $R(x,x) = 0$, then, by Lemma 3.6 one gets

$R'(x,x) = R(x,x) \vee [\bar{R}(x,p) * \bar{R}(q,x)] = \bar{R}(x,p) * \bar{R}(q,x) = \bar{R}(q,x) * \bar{R}(x,p) \leq \bar{R}(q,p) = R(q,p) = 0.$ ∎

Corollary 3.1 *Let R be a strict partial $*$-order on X, $p,q \in X, p \neq q$ and $R(p,q) = R(q,p) = 0$. Then there exists a strict partial $*$-order R' such that $R \subseteq R'$ and $R'(p,q) = 1$.*

Theorem 3.6 *Let R be a $*$-transitive fuzzy relation on X. Then there exists a total $*$-transitive fuzzy relation R^* on X such that (i) $R \subseteq R^*$; (ii) R^* preserves the irreflexivity of R.*

Proof. Let $\mathcal{C}$ be the family of $*$-transitive fuzzy relations Q on X such that $R \subseteq Q$ and Q preserves the irreflexivity of R. Obviously $R \in \mathcal{C}$. $\mathcal{C}$ is partially ordered w.r.t. the inclusion $\subseteq$ of fuzzy relations. We shall prove that $(\mathcal{C}, \subseteq)$ is inductive. i.e. every totally ordered family $\{Q_i\}_{i \in I} \subseteq \mathcal{C}$ has an upper bound in $\mathcal{C}$. Let $\{Q_i\}_{i \in I}$ be a totally ordered family in $\mathcal{C}$, i.e. for all $i,j \in I$ we have $Q_i \subseteq Q_j$ or $Q_j \subseteq Q_i$. It suffices to prove that $Q = \bigcup\limits_{i \in I} Q_i$ belongs to $\mathcal{C}$.

Let $x,y,z \in X$. According to Lemma 3.2 (6) we have:

$$Q(x,y) * Q(y,z) = [\bigvee_{i \in I} Q_i(x,y)] * [\bigvee_{j \in I} Q_j(y,z)] = \bigvee_{i,j \in I} (Q_i(x,y) * Q_j(y,z)).$$

Let $i,j \in I$ so $Q_i \subseteq Q_j$ or $Q_j \subseteq Q_i$. Assume for example that $Q_i \subseteq Q_j$, hence $Q_i(x,y) * Q_j(y,z) \leq Q_j(x,y) * Q_j(y,z) \leq Q_j(x,z)$.

This inequality holds for any $i,j \in I$, hence $Q(x,y) * Q(y,z) \leq Q(x,z)$. Thus Q is $*$-transitive. If $R(x,x) = 0$ then $Q_i(x,x) = 0$ for any $i \in I$ hence $Q(x,x) = \bigvee\limits_{i \in I} Q_i(x,x) = 0$. It is obvious that $R \subseteq Q$ hence $Q \in \mathcal{C}$ and $(\mathcal{C}, \subseteq)$ is inductive. By Zorn's Lemma there exists a maximal member R^* in $\mathcal{C}$, i.e. $R^* \subseteq P$ and $P \in \mathcal{C}$ implies $R^* = P$. According to the definition of $\mathcal{C}$, R^* is $*$-transitive, $R \subseteq R^*$ and $R^*(x,x) = 0$ for each $x \in X$ such that $R(x,x) = 0$.

Now we shall prove that R^* is total, i.e. $R^*(x,y) > 0$ or $R^*(y,x) > 0$ for all distinct $x,y \in X$. By absurdum assume $R^*(p,q) = R^*(q,p) = 0$ for some distinct $p,q \in X$. By Lemma 3.7 there exists a $*$-transitive fuzzy relation R' on X such that $R^* \subseteq R', R'(p,q) = 1$ and R' preserves the irreflexivity of R^*. One remarks that R^* preserves the irreflexivity of R:

$R(x,x) = 0 \Rightarrow R^*(x,x) = 0 \Rightarrow R'(x,x) = 0.$

Hence $R^* \subsetneqq R' \in \mathcal{C}$. This contradicts the maximality of R^*, hence R^* is total. ∎

Corollary 3.2 *If R is a strict partial $*$-order on X then there exists a total strict $*$-order R^* on X such that $R \subseteq R^*$.*

This corollary is a fuzzy version of Szpilrajn theorem [59].

Chapter 4

Fuzzy Choice Functions

In this chapter we start the study of fuzzy choice functions.

Section 4.1 specifies the context in which we work. There exist plenty of contributions on fuzzy choice functions (see the references of [33]). Most of them have the premises that preferences are vague but the act of choice is exact (see [5, 6, 7]). In [4] Banerjee lifts this condition putting forth the idea of fuzzy choice functions. However Banerjee's approach is not complete because the range of his choice functions is made of fuzzy sets of alternatives.

In our definition of a fuzzy choice function, both the domain and the range consist of fuzzy sets. By identifying a crisp set with its characteristic function, our definition includes Banerjee's.

Section 4.1 contains the main fuzzy preference relations associated to a fuzzy choice function. We prove several lemmas that establish connections and properties of these fuzzy preference relations. We introduce congruence axioms $WFCA$, $SFCA$ (fuzzy versions of WCA, SCA) and revealed preference axioms $WAFRP$, $SAFRP$ (fuzzy versions of $WARP$, $SARP$). In their equivalent form [27, 54] $WARP$ and $SARP$ lead to other fuzzy revealed preference axioms $WAFRP^\circ$ and $SAFRP^\circ$. Hansson's revealed preference axiom $HARP$ leads to its fuzzy version $HAFRP$.

Section 4.2 is dedicated to M-rational and G-rational fuzzy choice functions, analogous to the classical notions (see Section 3.2). M-rationality and G-rationality are two ways to show how the behaviour of a fuzzy choice function is determined by fuzzy preference relations. When these fuzzy preference relations are canonically associated to the choice function we obtain the concepts of M-normality and G-normality. One result of this section (Proposition 4.2) establishes the equivalence between G-rationality and G-normality. Proposition 4.3 shows that in general $WAFRP^\circ$ implies M-normality.

The results of this chapter are based on our papers [20, 21].

4.1 Definitions and axioms

A vast literature has been dedicated to fuzzy choice functions. Most authors build their results on the thesis that social choice is governed by fuzzy preferences (hence modelled through fuzzy binary relations) but the act of choice is exact (hence choice functions are crisp) (see [5, 6, 7]). They study the crisp choice functions generated by fuzzy preference relations.

Let X be a set of alternatives and $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ a family of available sets. An agent wants to choose a non-empty set $C(S)$ from each available set $S \in \mathcal{B}$. Based on an expertise he has to find a way to evaluate how "an alternative x is preferred to an alternative y". Suppose n experts are consulted in saying without ambiguity that "alternative x is preferred to alternative y" (for any $y \in X$). Herewith each expert i produces a crisp preference relation R_i on X. The agent will combine the evaluations of the n experts. Following [38], for any $x, y \in X$ denote by $R(x, y)$ the fraction of the number of experts that preferred x to y in the total number n:

$R(x,y) = \frac{card\{i|(x,y) \in R_i\}}{n}$.

For the agent the real number $R(x, y) \in [0, 1]$ represents the *degree of preference* of x with respect to y. In this way we have obtained a fuzzy preference relation $R : X^2 \to [0, 1]$.

One assigns a crisp choice function to a preference relation built as above. The first choice function defined like this belongs to Orlovsky [38]. Recall the construction of the Orlovsky function:

Let X be a non-empty set, $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and R a fuzzy preference relation on X. For any $S \in \mathcal{B}$ and $x \in X$ denote

$OV(x, S, R) = \bigwedge_{y \in S} min(1 - R(y, x) + R(x, y), 1)$.

For each $S \in \mathcal{B}$ denote

$C(S, R) = \{x \in S | OV(x, S, R) \geq OV(y, S, R) \text{ for all } y \in S\}$.

The function $C(., R)$ defined by the mapping $S \mapsto C(S, R)$ is called Orlovsky function. Such fuzzy choice functions associated to fuzzy preference relations have been studied in the literature [16, 5, 6, 7, 33] etc.

In [43] to each fuzzy relation R on X there are associated four choice functions based on a t-norm. Such a choice function has the form: $C(X) : X \to [0, 1]$ where for any $y \in X$, $C(X)(y)$ is interpreted as "the degree to which the alternative y in X is the 'best' element in the set X". For example, if $X = \{x_1, \ldots, x_n\}$, $*$ is a t-norm and R is a fuzzy relation on X then the first of the four choice functions in [43] is defined by $C(X)(x_i) = D^R(x_i) = R(x_i, x_1) * \ldots * R(x_i, x_{i-1}) * R(x_i, x_{i+1}) * \ldots * R(x_i, x_n)$ for any $x_i \in X$. For Gödel t-norm one obtains the choice function studied by Svitalsky [58]. The real number $D^R(x_i)$ is called *the degree of domination* of x_i and can be interpreted as the truth value of the statement "$x_i R x_j$ for each $x_j \in X$" (see [43]).

In [4], Banerjee admits the vagueness of the act of choice and develops a theory of revealed preference for choice functions with a fuzzy behavior. We shall give a short description of Banerjee's paper.

Let X be a non-empty set of alternatives, $\mathcal{H}$ the family of all non-empty finite subsets of X and $\mathcal{F}$ the family of non-zero fuzzy subsets of X with finite support. A Banerjee fuzzy choice function is a function $C : \mathcal{H} \to \mathcal{F}$ such that *supp* $C(S) \subseteq S$ for any $S \in \mathcal{H}$. This notion has the following interpretation [4]:

"For all $S \in \mathcal{H}$, $C(S)(x)$ will be taken to represent the extent to which x belongs to the set of chosen alternatives when the available set of alternatives is S".

According to the previous definition the domain $\mathcal{H}$ of a Banerjee fuzzy choice function is the family of all non-empty finite subsets of X. Then any non-empty finite subset of X is an available set of alternatives. We have no information about the alternatives in S except that they can be chosen. We will change this hypothesis, varying the alternatives with their availability degree. The domain $\mathcal{B}$ of a fuzzy choice function is made of non-zero fuzzy subsets of X; if $S \in \mathcal{B}$ and $x \in X$ then $S(x)$ can be taken as the availability degree of alternative x.

Now we shall formalize this thesis.

Definition 4.1 *Let us consider a pair $\langle X, \mathcal{B} \rangle$ where X is a non-empty set and $\mathcal{B}$ is a non-empty family of non-zero fuzzy subsets of X. A* fuzzy choice function *(=fuzzy consumer) on $\langle X, \mathcal{B} \rangle$ is a function $C : \mathcal{B} \to \mathcal{F}(X)$ such that for each $S \in \mathcal{B}$, $C(S)$ is non-zero and $C(S) \subseteq S$.*

The same definition of a fuzzy choice function can be also found in [39].

In terms of fuzzy consumers, X is the set of *bundles* and $\mathcal{B}$ is the family of *fuzzy budgets* ; the pair $\langle X, \mathcal{B} \rangle$ is called *a fuzzy budget space* . Let $x \in X$ be a bundle and $S \in \mathcal{B}$ be a fuzzy budget; the real number $C(S)(x)$ can be interpreted as the degree to which the bundle x is chosen subject to the fuzzy budget S. The fuzzy set $S \in \mathcal{B}$ offers an availability degree $S(x)$ for each $x \in X$. The degree $C(S)(x)$ to which x is chosen subject to S naturally belongs to the interval $(0, S(x)]$. $C(S)(x) > 0$ expresses the fact that the possibility of choosing x may be taken into consideration.

Remark 4.1 *Let $\mathcal{H}$ be the family of all non-empty finite subsets of X. $\mathcal{H}$ can be identified with $\mathcal{B}_0 = \{\chi_K | K \in \mathcal{H}\}$ and $\langle X, \mathcal{B}_0 \rangle$ is a fuzzy budget space. A Banerjee fuzzy choice function $C : \mathcal{H} \to \mathcal{F}$ induces a fuzzy choice function $C' : \mathcal{B}_0 \to \mathcal{F}$ by putting $C'(\chi_K) = C(K)$ for each $K \in \mathcal{H}$. Thus C and C' can be identified hence our definition for fuzzy choice functions includes Banerjee's.*

Banerjee fuzzifies only the range of a choice function; in our approach both the domain and the range of a choice function are fuzzified. In this case the results on fuzzy choice functions have a much deeper meaning.

Although a part of fuzzy choice function theory will be developed in this general setting, in order to extend Uzawa-Arrow-Sen theory we need some natural hypotheses.

In this thesis we work with the following hypotheses:

$(H1)$ Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X;

(H_2) $\mathcal{B}$ includes all fuzzy sets $[x_1, \ldots, x_n]$, $n \geq 1$ and $x_1, \ldots, x_n \in X$.

For the crisp case ($\mathcal{B} \subseteq \mathcal{P}(X)$) hypothesis $(H1)$ asserts that any $S \in \mathcal{B}$ and $C(S)$ are non-empty, hence $(H1)$ is automatically fulfilled in accordance with the definition of choice function; for the same case, $(H2)$ asserts that $\mathcal{B}$ includes all non-empty finite subsets of X.

Now we fix a continuous t-norm $*$. The following definitions will be formulated in the fuzzy set theory associated to the t-norm $*$; we shall use the residuated structure $([0, 1], \vee, \wedge, *, \rightarrow, 0, 1)$.

The following three definitions introduce the main fuzzy revealed preference relations used in fuzzy choice function theory. They generalize to fuzzy sets the crisp revealed preference relations defined by (3.2.5)-(3.2.13).

Definition 4.2 *Let $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$. We define the following fuzzy revealed preference relations R, P, I on X by*

(i) $R(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y))$;

(ii) $P(x, y) = R(x, y) * \neg R(y, x)$;

(iii) $I(x, y) = R(x, y) * R(y, x)$

for any $x, y \in X$.

Definition 4.3 *Assume hypothesis $(H2)$ holds. Let C be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$. We define the following fuzzy revealed preference relations $\bar{R}, \bar{P}, \bar{I}$ on X by*

(i) $\bar{R}(x, y) = C([x, y])(x)$;

(ii) $\bar{P}(x, y) = \bar{R}(x, y) * \neg \bar{R}(y, x)$;

(iii) $\bar{I}(x, y) = \bar{R}(x, y) * \bar{R}(y, x)$

for any $x, y \in X$.

Definition 4.4 *Let C be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$. We define the following fuzzy revealed preference relations $\tilde{R}, \tilde{P}, \tilde{I}$ on X by*

(i) $\tilde{P}(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y))$;

(ii) $\tilde{R}(x, y) = \neg \tilde{P}(y, x)$;

(iii) $\tilde{I}(x, y) = \tilde{R}(x, y) * \tilde{R}(y, x)$

for any $x, y \in X$.

Preserving the results from the crisp case, we denote by W the transitive closure of R and by P^* the transitive closure of $\tilde{P}$.

An easy computation shows that the fuzzy revealed preference relations introduced by these definitions extend the corresponding crisp revealed preference relations. We kept the notations used for crisp choice functions in order to see how Uzawa-Arrow-Sen theory is generalized to fuzzy choice functions.

Remark 4.2 *Let $C : \mathcal{H} \to \mathcal{F}$ be a Banerjee fuzzy choice function and $C' : \mathcal{B}_0 \to \mathcal{F}$ the associated fuzzy choice function (cf. Remark 4.1). For any $K \in \mathcal{H}$ and $u \in X$ we have $C'(\chi_K)(u) \leq \chi_K(u)$ hence*

$u \notin K \Rightarrow C'(\chi_K)(u) = \chi_K(u) = 0.$

Assume $*$ is the Gödel t-norm $\wedge$. Thus the fuzzy revealed preference relation R associated with C' (cf. Definition 4.2 (i)) has the following form

$$R(x, y) = \bigvee_{K \in \mathcal{H}} (C'(\chi_K)(x) \wedge \chi_K(y))$$
$$= \bigvee \{C(K)(x) | K \in \mathcal{H}, x, y \in K\}.$$

In this case R coincides with the fuzzy revealed preference relation defined by Banerjee in [4] , Definition 2.5.

Remark 4.3 *Assume the conditions of Remark 4.2 are fulfilled but $*$ is the Lukasiewicz t-norm $*_L$. According to the definition of the Lukasiewicz t-norm, for any $a, b \in [0, 1]$ we have $a *_L \neg b = max\ (0, a - b)$.*

Let $\tilde{P}$ be the fuzzy revealed preference relation associated with C' according to Definition 4.4 (i). An easy computation shows that for any $x, y \in X$ we have

$$\tilde{P}(x, y) = \bigvee_{K \in \mathcal{H}} (C'(\chi_K)(x) *_L \chi_K(y) *_L \neg C'(\chi_K)(y))$$
$$= \bigvee \{max(0, C(K)(x) - C(K)(y)) | K \in \mathcal{H}, x, y \in K\}.$$

This shows that in this case $\tilde{P}$ coincides with the fuzzy relation defined in [4], Definition 2.6.

Remarks 4.2, 4.3 show that in [4] appear some fuzzy revealed preference relations corresponding to Gödel t-norm and some others corresponding to Lukasiewicz t-norm, resulting a framework that combines the reasonings and the two t-norms. In this thesis we adopt a different viewpoint, in the attempt to develop a theory of fuzzy choice functions for a fixed t-norm.

Definition 4.5 *Let C be a fuzzy choice function on $\langle X, \mathcal{B} \rangle$. We define the fuzzy preference relations R_* and R_1 on X by*

(i) $R_*(x, y) = \bigwedge_{S \in \mathcal{B}} [(S(x) * C(S)(y)) \to C(S)(x)];$

(ii) $R_1(x, y) = \bigwedge_{S \in \mathcal{B}} [\neg S(x) \vee C(S)(x) \vee \neg C(S)(y)]$

for any $x, y \in X$.

Remark 4.4 *By Lemma 3.1 (11), $R_*(x, y)$ can be written*

$$R_*(x, y) = \bigwedge_{S \in \mathcal{B}} [S(x) \to (C(S)(y) \to C(S)(x))].$$

An easy computation shows that both R_1 and R_* are fuzzy generalizations of the crisp relation defined by (3.2.14).

We observe that for each $x \in X$,

$$R_*(x, x) = \bigwedge_{S \in \mathcal{B}} [(S(x) * C(S)(x)) \to C(S)(x)] = 1.$$

Lemma 4.1 $R_1 \subseteq R_*$.

Proof. Let $x, y \in X$ and $S \in \mathcal{B}$. By Lemma 3.1 (9)

$[\neg S(x) \vee C(S)(x) \vee \neg C(S)(y)] * (S(x) * C(S)(y)) =$

$= [\neg S(x) * S(x) * C(S)(y)] \vee [C(S)(x) * S(x) * C(S)(y)] \vee$

$\vee [\neg C(S)(y) * S(x) * C(S)(y)] =$

$= C(S)(x) * S(x) * C(S)(y) \leq C(S)(x)$.

Therefore by Lemma 3.1 (1) we get

$\neg S(x) \vee C(S)(x) \vee \neg C(S)(y) \leq (S(x) * C(S)(y)) \to C(S)(x)$.

The last inequality holds for any $S \in \mathcal{B}$ hence by definitions of R_* and R_1, we obtain $R_1(x, y) \leq R_*(x, y)$. ∎

Example 4.1 *Assume $*$ is the Gödel t-norm $\wedge$.*

Let us take $X = \{x, y\}$ and $\mathcal{B} = \{A\}$ with $A = 0.2\chi\{x\} + \chi\{y\}$ and the fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$ defined by $C(A) = 0.1\chi\{x\} + \chi\{y\}$. In this case

$R_1(x, x) = \neg A(x) \vee C(A)(x) \vee \neg C(A)(x) = \neg 0.2 \vee 0.1 \vee \neg 0.1 = 0.1$. Since $R_*(x, x) = 1$ it follows that $R_1(x, x) \neq R_*(x, x)$, hence R_* and R_1 are distinct.

Lemma 4.2 $\tilde{P} \subseteq R$.

Proof. For any $x, y \in X$ we have:

$$\tilde{P}(x, y) = \bigvee_{S \in \mathcal{B}} [C(S)(x) * S(y) * \neg C(S)(y)] \leq \bigvee_{S \in \mathcal{B}} [C(S)(x) * S(y)] = R(x, y).$$ ∎

Lemma 4.3 *Assume hypotheses (H1), (H2) are fulfilled and $C : \mathcal{B} \to \mathcal{F}(X)$ is a fuzzy choice function.*

(i) $\bar{R} \subseteq R$;

(ii) R and $\bar{R}$ are reflexive and strongly total.

Proof. (i) By the definition of R and $\bar{R}$ for any $x, y \in X$ we have

$\bar{R}(x,y) = C([x,y])(x) = C([x,y])(x) * [x,y](y) \leq \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y)) = R(x,y)$.

(ii) For any $x \in X$, $C([x]) \subseteq [x]$ hence for each $y \in X$:

$C([x])(y) \leq [x](y) = \begin{cases} 1 & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x. \end{cases}$

Since $C([x])$ is a normal fuzzy set, $C([x])(x) = 1$. Hence $\bar{R}(x,x) = C([x])(x) = 1$ so $\bar{R}$ is reflexive.

Let x, y be two distinct alternatives. For any $z \in X$ we have

$C([x,y])(z) \leq [x,y](z) = \begin{cases} 1 & \text{if} \quad z \in \{x,y\} \\ 0 & \text{otherwise} \quad . \end{cases}$

Since $C([x,y])$ is a normal fuzzy set it follows that $\bar{R}(x,y) = C([x,y])(x) = 1$ or $\bar{R}(y,x) = C([x,y])(y) = 1$, hence $\bar{R}$ is strongly total.

Using $\bar{R} \subseteq R$ and $\bar{R}$ is reflexive and strongly total it follows immediately that R is reflexive and strongly total. ∎

Now we shall consider the following axioms of fuzzy revealed preference:

$WAFRP$ (Weak Axiom of Fuzzy Revealed Preference)
$\tilde{P}(x,y) \leq \neg R(y,x)$ for all $x, y \in X$;

$SAFRP$ (Strong Axiom of Fuzzy Revealed Preference)
$P^*(x,y) \leq \neg R(y,x)$ for all $x, y \in X$.

Now we shall state two axioms of congruence for fuzzy choice functions:

$WFCA$ (Weak Fuzzy Congruence Axiom)
For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds
$R(x,y) * C(S)(y) * S(x) \leq C(S)(x)$.

$SFCA$ (Strong Fuzzy Congruence Axiom)
For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds
$W(x,y) * C(S)(y) * S(x) \leq C(S)(x)$.

Remark 4.5 *Axioms $WAFRP$, $SAFRP$, $WFCA$, $SFCA$ are fuzzy versions of axioms $WARP$, $SARP$, WCA, SCA in crisp choice function theory. Fuzzy versions of the above axioms are found in [4] in the form imposed by the context.*

Remark 4.6 *Since $\tilde{P}(x,y) \leq P^*(x,y)$ and $R(x,y) \leq W(x,y)$ for any $x, y \in X$ the following implications hold true for any fuzzy choice function C: $SAFRP \Rightarrow WAFRP$; $SFCA \Rightarrow WFCA$.*

If $\tilde{P}$ (resp. R) is $*$-transitive then $\tilde{P} = P^*$ (resp. $R = W$), therefore in this case:

$SAFRP \Leftrightarrow WAFRP$ *(resp.* $SFCA \Leftrightarrow WFCA$*).*

The next two axioms $WAFRP^\circ$ and $SAFRP^\circ$ extend to fuzzy choice functions the form of $WARP$ and $SARP$ formulated in terms of C-connected sequences ([54]).

$WAFRP^\circ$ For any $x, y \in X$ and $S_1, S_2 \in \mathcal{B}$

(4.1.1) $[S_1(x) \wedge C(S_2)(x)] \wedge [S_2(y) \wedge C(S_1)(y)] \leq E(S_1 \cap C(S_2), S_2 \cap C(S_1))$.

$SAFRP^\circ$ For any $x_1, \ldots, x_n \in X$ and $S_1, \ldots, S_n \in \mathcal{B}$

(4.1.2) $\bigwedge_{k=1}^{n} [S_k(x_k) \wedge C(S_{k+1})(x_k)] \leq \bigvee_{j=1}^{n-1} E(S_j \cap C(S_{j+1}), S_{j+1} \cap C(S_j))$

where $S_{n+1} = S_1$.

Remark 4.7 *In the crisp case (4.1.1) says: if there exist an element* $x \in S_1 \cap C(S_2)$ *and an element* $y \in S_2 \cap C(S_1)$ *then* $S_1 \cap C(S_2) = S_2 \cap C(S_1)$*.*

This is exactly the axiom $WARP$ formulated in terms of C-connected sequences, hence $WAFRP^\circ$ is a generalization of $WARP$. Similarly, $SAFRP^\circ$ generalizes $SARP$.

We add the axiom $HAFRP$ as a fuzzy extension of $HARP$:

$HAFRP$ For any $x_1, \ldots, x_n \in X$ and $S_1, \ldots, S_n \in \mathcal{B}$:

$$\bigwedge_{k=1}^{n} [S_k(x_k) \wedge C(S_{k+1})(x_k)] \leq \bigwedge_{j=1}^{n-1} E(S_j \cap C(S_{j+1}), S_{j+1} \cap C(S_j))$$

where $S_{n+1} = S_1$.

It is clear that $HAFRP$ implies $SAFRP^\circ$.

4.2 Rational and normal fuzzy choice functions

In this section we study the concepts of M-rational and G-rational fuzzy choice functions extending some results proved by Suzumura for crisp choice functions [54].

Let Q be a fuzzy preference relation on the universal set of alternatives X and $\mathcal{B}$ a non-empty family of non-zero fuzzy subsets of X. For any $S \in \mathcal{B}$ let us define the fuzzy subsets $M(S, Q)$ and $G(S, Q)$ of X:

(4.2.1) $M(S,Q)(x) = S(x) * \bigwedge_{y \in X} [(S(y) * Q(y,x)) \to Q(x,y)]$

$$= S(x) * \bigwedge_{y \in X} [S(y) \to (Q(y,x) \to Q(x,y))];$$

(4.2.2) $G(S,Q)(x) = S(x) * \bigwedge_{y \in X} [S(y) \to Q(x,y)]$.

Remark 4.8 *Assume* Q *is a fuzzy preference relation and* $S = \chi_U$ *with* $U \in \mathcal{P}(X)$*. An easy computation shows that*

$$\textit{(4.2.3)}\ M(\chi_U, Q)(x) = \begin{cases} \bigwedge\limits_{y \in U} (Q(y,x) \to Q(x,y)) & \textit{if} \quad x \in U \\ 0 & \textit{if} \quad x \notin U. \end{cases}$$

$$\textit{(4.2.4)}\ G(\chi_U, Q)(x) = \begin{cases} \bigwedge\limits_{y \in U} Q(x,y) & \textit{if} \quad x \in U \\ 0 & \textit{if} \quad x \notin U. \end{cases}$$

If Q is a crisp relation on X ($Q : X^2 \to \{0,1\}$) then $Q(y,x) \to Q(x,y) = \neg P_Q(y,x)$ then by (4.2.3) and (4.2.4) we have

$M(\chi_U, Q)(x) = 1$ if and only if $\neg P_Q(y,x) = 1$ for all $y \in U$

$G(\chi_U, Q)(x) = 1$ if and only if $Q(x,y) = 1$ for all $y \in U$.

Therefore (4.2.1) and (4.2.2) generalize to fuzzy choice functions the Q-maximal point sets and the Q-greatest point sets (see (3.2.3) and (3.2.4)).

Definition 4.6 *A fuzzy choice function C on $\langle X, \mathcal{B} \rangle$ will be called G-rational (resp. M-rational) if there is a fuzzy preference relation Q on X such that*
$C(S) = G(S,Q)$ (resp. $C(S) = M(S,Q)$) for any $S \in \mathcal{B}$.

For the rest of this section assume that $*$ is the Gödel t-norm $\wedge$.

Lemma 4.4 *(i) $G(S,Q) \subseteq M(S,Q)$ for each $S \in \mathcal{B}$;*

(ii) If Q is strongly complete then $G(S,Q) = M(S,Q)$.

Proof. (i) Let $x \in X$. Then for each $y \in X$, $S(y) \wedge Q(y,x) \leq S(y)$ implies $S(y) \to Q(x,y) \leq (S(y) \wedge Q(y,x)) \to Q(x,y)$ (by Lemma 3.1 (10)); it follows immediately that $G(S,Q)(x) \leq M(S,Q)(x)$.

(ii) For any $x, y \in X$ we have $Q(x,y) = 1$ or $Q(y,x) = 1$. We notice that

$$S(y) \to (Q(y,x) \to Q(x,y)) = \begin{cases} 1 & \text{if} \quad Q(x,y) = 1 \\ S(y) \to Q(x,y) & \text{if} \quad Q(y,x) = 1 \end{cases}$$
$$= S(y) \to Q(x,y)$$

then $M(S,Q)(x) = G(Q,S)(x)$ for any $x \in X$. ∎

Remark 4.9 *Let $C : \mathcal{B} \to \mathcal{F}(X)$ be a G-rational fuzzy choice function and Q a fuzzy preference relation such that $C(S) = G(S,Q)$ for each $S \in \mathcal{B}$. By Lemma 4.4 (ii), if Q is strongly complete $M(S,Q) = G(S,Q)$ for each $S \in \mathcal{B}$; in this case we say that C is strongly complete rational.*

Lemma 4.5 *Any M-rational fuzzy choice function is G-rational.*

Proof. For any fuzzy relation Q on X let us define a fuzzy relation Q' by $Q'(x,y) = Q(y,x) \to Q(x,y)$ for any $x, y \in X$. Then for any $S \in \mathcal{B}$ and $x \in X$:

$$G(S,Q')(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \to Q'(x,y)]$$

$$= S(x) \wedge \bigwedge_{y \in X} [S(y) \to (Q(y,x) \to Q(x,y))]$$
$$= M(S,Q)(x).$$

Then $G(S,Q') = M(S,Q)$ for any $S \in \mathcal{B}$ and the lemma follows immediately. ∎

For any fuzzy choice function C on $\langle X, \mathcal{B} \rangle$ let us consider the associated revealed preference relation R (see Def. 4.2) and define the functions $G^* : \mathcal{B} \to \mathcal{F}(X)$ and $M^* : \mathcal{B} \to \mathcal{F}(X)$

$$G^*(S)(x) = G(S,R)(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \to R(x,y)];$$

$$M^*(S)(x) = M(S,R)(x) = S(x) \wedge \bigwedge_{y \in X} [(S(y) \wedge R(y,x)) \to R(x,y)]$$

for any $S \in \mathcal{B}$ and $x \in X$.

Lemma 4.6 *If C is a fuzzy choice function then $C(S) \subseteq G^*(S) \subseteq M^*(S)$ for any $S \in \mathcal{B}$.*

Proof. Let $S \in \mathcal{B}$ and $x \in X$. We must prove that $C(S)(x) \leq G^*(S)(x) \leq M^*(S)(x)$. By the definition of R, $C(S)(x) \wedge S(y) \leq R(x,y)$ for any $y \in X$. Then $C(S)(x) \leq S(y) \to R(x,y)$ for any $y \in X$, hence $C(S)(x) \leq \bigwedge_{y \in X} [S(y) \to R(x,y)]$.

Since $C(S)(x) \leq S(x)$ it follows that

$$C(S)(x) \leq S(x) \wedge \bigwedge_{y \in X} [S(y) \to R(x,y)] = G^*(S)(x).$$

The inequality $G^*(S)(x) \leq M^*(S)(x)$ follows by Lemma 4.4 (i). ∎

Definition 4.7 *A fuzzy choice function C is G-normal (resp. M-normal) if $C(S) = G^*(S)$ (resp. $C(S) = M^*(S)$) for any $S \in \mathcal{B}$.*

If C is G-normal (resp. M-normal) then it is G-rational (resp. M-rational).

Proposition 4.1 *Any M-normal fuzzy choice function C is G-normal.*

Proof. Let $S \in \mathcal{B}$. By Lemma 4.6, $C(S) \subseteq G^*(S) \subseteq M^*(S) = C(S)$ hence $C(S) = G^*(S)$. ∎

Proposition 4.2 *For a fuzzy choice function C the following are equivalent:*

(1) C is G-rational;

(2) C is G-normal.

Proof. (1) $\Rightarrow$ (2) Assume there is a fuzzy relation Q such that $C(S) = G(S,Q)$ for any $S \in \mathcal{B}$. For any $x, y \in X$:

$$R(x,y) = \bigvee_{S\in\mathcal{B}} [C(S)(x) \wedge S(y)]$$

$$= \bigvee_{S\in\mathcal{B}} [S(x) \wedge S(y) \wedge \bigwedge_{z\in X} (S(z) \to Q(x,z))].$$

For any $S \in \mathcal{B}$ we have:

$$S(x) \wedge S(y) \wedge \bigwedge_{z\in X} (S(z) \to Q(x,z)) \leq S(x) \wedge S(y) \wedge (S(y) \to Q(x,y)) =$$

$$= S(x) \wedge S(y) \wedge Q(x,y) \leq Q(x,y).$$

Thus $R(x,y) \leq Q(x,y)$. We have proved that $R \subseteq Q$. For any $x, y \in X$ we have $C(S)(x) \wedge S(y) \leq R(x,y)$, hence $C(S)(x) \leq S(y) \to R(x,y)$.

Hence $C(S)(x) \leq \bigwedge_{y\in X} [S(y) \to R(x,y)]$. Since $C(S)(x) \leq S(x)$ it follows that $C(S)(x) \leq S(x) \wedge \bigwedge_{y\in X} [S(y) \to R(x,y)]$.

Since $R(x,y) \leq Q(x,y)$ we have $S(y) \to R(x,y) \leq S(y) \to Q(x,y)$ for any $x, y \in X$, hence

$$S(x) \wedge \bigwedge_{y\in X} (S(y) \to R(x,y)) \leq S(x) \wedge \bigwedge_{y\in X} (S(y) \to Q(x,y)) = C(S)(x).$$

It follows that $C(S)(x) = S(x) \wedge \bigwedge_{y\in X} (S(y) \to R(x,y))$ for any $x \in X$, hence C is G-normal.

(2) $\Rightarrow$ (1) Obviously. ∎

Proposition 4.3 *Let C be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$ such that $C(S)$ is a normal fuzzy subset of X for any $S \in \mathcal{B}$. If C satisfies $WAFRP^\circ$ then C is M-normal.*

Proof. By Lemma 4.6 it suffices to prove that $M^*(S)(x) \leq C(S)(x)$ for any $S \in \mathcal{B}$ and $x \in X$. Let $y \in X$ such that $C(S)(y) = 1$; then $S(y) = 1$. By the definition of $M^*(S)$ and Lemma 3.2 (2) we have

$$M^*(S)(x) = S(x) \wedge \bigwedge_{z\in X} [(S(z) \wedge R(z,x)) \to R(x,z)] \leq$$

$$\leq S(x) \wedge [(S(y) \wedge R(y,x)) \to R(x,y)] =$$

$$= S(x) \wedge [R(y,x) \to R(x,y)] =$$

$$= S(x) \wedge [\bigvee_{B\in\mathcal{B}} (C(B)(y) \wedge B(x)) \to R(x,y)] =$$

$$= S(x) \wedge \bigwedge_{B\in\mathcal{B}} [(C(B)(y) \wedge B(x)) \to R(x,y)] \leq$$

$$\leq S(x) \wedge [(C(S)(y) \wedge S(x)) \to R(x,y)] =$$

$$= S(x) \wedge [S(x) \to R(x,y)] =$$

$$= S(x) \wedge R(x,y) =$$

$$= S(x) \wedge \bigvee_{T \in \mathcal{B}} [C(T)(x) \wedge T(y)] =$$
$$= \bigvee_{T \in \mathcal{B}} [S(x) \wedge C(T)(x) \wedge T(y)] =$$
$$= \bigvee_{T \in \mathcal{B}} [S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y)].$$

According to $WAFRP^\circ$, $S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y) \leq C(S)(x)$ for any $T \in \mathcal{B}$, hence $M^*(S)(x) \leq C(S)(x)$. ∎

Corollary 4.1 *Let C be a fuzzy choice function on $\langle X, \mathcal{B} \rangle$ such that $C(S)$ is a normal subset of X for any $S \in \mathcal{B}$. If C satisfies $WAFRP^\circ$ then C is G-rational as well as M-rational.*

Proof. By Propositions 4.1 and 4.3. ∎

Proposition 4.4 *Any M-normal fuzzy choice function C is strongly complete rational.*

Proof. For any $S \in \mathcal{B}$, $C(S) = M^*(S)$. Let us take the fuzzy relation R° on X defined by $R^\circ(x, y) = R(y, x) \to R(x, y)$ for any $x, y \in X$. If $R(y, x) \leq R(x, y)$ then $R^\circ(x, y) = R(y, x) \to R(x, y) = 1$; if $R(y, x) > R(x, y)$ then $R^\circ(y, x) = R(x, y) \to R(y, x) = 1$. Thus R° is strongly complete. By a similar argument used in the proof of Lemma 4.5 we have $M^*(S) = M(S, R) = G(S, R^\circ)$ and by Lemma 4.4 (ii), $G(S, R^\circ) = M(S, R^\circ)$. Then $C(S) = G(S, R^\circ) = M(S, R^\circ)$ for any $S \in \mathcal{B}$, hence C is strongly complete rational. ∎

Figure 4.1 summarizes the results of this paragraph.

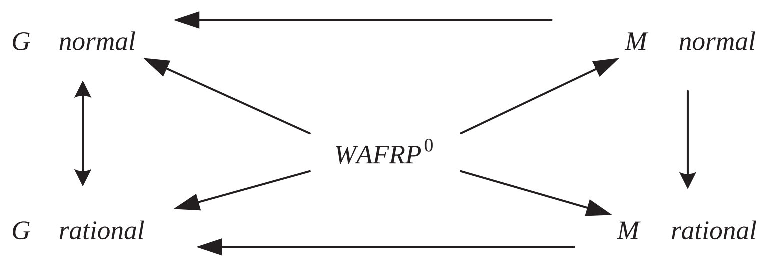

Figure 4.1: Properties of rationality and normality

Chapter 5

Fuzzy Revealed Preference and Consistency Conditions

In Chapter 4 we formulated the following axioms:

$WAFRP$ (Weak Axiom of Fuzzy Revealed Preference)

$SAFRP$ (Strong Axiom of Fuzzy Revealed Preference)

$WFCA$ (Weak Fuzzy Congruence Axiom)

$SFCA$ (Strong Fuzzy Congruence Axiom)

These are fuzzy versions of the revealed preference and congruence axioms ($WARP$, $SARP$, WCA, SCA) in classical consumer theory. We work in a fuzzy set theory corresponding to a continuous t-norm; thus the formulation of these axioms depends on the t-norm.

Chapter 5 studies various connections between these four axioms and properties of rationality and consistency. Generally we attempt to know to what extent classical consumer theory can be extended to our framework.

In our case the structure of residuated lattice offers a suitable setting.

The results of this chapter follow the line Uzawa-Arrow-Sen and will be obtained under hypotheses $(H1)$ and $(H2)$. We recall that under the hypothesis that the domain of the consumer contains the pairs and the triples of alternatives, Arrow-Sen theorem asserts the equivalence between $WARP$, $SARP$, WCA, SCA and other four conditions of rationality (see Theorem 3.1).

The first section of the chapter tries to answer the question to what degree Arrow-Sen theorem can be extended to fuzzy case. The equivalent properties of Arrow-Sen theorem lead to eight conditions for fuzzy choice functions. The main result of the section (Theorem 5.1) establishes equivalences or implications between these conditions. Some are true for an arbitrary continuous t-norm and others for Gödel or Lukasiewicz t-norm.

Section 5.2 deals with consistency conditions $F\alpha$, $F\beta$, the fuzzy forms of Sen's properties α, β [49]. The formulation of $F\alpha$ uses the subsethood function $I(.,.)$ and $F\beta$ is stated in terms of $\wedge$ and $\leftrightarrow$. We prove that a fuzzy choice function satisfies $F\alpha$, $F\beta$ if and only if $WFCA$ holds for C. We also study the fuzzy choice functions verifying a special hypothesis (U); in the crisp case, this hypothesis says that $C(S)$ is a singleton for each subset S. Under this hypothesis we prove that $WFCA$ and $F\alpha$ are equivalent.

Section 5.3 is concerned with condition $F\delta$, a fuzzy version of Sen's condition δ. The main result (Theorem 5.3) shows that for a normal choice function, the associated fuzzy preference relation R is quasi-transitive if and only if condition $F\delta$ holds.

In Section 5.4 we study some other consistency conditions $F\alpha 2$, $F\beta(+)$, $F\gamma 2$. We prove that a fuzzy choice function is normal if and only if it satisfies $F\alpha 2$ and $F\gamma 2$. Condition FPI is introduced as a fuzzy version of the path independence property PI [50]. The fuzzy preference relation $\bar{R}$ is quasi-transitive provided FPI holds.

In this chapter "rational" (resp. "normal") means G-rational (resp. G-normal).

Throughout this chapter assume hypotheses $(H1)$ and $(H2)$ hold.

The results of this chapter are based on our papers [20, 19].

5.1 A fuzzy approach to the Arrow-Sen theorem

In this section we will study various relations between the axioms of revealed preference and congruence.

Our aim is to investigate if Arrow-Sen Theorem (Theorem 3.1) is true for fuzzy choice functions. Some equivalences and implications of this theorem hold true in the fuzzy set theory based on a continuous t-norm, others for Gödel or Lukasiewicz t-norm. Particularly, for Gödel t-norm we prove that a fuzzy choice function C satisfies $SFCA$ if and only if the associated fuzzy relation R is a regular preference and C is normal. This shows that a strong form of Richter's Theorem [41] holds if hypotheses $(H1)$, $(H2)$ are fulfilled.

Let C be a fuzzy choice function. In this chapter we shall denote by $\hat{C}$ the fuzzy choice function G^* defined in Section 4.2. By Lemma 4.6 we have $C(S) \subseteq \hat{C}(S)$ for any $S \in \mathcal{B}$.

Proposition 5.1 *If the fuzzy choice function C satisfies $WFCA$ then R is a regular preference* [1] *on X.*

Proof. We shall prove the following assertions:

(a) R is reflexive.

(b) R is strongly total.

[1] We use 'regular preference' instead of 'regular fuzzy preference'.

These two assertions follow by Lemma 4.3.

(c) R is $*$-transitive.

Let $x, y, z \in X$. We shall prove that $R(x, y) * R(y, z) \leq R(x, z)$.

Denote $T = [x, y, x]$. Since

$C(T)(u) = \begin{cases} 1 & \text{if} \quad u \in \{x, y, z\} \\ 0 & \text{otherwise.} \end{cases}$

it follows that $C(T)(x) = 1$ or $C(T)(y) = 1$ or $C(T)(z) = 1$. We study separately these three cases.

(c_1) $C(T)(x) = 1$. Then

$$R(x, z) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(z)) \geq C(T)(x) * T(z) = 1.$$

Therefore $R(x, z) = 1$ and the inequality $R(x, y) * R(y, z) \leq R(x, z)$ is obviously satisfied.

(c_2) $C(T)(y) = 1$. By $WFCA$ we have

$R(x, y) * C(T)(y) * T(x) \leq C(T)(x)$.

Since $T(x) = C(T)(y) = 1$ we have $R(x, y) \leq C(T)(x)$ hence

$R(x, y) * R(y, z) \leq R(x, y) \leq C(T)(x) = C(T)(x) * T(z) \leq R(x, z)$.

(c_3) $C(T)(z) = 1$. By $WFCA$ we have

$R(y, z) = R(y, z) * C(T)(z) * T(y) \leq C(T)(y)$

$R(x, y) * C(T)(y) = R(x, y) * C(T)(y) * T(x) \leq C(T)(x)$

therefore

$R(x, y) * R(y, z) \leq R(x, y) * C(T)(y) \leq C(T)(x) * T(z) \leq R(x, z)$.

∎

Lemma 5.1 *Let C be a normal fuzzy choice function. Then $R = \bar{R}$.*

Proof. For any $x, y \in X$ we have

$\bar{R}(x, y) = C([x, y])(x) = \hat{C}([x, y])(x) = [x, y](x) * \bigwedge_{z \in X} ([x, y](z) \to R(x, z)) =$

$= ([x, y](x) \to R(x, x)) \wedge ([x, y](y) \to R(x, y)) = R(x, x) \wedge R(x, y) = R(x, y)$. ∎

Proposition 5.2 *Let C be a normal fuzzy choice function. If R is $*$-transitive then C satisfies $WFCA$.*

Proof. Let $S \in \mathcal{B}$ and $x, y \in X$. We must prove that $R(x, y) * C(S)(y) * S(x) \leq C(S)(x)$. Since

$C(S)(x) = \hat{C}(S)(x) = S(x) * \bigwedge_{z \in X} (S(z) \to R(x, z))$

it suffices to prove that
$R(x,y) * C(S)(y) \leq \bigwedge_{z \in X} (S(z) \to R(x,z))$.
Let $z \in X$. Since $C(S)(y) * S(z) \leq R(y,z)$ and R is $*$-transitive, then
$R(x,y) * C(S)(y) * S(z) \leq R(x,y) * R(y,z) \leq R(x,z)$.
Thus
$R(x,y) * C(S)(y) \leq S(z) \to R(x,z)$
for each $z \in X$, therefore the desired inequality follows. ∎

By Proposition 5.2, if C is a normal fuzzy choice function and R is a regular preference then C satisfies $WFCA$.

Proposition 5.3 *Assume that $*$ is the Lukasiewicz t-norm. If C is a fuzzy choice function such that $R = \tilde{R}$ then C satisfies $WFCA$.*

Proof. Suppose by absurdum that C does not satisfy $WFCA$ then there exist $S_0 \in \mathcal{B}$ and $x, y \in X$ such that

(a) $R(x,y) * C(S_0)(y) * S_0(x) \not\leq C(S_0)(x)$.
Since $R(x,y) \leq \tilde{R}(x,y) = \neg\tilde{P}(y,x)$ then we have

$$0 = R(x,y) * \tilde{P}(y,x) = R(x,y) * \bigvee_{S \in \mathcal{B}} (C(S)(y) * \neg C(S)(x) * S(x)) =$$
$$= \bigvee_{S \in \mathcal{B}} (R(x,y) * C(S)(y) * \neg C(S)(x) * S(x))$$

therefore $R(x,y) * C(S)(y) * S(x) * \neg C(S)(x) = 0$ for each $S \in \mathcal{B}$. But $\neg\neg C(S)(x) = C(S)(x)$ because $*$ is the Lukasiewicz t-norm, hence we get $R(x,y) * C(S)(y) * S(x) \leq C(S)(x)$, contradicting (a). ∎

Proposition 5.4 *Assume that $*$ is the Lukasiewicz t-norm. If C satisfies $WFCA$ then $R = \tilde{R}$.*

Proof. Using Lemma 3.2 (5) we get

$$R(x,y) * \tilde{P}(y,x) = R(x,y) * \bigvee_{S \in \mathcal{B}} (C(S)(y) * \neg C(S)(x) * S(x)) =$$
$$= \bigvee_{S \in \mathcal{B}} (R(x,y) * C(S)(y) * \neg C(S)(x) * S(x)).$$

By $WFCA$, $R(x,y) * C(S)(y) * S(x) \leq C(S)(x)$, hence, for any $S \in \mathcal{B}$ we have $R(x,y) * C(S)(y) * \neg C(S)(x) * S(x) = 0$.

It follows that $R(x,y) * \tilde{P}(y,x) = 0$. We have proved that $R(x,y) \leq \neg\tilde{P}(y,x) = \tilde{R}(x,y)$.

Now we shall prove the converse inequality $\tilde{R}(x,y) \leq R(x,y)$. First we shall establish the inequality

(a) $\tilde{R}(x,y) \leq C([x,y])(x)$.

Since

$C([x,y])(t) \leq [x,y](t) = \begin{cases} 1 & \text{if} \quad t \in \{x,y\} \\ 0 & \text{otherwise} \end{cases}$

and $C([x,y])$ is a normal fuzzy set we have $C([x,y])(x) = 1$ or $C([x,y])(y) = 1$. If $C([x,y])(x) = 1$ then (a) is obviously verified. Assume $C([x,y])(y) = 1$. Then

$$\tilde{R}(x,y) = \neg \tilde{P}(y,x) = \neg \bigvee_{S \in \mathcal{B}} (C(S)(y) * \neg C(S)(x) * S(x)) \leq$$
$$\leq \neg(C([x,y])(y) * \neg C([x,y])(x) * [x,y](x)) = \neg\neg C([x,y])(x) = C([x,y])(x)$$

and (a) is also verified. We remark that $C([x,y])(x) = C([x,y])(x) * [x,y](y) \leq R(x,y)$.

Thus, by (a), $\tilde{R}(x,y) \leq R(x,y)$. ∎

For a fuzzy choice function C let us consider the following statements

(i) R is a regular preference and C is normal;

(ii) $\bar{R}$ is a regular preference and C is normal;

(iii) C verifies $WFCA$;

(iv) C verifies $SFCA$;

(v) C verifies $WAFRP$;

(vi) C verifies $SAFRP$;

(vii) $R = \tilde{R}$;

(viii) $\bar{R} = \tilde{R}$ and C is normal.

We remark that the previous properties are the fuzzy versions of the equivalent conditions of Arrow-Sen theorem (see Theorem 3.1).

A natural problem is to obtain a fuzzy extension of Arrow-Sen theorem. The following theorem establishes some relations between the assertions (i)-(viii). This extends to fuzzy case a significant part of Arrow-Sen theorem.

Theorem 5.1 *Let C be a fuzzy choice function.*

(1) Conditions (i), (ii) are equivalent.

(2) Implication (i) $\Rightarrow$ (iii) holds; if $*$ is the Gödel t-norm $\wedge$ then the implication (iii) $\Rightarrow$ (i) also holds.

(3) Conditions (iii), (iv) are equivalent.

(4) If $*$ is the Lukasiewicz t-norm then conditions (iii),(v), (vi), (vii) are equivalent;

(5) Implication (viii) $\Rightarrow$ (vii) holds.

Proof. (1) By Lemma 5.1, $R = \bar{R}$, hence (i) $\Leftrightarrow$ (ii) holds.

(2) By Proposition 5.2, if C is normal and R is a regular preference then C satisfies $WFCA$, thus (i) $\Rightarrow$ (iii) holds. Now we shall prove that for the Gödel t-norm (iii) $\Rightarrow$ (i) is true. In accordance with Proposition 5.1, R is a regular preference. By Lemma 4.6, $C(S) \subseteq \hat{C}(S)$ for any $S \in \mathcal{B}$. Let $S \in \mathcal{B}$. We shall prove that $\hat{C}(S) \subseteq C(S)$, i.e. $\hat{C}(S)(x) \leq C(S)(x)$ for each $x \in X$. Since $C(S)$ is a normal fuzzy set $C(S)(z) = 1$ for some $z \in X$; we also have $S(z) = 1$. Thus

$$\hat{C}(S)(x) = S(x) \wedge \bigwedge_{y \in X} (S(y) \rightarrow R(x,y)) \leq S(x) \wedge (S(z) \rightarrow R(x,z)) =$$
$$S(x) \wedge R(x,z) = R(x,z) \wedge C(S)(z) \wedge S(x) \leq C(S)(x).$$

The last inequality follows according to $WFCA$. Therefore $\hat{C}(S) \subseteq C(S)$ so $C(S) = \hat{C}(S)$.

(3) By Remark 4.5, the implication (iv) $\Rightarrow$ (iii) is obvious. Since $WFCA$ implies the $*$-transitivity of R (see Proposition 5.1), it follows that R is equal to its transitive closure W. Thus $WFCA$ implies $SFCA$, hence the implication (iii) $\Rightarrow$ (iv) is proved.

(4) Assume that $*$ is the Lukasiewicz t-norm. First we will prove that (iii) $\Rightarrow$ (v) holds. Suppose by absurdum that C does not satisfy $WAFRP$, so $\tilde{P}(x,y) \not\leq \neg R(y,x)$ for some $x, y \in X$. Then, by Lemma 3.3 (1), $\tilde{P}(x,y) * R(y,x) > 0$. In accordance with Lemma 3.2 (5)

$$\tilde{P}(x,y) * R(y,x) = [\bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y))] * R(y,x) =$$
$$= \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y) * R(y,x))$$

therefore $C(S)(x) * S(y) * \neg C(S)(y) * R(y,x) > 0$ for some $S \in \mathcal{B}$. Since $\neg\neg C(S)(y) = C(S)(y)$, by Lemma 3.3 (1) we get $R(y,x) * C(S)(x) * S(y) \not\leq C(S)(y)$, contradicting $WFCA$.

In order to prove the converse implication (v) $\Rightarrow$ (iii) we assume by absurdum that C does not satisfy $WFCA$, i.e. there exist $S_0 \in \mathcal{B}$ and $x, y \in X$ such that $R(y,x) * C(S_0)(x) * S_0(y) \not\leq C(S_0)(y)$. By Lemma 3.3 (1) we obtain $R(y,x) * C(S_0)(x) * S_0(y) * \neg C(S_0)(y) > 0$. Thus

$$\tilde{P}(x,y) * R(y,x) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y) * R(y,x)) \geq$$
$$\geq C(S_0)(x) * S_0(y) * \neg C(S_0)(y) * R(y,x) > 0,$$

therefore, by Lemma 3.3 (1), $\tilde{P}(x,y) \not\leq \neg R(y,x)$, contradicting $WAFRP$.

The implication (vi) $\Rightarrow$ (v) is always true (see Remark 4.5). We shall establish the implication (v) $\Rightarrow$ (vi). If we assume (v), then (iii) holds, since we have proved that (iii) and (v) are equivalent. Thus C satisfies $WFCA$. By Proposition 5.1, R is a regular preference. Applying Proposition 5.4, we obtain $R = \tilde{R}$, hence $\tilde{R}$ is a regular preference. We shall prove that $\tilde{P}$ is $*$-transitive, i.e. $\tilde{P}(x,y) * \tilde{P}(y,z) \leq \tilde{P}(x,z)$. In accordance with the definition of $\tilde{P}$, this inequality can be written $\neg\tilde{R}(y,x) * \neg\tilde{R}(z,y) \leq \neg\tilde{R}(z,x)$. Since $*$ is the Lukasiewicz t-norm, by Lemma 3.3 (1), this last inequality is equivalent to

$\tilde{R}(z,x) * \neg\tilde{R}(y,x) * \neg\tilde{R}(z,y) = 0$.

Since $\tilde{R}$ is strongly total, we have $\tilde{R}(y,x) = 1$ or $\tilde{R}(x,y) = 1$. If $\tilde{R}(y,x) = 1$ then $\neg\tilde{R}(y,x) = 0$ and the previous inequality is obvious. Consider the case $\tilde{R}(x,y) = 1$. $\tilde{R}$ is $*$-transitive then $\tilde{R}(z,x) = \tilde{R}(z,x) * \tilde{R}(x,y) \leq \tilde{R}(z,y)$.

Therefore $\tilde{R}(z,x) * \neg\tilde{R}(y,x) * \neg\tilde{R}(z,y) \leq \tilde{R}(z,y) * \neg\tilde{R}(y,x) * \neg\tilde{R}(z,y) = 0$.

It follows that $\tilde{P}$ is $*$-transitive then $\tilde{P}$ is identical to its $*$-transitive closure P^*. Thus it is obvious that $WAFRP \Rightarrow SAFRP$.

(iii) $\Leftrightarrow$ (vii) By Propositions 5.3 and 5.4.

(5) If C is normal then $R = \bar{R}$ (by Lemma 5.1). Then $\bar{R} = \tilde{R}$ implies $R = \tilde{R}$. ∎

The equivalence (i) $\Leftrightarrow$ (iii) proved in Theorem 5.1 (2) for the case of Gödel t-norm can be viewed as a fuzzy extension of Richter theorem (assuming hypotheses $(H1)$, $(H2)$).

5.2 Conditions $F\alpha$ and $F\beta$

Conditions α and β were introduced by Sen [49] for crisp choice functions.

In this section we will consider fuzzy versions $F\alpha$ and $F\beta$ of these conditions and we will prove that a fuzzy choice function C satisfies $F\alpha$ and $F\beta$ if and only if C satisfies $WFCA$. We consider a class of fuzzy choice functions satisfying a new hypothesis (U). In the crisp case (U) expresses that $C(S)$ is a singleton for each fuzzy set S. Among results under hypothesis (U) there is the equivalence between $F\alpha$ and $WFCA$.

First we will recall the (crisp) conditions α and β.

Let $C : \mathcal{B} \to \mathcal{P}(X)$ be a crisp choice function.

Condition α . For any $S, T \in \mathcal{B}$ and for any $x \in X$, we have the implication

$x \in S, x \in C(T)$ and $S \subseteq T \Rightarrow x \in C(S)$.

Condition β . For any $S, T \in \mathcal{B}$ and for any $x, y \in X$, we have the implication
$x, y \in C(S)$ and $S \subseteq T \Rightarrow x \in C(T)$ if and only if $y \in C(T)$.

These two conditions can be extended to the fuzzy case.
Let $C : \mathcal{B} \to \mathcal{F}(X)$ be a fuzzy choice function on $\langle X, \mathcal{B} \rangle$.

Condition $F\alpha$. For any $S, T \in \mathcal{B}$ and $x \in X$ we have
$I(S,T) \wedge S(x) \wedge C(T)(x) \leq C(S)(x)$.

Condition $F\beta$. For any $S, T \in \mathcal{B}$ and $x, y \in X$ we have
$I(S,T) \wedge C(S)(x) \wedge C(S)(y) \leq C(T)(x) \leftrightarrow C(T)(y)$
where $\leftrightarrow$ is the biresiduum of the t-norm $\wedge$.

It is obvious that conditions $F\alpha$, $F\beta$ generalize α, β.
A weak form of conditions $F\alpha$, $F\beta$ can be given as:
Condition $F\alpha'$. For any $S, T \in \mathcal{B}$ and $x \in X$, if $S \subseteq T$ then
$S(x) \wedge C(T)(x) \leq C(S)(x)$.

Condition $F\beta'$. For any $S, T \in \mathcal{B}$ and $x, y \in X$, if $S \subseteq T$ then
$C(S)(x) \wedge C(S)(y) \leq C(T)(x) \leftrightarrow C(T)(y)$.
Since $S \subseteq T$ iff $I(S,T) = 1$ clearly $F\alpha \Rightarrow F\alpha'$, $F\beta \Rightarrow F\beta'$.
Although $F\alpha'$, $F\beta'$ are closer to α, β, the use of subsethood degree $I(A, B)$ in $F\alpha$ and $F\beta$ gives a better expression of the behaviour of fuzzy choice functions.

Proposition 5.5 *If C is a normal fuzzy choice function then $F\alpha$ is verified.*

Proof. Since $C = \hat{C}$ it suffices to show that for any $S, T \in \mathcal{B}$ and $x \in X$ the following inequality holds:
$I(S,T) \wedge S(x) \wedge \hat{C}(T)(x) \leq \hat{C}(S)(x)$.
By Lemma 3.1 (2) we have for any $u \in X$
$S(u) \wedge (S(u) \to T(u)) \wedge (T(u) \to R(x,u)) = S(u) \wedge T(u) \wedge (T(u) \to R(x,u)) = S(u) \wedge T(u) \wedge R(x,u) \leq R(x,u)$.

Using Lemma 3.1 (1) one infers
$(S(u) \to T(u)) \wedge (T(u) \to R(x,u)) \leq S(u) \to R(x,u)$.
Therefore

$I(S,T) \wedge S(x) \wedge \hat{C}(T)(x) =$
$= \bigwedge_{u \in X} (S(u) \to T(u)) \wedge S(x) \wedge T(x) \wedge \bigwedge_{u \in X} (T(u) \to R(x,u)) \leq$

$$\leq S(x) \wedge \bigwedge_{u \in X} [(S(u) \rightarrow T(u)) \wedge (T(u) \rightarrow R(x,u))] \leq$$
$$\leq S(x) \wedge \bigwedge_{u \in X} (S(u) \rightarrow R(x,u)) = \hat{C}(S)(x). \blacksquare$$

Proposition 5.6 *If the fuzzy choice function C fulfills condition $F\alpha$ then $R = \bar{R}$.*

Proof. By Lemma 4.3 (i) the inclusion $\bar{R} \subseteq R$ is always true. We prove that $R(x,y) \leq \bar{R}(x,y)$ for any $x, y \in X$. Let $S \in \mathcal{B}$. Then, by $F\alpha$

$I([x,y],S) \wedge [x,y](x) \wedge C(S)(x) \leq C([x,y])(x)$.

We remark that

$$I([x,y],S) = \bigwedge_{u \in X} ([x,y](u) \rightarrow S(u)) =$$
$$= ([x,y](x) \rightarrow S(x)) \wedge ([x,y](y) \rightarrow S(y)) =$$
$$= (1 \rightarrow S(x)) \wedge (1 \rightarrow S(y)) = S(x) \wedge S(y).$$

Thus the above inequality becomes

$S(x) \wedge S(y) \wedge C(S)(x) \leq C([x,y])(x)$.

Since $C(S)(x) \leq S(x)$ we have $C(S)(x) \wedge S(y) \leq C([x,y])(x)$ for each $S \in \mathcal{B}$ therefore

$$R(x,y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y)) \leq C([x,y])(x) = \bar{R}(x,y) \blacksquare$$

Proposition 5.7 *If a fuzzy choice function C satisfies $WFCA$ then conditions $F\alpha$, $F\beta$ are fulfilled.*

Proof. By Theorem 5.1, $WFCA$ implies the normality of C, hence, by Proposition 5.5, condition $F\alpha$ is verified. Assume by absurdum that C does not fulfill $F\beta$, hence there exist $S, T \in \mathcal{B}$ and $x, y \in X$ such that

$I(S,T) \wedge C(S)(x) \wedge C(S)(y) \not\leq C(T)(x) \leftrightarrow C(T)(y) =$

$= (C(T)(x) \rightarrow C(T)(y)) \wedge (C(T)(y) \rightarrow C(T)(x))$.

Therefore

$I(S,T) \wedge C(S)(x) \wedge C(S)(y) \not\leq C(T)(x) \rightarrow C(T)(y)$

or

$I(S,T) \wedge C(S)(x) \wedge C(S)(y) \not\leq C(T)(y) \rightarrow C(T)(x)$.

Assume the first case holds, hence, by Lemma 3.1 (1)

(a) $I(S,T) \wedge C(S)(x) \wedge C(S)(y) \wedge C(T)(x) \not\leq C(T)(y)$.

We remark that

$I(S,T) \wedge C(S)(x) \wedge C(S)(y) \wedge C(T)(x) \leq C(S)(y) \wedge S(x) \wedge C(T)(x) =$

$= C(S)(y) \wedge S(x) \wedge C(T)(x) \wedge S(y)$

because $C(S)(y) \wedge S(y) = C(S)(y)$. Since $C(S)(y) \wedge S(x) \leq R(y,x)$ one obtains

(b) $I(S,T) \wedge C(S)(x) \wedge C(S)(y) \wedge C(T)(x) \leq R(y,x) \wedge C(T)(x) \wedge S(y)$.

By (a) and (b) one infers

$R(y,x) \wedge C(T)(x) \wedge S(y) \not\leq C(T)(y)$

contradicting $WFCA$. ∎

Proposition 5.8 *If C fulfills conditions $F\alpha$, $F\beta$ then $WFCA$ holds.*

Proof. Let $S \in \mathcal{B}$ and $x, y \in X$. Since $I([x,y],S) = S(x) \wedge S(y)$ we have

$$S(x) \wedge C(S)(y) \wedge R(x,y) = S(x) \wedge S(y) \wedge C(S)(y) \wedge R(x,y) =$$
$$= I([x,y],S) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge R(x,y) =$$
$$= I([x,y],S) \wedge [x,y](y) \wedge C(S)(y) \wedge S(x) \wedge S(y) \wedge R(x,y) \leq$$
$$\leq C([x,y])(y) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge R(x,y)$$

because $I([x,y],S) \wedge [x,y](y) \wedge C(S)(y) \leq C([x,y])(y)$, by $F\alpha$.

Replacing $R(x,y)$ with its expression in Definition 4.2 we obtain

$$S(x) \wedge C(S)(y) \wedge R(x,y) \leq$$
$$\leq C([x,y])(y) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge \bigvee_{H\in\mathcal{B}} (C(H)(x) \wedge H(y)) =$$
$$= \bigvee_{H\in\mathcal{B}} [C([x,y])(y) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge C(H)(x) \wedge H(y)].$$

In accordance with $F\alpha$:

$C(H)(x) \wedge H(y) = C(H)(x) \wedge H(x) \wedge H(y) = I([x,y],H) \wedge [x,y](x) \wedge C(H)(x) \leq C([x,y])(x)$.

Therefore, by $F\beta$ and Lemma 3.1 (2):

$$C([x,y])(y) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge C(H)(x) \wedge H(y) \leq$$
$$\leq C([x,y])(y) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge C([x,y])(x) =$$
$$= I([x,y],S) \wedge C([x,y])(x) \wedge C([x,y])(y) \wedge C(S)(y) \leq$$
$$\leq [C(S)(x) \leftrightarrow C(S)(y)] \wedge C(S)(y) \leq$$
$$\leq C(S)(y) \wedge [C(S)(y) \to C(S)(x)] = C(S)(y) \wedge C(S)(x) \leq C(S)(x).$$

These inequalities hold for each $H \in \mathcal{B}$ therefore

$S(x) \wedge C(S)(y) \wedge R(x,y) \leq C(S)(x)$.

Hence the fuzzy choice function C verifies $WFCA$. ∎

Theorem 5.2 *For a fuzzy choice function C the following are equivalent:*

(1) C verifies conditions $F\alpha$, $F\beta$;

(2) $WFCA$ holds for C.

Proof. By Propositions 5.7 and 5.8. ∎

Remark 5.1 *The previous theorem generalizes to fuzzy choice functions a result of Sen (see [49], (T8)). This is one more argument that $F\alpha$, $F\beta$ are the appropriate versions of α, β and not conditions $F\alpha'$, $F\beta'$.*

In crisp consumer theory a special case is made by consumers $C : \mathcal{B} \to \mathcal{P}(X)$ with the property that $C(S)$ is a singleton for any $S \in \mathcal{B}$. We generalize this case considering fuzzy choice functions $C : \mathcal{B} \to \mathcal{F}(X)$ that verify:

(U) For any $S \in \mathcal{B}$, $C(S) = [x]$, for some $x \in X$.

For fuzzy choice functions C that verify (U) there is a unique $x \in X$ such that $C(S)(y) = \begin{cases} 1 & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x. \end{cases}$

Proposition 5.9 *If a normal fuzzy choice function C verifies (U) then R is a regular preference.*

Proof. Assume C is normal. By Lemma 4.3 we know that R is reflexive and strongly total; by Lemma 5.1, $R = \bar{R}$.

Let $x, y \in X$. Then

$R(x, y) = \bar{R}(x, y) = C([x, y])(x)$; $R(y, x) = C([x, y])(y)$.

But $C([x, y])$ is a normal fuzzy set so $C([x, y])(x) = 1$ or $C([x, y])(y) = 1$ because

$C([x, y])(t) \leq [x, y](t) = \begin{cases} 1 & \text{if} & t \in \{x, y\} \\ 0 & \text{otherwise.} \end{cases}$

Thus, by (U), we have exactly one of two cases:

$R(x, y) = 1$ and $R(y, x) = 0$.

$R(x, y) = 0$ and $R(y, x) = 1$.

Now we prove that R is $\wedge$-transitive. Let $x, y, z \in X$. We will establish the inequality $R(x, y) \wedge R(y, z) \leq R(x, z)$. The case $R(x, z) = 1$ is obvious. Consider $R(x, z) \neq 1$. Since $R(x, z) = C([x, z])(x)$ and C verifies (U), we must have $R(x, z) = 0$.

Assume by absurdum $R(x, y) \wedge R(y, z) \neq 0$ hence $R(x, y) = R(y, z) = 1$ and $R(y, x) = R(z, y) = 0$ because of (U). Therefore, since C is normal

$$C([x, y, z])(x) = [x, y, z](x) \wedge \bigwedge_{u \in X} ([x, y, z](u) \to R(x, u)) =$$
$$= R(x, x) \wedge R(x, y) \wedge R(x, z) = R(x, y) \wedge R(x, z) = 0$$

and similarly

$C([x, y, z])(y) = R(y, x) \wedge R(y, z) = 0$
$C([x, y, z])(z) = R(z, x) \wedge R(z, y) = 0.$
Since
$C([x, y, z])(u) \leq [x, y, z](u) = \begin{cases} 1 & \text{if} \quad u \in \{x, y, z\} \\ 0 & \text{otherwise} \end{cases}$
and $C([x, y, z])$ is a normal fuzzy set we have $C([x, y, z])(x) = 1$ or $C([x, y, z])(y) = 1$ or $C([x, y, z])(z) = 1$. Contradiction. Then R is transitive. ∎

Remark 5.2 *Assume C is a normal fuzzy choice function fulfilling (U). By the proof of Proposition 5.9, $R = \bar{R}$. Thus $P = \bar{P}$. We prove $\bar{P} = \bar{R}$.*
For any $x, y \in X$ we have
$\bar{P}(x, y) = \bar{R}(x, y) \wedge \neg\bar{R}(y, x) = C([x, y])(x) \wedge \neg C([x, y])(y)$
$\bar{R}(x, y) = C([x, y])(x).$
According to (U) we have one of the cases (see the proof of Proposition 5.9)
$C([x, y])(x) = 1, C([x, y])(y) = 0;$
$C([x, y])(x) = 0, C([x, y])(y) = 1.$
A simple computation shows that each of these cases leads to
$C([x, y])(x) = C([x, y])(x) \wedge \neg C([x, y])(y),$
so $\bar{R}(x, y) = \bar{P}(x, y)$. Hence $P = \bar{P} = \bar{R} = R$.

Proposition 5.10 *Let C be a fuzzy choice function satisfying (U). The following assertions are equivalent*

(1) $WFCA$ holds for C;

(2) C satisfies condition $F\alpha$.

Proof. (1) $\Rightarrow$ (2) By Theorem 5.1 $WFCA$ implies the normality of C. In accordance with Proposition 5.5 C satisfies $F\alpha$.
(2) $\Rightarrow$ (1) Let $x, y \in X$ and $S \in \mathcal{B}$. We prove that
(a) $R(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.
It is easy to see that $I([x, y], S) = S(x) \wedge S(y)$. Applying $F\alpha$ for $[x, y]$ and S we get
$C(S)(y) \wedge S(x) = S(x) \wedge S(y) \wedge C(S)(y) = I([x, y], S) \wedge [x, y](y) \wedge C(S)(y) \leq C([x, y])(y)$,
hence
$R(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(y) \wedge S(x) \leq C([x, y])(y)$.
Let $H \in \mathcal{B}$. Applying again $F\alpha$ for $[x, y]$ and H we get
$C(H)(x) \wedge H(y) = H(x) \wedge H(y) \wedge C(H)(x) = I([x, y], H) \wedge [x, y](x) \wedge C(H)(x) \leq C([x, y])(x)$
hence
$C(H)(x) \wedge H(y) \wedge C(S)(y) \wedge S(x) \leq C(H)(x) \wedge H(y) \leq C([x, y])(x)$.

Since these last inequalities hold for each $H \in \mathcal{B}$ we infer

$R(x,y) \wedge C(S)(y) \wedge S(x) = [\bigvee_{H\in\mathcal{B}} (C(H)(x) \wedge H(y))] \wedge C(S)(y) \wedge S(x) =$

$\bigvee_{H\in\mathcal{B}} [C(H)(x) \wedge H(y) \wedge C(S)(y) \wedge S(x)] \leq C([x,y])(x)$.

We conclude that

$R(x,y) \wedge C(S)(y) \wedge S(x) \leq C([x,y])(x) \wedge C([x,y])(y)$.

If $x \neq y$ then $C([x,y])(x) \wedge C([x,y])(y) = 0$ because of (U), then $R(x,y) \wedge C(S)(y) \wedge S(x) = 0 \leq C(S)(x)$. If $x = y$ then $R(x,y) \wedge C(S)(y) \wedge S(x) = C(S)(x) \wedge S(x) \leq C(S)(x)$. Hence the inequality (a) is proved. ∎

5.3 Quasi-transitivity and condition $F\delta$

This section deals with a fuzzy form $F\delta$ of Sen's condition δ. For a normal fuzzy choice function C we prove that the associated fuzzy preference relation R is quasi-transitive if and only if condition $F\delta$ holds.

Let Q be a fuzzy relation on X and P_Q be the fuzzy relation on X defined by $P_Q(x,y) = Q(x,y) \wedge \neg Q(y,x)$ for any $x, y \in X$. If R is the fuzzy revealed preference relation associated with a fuzzy choice function C (cf. Definition 4.2) then $P_R = P$.

We say that a fuzzy relation Q on X is *quasi-transitive* if

$P_Q(x,y) \wedge P_Q(y,z) \leq P_Q(x,z)$ for any $x, y, z \in X$.

Proposition 5.11 *Let Q be a reflexive and strongly total fuzzy relation on X. If Q is transitive then Q is quasi-transitive.*

Proof. By the definition of P_Q we have

$P_Q(x,y) \wedge P_Q(y,z) = Q(x,y) \wedge \neg Q(y,x) \wedge Q(y,z) \wedge \neg Q(z,y)$

for all $x, y, z \in X$. Hence $P_Q(x,y) \wedge P_Q(y,z) \leq Q(x,y) \wedge Q(y,z) \leq Q(x,z)$, Q being transitive. We remark that $Q(z,x) \wedge Q(x,y) \leq Q(z,y)$ hence

$Q(z,x) \wedge Q(x,y) \wedge \neg Q(z,y) \leq Q(z,y) \wedge \neg Q(z,y) = 0$

so $Q(z,x) \wedge Q(x,y) \wedge \neg Q(z,y) = 0$.

According to Lemma 3.3 (1), $Q(x,y) \wedge \neg Q(z,y) \leq \neg Q(z,x)$ hence

$P_Q(x,y) \wedge P_Q(y,z) \leq Q(x,y) \wedge \neg Q(z,y) \leq \neg Q(z,x)$.

Therefore $P_Q(x,y) \wedge P_Q(y,z) \leq Q(x,z) \wedge \neg Q(z,x) = P_Q(x,z)$, i.e. Q is quasi-transitive. ∎

Let $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ be a fuzzy choice function on $\langle X, \mathcal{B} \rangle$.

Definition 5.1 *We say that the fuzzy choice function C satisfies the condition $F\delta$ if for any $S = [a_1, \ldots, a_n]$, $T = [b_1, \ldots, b_m]$ in $\mathcal{B}$ and for any distinct $x, y \in X$ the following inequality holds*

$$I(S,T) \leq (C(S)(x) \wedge C(S)(y)) \rightarrow \neg(C(T)(x) \wedge \bigwedge_{t \neq x} \neg C(T)(t)).$$

Remark 5.3 *By Lemma 3.1 (1) the previous inequality is equivalent to*

$$I(S,T) \wedge C(S)(x) \wedge C(S)(y) \leq \neg(C(T)(x) \wedge \bigwedge_{t \neq x} \neg C(T)(t)).$$

An easy computation shows that in case of crisp choice functions condition $F\delta$ is exactly condition δ.

Proposition 5.12 *Assume the fuzzy choice function C is normal and satisfies condition $F\delta$. Then the associated fuzzy revealed preference relation R is quasi-transitive.*

Proof. We prove that for all $x, y, z \in X$ the following inequality holds

(a) $P(x,y) \wedge P(y,z) \leq P(x,z) = R(x,z) \wedge \neg R(z,x)$.

Recall that R is reflexive and strongly total in accordance with Lemma 4.3.

Since C is normal one gets

$C([x,y,z])(x) = \hat{C}([x,y,z])(x) = R(x,y) \wedge R(x,z)$;

$C([x,y,z])(y) = \hat{C}([x,y,z])(y) = R(y,x) \wedge R(y,z)$;

$C([x,y,z])(z) = \hat{C}([x,y,z])(z) = R(z,x) \wedge R(z,y)$.

By the definition of a fuzzy choice function, $C([x,y,z])(x) = 1$ or $C([x,y,z])(y) = 1$ or $C([x,y,z])(z) = 1$.

If $C([x,y,z])(y) = 1$ then $1 = R(y,x) \wedge R(y,z) \leq R(y,x), R(y,z)$ hence $R(y,x) = R(y,z) = 1$. One gets $P(x,y) = R(x,y) \wedge \neg R(y,x) = 0$ so (a) is trivially satisfied. Similarly, if $C([x,y,z])(z) = 1$ then $P(y,z) = 0$ and (a) is satisfied.

Let us consider the case $C([x,y,z])(x) = 1$ hence $R(x,y) = R(x,z) = 1$. Then (a) is equivalent to $P(x,y) \wedge P(y,z) \leq \neg R(z,x)$. By Lemma 3.3 (1), this last inequality is equivalent to

(b) $P(x,y) \wedge P(y,z) \wedge R(z,x) = 0$.

Since C is normal we have $C([x,z])(x) = R(x,z) = 1$ and $C([x,z])(z) = R(z,x)$. In accordance with condition $F\delta$ and Lemma 3.3 (7), (8) the following hold

$$
\begin{aligned}
R(z,x) &= R(x,z) \wedge R(z,x) \\
&= I([x,z],[x,y,z]) \wedge C([x,z])(x) \wedge C([x,z])(z) \\
&\leq \neg(C([x,y,z])(x) \wedge \bigwedge_{t\neq x} \neg C([x,y,z])(t)) \\
&= \neg(\bigwedge_{t\neq x} \neg C([x,y,z])(t)) \\
&= \neg(\neg C([x,y,z])(y) \wedge \neg C([x,y,z])(z)) \\
&= \neg(\neg(R(y,x) \wedge R(y,z)) \wedge \neg(R(z,x) \wedge R(z,y))) \\
&\leq \neg(\neg R(y,x) \wedge \neg R(z,y)) = \neg\neg R(y,x) \vee \neg\neg R(z,y)
\end{aligned}
$$

Thus one gets
$P(x,y) \wedge P(y,z) \wedge R(z,x) \leq P(x,y) \wedge P(y,z) \wedge (\neg\neg R(y,x) \vee \neg\neg R(z,y)) \leq$
$\leq \neg R(y,x) \wedge \neg R(z,y) \wedge (\neg\neg R(y,x) \vee \neg\neg R(z,y)) =$
$= (\neg R(y,x) \wedge \neg R(z,y) \wedge \neg\neg R(y,x)) \vee (\neg R(y,x) \wedge \neg R(z,y) \wedge \neg\neg R(z,y)) = 0$
and the inequality (b) was proved. ∎

Proposition 5.13 *If C is a normal fuzzy choice function and R is quasi-transitive then condition $F\delta$ holds.*

Proof. Suppose by absurdum that $F\delta$ does not hold, i.e. there exist $S = [a_1, \ldots, a_n]$, $T = [b_1, \ldots, b_m] \in \mathcal{B}$ and $x, y \in X$ such that

(a) $I(S,T) \wedge C(S)(x) \wedge C(S)(y) \not\leq \neg(C(T)(x) \wedge \bigwedge_{t\neq x} \neg C(T)(t))$.

We observe that

(b) $C(S)(y) \wedge C(S)(x) \leq C(S)(y) \wedge S(x) \leq R(y,x)$.

If $R(y,x) = 0$ then, by (b), $C(S)(x) \wedge C(S)(y) = 0$, contradicting (a); hence $R(y,x) > 0$. Thus $\neg R(y,x) = 0$ and $P(x,y) = R(x,y) \wedge \neg R(y,x) = 0$.

Since $S(t), T(t) \in \{0,1\}$ we have $S(t) \to T(t) \in \{0,1\}$ for each $t \in X$ hence $I(S,T) = \bigwedge_{t\in X} (S(t) \to T(t)) \in \{0,1\}$.

$I(S,T) = 0$ contradicts (a), hence $I(S,T) = 1$. Then $S \subseteq T$ so $\{a_1, \ldots, a_n\} \subseteq \{b_1, \ldots, b_m\}$. If $x \notin \{a_1, \ldots, a_n\}$ then $C(S)(x) \leq S(x) = 0$ contradicting (a); then $x \in \{a_1, \ldots, a_n\}$. Similarly, $y \in \{a_1, \ldots, a_n\}$.

By (a), $\neg C(T)(t) > 0$ for each $t \neq x$. Since

$$\neg C(T)(t) = \begin{cases} 1 & \text{if } C(T)(t) = 0 \\ 0 & \text{if } C(T)(t) > 0 \end{cases}$$

we must have $\neg C(T)(t) = 1$, hence $C(T)(t) = 0$ for each $t \neq x$.

But C is normal, so

$C(T)(y) = T(y) \wedge \bigwedge_{z\in X} (T(z) \to R(y,z))$.

Since $T(y) = 1$ and $T(z) \to R(y,z) = \begin{cases} R(y,z) & \text{if} \quad z \in \{b_1, \dots, b_m\} \\ 1 & \text{otherwise} \end{cases}$
it follows that $0 = C(T)(y) = R(y, b_1) \wedge \ldots \wedge R(y, b_m)$. Thus $R(y, z_1) = 0$ for some $z_1 \in \{b_1, \dots, b_m\}$. $R(y,x) > 0$ implies $z_1 \neq x$; of course $z_1 \neq y$. R being strongly total, $R(z_1, y) = 1$, therefore $P(z_1, y) = R(z_1, y) \wedge \neg R(y, z_1) = 1$.

Applying the same procedure with y instead of x and z_1 instead of y there exists $z_2 \in \{b_1, \dots, b_m\} \setminus \{y, z_1\}$ such that $P(z_2, z_1) = 1$. If $z_2 = x$ then according to the hypothesis of quasi-transitivity, $1 = P(x, z_1) \wedge P(z_1, y) \leq P(x, y)$. This contradicts $P(x,y) = 0$, so $z_2 \neq x$. In conclusion, $z_2 \in \{b_1, \dots, b_m\} \setminus \{x, y, z_1\}$.

By induction there exists a sequence $z_1, z_2, \dots, z_k, \dots$ of elements of $\{b_1, \dots, b_m\}$ such that $z_k \in \{b_1, \dots, b_m\} \setminus \{x, y, z_1, \dots, z_{k-1}\}$ for any $k = 1, 2, \dots$ (z_0 is taken y). Thus the terms of the sequence (z_k) will be distinct, contradicting the finitude of $\{b_1, \dots, b_m\}$. The contradiction shows that $F\delta$ is verified. ∎

Summing the two previous propositions we get

Theorem 5.3 *If C is a normal fuzzy choice function, then R is quasi-transitive if and only if condition $F\delta$ is verified.*

Remark 5.4 *Theorem 5.3 is the generalization for fuzzy choice functions of a Sen result (see [49], ($T10$)).*

5.4 Other consistency conditions

In [50] there exist other consistency conditions besides properties α, β, γ, δ. In this section we study conditions $F\alpha2$, $F\beta(+)$ and $F\gamma2$, fuzzy versions of conditions $\alpha2$, $\beta(+)$ and $\gamma2$ for crisp choice functions [50]. Then a fuzzy choice function is normal if and only if conditions $F\alpha2$ and $F\gamma2$ are satisfied. If a fuzzy choice function C verifies $F\beta(+)$ then the associated fuzzy preference relation R is transitive. We will prove that for a fuzzy choice function C, the path independence condition $C(S \cup T) = C(C(S) \cup C(T))$ implies the quasi-transitivity of $\bar{R}$.

Throughout this section C will denote a fuzzy choice function on $\langle X, \mathcal{B} \rangle$.

We introduce condition $F\alpha2$, a fuzzy form of the property $\alpha2$ in [50].

Condition $F\alpha2$
For any $S \in \mathcal{B}$ and for any $x, y \in X$, $C(S)(x) \wedge S(y) \leq C([x, y])(x)$.

The following proposition shows that condition $F\alpha2$ is obtained by relaxing condition $F\alpha$.

Proposition 5.14 *Condition $F\alpha$ implies condition $F\alpha2$.*

Proof. By condition $F\alpha$ we have
$I([x,y],S) \wedge C(S)(x) \wedge [x,y](x) \leq C([x,y])(x)$
for any $S \in \mathcal{B}$ and $x, y \in X$. Observing that
$I([x,y],S) \wedge C(S)(x) \wedge [x,y](x) = S(x) \wedge S(y) \wedge C(S)(x) = C(S)(x) \wedge S(y)$
one gets exactly condition $F\alpha 2$. ∎

Proposition 5.15 *The following statements are equivalent:*

(a) Condition $F\alpha 2$;

(b) For any $S \in \mathcal{B}$ and $x \in X$, $C(S)(x) \leq \bigwedge_{y \in X} (S(y) \to C([x,y])(x))$;

(c) $R = \bar{R}$.

Proof. (a) ⇔ (b) For any $S \in \mathcal{B}$ the following conditions are equivalent:

- for any $x, y \in X$, $C(S)(x) \wedge S(y) \leq C([x,y])(x)$;
- for any $x, y \in X$, $C(S)(x) \leq S(y) \to C([x,y])(x)$;
- for any $x \in X$, $C(S)(x) \leq \bigwedge_{y \in X} (S(y) \to C([x,y])(x))$.

Thus (a), (b) are equivalent.

(a) ⇔ (c) Since $\bar{R}(x,y) = C([x,y])(x)$ and $\bar{R} \subseteq R$ (cf. Lemma 4.3 (i)) the following properties are equivalent:

- Condition $F\alpha 2$;
- for any $S \in \mathcal{B}$ and $x, y \in X$, $C(S)(x) \wedge S(y) \leq \bar{R}(x,y)$;
- for any $x, y \in X$, $\bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y)) \leq \bar{R}(x,y)$;
- for any $x, y \in X$, $R(x,y) \leq \bar{R}(x,y)$;
- for any $x, y \in X$, $R(x,y) = \bar{R}(x,y)$.

In conclusion, condition $F\alpha 2$ and (c) are equivalent. ∎

Now let us consider the following property.

Condition $F\gamma 2$
For any $S \in \mathcal{B}$ and $x \in X$, $S(x) \wedge \bigwedge_{y \in X} (S(y) \to C([x,y])(x)) \leq C(S)(x)$.

Since for any $S \in \mathcal{B}$ and $x \in X$ we have

$$G(S,\bar{R})(x) = S(x) \wedge \bigwedge_{y\in X}(S(y) \to \bar{R}(x,y))$$
$$= S(x) \wedge \bigwedge_{y\in X}(S(y) \to C([x,y])(x,y))$$
condition $F\gamma2$ can be expressed

• For any $S \in \mathcal{B}$ and $x \in X$, $G(S,\bar{R})(x) \leq C(S)(x)$.

Proposition 5.16 *The following properties are equivalent:*

(a) C is normal;

(b) C satisfies conditions $F\alpha2$ and $F\gamma2$.

Proof. (a) $\Rightarrow$ (b) Since C is normal, by Lemma 5.1 $R = \bar{R}$ hence condition $F\alpha2$ is verified.

For each $S \in \mathcal{B}$ and $x, y \in X$ we have

$$\begin{aligned} S(x) \wedge \bigwedge_{y\in X}(S(y) \to C([x,y])(x)) &= S(x) \wedge \bigwedge_{y\in X}(S(y) \to \bar{R}(x,y)) \\ &= S(x) \wedge \bigwedge_{y\in X}(S(y) \to R(x,y)) \\ &= \hat{C}(S)(x) \\ &= C(S)(x). \end{aligned}$$

Then condition $F\gamma2$ is verified.

(b) $\Rightarrow$ (a) Since $R = \bar{R}$ (by Proposition 5.15) and $F\gamma2$ holds we get

$$\begin{aligned} \hat{C}(S)(x) &= S(x) \wedge \bigwedge_{y\in X}(S(y) \to R(x,y)) \\ &= S(x) \wedge \bigwedge_{y\in X}(S(y) \to \bar{R}(x,y)) \leq C(S)(x). \end{aligned}$$

The inequality $C(S)(x) \leq \hat{C}(S)(x)$ always holds, then $\hat{C}(S)(x) = C(S)(x)$ for each $S \in \mathcal{B}$ and $x \in X$. ∎

Now we consider a fuzzy version of the property $\beta(+)$ in [50].

Condition $F\beta(+)$

For any $S, T \in \mathcal{B}$ and $x, y \in X$, $I(S,T) \wedge C(S)(x) \wedge S(y) \leq C(T)(y) \to C(T)(x)$.

Proposition 5.17 *$F\beta(+)$ implies $F\beta$.*

Proof. For any $S, T \in \mathcal{B}$ and $x, y \in X$

$$\begin{aligned} I(S,T) \wedge C(S)(x) \wedge C(S)(y) &\leq I(S,T) \wedge C(S)(x) \wedge S(y) \\ &\leq C(T)(y) \to C(T)(x) \end{aligned}$$

and similarly, $I(S,T) \wedge C(S)(x) \wedge C(S)(y) \leq C(T)(x) \to C(T)(y)$.

Then condition $F\beta$ follows immediately. ∎

Proposition 5.18 *If $F\beta(+)$ holds then for any $S, T \in \mathcal{B}$ and $x \in X$ we have $C(S)(x) \wedge C(T)(x) \leq C(S \cup T)(x)$.*

Proof. Let $y \in X$ such that $C(S \cup T)(y) = 1$; then $S(y) \vee T(y) = 1$ hence $S(y) = 1$ or $T(y) = 1$. Assume $S(y) = 1$. Then, by $F\beta(+)$:

$C(S)(x) = I(S, S \cup T) \wedge C(S)(x) \wedge S(y) \leq C(S \cup T)(y) \to C(S \cup T)(x) = 1 \to C(S \cup T)(x) = C(S \cup T)(x)$.

Similarly, if $T(y) = 1$, then $C(T)(x) \leq C(S \cup T)(x)$; therefore $C(S)(x) \wedge C(T)(x) \leq C(S \cup T)(x)$. ∎

Proposition 5.19 *If C satisfies $F\beta(+)$ then R is transitive.*

Proof. Let $S, T \in \mathcal{B}$ and $x, y, z \in X$. Take $w \in X$ such that $C(S \cup T)(w) = 1$; since $C(S \cup T)(w) \leq S(w) \vee T(w)$ we have $S(w) = 1$ or $T(w) = 1$. Assume $T(w) = 1$. Exactly as in the proof of Proposition 5.18 one gets

(a) $C(T)(y) \leq C(S \cup T)(y)$.

Applying $F\beta(+)$ it follows that

$$\begin{aligned} C(S)(x) \wedge S(y) &= I(S, S \cup T) \wedge C(S)(x) \wedge S(y) \\ &\leq C(S \cup T)(y) \to C(S \cup T)(x) \end{aligned}$$

hence by Lemma 3.1 (1)

(b) $C(S)(x) \wedge S(y) \wedge C(S \cup T)(y) \leq C(S \cup T)(x)$.

In accordance with (a) and (b) the following inequalities hold:

$$\begin{aligned} C(S)(x) \wedge S(y) \wedge C(T)(y) \wedge T(z) &\leq C(S)(x) \wedge S(y) \wedge C(S \cup T)(y) \wedge T(z) \\ &\leq C(S \cup T)(x) \wedge T(z) \\ &\leq C(S \cup T)(x) \wedge (S \cup T)(z) \\ &\leq R(x, z). \end{aligned}$$

These inequalities are true for all $S, T \in \mathcal{B}$, hence

$R(x,y) \wedge R(y,z) = [\bigvee_{S\in\mathcal{B}} (C(S)(x) \wedge S(y))] \wedge [\bigvee_{T\in\mathcal{B}} (C(T)(y) \wedge T(z))]=$
$= \bigvee_{S,T\in\mathcal{B}} (C(S)(x) \wedge S(y) \wedge C(T)(y) \wedge T(z)) \leq R(x,z)$. ∎

We say that the fuzzy choice function C is *path independent* if for any $S, T \in \mathcal{B}$ the following equality holds

(FPI) $C(S \cup T) = C(C(S) \cup C(T))$.

Condition FPI extends to fuzzy setting the path independence property PI for crisp choice functions (see [50], p. 68).

Proposition 5.20 *If FPI holds then the fuzzy relation $\bar{R}$ is quasi-transitive.*

Proof. Let $x, y, z \in X$. We must prove that $\bar{P}(x,y) \wedge \bar{P}(y,z) \leq \bar{P}(x,z)$, i.e.

(a) $\bar{R}(x,y) \wedge \neg\bar{R}(y,x) \wedge \bar{R}(y,z) \wedge \neg\bar{R}(z,y) \leq \bar{R}(x,z) \wedge \neg\bar{R}(z,x)$.

If $\bar{R}(y,x) > 0$ or $\bar{R}(z,y) > 0$ then $\neg\bar{R}(y,x) = 0$ or $\neg\bar{R}(z,y) = 0$ so (a) is trivially verified. Assume $\bar{R}(y,x) = \bar{R}(z,y) = 0$ then, since $\bar{R}$ is strongly total, $\bar{R}(x,y) = \bar{R}(y,z) = 1$. Thus

$C([x,y])(x) = \bar{R}(x,y) = 1$; $C([x,y])(y) = \bar{R}(y,x) = 0$

hence $C([x,y]) = [x]$. Similarly, $C([y,z]) = [y]$. According to FPI we can write

$$\begin{aligned} C([x,y,z]) = C([x,y] \cup [y,z]) &= C(C([x,y]) \cup C([y,z])) \\ &= C([x] \cup [y]) = C([x,y]) = [x]; \end{aligned}$$

$$\begin{aligned} C([x,y,z]) = C([x,y] \cup [z]) &= C(C([x,y]) \cup C([z])) \\ &= C([x] \cup [z]) = C([x,z]) \end{aligned}$$

hence $C([x,z]) = [x]$. This yields

$\bar{R}(x,z) = C([x,z])(x) = [x](x) = 1$

$\bar{R}(z,x) = C([x,z])(z) = [x](z) = 0$.

Then $\bar{R}(x,z) \wedge \neg\bar{R}(z,x) = 1$ and the inequality (a) holds. ∎

Proposition 5.21 *If $F\alpha$ holds then $C(S \cup T) \leq C(C(S) \cup C(T))$ for all $S, T \in \mathcal{B}$.*

Proof. Let $S, T \in \mathcal{B}$ and $x \in X$. First we prove that

(a) $C(S \cup T)(x) \leq C(S)(x) \vee C(T)(x)$.

Assume $T(x) \leq S(x)$ then $C(S \cup T)(x) \leq S(x) \vee T(x) = S(x)$. By $F\alpha$ we get

$C(S \cup T)(x) = I(S, S \cup T) \wedge C(S \cup T)(x) \wedge S(x) \leq C(S)(x) \leq C(S)(x) \vee C(T)(x)$.

If $S(x) \leq T(x)$ then (a) follows similarly. We apply again $F\alpha$:

$$\begin{aligned} C(S \cup T)(x) &= C(S \cup T)(x) \wedge (C(S) \cup C(T))(x) \\ &= I(C(S) \cup C(T), S \cup T) \wedge C(S \cup T)(x) \wedge (C(S) \cup C(T))(x) \\ &\leq C(C(S) \cup C(T))(x) \end{aligned}$$

Thus $C(S \cup T) \leq C(C(S) \cup C(T))$. ∎

Remark 5.5 *In case of crisp choice functions the converse of Proposition 5.21 also holds (see [50], Proposition 17). An open problem is whether the converse of Proposition 5.21 holds true.*

5.5 An example

As seen above, Theorem 5.1 establishes some connections between conditions (i)-(viii) in Section 5.1.

The example presented in this section is constructed with the aim to clarify these connections and the limitations of Theorem 5.1.

Let $X = \{x, y\}$ and $\mathcal{B} = \{[x], [y], [x, y], A\}$ where $A \in \mathcal{F}(X)$ is defined by $A = 0.3\chi\{x\} + \chi\{y\}$.

Consider function $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ defined by

$C([x]) = \chi\{x\}; C([y]) = \chi\{y\}; C([x, y]) = 0.25\chi\{x\} + \chi\{y\}; C(A) = 0.25\chi\{x\} + \chi\{y\}$.

C is a fuzzy choice function fulfilling $(H1)$ and $(H2)$. We determine first the fuzzy relation R associated to C. According to Definition 4.2 (i)

$R(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y))=$

$= (C([x])(x) * [x](y)) \vee (C([y])(x) * [y](y)) \vee (C([x, y])(x) * [x, y](y)) \vee (C(A)(x) * A(y)) =$

$= 1 * 0 \vee 0 * 1 \vee 0.25 * 1 \vee 0.25 * 1 = 0.25$.

Analogously $R(x, x) = R(y, y) = R(y, x) = 1$, thus

$$R = \begin{pmatrix} 1 & 0.25 \\ 1 & 1 \end{pmatrix}$$

It is clear that R is $*$-transitive, reflexive and strongly total.

We check now if the fuzzy choice function C verifies $WFCA$. For all $a, b \in X$ and $S \in \mathcal{B}$ we must have the inequality

(a) $R(a, b) * C(S)(b) * S(a) \leq C(S)(a)$.

For $a = b$ this inequality is always true; consider only the cases $a = x, b = y$ and $a = y, b = x$.

Case $a = x, b = y$. Since $R(x, y) = 0.25$ we have to prove that for any $S \in \mathcal{B}$ the following inequality holds:

(b) $0.25 * C(S)(y) * S(x) \leq C(S)(x)$.

$-S = [x]$. We have $C(S)(y) = C([x])(y) = 0$, hence (b) is verified.

$-S = [y]$. We have $0.25 * C([y])(y) * [y](x) = 0$, hence (b) is verified.

$-S = [x, y]$. We have $0.25 * C([x, y])(y) * [x, y](x) = 0.25 = C([x, y])(x)$.

$-S = A$. We have $0.25 * C(A)(y) * A(x) = 0.25 * 1 * 0.3 \leq C(A)(x) = 0.25$.

The case $a = y, b = x$ follows similarly.

In conclusion C verifies $WFCA$. The fuzzy relation R being $*$-transitive, $R = W$ then C satisfies also $SFCA$.

We check now if condition (i) is verified for C. R is a regular preference, then we investigate if C is normal:

$\hat{C}([x])(x) = 1 = C([x])(x)$

$\hat{C}([x])(y) = 0 = C([x])(y)$

$\hat{C}([y])(x) = 0 = C([y])(x)$

$\hat{C}([y])(y) = 1 = C([y])(y)$

$\hat{C}([x, y])(x) = [x, y](x) * [([x, y](x) \to R(x, x)) \wedge ([x, y](y) \to R(x, y))] = R(x, y) = 0.25 = C([x, y])(x)$

$\hat{C}([x, y])(y) = R(y, x) = 1 = C([x, y])(y)$

$\hat{C}(A)(y) = A(y) * [((A(x) \to R(y, x)) \wedge (A(y) \to R(y, y))] = 0.3 \to R(y, x) = 1 = C(A)(y)$

$\hat{C}(A)(x) = A(x) * [((A(x) \to R(x, x)) \wedge (A(y) \to R(x, y))] = 0.3 * R(x, y) = 0.3 * 0.25$; $C(A)(x) = 0.25$.

For the Gödel t-norm we have

$\hat{C}(A)(x) = 0.3 \wedge 0.25 = 0.25 = C(A)(x)$,

hence $\hat{C} = C$.

For the Lukasiewicz t-norm $*_L$

$\hat{C}(A)(x) = 0.3 *_L 0.25 = \max(0.3 + 0.25 - 1, 0) = 0 \neq C(A)(x)$

and for the product t-norm $*_P$

$\hat{C}(A)(x) = 0.3 *_P 0.25 = 0.3 \times 0.25 \neq C(A)(x)$.

This example shows that the implication (iii) $\Rightarrow$ (i) in Theorem 5.1 is false in the case of Lukasiewicz or product t-norms; meanwhile it confirms that the equivalence (i) $\Leftrightarrow$ (iii) is true for Gödel t-norm .

We compute now $\tilde{P}$ and $\tilde{R}$. According to Definition 4.4 (i) we have

$$\tilde{P}(y, x) = \bigvee_{S \in \mathcal{B}} (C(S)(y) * S(x) * \neg C(S)(x)) =$$

$$= [C([x])(y) * [x](x) * \neg C([x])(x)] \vee [C([y])(y) * [y](x) * \neg C([y])(x)] \vee$$

$\vee[C([x,y])(y) * [x,y](x) * \neg C([x,y])(x)] \vee [C(A)(y) * A(x) * \neg C(A)(x)] =$
$= (\neg 0.25) \vee (0.3 * \neg 0.25).$

Similarly, $\tilde{P}(x,y) = 0$. Hence

$$\tilde{P} = \begin{pmatrix} 0 & 0 \\ (\neg 0.25) \vee (0.3 * \neg 0.25) & 0 \end{pmatrix}$$

For Lukasiewicz t-norm $*_L$ we have
$\tilde{P}(y,x) = (1 - 0.25) \vee \max\ (0.3 + 0.75 - 1, 0) = 0.75$, hence

$$\tilde{P} = \begin{pmatrix} 0 & 0 \\ 0.75 & 0 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & 0.25 \\ 1 & 1 \end{pmatrix} = R.$$

For Gödel and product t-norms

$$\tilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq R.$$

It follows that for Gödel and product t-norms the equivalence (iv) $\Leftrightarrow$ (vii) in Theorem 5.1 does not take place.

Chapter 6

General Results

In the previous chapter we studied under hypotheses $(H1)$ and $(H2)$ revealed preference axioms $WAFRP$, $SAFRP$, congruence axioms $WFCA$, $SFCA$ and some consistency conditions. This line of enquiry follows Uzawa-Arrow-Sen theory that starts from the assumption that the domain of the choice function contains the finite sets of alternatives.

Richter, Hansson, Suzumura and their followers investigated the choice functions'rationality and revealed preference and congruence conditions without this hypothesis.

The purpose of this chapter is a fuzzy approach of revealed preference and congruence axioms for fuzzy choice functions in the general case, following Richter-Hansson-Suzumura theory; we will work ignoring hypotheses $(H1)$ and $(H2)$.

In the first two sections we fix Gödel t-norm and in the last section we work with an arbitrary continuous t-norm $*$. As we have seen in Section 4.1, axioms $WARP$ and $SARP$ written in terms of C-connected sequences lead to new revealed preference axioms $WAFRP^{\circ}$, $SAFRP^{\circ}$. We also introduced axiom $HAFRP$.

In Section 6.1 we prove two main theorems:

1. The axioms $WFCA$ and $WAFRP^{\circ}$ are equivalent.
2. The axioms $SFCA$ and $HAFRP$ are equivalent.

Another proposition asserts that $WFCA \Rightarrow WAFRP$ and $SFCA \Rightarrow SAFRP$. According to the first theorem we have $WAFRP^{\circ} \Rightarrow WAFRP$. These results are summarized in a diagram that illustrates the hierarchy of axioms and other conditions of rationality.

Section 6.2 deals with a particular case of fuzzy choice functions. For them we establish the equivalence between $WAFRP^{\circ}$, G-normality, M-normality and other two properties expressed by algebraic identities. Particularly the implication $WAFRP \Rightarrow WAFRP^{\circ}$ fails hence $WAFRP$ and $WAFRP^{\circ}$ are not equivalent.

A classical Richter theorem [41] asserts that a classical choice function satisfies SCA if and only if it is rationalized by a reflexive, transitive and total preference relation.

Section 6.3 investigates to what extent Richter's theorem can be extended to fuzzy choice functions. We work here in a fuzzy set theory based on an arbitrary continuous t-norm $*$. We introduce the notion of totally $*$-rational fuzzy choice function, i.e. a G-rational fuzzy choice function which is rationalized by a reflexive, $*$-transitive and total fuzzy preference relation. A fuzzy choice function is said to be $*$-congruous if it satisfies $SFCA$.

First we prove that every totally $*$-rational choice function is $*$-congruous. To prove the converse implication is an open question. We need to define the notion of $*$-semirational fuzzy choice function in order to construct the proof on the fuzzy level. We find a surprising result: any fuzzy choice function is $*$-semirational; to prove that we essentially apply Theorem 3.8.

The results of this chapter are based on our papers [18, 17, 21].

6.1 The hierarchy of axioms

In this section we shall establish some connections between various axioms introduced in the previous paragraph. The results will be summarized in the final diagram.

Proposition 6.1 *(i)* $WFCA \Rightarrow WAFRP$*;*
(ii) $SFCA \Rightarrow SAFRP$*.*

Proof. (i) Let $C : \mathcal{B} \to \mathcal{F}(X)$ be a fuzzy choice function fulfilling $WFCA$. For any $x, y \in X$ we have

$$\begin{aligned}\tilde{P}(x,y) \wedge R(y,x) &= [\bigvee_{S\in\mathcal{B}} (C(S)(x) \wedge S(y) \wedge \neg C(S)(y)] \wedge R(y,x) \\ &= \bigvee_{S\in\mathcal{B}} [R(y,x) \wedge C(S)(x) \wedge S(y) \wedge \neg C(S)(y)] \\ &\leq \bigvee_{S\in\mathcal{B}} (C(S)(y) \wedge \neg C(S)(y)) = 0\end{aligned}$$

because $R(y,x) \wedge C(S)(x) \wedge S(y) \leq C(S)(y)$, by $WFCA$. Thus $\tilde{P}(x,y) \wedge R(y,x) = 0$, hence $\tilde{P}(x,y) \leq \neg R(y,x)$.

(ii) Suppose C satisfies $SFCA$. For any $x, y \in X$:

$T(\tilde{P})(x,y) \wedge R(y,x) =$

$= [\tilde{P}(x,y) \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n\in X} (\tilde{P}(x,t_1) \wedge \ldots \wedge \tilde{P}(t_n,y))] \wedge R(y,x) =$

$= [\tilde{P}(x,y) \wedge R(y,x)] \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n\in X} [\tilde{P}(x,t_1) \wedge \ldots \wedge \tilde{P}(t_n,y) \wedge R(y,x)].$

But $SFCA$ implies $WFCA$, hence by (i), $\tilde{P}(x,y) \wedge R(y,x) = 0$. For any $n \geq 1$ and $t_1, \ldots, t_n \in X$ we have, according to Lemma 4.2:

$\tilde{P}(x,t_1) \wedge \ldots \wedge \tilde{P}(t_n,y) \wedge R(y,x) \leq R(x,t_1) \wedge \ldots \wedge R(t_{n-1},t_n) \wedge \tilde{P}(t_n,y) \wedge R(y,x) =$

$= R(y,x) \wedge R(x,t_1) \wedge \ldots \wedge R(t_{n-1},t_n) \wedge \tilde{P}(t_n,y) \leq$

$\leq T(R)(y,t_n) \wedge \tilde{P}(t_n,y) =$

$= T(R)(y,t_n) \wedge \bigvee_{S \in \mathcal{B}} [C(S)(t_n) \wedge S(y) \wedge \neg C(S)(y)]$

$= \bigvee_{S \in \mathcal{B}} [T(R)(y,t_n) \wedge C(S)(t_n) \wedge S(y) \wedge \neg C(S)(y)] \leq$

$\leq \bigvee_{S \in \mathcal{B}} (C(S)(y) \wedge \neg C(S)(y)) = 0$

because $T(R)(y,t_n) \wedge C(S)(t_n) \wedge S(y) \leq C(S)(y)$ (by $SFCA$). Thus

$\tilde{P}(x,t_1) \wedge \ldots \wedge \tilde{P}(t_n,y) \wedge R(y,x) = 0$ for any $n \geq 1$ and $t_1, \ldots, t_n \in X$, therefore $T(\tilde{P})(x,y) \wedge R(y,x) = 0$. By Lemma 3.1 (1), $T(\tilde{P})(x,y) \leq \neg R(y,x)$. ∎

Remark 6.1 *Let C be a fuzzy choice function, $S \in \mathcal{B}$ and $x, y \in X$. By the definition of R*

$R(x,y) \wedge S(x) \wedge C(S)(y) = [\bigvee_{T \in \mathcal{B}} (C(T)(x) \wedge T(y))] \wedge S(x) \wedge C(S)(y)$

$= \bigvee_{T \in \mathcal{B}} [S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y)].$

Then $WFCA$ is equivalent with the following statement

- *For any $S, T \in \mathcal{B}$ and $x, y \in X$*

$S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y) \leq C(S)(x).$

Theorem 6.1 *For a fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$ the following are equivalent:*

(i) C satisfies $WFCA$;

(ii) $R \subseteq R_*$;

(iii) C satisfies $WAFRP^\circ$.

Proof. (i) ⇔ (ii). The following assertions are equivalent:

- $R \subseteq R_*$;
- For any $x, y \in X$:

$\bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y)) \leq \bigwedge_{T \in \mathcal{B}} [(T(x) \wedge C(T)(y)) \to C(T)(x)]$;

- For any $x, y \in X$ and $S, T \in \mathcal{B}$:

$C(S)(x) \wedge S(y) \leq (T(x) \wedge C(T)(y)) \to C(T)(x).$

- For any $x, y \in X$ and $S, T \in \mathcal{B}$:

$C(S)(x) \wedge S(y) \wedge T(x) \wedge C(T)(y) \leq C(T)(x).$

In accordance with Remark 6.1, (i) and (ii) are equivalent.

(iii) $\Rightarrow$ (i) Assume that C satisfies $WAFRP^\circ$. Let $x, y \in X$ and $S, T \in \mathcal{B}$. By $WAFRP^\circ$ one gets

$S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y) \leq E(S \cap C(T), T \cap C(S)) =$
$= \bigwedge_{u \in X} [(S(u) \wedge C(T)(u)) \leftrightarrow (T(u) \wedge C(S)(u))] \leq$
$\leq (S(x) \wedge C(T)(x)) \leftrightarrow (T(x) \wedge C(S)(x)) \leq$
$\leq (S(x) \wedge C(T)(x)) \rightarrow (T(x) \wedge C(S)(x)) =$
$= [(S(x) \wedge C(T)(x)) \rightarrow T(x)] \wedge [(S(x) \wedge C(T)(x)) \rightarrow C(S)(x)] =$
$= (S(x) \wedge C(T)(x)) \rightarrow C(S)(x)$

because $(S(x) \wedge C(T)(x)) \rightarrow T(x) = 1$ (by Lemma 3.1 (4)). It follows that

$S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y) =$
$= [S(x) \wedge T(y) \wedge C(T)(x) \wedge C(S)(y)] \wedge (S(x) \wedge C(T)(x)) \leq C(S)(x)$

in accordance with Lemma 3.1 (1). According to Remark 6.1, C satisfies $WFCA$.

(i) $\Rightarrow$ (iii) Assume C fulfills $WFCA$. By Remark 6.1, for any $S, T \in \mathcal{B}$ and $x, y, u \in X$ we have

$S(x) \wedge T(y) \wedge C(S)(y) \wedge C(T)(x) \wedge S(u) \wedge C(T)(u) \leq S(u) \wedge T(y) \wedge C(T)(u) \wedge C(S)(y) \leq C(S)(u)$.

Thus, by Lemma 3.1 (1):

$S(x) \wedge T(y) \wedge C(S)(y) \wedge C(T)(x) \leq (S(u) \wedge C(T)(u)) \rightarrow C(S)(u) =$
$= (S(u) \wedge C(T)(u)) \rightarrow (T(u) \wedge C(S)(u))$

and similarly,

$S(x) \wedge T(y) \wedge C(S)(y) \wedge C(T)(x) \leq (T(u) \wedge C(S)(u)) \rightarrow (S(u) \wedge C(T)(u))$

The last two inequalities give

$S(x) \wedge T(y) \wedge C(S)(y) \wedge C(T)(x) \leq (S(u) \wedge C(T)(u)) \leftrightarrow (T(u) \wedge C(S)(u))$.

This inequality is true for each $u \in X$, hence

$S(x) \wedge T(y) \wedge C(S)(y) \wedge C(T)(x) \leq E(S \cap C(T), T \cap C(S))$

so C satisfies the axiom $WAFRP^\circ$. ∎

Corollary 6.1 *$WAFRP^\circ \Rightarrow WAFRP$.*

Proof. By Proposition 6.1 and Theorem 6.1. ∎

Theorem 6.2 *For any fuzzy choice function $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ the following are equivalent:*

(i) C satisfies $HAFRP$;

(ii) C satisfies $SFCA$.

Proof. (i) $\Rightarrow$ (ii) Assume C satisfies $HAFRP$. For any $S \in \mathcal{B}$ and $x, y \in X$ we shall prove that

(a) $T(R)(x,y) \wedge S(x) \wedge C(S)(y) \leq C(S)(x)$

The left hand-side in (a) can be written

$T(R)(x,y) \wedge S(x) \wedge C(S)(y) =$

$= [R(x,y) \vee \bigvee_{n=1}^{\infty} \bigvee_{z_1,\ldots,z_n \in X} (R(x,z_1) \wedge \ldots \wedge R(z_n,y))] \wedge S(x) \wedge C(S)(y) =$

$= [R(x,y) \wedge S(x) \wedge C(S)(y)] \vee \bigvee_{n=1}^{\infty} \bigvee_{z_1,\ldots,z_n \in X} [R(x,z_1) \wedge \ldots \wedge R(z_n,y) \wedge S(x) \wedge C(S)(y)]$.

Then proving (a) is equivalent to establishing the following two inequalities:

(b) $R(x,y) \wedge S(x) \wedge C(S)(y) \leq C(S)(x)$;

(c) For any integer $n \geq 1$ and $z_1, \ldots, z_n \in X$:

$R(x,z_1) \wedge R(z_1,z_2) \wedge \ldots \wedge R(z_n,y) \wedge S(x) \wedge C(S)(y) \leq C(S)(x)$.

The axiom $HAFRP$ implies $WAFRP^{\circ}$ then, by Theorem 6.1, C satisfies $WFCA$. Then (b) is verified. According to the definition of R

$R(x,z_1) \wedge R(z_1,z_2) \wedge \ldots \wedge R(z_n,y) \wedge S(x) \wedge C(S)(y) =$

$= [\bigvee_{S_1 \in \mathcal{B}} (C(S_1)(x) \wedge S_1(z_1))] \wedge \ldots \wedge [\bigvee_{S_{n+1} \in \mathcal{B}} (C(S_{n+1})(z_n) \wedge S_{n+1}(y))] \wedge S(x) \wedge C(S)(y) =$

$= \bigvee_{S_1,\ldots,S_{n+1}} \mathcal{T}(S_1, \ldots, S_{n+1})$

where

$\mathcal{T}(S_1, \ldots, S_{n+1}) = C(S_1)(x) \wedge S_1(z_1) \wedge C(S_2)(z_1) \wedge \ldots$
$\wedge C(S_{n+1})(z_n) \wedge S_{n+1}(y) \wedge S(x) \wedge C(S)(y) =$
$= [S(x) \wedge C(S_1)(x)] \wedge [S_1(z_1) \wedge C(S_2)(z_1)] \wedge \ldots \wedge [S_n(z_n) \wedge C(S_{n+1})(z_n)] \wedge$
$\wedge [S_{n+1}(y) \wedge C(S)(y)] = \bigwedge_{k=0}^{n+1} [S_k(z_k) \wedge C(S_{k+1})(z_k)]$

with $S_0 = S$, $z_0 = x$ and $z_{n+1} = y$. According to $HAFRP$:

$$\begin{aligned}
\mathcal{T}(S_1, \ldots, S_{n+1}) &\leq E(S \cap C(S_1), S_1 \cap C(S)) \leq \\
&\leq (S(x) \wedge C(S_1)(x)) \leftrightarrow (S_1(x) \wedge C(S)(x)) \leq \\
&\leq (S(x) \wedge C(S_1)(x)) \rightarrow (S_1(x) \wedge C(S)(x)) = \\
&= (S(x) \wedge C(S_1)(x)) \rightarrow C(S)(x).
\end{aligned}$$

We also have $\mathcal{T}(S_1, \ldots, S_{n+1}) \leq S(x) \wedge C(S_1)(x)$ hence by Lemma 3.1 (2)

$$\begin{aligned}
\mathcal{T}(S_1, \ldots, S_{n+1}) &\leq (S(x) \wedge C(S_1)(x_1)) \wedge [(S(x) \wedge C(S_1)(x)) \rightarrow C(S)(x)] = \\
&= S(x) \wedge C(S_1)(x) \wedge C(S)(x) \leq C(S)(x).
\end{aligned}$$

This inequality holds for all $S_1, \ldots, S_{n+1} \in \mathcal{B}$ hence

$R(x,z_1) \wedge R(z_1,z_2) \wedge \ldots \wedge R(z_n,y) \wedge S(x) \wedge C(S)(y) =$

$= \bigvee_{S_1,\ldots,S_{n+1}} \mathcal{T}(S_1, \ldots, S_{n+1}) \leq C(S)(x)$.

Condition (c) was proved so the proof of (i) $\Rightarrow$ (ii) is finished.

(ii) Assume C fulfills $SFCA$. Let $S_1,\dots,S_n \in \mathcal{B}$ and $z_1,\dots,z_n \in X$. We shall prove that the following inequality holds:

(d) $\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)] \le E(S_1\cap C(S_2), S_2\cap C(S_1))$

where $S_{n+1}=S_1$. We remark that

(e) $\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)] =$

$= S_1(z_1)\wedge[C(S_2)(z_1)\wedge S_2(z_2)]\wedge\ldots\wedge[C(S_n)(z_{n-1})\wedge S_n(z_n)]\wedge C(S_1)(z_n) \le$

$\le S_1(z_1)\wedge R(z_1,z_2)\wedge\ldots\wedge R(z_{n-1},z_n)\wedge C(S_1)(z_n)\le$

$\le S_1(z_1)\wedge T(R)(z_1,z_n)\wedge C(S_1)(z_n)$

because $C(S_2)(z_1)\wedge S_2(z_2) \le R(z_1,z_2),\dots,C(S_n)(z_{n-1})\wedge S_n(z_n) \le R(z_{n-1},z_n)$.

Let $z\in X$. We shall establish the inequality

(f) $\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)] \le (S_1(z)\wedge C(S_2)(z))\to C(S_1)(z)$

$$= [S_1(z)\wedge C(S_2)(z)]\to[S_2(z)\wedge C(S_1)(z))].$$

By Lemma 3.1 (1), (f) is equivalent to

(g) $\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)]\wedge S_1(z)\wedge C(S_2)(z)\le C(S_1)(z).$

Let us denote by ϵ the left hand side member of the inequality (g). It is obvious that

$\epsilon \le C(S_2)(z_1)\wedge C(S_2)(z)\le S_2(z_1)\wedge C(S_2)(z)\le R(z,z_1)$.

By (e) we have $\epsilon\le T(R)(z_1,z_n)$ therefore

$\epsilon \le R(z,z_1)\wedge T(R)(z_1,z_n)\le T(R)(z,z_1)\wedge T(R)(z_1,z_n)\le T(R)(z,z_n)$

because $T(R)$ is transitive. According to (e), $\epsilon\le C(S_1)(z_n)$. We also have $\epsilon\le S_1(z)$, therefore $\epsilon\le T(R)(z,z_n)\wedge S_1(z)\wedge C(S_1)(z_n)\le C(S_1)(z)$, the last inequality following by $SFCA$. Then (g) was proved.

In accordance with (e) we can write

$\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)]\wedge S_2(z)\wedge C(S_1)(z)\le$

$\le S_1(z_1)\wedge T(R)(z_1,z_n)\wedge C(S_1)(z_n)\wedge S_2(z)\wedge C(S_1)(z)\le$

$\le S_2(z)\wedge(C(S_1)(z)\wedge S_1(z_1))\le R(z,z_1)\wedge S_2(z)$.

It is obvious that $\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)]\le C(S_2)(z_1)$ hence by using $WFCA$

$\displaystyle\bigwedge_{k=1}^{n}[S_k(z_k)\wedge C(S_{k+1})(z_k)]\wedge S_2(z)\wedge C(S_1)(z)\le$

$\le R(z,z_1)\wedge S_2(z)\wedge C(S_2)(z_1)\le C(S_2)(z)$.

By this last inequality one infers

(h) $\bigwedge_{k=1}^{n} [S_k(z_k) \wedge C(S_{k+1})(z_k)] \leq (S_2(z) \wedge C(S_1)(z)) \to C(S_2)(z) =$

$= (S_2(z) \wedge C(S_1)(z)) \to (S_1(z) \wedge C(S_2)(z)).$

From (f) and (h) we get for each $z \in X$:

$$\bigwedge_{k=1}^{n} [S_k(z_k) \wedge C(S_{k+1})(z_k)] \leq (S_1(z) \wedge C(S_2)(z)) \leftrightarrow (S_2(z) \wedge C(S_1)(z)).$$

Then (d) follows immediately. In a similar way we can show that

$$\bigwedge_{k=1}^{n} [S_k(z_k) \wedge C(S_{k+1})(z_k)] \leq E(S_2 \cap C(S_3), S_3 \cap C(S_2))$$

$$\vdots$$

$$\bigwedge_{k=1}^{n} [S_k(z_k) \wedge C(S_{k+1})(z_k)] \leq E(S_{n-1} \cap C(S_n), S_n \cap C(S_{n-1}))$$

then $HAFRP$ follows. The proof is finished. ∎

Remark 6.2 *The results of this section can be summarized in Figure 6.1.*

Figure 6.1: Hierarchy of axioms

6.2 A particular class of fuzzy choice functions

In this section a particular class of fuzzy choice function is studied with emphasis on the rationality conditions discussed in the previous section.

Let $X = \{x, y\}$ and $\mathcal{B} = \{A, B\}$ where $A, B \in \mathcal{F}(X)$ are given by
$A = \alpha\chi\{x\} + \chi\{y\}$; $B = \chi\{x\} + \beta\chi\{y\}$ $(0 \leq \alpha, \beta \leq 1)$.

Consider the function $C : \mathcal{B} \to \mathcal{F}(X)$ defined by:

(6.2.1) $C(A) = \gamma\chi\{x\} + \chi\{y\}$; $C(B) = \chi\{x\} + \delta\chi\{y\}$ $(0 \leq \gamma \leq \alpha, 0 \leq \delta \leq \beta)$.

For any $\alpha, \beta, \gamma, \delta$ as above, C is a fuzzy choice function on $\langle X, \mathcal{B} \rangle$. In this section we will investigate some properties of the fuzzy choice functions introduced by (6.2.1). We will obtain some new information on the connections between various conditions studied in the previous sections.

First we compute the fuzzy preference relation R:

$R(x,x) = R(y,y) = 1$;

$R(x,y) = (C(A)(x) \wedge A(y)) \vee (C(B)(x) \wedge B(y)) = \beta \vee \gamma$;

$R(y,x) = (C(A)(y) \wedge A(x)) \vee (C(B)(y) \wedge B(x)) = \alpha \vee \delta$.

Thus

(6.2.2) $R = \begin{pmatrix} 1 & \beta \vee \gamma \\ \alpha \vee \delta & 1 \end{pmatrix}$.

We also compute the fuzzy relation R_*:

$R_*(x,x) = R_*(y,y) = 1$;

$R_*(x,y) = [(A(x) \wedge C(A)(y)) \rightarrow C(A)(x)] \wedge [(B(x) \wedge C(B)(y)) \rightarrow C(B)(x)] =$
$= \alpha \rightarrow \gamma$;

$R_*(y,x) = [(A(y) \wedge C(A)(x)) \rightarrow C(A)(y)] \wedge [(B(y) \wedge C(B)(x)) \rightarrow C(B)(y)] =$
$= \beta \rightarrow \delta$.

Hence

(6.2.3) $R_* = \begin{pmatrix} 1 & \alpha \rightarrow \gamma \\ \beta \rightarrow \delta & 1 \end{pmatrix}$.

Proposition 6.2 *The fuzzy choice function C defined by (6.2.1) satisfies $WAFRP^\circ$ if and only if $\beta \wedge (\alpha \vee \delta) = \delta$ and $\alpha \wedge (\beta \vee \gamma) = \gamma$.*

Proof. The following conditions are equivalent:

- $R \subseteq R_*$;
- $\alpha \vee \delta \leq \beta \rightarrow \delta$ and $\beta \vee \gamma \leq \alpha \rightarrow \gamma$;
- $\beta \wedge (\alpha \vee \delta) \leq \delta$ and $\alpha \wedge (\beta \vee \gamma) \leq \gamma$ (by Lemma 3.1 (1));
- $\beta \wedge (\alpha \vee \delta) = \delta$ and $\alpha \wedge (\beta \vee \gamma) = \gamma$ (since $\gamma \leq \alpha, \delta \leq \beta$).

According to Theorem 6.1 the desired equivalence follows. ∎

Let us compute the fuzzy relation $\tilde{P}$:

$\tilde{P}(x,x) = \tilde{P}(y,y) = 0$;

$\tilde{P}(x,y) = [C(A)(x) \wedge A(y) \wedge \neg C(A)(y)] \vee [C(B)(x) \wedge B(y) \wedge \neg C(B)(y)] =$
$= \beta \wedge \neg\delta$;

$\tilde{P}(y,x) = [C(A)(y) \wedge A(x) \wedge \neg C(A)(x)] \vee [C(B)(y) \wedge B(x) \wedge \neg C(B)(x)] =$
$= \alpha \wedge \neg\gamma$.

Therefore

(6.2.4) $\tilde{P} = \begin{pmatrix} 0 & \beta \wedge \neg\delta \\ \alpha \wedge \neg\gamma & 0 \end{pmatrix}$.

Proposition 6.3 *The fuzzy choice function C verifies $WAFRP$ if and only if $\alpha \wedge \beta \wedge \neg\delta = \alpha \wedge \beta \wedge \neg\gamma = 0$.*

Proof. The following assertions are equivalent:

- C satisfies $WAFRP$;
- $\tilde{P}(x, y) \leq \neg R(y, x)$ and $\tilde{P}(y, x) \leq \neg R(x, y)$;
- $\beta \wedge \neg\delta \leq \neg(\alpha \vee \delta)$ and $\alpha \wedge \neg\gamma \leq \neg(\beta \vee \gamma)$;
- $\alpha \wedge \beta \wedge \neg\delta = 0$ and $\alpha \wedge \beta \wedge \neg\gamma = 0$. ∎

Remark 6.3 *Let us take $\alpha = 1, \gamma = \frac{1}{2}, \beta = \delta = \frac{2}{3}$. Since $\neg\delta = \neg\gamma = 0$, by Proposition 6.3 one can infer that C satisfies $WAFRP$. We remark that $\alpha \wedge (\beta \vee \gamma) = 1 \wedge (\frac{2}{3} \vee \frac{1}{2}) = \frac{2}{3} \neq \gamma$, hence, by Proposition 6.2, the axiom $WAFRP^\circ$ does not hold. In conclusion the implication*

$WAFRP \Rightarrow WAFRP^\circ$

fails and $WAFRP$, $WAFRP^\circ$ are not equivalent conditions.

Remark 6.4 *We observe that*

(6.2.5) $T(\tilde{P}) = \begin{pmatrix} \alpha \wedge \beta \wedge \neg\gamma \wedge \neg\delta & \beta \wedge \neg\delta \\ \alpha \wedge \neg\gamma & \alpha \wedge \beta \wedge \neg\gamma \wedge \neg\delta \end{pmatrix}.$

Since $\alpha \wedge \beta \wedge \neg\delta = 0$ and $\alpha \wedge \beta \wedge \neg\gamma = 0$ implies $\alpha \wedge \beta \wedge \neg\gamma \wedge \neg\delta = 0$, $T(\tilde{P}) = \tilde{P}$, therefore by Proposition 6.3 we also obtain the equivalence

C satisfies $SAFRP$ if and only if $\alpha \wedge \beta \wedge \neg\delta = \alpha \wedge \beta \wedge \neg\gamma = 0$.

In our case $\mathcal{B}$ has two members A, B then the axioms $WAFRP^\circ$, $SAFRP^\circ$ and $HAFRP$ are equivalent. Then using again the argument in Remark 6.3 the implication

$SAFRP \Rightarrow HAFRP$

does not hold.

Proposition 6.4 *For the fuzzy choice function C defined by (6.2.1) the following are equivalent:*

(i) C is G-normal;

(ii) C is M-normal;

(iii) $\alpha \wedge \beta = \gamma \wedge \delta$.

Proof. We compute the values of $G^*(A)$ and $G^*(B)$:

$G^*(A)(x) = A(x) \wedge [A(x) \rightarrow R(x, x)] \wedge [A(y) \rightarrow R(x, y)] = \alpha \wedge (\beta \vee \gamma)$

$G^*(A)(y) = A(y) \wedge [A(x) \rightarrow R(y, x)] \wedge [A(y) \rightarrow R(y, y)] = \alpha \rightarrow (\alpha \vee \delta) = 1$

and similarly, $G^*(B)(x) = 1$ and $G^*(B)(y) = \beta \wedge (\alpha \vee \delta)$.

Applying Lemma 3.1 we also compute the values of $M^*(A)$ and $M^*(B)$:

$M^*(A)(x) = A(x) \wedge [(A(x) \wedge R(x, x)) \rightarrow R(x, x)] \wedge [(A(y) \wedge R(y, x)) \rightarrow R(x, y)] =$

$= \alpha \wedge [(\alpha \vee \delta) \rightarrow (\beta \vee \gamma)] =$

$= \alpha \wedge [\alpha \rightarrow (\beta \vee \gamma)] \wedge [\delta \rightarrow (\beta \vee \gamma)]$

$= \alpha \wedge (\beta \vee \gamma) \wedge (\delta \rightarrow (\beta \vee \gamma))$

$= \alpha \wedge (\beta \vee \gamma).$

$M^*(A)(y) = A(y) \wedge [(A(x) \wedge R(x,y)) \to R(y,x)] \wedge [(A(y) \wedge R(y,y)) \to R(y,y)]$

$= (\alpha \wedge (\beta \vee \gamma)) \to (\alpha \vee \delta) = 1$

and similarly $M^*(B)(x) = 1$ and $M^*(B)(y) = \beta \wedge (\alpha \vee \delta)$.

It follows that $G^*(A) = M^*(A)$ and $G^*(B) = M^*(B)$ and each of (i), (ii) are equivalent to

- $\alpha \wedge (\beta \vee \gamma) = \gamma$ and $\beta \wedge (\alpha \vee \delta) = \delta$
- $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = \gamma$ and $(\beta \wedge \alpha) \vee (\beta \wedge \delta) = \delta$
- $\alpha \wedge \beta \leq \gamma$ and $\alpha \wedge \beta \leq \delta$
- $\alpha \wedge \beta \leq \gamma \wedge \delta$
- $\alpha \wedge \beta = \gamma \wedge \delta$. ∎

Theorem 6.3 *Let C be the fuzzy choice function defined by (6.2.1). Then the following assertions are equivalent*

(i) C satisfies $WAFRP^\circ$;

(ii) C is G-normal;

(iii) C is M-normal;

(iv) $\alpha \wedge \beta = \gamma \wedge \delta$;

(v) $\alpha \wedge (\beta \vee \gamma) = \gamma$ and $\beta \wedge (\alpha \vee \delta) = \delta$.

Proof. According to Propositions 6.2 and 6.4 it suffices to establish the equivalence of (iv) and (v).

Suppose first $\alpha \leq \beta$. Then $\alpha \wedge \beta = \gamma \wedge \delta$ if and only if $\alpha = \gamma \wedge \delta$ if and only if ($\alpha = \gamma$ and $\alpha \leq \delta$). Since $\alpha \leq \beta$ we have

$\alpha \wedge (\gamma \vee \beta) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = \alpha \vee \gamma = \alpha$

$\beta \wedge (\alpha \vee \delta) = (\alpha \wedge \beta) \vee (\beta \wedge \delta) = \alpha \vee \delta$

hence $[\alpha \wedge (\gamma \vee \beta) = \gamma$ if and only if $\alpha = \gamma]$ and $[\beta \wedge (\alpha \vee \delta) = \delta$ if and only if $\alpha \vee \delta = \delta$ if and only if $\alpha \leq \delta]$. It follows that for $\alpha \leq \beta$, (iv) if and only if (v); the case $\beta \leq \alpha$ is similar. ∎

6.3 A fuzzy analysis of the Richter theorem

A *strict partial $*$-order* on X is an irreflexive and $*$-transitive fuzzy relation on X. The notion of similarity relation has a crucial role in the analysis of fuzzy phenomena. A *$*$-similarity relation* E on X is a reflexive, symmetric and $*$-transitive fuzzy relation. If E is a $*$-similarity relation on X and R a fuzzy relation on X then E is called a *congruence* w.r.t. R if $E(x,u) *$

$E(y,v) * R(x,y) \leq R(u,v)$ for all $x,y,u,v \in X$. If E is a congruence w.r.t. R then $E(x,u) * R(x,y) \leq R(u,y)$ and $E(y,v) * R(x,y) \leq R(x,v)$ for all $x,y,u,v \in X$.

Let E be a congruence w.r.t. R. Define a (crisp) binary relation on X: $x \approx y \Leftrightarrow E(x,y) = 1$. It is easy to see that if $\approx$ is an equivalence relation on X, then one can consider the quotient set $Y = X/_{\approx}$. $[x]$ will denote the equivalence class of $x \in X$.

Let us consider the fuzzy relation $\tilde{R}$ on Y defined by

(6.3.1) $\tilde{R}([x],[y]) = \bigvee_{u,v \in X} (E(u,x) * E(v,y) * R(u,v))$ for any $x,y \in X$.

Lemma 6.1 *The fuzzy relation $\tilde{R}$ is correctly defined, i.e.*
(6.3.2) $\bigvee_{u,v \in X} (E(u,x) * E(v,y) * R(u,v)) = \bigvee_{u,v \in X} (E(u,x') * E(v,y') * R(u,v))$ *for all $x,y,x',y' \in X$ such that $x \approx x'$ and $y \approx y'$.*

Proof. Let us denote by α and β the left and the right members in (6.3.2) respectively.

Let $u,v \in X$. Then
$E(u,x) = E(u,x) * E(x,x') \leq E(u,x')$
$E(v,y) = E(v,y) * E(y,y') \leq E(v,y')$
because $E(x,x') = E(y,y') = 1$. Therefore
$E(u,x) * E(v,y) * R(u,v) \leq E(u,x') * E(v,y') * R(u,v) \leq \beta$.

This inequality holds for any $u,v \in X$ hence $\alpha \leq \beta$. The converse inequality follows similarly. ∎

Proposition 6.5 *If R is a $*$-transitive fuzzy relation on X then $\tilde{R}$ is also $*$-transitive.*

Proof. Let $x,y,z \in X$. By Lemma 3.2 (5)

$\tilde{R}([x],[y]) * \tilde{R}([y],[z]) = [\bigvee_{u,v \in X} (E(u,x) * E(v,y) * R(u,v))] * [\bigvee_{s,t \in X} (E(s,y) * E(t,z) * R(s,t))] = \bigvee_{u,v,s,t \in X} (E(u,x) * E(v,y) * R(u,v) * E(s,y) * E(t,z) * R(s,t))$

and

$\tilde{R}([x],[z]) = \bigvee_{u,t \in X} (E(u,x) * E(t,z) * R(u,t))$.

Let $u,v,s,t \in X$. Since

$E(v,y) * E(s,y) = E(v,y) * E(y,s) \leq E(v,s)$
$E(v,s) * R(u,v) \leq R(u,s)$

$R(u,s) * R(s,t) \leq R(u,t)$

it follows that

$E(u,x) * E(v,y) * R(u,v) * E(s,y) * E(t,z) * R(s,t) = E(u,x) * E(v,y) * E(s,y) * E(t,z) * R(u,v) * R(s,t) \leq E(u,x) * E(v,s) * E(t,z) * R(u,v) * R(s,t) \leq E(u,x) * R(u,s) * E(t,z) * R(s,t) \leq E(u,x) * E(t,z) * R(u,t) \leq \tilde{R}([x],[z])$.

But this inequality holds for all $u,v,s,t \in X$ therefore
$\tilde{R}([x],[y]) * \tilde{R}([y],[z]) \leq \tilde{R}([x],[z])$. ∎

Proposition 6.6 *If $R(x,x) = 0$ then $\tilde{R}([x],[x]) = 0$ for all $x \in X$.*

Proof. For $x \in X$ we have

$$\tilde{R}([x],[x]) = \bigvee_{u,v \in X} (E(u,x) * E(v,x) * R(u,v)) = 0$$

because $E(u,x) * E(v,x) * R(u,v) \leq R(x,x) = 0$. ∎

Corollary 6.2 *If R is a strict partial $*$-order on X then $\tilde{R}$ is a strict partial $*$-order on Y.*

A fuzzy choice function C is *totally $*$-rational* if there exists a fuzzy relation G on X which is reflexive, $*$-transitive and total, and such that

(6.3.3) $C(B)(x) = B(x) * \bigwedge_{y \in X} (B(y) \to G(x,y))$

for any $B \in \mathcal{B}$ and $x \in X$.

Let C be a fuzzy choice function. Consider the fuzzy relation R on X introduced in Definition 4.2 by

(6.3.4) $R(x,y) = \bigvee_{B \in \mathcal{B}} (C(B)(x) * B(y))$

for all $x, y \in X$. Denote by W the $*$-transitive closure of R. The fuzzy choice function C is *$*$-congruous* if for any $B \in \mathcal{B}$ and $x, y \in X$ we have

(6.3.5) $C(B)(x) * B(y) * W(y,x) \leq C(B)(y)$.

The following result generalizes a part of Richter's theorem.

Theorem 6.4 *Every totally $*$-rational fuzzy choice function is $*$-congruous.*

Proof. Assume that the fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$ is totally $*$-rational, i.e. there exists a fuzzy relation G on X which is reflexive, $*$-transitive and total, and such that

$$C(B)(x) = B(x) * \bigwedge_{v \in X} (B(v) \to G(x,v))$$

for all $B \in \mathcal{B}$ and $x \in X$.

Firstly we will prove that for any $x, y \in X$, the following inequality holds:

(a) $R(x, y) \leq G(x, y)$.

Let $B \in \mathcal{B}$. The total $*$-rationality of C yields

$$C(B)(x) \leq \bigwedge_{v \in X} (B(v) \to G(x, v)),$$

hence $C(B)(x) \leq B(v) \to G(x, v)$ for each $v \in X$. Particularly, $C(B)(x) \leq B(y) \to G(x, y)$, hence, by Lemma 3.1 (1), we get $C(B)(x) * B(y) \leq G(x, y)$. This last inequality holds for any $B \in \mathcal{B}$, therefore

$$R(x, y) = \bigvee_{B \in \mathcal{B}} (C(B)(x) * B(y)) \leq G(x, y).$$

Let $B \in \mathcal{B}$ and $x, y \in X$. We must prove that

(b) $C(B)(x) * B(y) * W(y, x) \leq C(B)(y)$.
Using Lemma 3.5 we compute the left term of (b):

$$C(B)(x) * B(y) * W(y, x) =$$

$$= C(B)(x) * B(y) * [R(y, x) \vee \bigvee_{n=1}^{\infty} \bigvee_{u_1, \ldots, u_n \in X} (R(y, u_1) * \ldots * R(u_n, x))].$$

In accordance with (a) we get the inequality:

(c) $C(B)(x) * B(y) * W(y, x) \leq$

$$\leq C(B)(x) * B(y) * [G(y, x) \vee \bigvee_{n=1}^{\infty} \bigvee_{u_1, \ldots, u_n \in X} (G(y, u_1) * \ldots * G(u_n, x))].$$

Since G is $*$-transitive, $G(y, u_1) * \ldots * G(u_n, x) \leq G(y, x)$ for all $u_1, \ldots, u_n \in X$ therefore

$$\bigvee_{u_1, \ldots, u_n \in X} (G(y, u_1) * \ldots * G(u_n, x)) \leq G(y, x).$$

This inequality holds for any $n \geq 1$, hence

$$\bigvee_{n=1}^{\infty} \bigvee_{u_1, \ldots, u_n \in X} (G(y, u_1) * \ldots * G(u_n, x)) \leq G(y, x).$$

Then

$$G(y,x) \vee \bigvee_{n=1}^{\infty} \bigvee_{u_1,\ldots,u_n \in X} (G(y,u_1) * \ldots * G(u_n,x)) = G(y,x)$$

hence the inequality (c) becomes

(d) $C(B)(x) * B(y) * W(y,x) \leq C(B)(x) * B(y) * G(y,x)$.

Now we will establish the inequality

(e) $C(B)(x) * G(y,x) \leq \bigwedge_{v \in X} (B(v) \to G(y,v))$.

It suffices to show that for any $v \in X$ we have

(f) $C(B)(x) * G(y,x) \leq B(v) \to G(y,v)$.

Let $v \in X$. We notice that

$C(B)(x) * G(y,x) * B(v) =$

$= B(x) * [\bigwedge_{u \in X} (B(u) \to G(x,u))] * G(y,x) * B(v) \leq$

$\leq B(x) * [B(v) \to G(x,v)] * G(y,x) * B(v) =$

$= B(x) * B(v) * [B(v) \to G(x,v)] * G(y,x) \leq$

$\leq B(x) * G(x,v) * G(y,x) \leq$

$\leq G(y,x) * G(x,v) \leq G(y,v)$

because $B(v) * [B(v) \to G(x,v)] = B(v) \wedge G(x,v) \leq G(x,v)$ and G is $*$-transitive. In accordance with Lemma 3.1 (1) we obtain the inequality (f). Thus the inequality (e) was proved.

From (e) we can infer

(g) $C(B)(x) * B(y) * G(y,x) \leq B(y) * \bigwedge_{v \in X} (B(v) \to G(y,v)) = C(B)(y)$.

Now the desired inequality (b) follows from (d) and (g). Thus C is $*$-congruous. ∎

Let C be a (crisp) choice function on $\langle X, \mathcal{B} \rangle$ with $\mathcal{B} \subseteq \mathcal{P}(X)$. We shall relax the criterion in the definition of the rational choice function. We shall say that C is *semirational* if there exists a binary relation G on X which is reflexive, transitive and total, and such that for any $B \in \mathcal{B}$ we have

(6.3.6) $C(B) \subseteq B \cap \{x|(x,y) \in G$ for all $y \in B\}$.

But $C(B) \subseteq B$ hence the previous relation is equivalent to the condition

(6.3.7) $C(B) \subseteq \{x|(x,y) \in G$ for all $y \in B\}$.

We shall extend this new concept to the fuzzy setting.

Let C be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$. C is called *$*$-semirational* if there exists a fuzzy relation G on X reflexive, $*$-transitive and total, and such that for any $B \in \mathcal{B}$ and $x \in X$ we have

(6.3.8) $C(B)(x) \leq \bigwedge\limits_{y \in X} (B(y) \rightarrow G(x,y))$.

Recall that $*_L$ is the Lukasiewicz t-norm. Let us consider the Lukasiewicz t-conorm $\oplus$ (see [31], p. 11):

(6.3.9) $a \oplus b = min(a+b, 1)$.

Then the operation $\oplus$ is associative, commutative and $a \oplus 0 = 0 \oplus a = a$ for any $a \in [0,1]$. In this case the negation is given by $\neg a = 1 - a$ hence $\neg\neg a = a$ for any $a \in [0,1]$.

Lemma 6.2 *([26]) For any $a, b, c \in [0,1]$ the following hold:*

(a) $a \leq b$ implies $a \oplus c \leq b \oplus c$;

(b) $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$;

(c) $a \oplus (b \vee c) = (a \oplus b) \vee (a \oplus c)$;

(d) $a \oplus \neg a = 1$.

The following result seems to be surprising.

Theorem 6.5 *Assume that $*$ is a continuous t-norm. Then every fuzzy choice function is $*$-semirational.*

Proof. Assume $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ is a fuzzy choice function. Let us consider the fuzzy relation P on X defined by

(a) $P(x,y) = W(x,y) \wedge \neg W(y,x)$

for any $x, y \in X$.

We shall prove that P is $*$-transitive. Let $x, y, z \in X$. Then

$P(x,y) * P(y,z) = [W(x,y) \wedge \neg W(y,x)] * [W(y,z) \wedge \neg W(z,y)] \leq W(x,y) * W(y,z) \leq W(x,z)$

because W is $*$-transitive. We also have

$P(x,y) * P(y,z) * W(z,x) = [W(x,y) \wedge \neg W(y,x)] * [W(y,z) \wedge \neg W(z,y)] * W(z,x) \leq W(x,y) * \neg W(z,y) * W(z,x) = W(z,x) * W(x,y) * \neg W(z,y) \leq$

$W(z,y) * \neg W(z,y) = 0$.

In accordance with the property of negation these inequalities infer $P(x,y) * P(y,z) \leq \neg W(z,x)$ hence $P(x,y) * P(y,z) \leq W(x,z) \wedge \neg W(z,x) = P(x,z)$. Thus P is $*$-transitive.

Now we shall define the fuzzy relation J:

(b) $J(x,y) = \begin{cases} 1 & \text{if} \quad x = y \\ W(x,y) \wedge W(y,x) & \text{if} \quad x \neq y \end{cases}$.

We shall prove that J is a $*$-similarity relation on X. It is clear that $J(x,x) = 1$ and $J(x,y) = J(y,x)$ for any $x, y \in X$. For any $x, y, z \in X$ such that $x \neq y$, $y \neq z$ and $x \neq z$ we have

$J(x,y) * J(y,z) = [W(x,y) \wedge W(y,x)] * [W(y,z) \wedge W(z,y)] \leq W(x,y) * W(y,z) \leq W(x,z)$

and similarly, $J(x,y) * J(y,z) \leq W(z,x)$. Thus $J(x,y) * J(y,z) \leq W(x,z) \wedge W(z,x) = J(x,z)$. This inequality is obviously true for the other cases, hence J is $*$-transitive. Then J is a $*$-similarity relation.

Now we shall prove that J is a congruence w.r.t. P, i.e. for all $x, y, u, v \in X$, $J(x,u) * J(y,v) * P(x,y) \leq P(u,v)$. Assume $x \neq u, y \neq v$ hence

$J(x,u) * J(y,v) * P(x,y) = [W(x,u) \wedge W(u,x)] * [W(y,v) \wedge W(v,y)] * [W(x,y) \wedge \neg W(y,x)] \leq W(u,x) * W(y,v) * W(x,y) = W(u,x) * W(x,y) * W(y,v) \leq W(u,v)$,

because W is $*$-transitive. We also have

$J(x,u) * J(y,v) * P(x,y) * W(v,u) \leq W(u,x) * W(y,v) * \neg W(y,x) * W(v,u) = W(y,v) * W(v,u) * W(u,x) * \neg W(y,x) \leq W(y,x) * \neg W(y,x) = 0$

hence $J(x,u) * J(y,v) * P(x,y) \leq \neg W(v,u)$. Thus

$J(x,u) * J(y,v) * P(x,y) \leq W(u,v) \wedge \neg W(v,u) = P(u,v)$.

Hence J is a congruence w.r.t. P.

Now let us consider the equivalence relation $\approx$ on X: $x \approx y \Leftrightarrow J(x,y) = 1$. Let $Y = X_{/\approx}$ be the quotient set of X w.r.t. $\approx$.

By Proposition 6.5 we can consider the following $*$-transitive fuzzy relation $\tilde{P}$ on Y:

(c) $\tilde{P}([x],[y]) = \bigvee_{u,v \in X} (J(u,x) * J(v,y) * P(u,v))$

for any $x, y \in X$. In accordance with Theorem 3.6 there exists a total $*$-transitive fuzzy relation R on Y such that $\tilde{P} \subseteq R$. Let us define the following fuzzy relation H on X:

(d) $H(x,y) = J(x,y) \oplus R([x],[y])$

for any $x, y \in X$. Let G be the $*$-transitive closure of H. Since $H(x,x) \geq J(x,x) = 1$ for each $x \in X$, H is reflexive so G is also reflexive. We also have

$$G(x,y) \vee G(y,x) \geq H(x,y) \vee H(y,x) \geq R([x],[y]) \vee R([y],[x]) > 0$$

because R is total. Hence G is also total. Of course G is $*$-transitive. For any $B \in \mathcal{B}$ and $x \in B$ we shall establish the inequality

(e) $C(B)(x) \leq \bigwedge_{v \in X} (B(v) \to G(x,v))$.

In order to prove (e) it suffices to show that for any $v \in X$ the following inequality holds:

(f) $C(B)(x) \leq B(v) \to G(x,v)$.

By Lemma 3.1 (1) the inequality (f) is equivalent to the condition

(g) $C(B)(x) * B(v) \leq G(x,v)$.

Let $v \in X$. In accordance with the definition of R we have

(h) $C(B)(x) * B(v) \leq R(x,v) \leq W(x,v)$.

But

$H(x,v) = J(x,v) \oplus R([x],[v]) \geq J(x,v) \oplus \tilde{P}([x],[v]) = J(x,v) \oplus \bigvee_{s,t \in X} [J(x,s) * J(t,v) * P(s,t)] \geq J(x,v) \oplus [J(x,x) * J(v,v) * P(x,v)] = J(x,v) \oplus P(x,v)$ because $J(x,x) = J(v,v) = 1$. Using Lemma 6.2 one gets

$J(x,v) \oplus P(x,v) = [W(x,v) \wedge W(v,x)] \oplus [W(x,v) \wedge \neg W(v,x)] = (W(x,v) \oplus [W(x,v) \wedge \neg W(v,x)]) \wedge (W(v,x) \oplus [W(x,v) \wedge \neg W(v,x)]) \geq W(x,v) \wedge [W(v,x) \oplus W(x,v)] \wedge [W(v,x) \oplus \neg W(v,x)] = W(x,v) \wedge [W(v,x) \oplus W(x,v)] = W(x,v)$.
Thus $W(x,v) \leq J(x,v) \oplus P(x,v) \leq H(x,v)$, hence, by (h), we obtain

$C(B)(x) * B(v) \leq W(x,v) \leq H(x,v) \leq G(x,v)$.
Hence (g) was proved and C is $*$-semirational. ∎

Chapter 7

Degree of Dominance

In the literature of fuzzy preference relations there are several ways to define the dominance (see [16, 33, 43] for a detailed discussion). In general the dominance is related to a fuzzy preference relation. For a fuzzy preference relation there exist a lot of ways to define the degree of dominance of an alternative [4, 5, 6, 7, 16, 33, 38, 43].

The concept of dominance in [4] is related to the act of choice and is expressed in terms of the fuzzy choice function.

This chapter aims at introducing a notion of *degree of dominance* of an alternative x with respect to an available fuzzy subset S of the universe X of alternatives. The degree of dominance defined here refines Banerjee's notion of dominance [4]. Banerjee's notion of dominance expresses the dominant position of some alternatives in the set of alternatives. In the decision making processes a differentiation of the alternatives according to various criteria is most of the times necessary. In the real world there are cases when these criteria are vague due to the partial information that the decision-maker possesses. The representation of these vague criteria within the choice problems is done by the available fuzzy sets. If x is an alternative and S is an available fuzzy set that corresponds to a criterion then the degree of dominance $D_S(x)$ is a number that belongs to the unit interval. This number expresses the position of alternative x with respect to the other alternatives as a result of the act of choice. If an alternative has the degree of dominance equal to 1 then it will be dominant with respect to criterion S. With the degree of dominance one can establish a hierarchy of alternatives with respect to the criterion defined by S. The difference between Banerjee's notion of dominance and the degree of dominance introduced in this chapter is that our notion takes into account all the alternatives, not only the dominant ones. At the same time, when there are no dominant alternatives ($D_S(x) \neq 1$ for any $x \in X$) one can select the alternatives with the maximum degree of dominance.

Briefly, the economic motivation of introducing the degree of dominance resides in:

• the degree of dominance is a concept that allows for a direct hierarchy of alternatives in accordance with the criteria of choice;

• it is an instrument by which all available alternatives, not only the dominant ones can be ranked;

• it helps to formulate new axioms of congruence that refine Banerjee's [4];

• it offers simple computations for the ranking of alternatives in concrete problems (see Chapter 8).

The results of the chapter are true for Gödel t-norm.

Section 7.1 is an overview of the context in which Banerjee's concept of degree of dominance was formulated. Banerjee [4] formulates a fuzzy revealed preference theory for his fuzzy choice functions. He studies three congruence axioms $FC1$, $FC2$, $FC3$. In the same setting, Wang [62] establishes deeper connections between $FC1$, $FC2$, $FC3$. These three axioms are formulated in terms of dominance of an alternative x in an available set S of alternatives.

In Section 7.2 we introduce our notion of degree of dominance. One result of this section shows that under very loose conditions (particularly in Banerjee's case) the degree of dominance of x with respect to S can be expressed function of degree of dominance of x with respect to fuzzy subsets of type $[x, y]$, $y \in X$.

In Section 7.3, starting from the concept of degree of dominance defined in the previous section, the congruence axioms FC^*1, FC^*2, FC^*3 for the class of fuzzy choice functions defined in this thesis are formulated. In Banerjee's context, conditions $FC1$, $FC2$, $FC3$ are implied by these three axioms.

We prove that FC^*1 implies FC^*3 and FC^*2 implies FC^*3.

The degree of dominance allows us to formulate a new revealed preference axiom $WAFRP_D$. We prove that the axioms $WAFRP_D$ and FC^*1 are equivalent. This theorem parallels a result of Theorem 6.1 which asserts that the revealed preference axiom $WAFRP^\circ$ is equivalent to FC^*3.

Section 7.3 concludes with an example that shows the relevance of the concept of degree of dominance for the process of decision making, by establishing a ranking of alternatives with respect to multiple criteria.

The results of this chapter are based on our paper [22].

7.1 Dominance in Banerjee's framework

The process of decision making deals in real life with vague preferences, modelled by fuzzy relations. Orlovsky initiated a theory of choice based on fuzzy preference relations [38]. He defined a notion of degree of dominance as a mode of selecting the best alternatives. Several authors have proposed other notions that express the dominance of an alternative [5, 6, 7, 16, 33, 38, 43]. These notions start from a fuzzy preference relation.

In [4] Banerjee develops a theory of revealed preference for a class of choice function whose domain is the family of all non-empty finite subsets of a universe of alternatives X and whose range consists of non-zero fuzzy subsets of X. We need to emphasize that Banerjee's notion of dominance is directly related to the choice function, *not* to the fuzzy preference relation. The degree of dominance defined in this thesis extends Banerjee's notion of dominance.

In this section we make a short overview of the results in [4, 62].

Let X be a universal set of alternatives, $\mathcal{H}$ the family of non-empty finite subsets of X and $\mathcal{F}$ all non-zero fuzzy subsets of X with finite support.

Recall that a *Banerjee fuzzy choice function* is a function $C : \mathcal{H} \to \mathcal{F}$ such that *supp* $C(S) \subseteq S$ for any $S \in \mathcal{H}$. The *fuzzy revealed preference relation* R associated with a fuzzy choice function C is defined by ([4]):

(7.1.1) $R(x, y) = \bigvee\{C(S)(x) | S \in \mathcal{H}, x, y \in S\}$

for any $x, y \in X$. It is obvious that $C(S)(x) \leq R(x, y)$ for any $S \in \mathcal{H}$ and $x, y \in X$.

Let C be a fuzzy choice function, $S \in \mathcal{H}$ and $x \in S$. x is said to be *dominant* in S if $C(S)(y) \leq C(S)(x)$ for any $y \in S$. The dominance of x in S means that x has a higher potentiality of being chosen than the other elements of S. It is obvious that this definition of dominance is related to the act of choice, not to a preference relation.

Banerjee also considers a second type of dominance, associated to a fuzzy preference relation.

Let R be a fuzzy preference relation on X, $S \in \mathcal{H}$ and $x \in X$. x is said to be relation dominant in S in terms of R if $R(x, y) \geq R(y, x)$ for all $y \in S$.

Let $S \in \mathcal{H}$, $S = \{x_1, \ldots, x_n\}$. The *restriction* of R to S is $R|_S = (R(x_i, x_j))_{n \times n}$. Then we have the *composition* $R|_S \circ C(S) = \bigvee\limits_{j=1}^{n} (R(x_i, x_j) \wedge C(S)(x_j))$.

In [4] Banerjee introduced the following *congruence axioms* for a fuzzy choice function C:

$FC1$ For any $S \in \mathcal{H}$ and $x, y \in S$, if y is dominant in S then $C(S)(x) = R(x, y)$.

$FC2$ For any $S \in \mathcal{H}$ and $x, y \in S$, if y is dominant in S and $R(y, x) \leq R(x, y)$ then x is dominant in S.

$FC3$ For any $S \in \mathcal{H}$, $\alpha \in (0, 1]$ and $x, y \in S$, $\alpha \leq C(S)(y)$ and $\alpha \leq R(x, y)$ imply $\alpha \leq C(S)(x)$.

In [62], Wang proved that $FC3$ holds iff for any $S \in \mathcal{H}$, $R|_S \circ C(S) \subseteq C(S)$. Then $FC3$ is equivalent with any of the following statements:

∘ For any $S \in \mathcal{H}$ and $x \in S$, $\bigvee\limits_{y \in S} (R(x, y) \wedge C(S)(y)) \leq C(S)(x)$;

$\circ$ For any $S \in \mathcal{H}$ and $x, y \in S$, $R(x, y) \wedge C(S)(y) \leq C(S)(x)$.

In [62] it is proved that $FC1$ implies $FC2$, $FC3$ implies $FC2$ and $FC1$, $FC3$ are independent.

7.2 Degree of dominance

In this section we shall define a notion of degree of dominance in the framework of the fuzzy choice functions introduced in this thesis. This kind of dominance is attached to a fuzzy choice function and not to a fuzzy preference relation. It shows to what extent, as the result of the act of choice, an alternative has a dominant position among others.

We fix a fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$.

Recall that the fuzzy revealed preference relation R on X associated with C is defined by

$$R(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y))$$

for any $x, y \in X$ (cf. Definition 4.2 (i)).

Particularizing this definition for the case when C is a Banerjee choice function we obtain the fuzzy relation defined by (7.1.1).

As seen in the previous section, the concept of dominance appears essentially in the expression of congruence axioms $FC1$-$FC3$. We define now the *degree of dominance* of an alternative x with respect to a fuzzy subset S. This will be a real number that shows the position of x among the other alternatives.

Definition 7.1 *Let $S \in \mathcal{B}$ and $x \in X$. The degree of dominance* of x in S is given by

$$D_S(x) = S(x) \wedge \bigwedge_{y \in X} [C(S)(y) \to C(S)(x)]$$

$$= S(x) \wedge [(\bigvee_{y \in X} C(S)(y)) \to C(S)(x)].$$

If $D_S(x) = 1$ then we say that x is *dominant* in S.

Remark 7.1 *Let S be a crisp subset of X. Identifying S with its characteristic function we have the equivalences:*

$D_S(x) = 1$ *iff* $S(x) = 1$ *and* $C(S)(y) \leq C(S)(x)$ *for any* $y \in X$

iff $x \in S$ *and* $C(S)(y) \leq C(S)(x)$ *for any* $y \in S$.

This shows that in this case we obtain exactly the notion of dominance of Banerjee.

Remark 7.2 *In accordance with Definition 7.1, x is dominant in S iff* $S(x) = 1$ *and* $\bigvee_{y \in X} C(S)(y) = C(S)(x)$.

Remark 7.3 *Assume that C satisfies (H1), i.e. $C(S)(y_0) = 1$ for some $y_0 \in X$. In this case $\bigvee_{y\in X} C(S)(y) = 1$ therefore $D_S(x) = C(S)(x)$.*

Lemma 7.1 *If $[x, y] \in \mathcal{B}$ then $D_{[x,y]}(x) = C([x, y])(y) \to C([x, y])(x)$.*

Proof. Since $C([x, y]) \subseteq [x, y]$ we have $C([x, y])(z) = 0$ for $z \notin \{x, y\}$. Then

$$D_{[x,y]}(x) = [x, y](x) \wedge \bigwedge_{z\in X} [C([x, y])(z) \to C([x, y])(x)]$$
$$= [C([x, y])(x) \to C([x, y])(x)] \wedge [C([x, y])(y) \to C([x, y])(x)]$$
$$= C([x, y])(y) \to C([x, y])(x). \blacksquare$$

Proposition 7.1 *For any $S \in \mathcal{B}$ and $x, y \in X$ we have*
(i) $C(S)(x) \leq D_S(x) \leq S(x)$;
(ii) $S(x) \wedge D_S(y) \wedge [C(S)(y) \to C(S)(x)] \leq D_S(x)$.

Proof. (i) According to Lemma 3.1 (4), $C(S)(x) \leq (\bigvee_{y\in X} C(S)(y)) \to C(S)(x)$
hence $C(S)(x) \leq S(x) \wedge [(\bigvee_{y\in X} C(S)(y)) \to C(S)(x)] = D_S(x)$.

(ii) By Lemma 3.1 (12) the following inequality holds for any $z \in X$:
$[C(S)(z) \to C(S)(y)] \wedge [C(S)(y) \to C(S)(x)] \leq C(S)(z) \to C(S)(x)$.
Thus

$$[C(S)(y) \to C(S)(x)] \wedge D_S(y) \wedge S(x) =$$
$$= [C(S)(y) \to C(S)(x)] \wedge \bigwedge_{z\in X} [C(S)(z) \to C(S)(y)] \wedge S(x) \wedge S(y) =$$
$$= S(y) \wedge S(x) \wedge \bigwedge_{z\in X} ([C(S)(z) \to C(S)(y)] \wedge [C(S)(y) \to C(S)(x)]) \leq$$
$$\leq S(x) \wedge \bigwedge_{z\in X} [C(S)(z) \to C(S)(x)] = D_S(x). \blacksquare$$

Remark 7.4 *By Proposition 7.1, $D_S(x) > 0$ for some $x \in X$. Then the assignment $S \mapsto D_S$ is a fuzzy choice function $D : \mathcal{B} \to \mathcal{F}(X)$. According to Remark 7.3, if C satisfies (H1) then $C = D$. It implies that the study of the degree of dominance is interesting for the case when hypothesis (H1) does not hold.*

Remark 7.5 *For $S \in \mathcal{B}$ and $x \in X$ we define the sequence $(D^n_S(x))_{n\geq 1}$ by induction:*
$D^1_S(x) = D_S(x)$; $D^{n+1}_S(x) = S(x) \wedge \bigwedge_{y\in X} [D^n_S(y) \to D^n_S(x)]$.

By Proposition 7.1 (i) we have $C(S)(x) \leq D_S^1(x) \leq \ldots \leq D_S^n(x) \leq \ldots \leq D_S^\infty(x) \leq S(x)$, *where* $D_S^\infty(x) = \bigvee_{n=1}^{\infty} D_S^n(x)$. *The assignments* $S \mapsto D_S^n$, $n \geq 1$ *and* $S \mapsto D_S^\infty$ *provide new fuzzy choice functions.*

The following definition generalizes Banerjee's notion of dominant relation in S in terms of R.

Definition 7.2 *Let* Q *be a fuzzy preference relation on* X, $S \in \mathcal{B}$ *and* $x \in X$. *The degree of dominance* of x in S in terms of Q is defined by

$$D_S^Q(x) = S(x) \wedge \bigwedge_{y \in X} [(S(y) \wedge Q(y,x)) \to Q(x,y)]$$

If $D_S^Q(x) = 1$ then we say that x is *dominant in* S *in terms of* Q .

Example 7.1 Consider the set of alternatives $X = \{x, y\}$ and the criterion $S = a\chi\{x\} + b\chi\{y\}$ for which the choice function is given by $C(S) = \alpha\chi\{x\} + \beta\chi\{y\}$, $0 < \alpha < a < 1$, $0 < \beta < b < 1$.

We intend to calculate the sequences $(D_S^n(x))_{n \geq 1}$, $(D_S^n(y))_{n \geq 1}$.

Case $\beta \leq \alpha$

$D_S^1(x) = D_S(x) = a \wedge (\beta \to \alpha) = a$

$D_S^1(y) = D_S(y) = b \wedge (\alpha \to \beta) = b \wedge \beta = \beta$

$D_S^2(x) = a \wedge (\beta \to a) = a$, $D_S^2(y) = b \wedge (a \to \beta)$

$D_S^3(x) = a$, $D_S^3(y) = b \wedge (a \to \beta)$

In general $D_S^n(x) = a$, $D_S^n(y) = b \wedge (a \to \beta)$ for $n \geq 2$.

The case $\alpha \leq \beta$ is treated analogously.

7.3 New congruence axioms

The congruence axioms $FC1$, $FC2$, $FC3$ play an important role in Banerjee's theory of revealed preference. The formulation of $FC1$, $FC2$ uses the notion of dominance and $FC3$ is a generalization of Weak Congruence Axiom (WCA).

In this section we introduce the congruence axioms FC^*1, FC^*2, FC^*3 which are refinements of axioms $FC1$, $FC2$, $FC3$. Axioms FC^*1 and FC^*2 are formulated in terms of degree of dominance. FC^*3 is Weak Fuzzy Congruence Axiom ($WFCA$) defined in Section 4.1.

FC^*1 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$S(x) \wedge D_S(y) \leq R(x,y) \to C(S)(x)$.

FC^*2 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$S(x) \wedge D_S(y) \wedge (R(y,x) \to R(x,y)) \leq D_S(x)$.

FC^*3 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$S(x) \wedge C(S)(y) \wedge R(x,y) \leq C(S)(x)$.

The form FC^*1 is derived from FC^*3 by replacing $D_S(y)$ by $C(S)(y)$. By Remarks 7.3 and 7.4, $D_S(x)$ (resp. $D_S(y)$) can be viewed as a substitute of $C(S)(x)$ (resp. $C(S)(y)$).

If hypothesis $(H1)$ holds, then by Remark 7.3, $D_S(y) = C(S)(y)$ and by Lemma 3.1 (1) axioms FC^*1 and FC^*3 are equivalent.

Remark 7.6 *Let $S \in \mathcal{B}$ and $x, y \in X$. Thus*

$C(S)(x) \wedge S(x) \wedge D_S(y) = C(S)(x) \wedge D_S(y)$

$= C(S)(x) \wedge S(y) \wedge \bigwedge_{z \in X} [C(S)(z) \to C(S)(y)]$

$\leq C(S)(x) \wedge S(y) \leq R(x,y)$

hence, by Lemma 3.1 (1), $S(x) \wedge D_S(y) \leq C(S)(x) \to R(x,y)$. Therefore, if FC^*1 holds then $S(x) \wedge D_S(y) \leq R(x,y) \leftrightarrow C(S)(x)$.

Remark 7.7 *Assume FC^*3 holds. Then for any $S \in \mathcal{B}$ and $x \in X$:*

$\bigvee_{y \in X} [S(x) \wedge C(S)(y) \wedge R(x,y)] \leq C(S)(x)$.

Assume that R is reflexive (for example if $(H1)$, $(H2)$ hold).

Since $C(S)(x) \leq S(x) \wedge C(S)(x) \wedge R(x,x)$ it follows that

$C(S)(x) = \bigvee_{y \in X} [S(x) \wedge C(S)(y) \wedge R(x,y)]$.

Remark 7.8 *Notice that FC^*3 appears under the name $WFCA$ (Weak Fuzzy Congruence Axiom).*

Proposition 7.2 *$FC^*1 \Rightarrow FC^*3$.*

Proof. $S(x) \wedge C(S)(y) \wedge R(x,y) \leq S(x) \wedge D_S(y) \wedge R(x,y) \leq C(S)(x)$, hence $FC^*1 \Rightarrow FC^*3$. ∎

Example 7.2 shows that FC^*3 does not necessarily imply FC^*1.

Example 7.2 Let $X = \{a, b\}$ and $A = a\chi\{x\} + b\chi\{y\}$, $C(A) = s\chi\{x\} + t\chi\{y\}$, $B = c\chi\{x\} + d\chi\{y\}$, $C(B) = u\chi\{x\} + w\chi\{y\}$, where $0 < s \leq a$, $0 < t \leq b$, $0 < u \leq c$, $0 < w \leq d$.

Then C is a fuzzy choice function on $\langle X, \mathcal{B} \rangle$ where $\mathcal{B} = \{A, B\}$.

Applying Definition 4.2 (i) one obtains:

$R(x,x) = s \vee u$, $R(x,y) = (s \wedge b) \vee (u \wedge d)$,

$R(y,x) = (t \wedge a) \vee (w \wedge c)$, $R(y,y) = t \vee w$.

We compute now the degrees of dominance:

$D_A(x) = a \wedge (t \to s)$

$D_A(y) = b \wedge (s \to t)$

$D_B(x) = c \wedge (w \to u)$

$D_B(y) = d \wedge (u \to w)$.

If FC^*1 holds then $D_A(x) \wedge A(x) \wedge R(x,x) \leq C(A)(x)$, i.e. $a \wedge (t \to s) \wedge (s \vee u) \leq s$. If we assume $t = w < s < u < a$ then $a \wedge (t \to s) \wedge (s \vee u) = a \wedge 1 \wedge u = u > s$, hence FC^*1 does not hold.

The axiom FC^*3 holds iff the following inequalities are verified:

(1) $A(x) \wedge C(A)(y) \wedge R(x,y) \leq C(A)(x)$

(2) $A(y) \wedge C(A)(x) \wedge R(y,x) \leq C(A)(y)$

(3) $B(x) \wedge C(B)(y) \wedge R(x,y) \leq C(B)(x)$

(4) $B(y) \wedge C(B)(x) \wedge R(y,x) \leq C(B)(y)$.

The first condition can be written: $a \wedge t \wedge [(s \wedge b) \vee (u \wedge d)] \leq s$. By distributivity this inequality is equivalent to $a \wedge t \wedge u \wedge d \leq s$. But $s \leq a$ hence $a \wedge t \wedge u \wedge d \leq s$ iff $t \wedge u \wedge d \leq s$.

We have proved that (1) is equivalent to

(1') $t \wedge u \wedge d \leq s$.

In a similar way (2), (3) and (4) are equivalent to

(2') $s \wedge w \wedge c \leq t$

(3') $w \wedge s \wedge b \leq u$

(4') $u \wedge t \wedge b \leq w$.

If $t = w < s < u < a$ the inequalities (1')-(4') are verified, hence FC^*3 holds. Thus FC^*3 does not necessarily imply FC^*1.

Proposition 7.3 *$FC^*3 \Rightarrow FC^*2$.*

Proof. Let $S \in \mathcal{B}$ and $x, y \in X$. By $R(y,x) \geq C(S)(y) \wedge S(x)$, Lemma 3.1 (10), (2) and FC^*3 we get for any $z \in X$:

$C(S)(z) \wedge S(x) \wedge [C(S)(z) \to C(S)(y)] \wedge [R(y,x) \to R(x,y)] =$
$= S(x) \wedge C(S)(z) \wedge C(S)(y) \wedge [R(y,x) \to R(x,y)] \leq$
$\leq C(S)(z) \wedge (C(S)(y) \wedge S(x)) \wedge [(C(S)(y) \wedge S(x)) \to R(x,y)] =$
$= C(S)(z) \wedge S(x) \wedge C(S)(y) \wedge R(x,y) \leq$
$\leq S(x) \wedge C(S)(y) \wedge R(x,y) \leq C(S)(x)$.

Hence, by Lemma 3.1 (1):

$S(x) \wedge [C(S)(z) \to C(S)(y)] \wedge [R(y,x) \to R(x,y)] \leq C(S)(z) \to C(S)(x)$.

Since this inequality holds for any $z \in X$ we obtain:

$$S(x) \wedge D_S(y) \wedge [R(y,x) \to R(x,y)] =$$
$$= S(x) \wedge S(y) \wedge \bigwedge_{z \in X} [C(S)(z) \to C(S)(y)] \wedge [R(y,x) \to R(x,y)] \leq$$
$$\leq S(x) \wedge \bigwedge_{z \in X} (S(x) \wedge [C(S)(z) \to C(S)(y)] \wedge [R(y,x) \to R(x,y)]) \leq$$
$$\leq S(x) \wedge \bigwedge_{z \in X} [C(S)(z) \to C(S)(x)] = D_S(x). \blacksquare$$

Proposition 7.4 *If FC^*1 holds then $D_S(x) \leq D_S^R(x)$ for any $S \in \mathcal{B}$ and $x \in X$.*

Proof. By absurdum, assume that there exist $S \in \mathcal{B}$ and $x \in X$ such that

$D_S(x) \not\leq D_S^R(x) = S(x) \wedge \bigwedge_{y \in X} [(S(y) \wedge R(y,x)) \to R(x,y)]$.

Since $D_S(x) \leq S(x)$, there exists $y \in X$ such that $D_S(x) \not\leq (S(y) \wedge R(y,x)) \to R(x,y)$ hence

$$D_S(x) \wedge S(y) \wedge R(y,x) \not\leq R(x,y), \text{ i.e. } R(x,y) < D_S(x) \wedge S(y) \wedge R(y,x).$$

According to FC^*1 we have $D_S(x) \wedge S(y) \wedge R(y,x) \leq C(S)(y)$. By the definition of $D_S(x)$, $D_S(x) \leq C(S)(y) \to C(S)(x)$, therefore

$$\begin{aligned} D_S(x) \wedge S(y) \wedge R(y,x) &\leq C(S)(y) \wedge [C(S)(y) \to C(S)(x)] \\ &= C(S)(x) \wedge C(S)(y) \leq C(S)(x) \wedge S(y) \\ &\leq R(x,y). \end{aligned}$$

We have obtained the contradiction $R(x,y) < R(x,y)$, hence the proposition is proved.

∎

Theorem 7.1 *Assume that the fuzzy choice function C fulfills (H2). Then axiom FC^*1 implies that for any $S \in \mathcal{B}$ and $x \in X$ we have*

$D_S(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \to D_{[x,y]}(x)]$.

Proof. By Proposition 7.4 one gets

$D_{[x,y]}(x) \leq D_{[x,y]}^R(x) \leq R(y,x) \to R(x,y)$

for any $x, y \in X$. Hence, by Lemma 3.1 (10), (11)

$S(y) \to D_{[x,y]}(x) \leq S(y) \to (R(y,x) \to R(x,y)) = ((S(y) \wedge R(y,x)) \to R(x,y)$.

Thus using Lemma 3.1 (11), (2)

$C(S)(y) \wedge [S(y) \to D_{[x,y]}(x)] \leq C(S)(y) \wedge [(S(y) \wedge R(y,x)) \to R(x,y)]$

$= C(S)(y) \wedge S(y) \wedge [S(y) \to (R(y,x) \to R(x,y))]$

$= C(S)(y) \wedge S(y) \wedge [R(y,x) \to R(x,y)]$

$= C(S)(y) \wedge [R(y,x) \to R(x,y)]$.

Since $R(y,x) \geq C(S)(y) \wedge S(x)$ we have $S(x) \wedge C(S)(y) = S(x) \wedge C(S)(y) \wedge R(y,x)$ hence, by Lemma 3.1 (11), (2) and Proposition 7.2 one gets

$S(x) \wedge C(S)(y) \wedge [S(y) \to D_{[x,y]}(x)]$

$= S(x) \wedge C(S)(y) \wedge R(y,x) \wedge [S(y) \to D_{[x,y]}(x)]$

$\leq S(x) \wedge C(S)(y) \wedge R(y,x) \wedge [R(y,x) \to R(x,y)]$

$= S(x) \wedge C(S)(y) \wedge R(y,x) \wedge R(x,y)$

$\leq S(x) \wedge C(S)(y) \wedge R(x,y) \leq C(S)(x)$.

By Lemma 3.1 (1) this yields

$S(x) \wedge [S(y) \to D_{[x,y]}(x)] \leq C(S)(y) \to C(S)(x)$.

This last inequality holds for each $y \in X$ hence

$$S(x) \wedge \bigwedge_{y\in X} [S(y) \to D_{[x,y]}(x)] = S(x) \wedge \bigwedge_{y\in X} (S(x) \wedge [S(y) \to D_{[x,y]}(x)]) \leq$$

$$\leq S(x) \wedge \bigwedge_{y\in X} [C(S)(y) \to C(S)(x)] = D_S(x).$$

Now we shall establish the converse inequality. We know that $C(S)(x) \wedge S(y) \leq R(x,y)$. By Proposition 7.4 the following inequalities hold

$D_S(x) \leq D_S^R(x) \leq (S(y) \wedge R(y,x)) \to R(x,y)$ for any $y \in X$.

Then by Lemma 3.1 (1):

$D_S(x) \wedge S(y) \wedge R(y,x) \leq R(x,y)$.

Since $C([x,y])(y) \leq R(y,x)$ we get

$D_S(x) \wedge S(y) \wedge C([x,y])(y) \leq D_S(x) \wedge S(y) \wedge R(y,x) \leq R(x,y)$.

Thus by FC^*3 we obtain

$D_S(x) \wedge S(y) \wedge C([x,y])(y) \leq C([x,y])(y) \wedge R(x,y) = [x,y](x) \wedge C([x,y])(y) \wedge R(x,y) \leq C([x,y])(x)$.

It follows that

$D_S(x) \wedge S(y) \wedge C([x,y])(y) \leq C([x,y])(x)$

hence, by Lemma 3.1 (1) and Lemma 7.1

$D_S(x) \wedge S(y) \leq C([x,y])(y) \to C([x,y])(x) = D_{[x,y]}(x)$.

Applying again Lemma 3.1 (1) we obtain $D_S(x) \leq S(y) \to D_{[x,y]}(x)$ for each $y \in X$ hence

$$D_S(x) \leq S(x) \wedge \bigwedge_{y\in X} [S(y) \to D_{[x,y]}(x)]. \ \blacksquare$$

The formulation of axiom FC^*3 has Lemma 2.1 in [62] as starting point. The following result establishes the equivalence of FC^*3 with a direct generalization of $FC3$.

Proposition 7.5 *The following assertions are equivalent:*

(1) The axiom FC^*3 holds;

(2) For any $S \in \mathcal{B}$, $x, y \in X$ and $\alpha \in (0,1]$,

$S(x) \wedge S(y) \wedge [\alpha \to C(S)(y)] \wedge [\alpha \to R(x,y)] \leq \alpha \to C(S)(x)$.

Proof. By Lemma 3.2 (1)

$S(x) \wedge S(y) \wedge [\alpha \to C(S)(y)] \wedge [\alpha \to R(x,y)] = S(x) \wedge S(y) \wedge [\alpha \to (C(S)(y) \wedge R(x,y))]$,

hence, by Lemma 3.1 (1) the inequality in (2) is equivalent to

$S(x) \wedge S(y) \wedge \alpha \wedge [\alpha \to (C(S)(y) \wedge R(x,y))] \leq C(S)(x)$.

According to Lemma 3.1 (2)

$S(x) \wedge S(y) \wedge \alpha \wedge [\alpha \to (C(S)(y) \wedge R(x,y))] = S(x) \wedge S(y) \wedge \alpha \wedge C(S)(y) \wedge R(x,y) = S(x) \wedge C(S)(y) \wedge R(x,y) \wedge \alpha$

because $C(S)(y) \leq S(y)$. Thus the inequality in (2) is equivalent to

(a) $S(x) \wedge C(S)(y) \wedge R(x,y) \wedge \alpha \leq C(S)(x)$.

Assuming FC^*3 holds, the inequality (a) also holds since $\alpha \leq 1$. Conversely, if in (a) one takes $\alpha = 1$ one obtains FC^*3. ∎

Definition 7.3 *Let C be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$. We define the fuzzy relation R_2 on X by*
$$R_2(x, y) = \bigwedge_{S \in \mathcal{B}} [(S(x) \wedge D_S(y)) \to C(S)(x)].$$

Remark 7.9 *Let C be a fuzzy choice function, $S \in \mathcal{B}$ and $x, y \in X$. By Definition 4.2 (i)*
$$R(x, y) \wedge S(x) \wedge D_S(y) = [\bigvee_{T \in \mathcal{B}} (C(T)(x) \wedge T(y))] \wedge S(x) \wedge D_S(y)$$
$$= \bigvee_{T \in \mathcal{B}} [S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y)].$$
Then FC^*1 is equivalent to the following statement
• For any $S, T \in \mathcal{B}$ and $x, y \in X$
$S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y) \leq C(S)(x)$.

In Section 4.1 the following revealed preference axiom was considered:
$WAFRP^\circ$ For any $S, T \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$[S(x) \wedge C(T)(x)] \wedge [T(x) \wedge C(S)(x)] \leq E(S \cap C(T), T \cap C(S))$.
Theorem 6.1 asserts that $WAFRP^\circ$ and $FC^*3 = WFCA$ are equivalent.

A problem is if we can find a similar result for condition FC^*1. In order to obtain an answer to this problem we introduce the following axiom:
$WAFRP_D$ For any $x, y \in X$ and $S, T \in \mathcal{B}$,
$[S(x) \wedge C(T)(x)] \wedge [T(y) \wedge D_S(y)] \leq I(S \cap C(T), T \cap C(S))$.

Theorem 7.2 *For a fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$ the following are equivalent:*

(i) C satisfies FC^*1;

(ii) $R \subseteq R_2$;

(iii) C satisfies $WAFRP_D$.

Proof. (i) ⇔ (ii). The following assertions are equivalent:
• $R \subseteq R_2$;
• For any $x, y \in X$:
$$\bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y)) \leq \bigwedge_{T \in \mathcal{B}} [(T(x) \wedge D_T(y)) \to C(T)(x)];$$
• For any $x, y \in X$ and $S, T \in \mathcal{B}$:
$C(S)(x) \wedge S(y) \leq (T(x) \wedge D_T(y)) \to C(T)(x)$.
• For any $x, y \in X$ and $S, T \in \mathcal{B}$:
$C(S)(x) \wedge S(y) \wedge T(x) \wedge D_T(y) \leq C(T)(x)$.

In accordance with Remark 7.9, (i) and (ii) are equivalent.

(iii) $\Rightarrow$ (i) Assume that C satisfies $WAFRP_D$. Let $x, y \in X$ and $S, T \in \mathcal{B}$. By $WAFRP_D$ one gets

$S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y) \leq I(S \cap C(T), T \cap C(S)) =$

$= \bigwedge_{u \in X} [(S(u) \wedge C(T)(u)) \to (T(u) \wedge C(S)(u))] \leq$

$\leq (S(x) \wedge C(T)(x)) \to (T(x) \wedge C(S)(x)) =$

$= [(S(x) \wedge C(T)(x)) \to T(x)] \wedge [(S(x) \wedge C(T)(x)) \to C(S)(x)] =$

$= (S(x) \wedge C(T)(x)) \to C(S)(x)$

because $(S(x) \wedge C(T)(x)) \to T(x) = 1$ (by Lemma 3.1 (5)). It follows that

$S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y) =$

$= [S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y)] \wedge (S(x) \wedge C(T)(x)) \leq C(S)(x)$

in accordance with Lemma 3.1 (1). According to Remark 7.9, C satisfies FC^*1.

(i) $\Rightarrow$ (iii) Assume C fulfills FC^*1. By Remark 7.9, for any $S, T \in \mathcal{B}$ and $x, y, u \in X$ we have

$S(x) \wedge T(y) \wedge D_S(y) \wedge C(T)(x) \wedge S(u) \wedge C(T)(u) \leq S(u) \wedge T(y) \wedge C(T)(u) \wedge D_S(y) \leq C(S)(u)$.

Thus, by Lemma 3.1 (1):

$S(x) \wedge T(y) \wedge D_S(y) \wedge C(T)(x) \leq (S(u) \wedge C(T)(u)) \to C(S)(u) =$

$= (S(u) \wedge C(T)(u)) \to (T(u) \wedge C(S)(u))$

This inequality is true for each $u \in X$, hence

$S(x) \wedge T(y) \wedge D_S(y) \wedge C(T)(x) \leq I(S \cap C(T), T \cap C(S))$

so C satisfies the axiom $WAFRP_D$. ∎

Let W be the transitive closure of the fuzzy preference relation R. We notice that axioms FC^*1-FC^*3 are expressed in terms of R. If R is replaced with W the following three congruence axioms are obtained:

SFC^*1 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$S(x) \wedge D_S(y) \leq W(x, y) \to C(S)(x)$.

SFC^*2 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$S(x) \wedge D_S(y) \wedge (W(y, x) \to W(x, y)) \leq D_S(x)$.

SFC^*3 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$S(x) \wedge C(S)(y) \wedge W(x, y) \leq C(S)(x)$.

SFC^*3 is exactly the congruence axiom $SFCA$ (Strong Fuzzy Congruence Axiom) defined in Section 4.1.

Proposition 7.6 $SFC^*1 \Rightarrow SFC^*3$.

Proof. Similar to the proof of Proposition 7.2. ∎

Proposition 7.7 $SFC^*3 \Rightarrow SFC^*2$.

Proof. Let $S \in \mathcal{B}$ and $x, y \in X$. We observe that $W(x,y) \geq R(x,y) \geq C(S)(x) \wedge S(y)$. Therefore, using $SFC^{*}3$ we get for any $z \in X$:

$$C(S)(z) \wedge S(x) \wedge [C(S)(z) \to C(S)(y)] \wedge [W(y,x) \to W(x,y)] =$$
$$= S(x) \wedge C(S)(z) \wedge C(S)(y) \wedge [W(y,z) \to W(x,y)] \leq$$
$$\leq C(S)(z) \wedge (C(S)(y) \wedge S(x)) \wedge [(C(S)(y) \wedge S(x)) \to W(x,y)] =$$
$$= C(S)(z) \wedge (C(S)(y) \wedge S(x)) \wedge W(x,y) \leq$$
$$\leq S(x) \wedge C(S)(y) \wedge W(x,y) \leq C(S)(x).$$

We proceed next as in the proof of Proposition 7.3. ∎

Extending our results from Chapter 6 we have introduced new axioms of congruence expressed in terms of the degree of dominance and we have established relationships between them and some axioms of revealed preference. These results can be summarized in Figure 7.1.

$$\begin{tikzcd}
SFC^{*}1 \arrow[r] \arrow[d] & SFC^{*}3 \arrow[r] \arrow[d] & SFC^{*}2 \arrow[d] \\
FC^{*}1 \arrow[r] \arrow[d, leftrightarrow] & FC^{*}3 \arrow[r] \arrow[d, leftrightarrow] & FC^{*}2 \arrow[d, leftrightarrow] \\
WAFRP_{D} \arrow[r] & WAFRP^{0} \arrow[r, dash] & ?
\end{tikzcd}$$

Figure 7.1: Axioms of revealed preference and congruence

The bottom line of the diagram contains axioms of revealed preference; the other lines contain axioms of congruence. An open problem is to complete the above diagram with an axiom of revealed preference equivalent to $FC^{*}2$.

Example 7.3 will clarify the notion of degree of dominance. Given a set of alternatives and a set of criteria we want to establish the hierarchical structure induced by each criterion. Finally we define an aggregated degree of dominance and we determine the overall hierarchy.

Example 7.3 Consider a universe of alternatives $X = \{x_1, x_2, x_3, x_4, x_5\}$ and a set of criteria $\mathcal{B} = \{S_1, S_2, S_3\}$, where

$S_1 = 0.3\chi\{x_1\} + 0.5\chi\{x_2\} + 0.2\chi\{x_3\} + 0.2\chi\{x_4\} + 0.6\chi\{x_5\}$, $C(S_1) = 0.1\chi\{x_1\} + 0.2\chi\{x_2\} + 0.2\chi\{x_3\} + 0.1\chi\{x_4\} + 0.3\chi\{x_5\}$;

$S_2 = 0.2\chi\{x_1\} + 0.4\chi\{x_2\} + 0.3\chi\{x_3\} + 0.5\chi\{x_4\} + 0.4\chi\{x_5\}$, $C(S_2) = 0.1\chi\{x_1\} + 0.3\chi\{x_2\} + 0.2\chi\{x_3\} + 0.4\chi\{x_4\} + 0.1\chi\{x_5\}$;

$S_3 = 0.4\chi\{x_1\} + 0.3\chi\{x_2\} + 0.2\chi\{x_3\} + 0.2\chi\{x_4\} + 0.6\chi\{x_5\}$, $C(S_3) = 0.3\chi\{x_1\} + 0.2\chi\{x_2\} + 0.2\chi\{x_3\} + 0.1\chi\{x_4\} + 0.4\chi\{x_5\}$.

The degrees of dominance of alternative $x_i, i = 1, \ldots, 5$ with respect to criterion $S_j, j = 1, \ldots, 3$ calculated according to Definition 7.1 are represented in the following table:

$D_{S_i}(x_j)$	x_1	x_2	x_3	x_4	x_5
S_1	0.1	0.2	0.2	0.1	0.6
S_2	0.1	0.3	0.2	0.5	0.1
S_3	0.3	0.2	0.2	0.1	0.6

Figure 7.2 indicates the hierarchy of alternatives for each of the three criteria. According to criterion S_1, the alternative with the greatest potentiality of being chosen is x_5, alternatives x_2 and x_3 can be equally chosen with the degree 0.2 and alternatives x_1 and x_4 have the least chance of being chosen 0.1, etc.

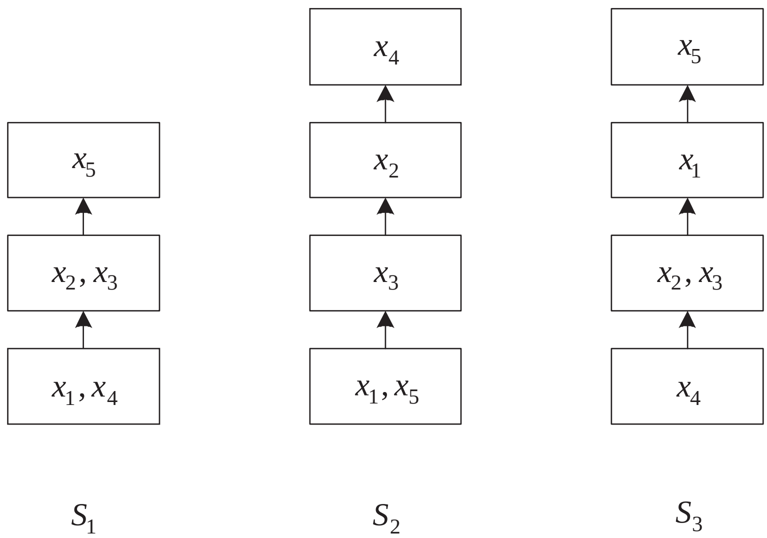

Figure 7.2: The hierarchy of alternatives induced by criteria S_1, S_2, S_3

Define the aggregated degree of dominance of an alternative x: $D(x) = \frac{1}{n}\sum\limits_{S\in\mathcal{B}} D_S(x)$, where n=card $\mathcal{B}$.

In our case:

$D(x_1) = \frac{0.5}{3}$, $D(x_2) = \frac{0.7}{3}$, $D(x_3) = 0.2$, $D(x_4) = \frac{0.7}{3}$, $D(x_5) = \frac{1.3}{3}$

The hierarchy of alternatives determined by the aggregated degree of dominance is given by $D(x_1) < D(x_3) < D(x_2) = D(x_4) < D(x_5)$. Overall, we notice that alternative x_5 has the greatest chances of being selected and alternative x_1 the least.

Chapter 8

Applications

In making a choice, a set of alternatives and a set of criteria are usually needed.

According to [67], the alternatives and the criteria are defined as follows:

"Alternatives are usually *mutually exclusive* activities, objects, projects, or models of behaviour among which a choice is possible".

"Criteria are measures, rules and standards that guide decision making. Since decision making is conducted by selecting or formulating different attributes, objectives or goals, all three categories can be referred as criteria. That is, criteria are all those attributes, objectives or goals which have been judged relevant in a given decision situation by a particular decision maker (individual or group)".

In the real world the human vagueness and imprecision prevail and fuzzy logic is introduced as a tool for modelling the acts of choice.

In this chapter we shall present three possible applications of fuzzy revealed preference theory. They represent models of decision making based on the ranking of alternatives according to fuzzy choices. An agent's decision is based on the ranking of alternatives according to different criteria. This ranking is obtained by using fuzzy choice problems and the instrument by which it is established is the degree of dominance associated with a fuzzy choice function.

8.1 Application 1

8.1.1 Negotiations on electronic markets

Electronic markets are online markets whose actors and actions correspond to the conventional ones. An electronic market is analogous to a shop where a trading activity takes place. Electronic markets exist in various forms, the most common being the online shopping market. The main actors on an electronic market are buyers and sellers. They meet online, exchange information, negotiate and trade.

There are several distinctions between the physical and the virtual market. They differ in size and in the range of services. The main advantage of an electronic market over a physical one is that the participants (especially buyers) have a larger diversity of information on existing products and services and quick and easy access to them. This gives the opportunity to the buyer (=consumer) to specify his preferences and to determine his preference-based choices. Consequently the seller will adapt to the consumer's preferences and will try to find the best mechanisms to satisfy customer's demand.

In real life usually the preferences and the choices of both buyers and sellers are vague, therefore the contact between buyers and sellers on the electronic market takes the form of a multi-stage negotiation, characterized by repeated offers and counter-offers. At the very end, the consumer will make an exact choice, but in the process of decision making, this choice is vague. This vagueness of preferences and choices arises from various reasons, for example the incapacity of the buyer to choose among different products due to subjective reasons, or for negotiation issues (attributes of products or value-added services e.g. price, warranties, delivery times, return policies cf. Kurbel et al. [30]). Papers [30, 34] make an exhaustive analysis and classification of multi-agent electronic markets and different types of negotiations with vague preferences.

There are several criteria to classify multi-agent electronic markets [35].

A first classification of multi-agent electronic markets is according to the number of actors that are involved in the process of negotiation. There can be one buyer and multiple sellers, one seller and multiple buyers, multiple buyers and multiple sellers. These possible three situations are illustrated in Figure 8.1, where B denotes the buyer and S denotes the seller [34, 25].

A second classification regards the type of the electronic market: business-to-business, business-to-consumer and consumer-to-consumer.

A third classification takes into account the number of negotiation issues. There exist negotiations over one issue (usually this is the price) or several issues.

A fourth classification concerns the level of vagueness of the preferences on the negotiation issues. These preferences can be crisp or fuzzy; the fuzziness of the preferences generates the offers and the counter-offers in a negotiation [34]. If all preferences are crisp, the negotiation becomes very simple, taking the form of a comparison of preferences from each party's side; in this case counter-offers are not present. In complex negotiations preferences of consumers and sellers are usually fuzzy or combined.

Papers [34, 35] overview different types of electronic markets, with crisp and/or vague preferences of buyers and/or sellers. We recall here one example of electronic markets as it has been described by [34, 30, 35].

This example refers to *Frictionless Commerce* online shopping market (wwww.frictionless.com), an electronic market based on search and comparison of products according to their price and value. The negotiation in the

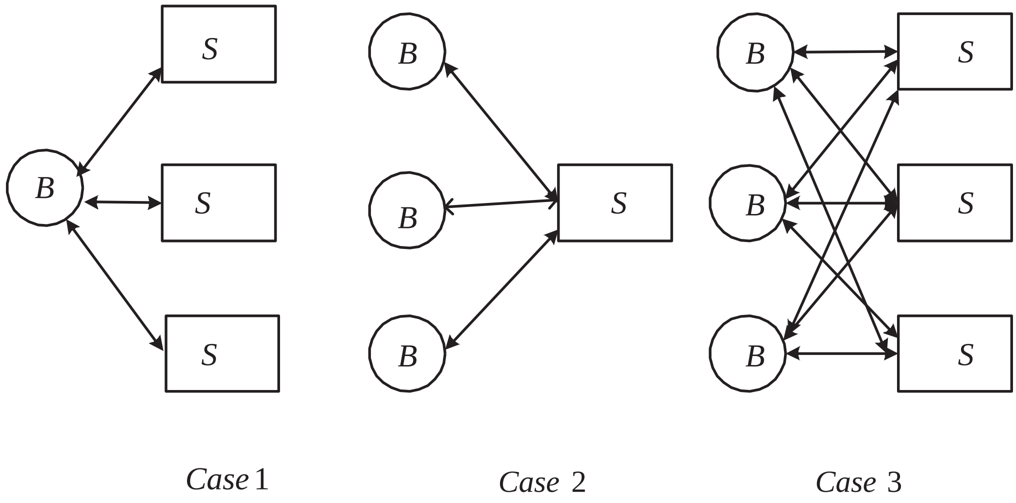

Figure 8.1: Three types of negotiation on electronic markets (source [34, 25])

framework of *Frictionless Commerce* involves one seller and multiple buyers. Consumers' preferences are vague, while the seller's proposals are crisp. This electronic market will help consumers rank their fuzzy preferences by means of multi-attribute utility theory.

8.1.2 A multi-issue negotiation model

In [15] Van de Walle et al. discuss a negotiation situation on an electronic market with one seller and multiple buyers (Case 2 in Figure 8.1). The negotiation takes place according to multiple criteria that regard attributes of the products or services negotiated. In this subsection we will give a short description of their model.

A person wants to sell a product on an electronic market and he has to choose one buyer among several buyers. The negotiation issues here refer to the multiple criteria that consist of the attributes of the product (e. g. price, delivery times etc.). The negotiator (=seller) receives offers from the potential buyers and consequently, he defines his preferences, represented by a fuzzy preference relation on the set of alternatives (=buyers). By analyzing this fuzzy preference relation by means of α-cut levels, at different stages of the negotiation, the offers of the buyers will be ranked.

The simplest and most common solution for choosing the best buyer is to evaluate each offer by associating a weight to every negotiable issue, to calculate a weighted sum for each offer and to choose the offer with the highest value. This solution assumes that all offers are comparable. In the model proposed in [15] the ranking of the offers is partial, based on pairwise comparisons, compared to other approaches where the rank order was linear. This approach is closer to real life situations.

Next we give the main steps of the approach in [15].

Suppose there are m criteria $C_1, \ldots, C_m$ and n offers $x_1, \ldots, x_n$ of n buyers. The preference of the seller with respect to the values specified by the buyers are represented in a matrix $P = (p_{ij})_{n \times m}$. The real number $p_{ij} \in [0, 1]$ shows the degree of preference of the seller with respect to the value offered by x_i on criterion C_j. The row $P_i = (p_{i1}, \ldots, p_{im})$ is the vector of the seller's preferences with respect to the i-th buyer. A pairwise comparison between vectors P_i and P_j, $i, j = 1, \ldots, n$ is required in order to establish a ranking of the offers. For such vectors P_i and P_j the real number $Inc(P_i, P_j)$ shows the degree of inclusion of P_i in P_j:

(8.2.1) $Inc(P_i, P_j) = \frac{1}{m} \sum_{k=1}^{m} min(1, 1 - P_{ik} + P_{jk}).$

This degree of inclusion is summarized in the matrix $D = (Inc(P_i, P_j))_{n \times n}$ that is a fuzzy preference relation on the set of alternatives $X = \{x_1, \ldots, x_n\}$. Notice that P is reflexive but not transitive. By computing its transitive closure Q one obtains a fuzzy preorder. The properties of matrix Q allow for a better interpretation of the offers [15].

Starting from Q, we consider the α-cuts at various levels, obtaining crisp preorders Q_α, $\alpha \in (0, 1]$ [16]. Each preorder Q_α leads to an equivalence relation E_α:

(8.2.2) $(x, y) \in E_\alpha$ iff $(x, y) \in Q_\alpha \wedge (y, x) \in Q_\alpha$

Denote by $[x]_\alpha$ the equivalence class of $x \in X$ with respect to E_α. Define the order relation $\leq_\alpha$ on the quotient set X/E_α of the equivalence classes:

(8.2.3) $[x]_\alpha \leq_\alpha [y]_\alpha$ iff $(x, y) \in Q_\alpha$.

With $\leq_\alpha$ one can establish a hierarchy of offers for each criterion.

[15] gives a numerical illustration of this model by Hasse diagrams.

8.1.3 A negotiation model based on choices

This subsection studies a modified form of the problem described in the previous subsection. If there the ranking of alternatives is based on a fuzzy preference relation, here this ranking will result from a fuzzy choice function. We will arrive at a choice problem, and for the ranking of offers for each criterion we will apply the degree of dominance introduced in Chapter 7. Our results build on the Gödel t-norm.

We start from a similar situation as above and we indicate a way to obtain the matrix P of fuzzy preferences. A person wants to sell a product described by some attributes and he registers it on an electronic market. He has to choose one buyer among several buyers for his product, and his choice will be made according to the offers that he gets. The offers are made with respect to the criteria given by the attributes of the product. Therefore we are again in face of a multiple-criteria negotiation problem with one seller and multiple buyers.

Our treatment of the problem differs from [15]. In [15] the ranking of alternatives is derived from a somehow given fuzzy preference relation and not from the act of choice. In our case the ranking of alternatives is based on a choice function associated with a preference relation P and on some (crisp) available sets of alternatives defined by the thresholds $e_1, \ldots, e_n$.

The negotiation issues (=criteria) are denoted by $C_1, \ldots, C_m$. The seller proposes the values $b_1, \ldots, b_m$ for his product. The i-th potential buyer of the product responds with the offer $a_{i1}, \ldots, a_{im}$. In order to have a fuzzy choice problem, we can assume that $0 < a_{ij} \leq 1$, respectively $0 < b_j \leq 1$ for any $i = 1, \ldots, n$, $j = 1, \ldots, m$. Otherwise, this can be obtained by dividing all values a_{ij}, b_j by a convenient power of 10. In this way the values a_{ij}, b_j preserve their initial values, hence the preferences remain the same.

These values are summarized in the following table:

	C_1	C_2	$\ldots$	C_m
Seller's offer	b_1	b_2	$\ldots$	b_m
First buyer's offer	a_{11}	a_{12}	$\ldots$	a_{1m}
Second buyer's offer	a_{21}	a_{22}	$\ldots$	a_{2m}
$\ldots$				
n-th buyer's offer	a_{n1}	a_{n2}	$\ldots$	a_{nm}

The values proposed by the seller generally differ from the values proposed by the buyers. The distance $|b_j - a_{ij}|$ measures the closeness of the values that belong to the buyers and to the seller with respect to criterion j. If $|b_j - a_{ij}| < |b_j - a_{kj}|$ then obviously the seller will prefer the offer a_{ij} to the offer a_{kj}. The number $p_{ij} = 1 - |b_j - a_{ij}|$ will represent the degree to which the seller prefers his value b_j to the value a_{ij} suggested by the buyer i on criterion j. If the number p_{ij} tends to 1, then a_{ij} reaches b_j therefore the seller and the buyer i reach consensus with respect to negotiation issue j. When p_{ij} reaches 0 then there will be a disensus between them. The values p_{ij} are represented in a matrix P with n rows and m columns .

Consider now similarly as in the previous model [15] the matrix $D = (Inc(P_i, P_j))_{n\times n}$, where P_i is the i-th row and $Inc(P_i, Pj)$ is given by (8.2.1). According to [15], matrix D on the set of the buyers comprises the inclusion degrees of the rows P_i. Denote by $Q = (q_{ij})_{n\times n}$ the transitive closure of D.

In different stages of negotiation, it is possible that the buyers'offers are exaggerated for the seller. Therefore we need to introduce the thresholds $e_1, \ldots, e_n \in (0, 1)$. If $|b_j - a_{ij}| \leq e_j$ then we say that the values a_{ij} are admissible. By introducing these thresholds, the buyers whose offers differ to a large extent from the seller's proposal are eliminated.

Define the crisp subsets $S_1, \ldots, S_m$ of X:

(8.3.1) $S_j = \{x_i \in X | |b_j - a_{ij}| \leq e_j\}$.

Take $\mathcal{B} = \{S_1, \ldots, S_m\}$ and $\langle X, \mathcal{B}\rangle$ is the *choice space* . In this manner we have obtained a context similar to Banerjee's [4], with the difference that here $\mathcal{B}$ contains only $S_1, \ldots, S_m$.

On the choice space $\langle X, \mathcal{B}\rangle$ define the fuzzy choice function C by

(8.3.2) $C(S_j)(x_i) = \bigwedge\limits_{y\in S_j} Q(x_i, y)$

for any $j = 1, \ldots, m$ and $x_i \in S_j$.

Remark 8.1 *Note that by identifying a crisp subset of X with its characteristic function, the fuzzy choice function defined in (8.3.2) is a particular case of the class of fuzzy choice functions introduced by (4.2.2) (see also Remark 4.8).*

In this context the degree of dominance $D_{S_j}(x_i)$ gets the form:

(8.3.3) $$D_{S_j}(x_i) = \bigwedge_{y\in X} [C(S_j)(y) \to C(S_j)(x_i)] = \bigwedge_{y\in S_j} [C(S_j)(y) \to C(S_j)(x_i)].$$

The degree of dominance $D_{S_j}(x_i)$ reflects the dominance of alternative x_i with respect to criterion j. Ordering the set $\{D_{S_j}(x_1), \ldots, D_{S_j}(x_n)\}$ one obtains a ranking of alternatives $x_1, \ldots, x_n$ with respect to criterion j.

The fuzzy choice function defined by us is based on the preferences p_{ij} derived from the buyers'offers and on the values b_j proposed by the seller; obviously the seller's choices are potential. The fuzzy choice function gives the seller information about the ranking of alternatives by means of potential choices. This information helps the seller to negotiate with those buyers situated in a superior position in this hierarchy.

8.1.4 A numerical illustration

In this subsection we shall illustrate the theoretical analysis from above with a simple example.

Consider the case of a seller that registers a product on an electronic market and describes it according to three criteria, for example *Price* , *Delivery time* and *Warranty* . Five buyers are interested in buying this product. The following table summarizes the reservation prices proposed by the seller and the reservation prices of the buyers:

	C_1	C_2	C_3
Seller's offer	0.5	0.6	0.7
First buyer's offer	0.47	0.57	0.41
Second buyer's offer	0.45	0.55	0.55
Third buyer's offer	0.39	0.56	0.68
Fourth buyer's offer	0.48	0.58	0.67
Fifth buyer's offer	0.46	0.53	0.66

The values of the thresholds are $e_1 = 0.05$, $e_2 = 0.06$ and $e_3 = 0.07$.

Next we present the main results.

The matrix P of the preferences of the seller has the form

$$P = \begin{pmatrix} 0.97 & 0.97 & 0.71 \\ 0.95 & 0.95 & 0.85 \\ 0.89 & 0.96 & 0.98 \\ 0.98 & 0.98 & 0.97 \\ 0.96 & 0.93 & 0.96 \end{pmatrix}.$$

For example the value $p_{11} = 0.97$ represents the degree to which the seller prefers his reservation price to the first buyer's reservation price with respect to the first criterion (in this case the degree of preference is very high). If we compare the reservation prices of the five potential buyers with respect to the first criterion of the product and look at the first column of P, to a high degree the seller will prefer to keep his reservation price to the offers of the buyers.

We compute next the degrees of inclusion $Inc(P_i, P_j)$ according to (8.2.1) and we obtain the matrix D:

$$D = \begin{pmatrix} 1 & 0.98 & 0.97 & 1 & 0.98 \\ 0.95 & 1 & 0.98 & 1 & 0.99 \\ 0.91 & 0.95 & 1 & 0.99 & 0.98 \\ 0.90 & 0.94 & 0.96 & 1 & 0.97 \\ 0.91 & 0.96 & 0.97 & 1 & 1 \end{pmatrix}.$$

The transitive closure of D is matrix Q:

$$Q = \begin{pmatrix} 1 & 0.98 & 0.98 & 1 & 0.98 \\ 0.95 & 1 & 0.98 & 1 & 0.99 \\ 0.95 & 0.96 & 1 & 0.99 & 0.98 \\ 0.95 & 0.96 & 0.97 & 1 & 0.97 \\ 0.95 & 0.96 & 0.97 & 1 & 1 \end{pmatrix}.$$

The crisp sets S_1, S_2 and S_3 corresponding to the three criteria that characterize the product are given by (8.3.1):

$S_1 = \{x_1, x_2, x_4, x_5\}$
$S_2 = \{x_1, x_2, x_3, x_4\}$
$S_3 = \{x_1, x_2, x_3\}$

The corresponding fuzzy choice functions are:

$C(S_1) = 0.98\chi\{x_1\} + 0.95\chi\{x_2\} + 0.95\chi\{x_4\} + 0.95\chi\{x_5\}$
$C(S_2) = 0.98\chi\{x_1\} + 0.95\chi\{x_2\} + 0.95\chi\{x_3\} + 0.95\chi\{x_4\}$
$C(S_3) = 0.98\chi\{x_1\} + 0.95\chi\{x_2\} + 0.95\chi\{x_3\}$.

The corresponding degrees of dominance are represented in the table:

$D_{S_i}(x_j)$	x_1	x_2	x_3	x_4	x_5
S_1	1	0.95	0	0.95	0.95
S_2	1	0.95	0.95	0.95	0
S_3	1	0.95	0.95	0	0

The aggregated degrees of dominance are:

$D(x_1) = 1$, $D(x_2) = 0.95$, $D(x_3) = 0.63$, $D(x_4) = 0.63$, $D(x_5) = 0.31$.
Therefore the order of the offers is
$D(x_5) < D(x_3) = D(x_3) < D(x_2) < D(x_1)$.
According to each of the three criteria, the seller will choose the first offer that is dominant. This option will remain if the overall degree of dominance is considered.

8.1.5 Discussion

In this section we have developed a simple multi-issue negotiation model with one seller and multiple buyers.

The vagueness of the preferences and choices that characterizes the transactions on electronic markets and that is specific to the process of decision-making is very difficult to model. So far there exist some agent-based e-marketplaces that try to represent this vagueness. Comprehensive classifications and descriptions of existing agent-based e-marketplaces with crisp and/or fuzzy preferences of sellers and buyers are made by [34, 35, 30].

In the negotiation stages between the buyers'offers and the seller's proposals big differences might exist. By introducing the thresholds $e_1, \ldots, e_n$, these differences will be eliminated. By the presence of the thresholds our model differs from [15]. Another difference consists in the way the alternatives are ranked. In [15] the ranking is based on a preference relation and here the ranking is based on the degree of dominance, that directly expresses the act of choice.

8.2 Application 2

8.2.1 Adverse selection

One characteristic of the markets is the asymmetry of information existing between two parties in a transaction. Together with moral hazard, a typical problem that usually appears as a result of information asymmetry is known as *adverse selection* .

We shortly describe this typical situation. Asymmetry of information appears when one party in a transaction is better informed than the other party. It mainly concerns the attributes of a product.

Asymmetry of information is the main reason for the phenomenon known in the literature as adverse selection. The most well-known situation of adverse selection has been analyzed by Akerlof [1] for the market of used cars(=lemons). Two types of cars are sold on a market at the same price: used cars and new cars. The sellers perfectly know the quality of the cars while the buyers do not know what type of cars they buy. Consequently only the used cars will dominate the market and they will drive away the good-quality cars. This situation has negative consequences, in the sense that the market cannot allocate the products efficiently and that leads to market failures.

In this section we propose a model of decision making based on the ranking of alternatives according to fuzzy choices. Here the criteria are derived from the partial information existing in the model.

Our model tries to correct the adverse selection that appears as the result of asymmetric information between buyers and sellers by interpreting the vectors that contain the partial information(=quality) as criteria for the buyers in decision-making.

8.2.2 The mathematical model

A company wants to buy m types of products $P_1, \ldots, P_m$. To acquire them n potential sellers present their offers $x_1, \ldots, x_n$. On these products certain information such as the cost, the delivery time etc. exists, but also partial information such as the quality of the products exists. The partial information on the quality of the products is summarized in the following table:

	P_1	P_2	$\ldots$	P_m
First seller's information	a_{11}	a_{12}	$\ldots$	a_{1m}
Second seller's information	a_{21}	a_{22}	$\ldots$	a_{2m}
$\ldots$				
n-th seller's information	a_{n1}	a_{n2}	$\ldots$	a_{nm}

The number $a_{ij} \in (0, 1]$, $i = 1, \ldots n$, $j = 1, \ldots, m$ describes the quality of the product P_j in offer x_i. From the n offers (=alternatives) only one has to be chosen. For a better decision, the company hires m experts, one for each product. The experts can say an opinion only on the certain information regarding the product. In making its choice, the company should also take into consideration the partial information, reflected by the vectors $\mathbf{a_j} = (a_{1j}, \ldots, a_{nj})$, $j = 1, \ldots, m$. When deciding on the offer x_i, the company will consider the aggregated opinion of the experts and the information in vectors $\mathbf{a_j}$. This is the reason why we might consider these vectors as criteria in the process of decision making. This fact will appear later in the analysis.

The result of the expertise of the k-th expert is given in a matrix $Q_k = (q_{ij}^k)$ of dimension $n \times n$. For simplicity we assume that the matrices $Q_1, \ldots, Q_m$ are Boolean. In interpretation, $q_{ij}^k = 1$ means that the k-th expert considers the offer x_i at least as good as offer x_j as far as the quality of product P_k is concerned.

Two natural conditions are imposed on the elements of Q_k:

(a) $q_{ii}^k = 1$ for any $i = 1, \ldots, n$

(b) Either $q_{ij}^k = 1$ or $q_{ji}^k = 1$ for any $i, j = 1, \ldots, n$, $i \neq j$.

Condition (a) says that any offer should be at least as good as itself. Condition (b) says that any two offers x_i and x_j are comparable, so only one has to be chosen. This way the opinion of each expert is modelled as a crisp preference relation, reflexive and total.

The information collected from all experts is aggregated in the matrix $Q = (q_{ij})_{n \times n}$:

$$q_{ij} = \frac{1}{m} \sum_{k=1}^{m} q_{ij}^k \ , \ i, j = 1 \ldots, n.$$

The information given by matrix Q regards the m products.

The sum $\sum_{k=1}^{m} q_{ij}^k$ in the expression of q_{ij} represents the information from the experts that think that offer x_i is at least as good as offer x_j

Notice that $0 < q_{ij} \leq 1$ for all $i, j = 1, \ldots, n$, hence Q is a fuzzy preference relation. The number q_{ij} shows the degree to which the expertise overall decided that alternative x_i is at least as good as alternative x_j. Obviously Q is reflexive.

So far we have a preference fuzzy relation given by Q and m vectors $\mathbf{a_j} = (a_{1j}, \ldots, a_{nj})$, $j = 1, \ldots, m$. As discussed above, the vectors $\mathbf{a_1}, \ldots, \mathbf{a_m}$ can be considered criteria that the company should consider in choosing the best offer.

Now we are in the position to formulate a problem with fuzzy choices. For simplicity we will use the Gödel t-norm.

Denote by $S = \{x_1, \ldots, x_n\}$ the set of alternatives (=offers) and by $S_1, \ldots, S_m$ the fuzzy subsets of X:

$S_j(x_i) = a_{ij}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Denote $\mathcal{B} = \{S_1, \ldots, S_m\}$. $\langle X, \mathcal{B} \rangle$ represents the *fuzzy choice space* .

Next we investigate this choice problem.

To the fuzzy preference relation Q on X we assign a fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$ defined by $C(S) = G(S, Q)$ for any $S \in \mathcal{B}$ (Definition 4.6 and (4.2.2)). Then for any $S \in \mathcal{B}$ and $x \in X$ we have $C(S)(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \to Q(x, y)]$.

Recall that for this fuzzy choice function, $C(S)(x)$ is the degree of truth of the statement "x is one of the Q-greatest alternatives satisfying criterion S".

In theory it is possible to find $S \in \mathcal{B}$, such that $C(S)(x_i) = 0$ for $i = 1, \ldots, n$. In this case C is no longer a fuzzy choice function. As we cannot repeat the expertise until C becomes a fuzzy choice function, we have to make the assumption that for any $S \in \mathcal{B}$, $C(S)(x_i) > 0$, for any $i = 1, \ldots, n$. This assumption is very realistic in practice: for each of the m products there will always exist some information regarding its quality from each seller.

Once the fuzzy choice function has been constructed, two issues are raised:

1) To find to what extent the mathematical form of the choice (given by C) matches the expertise;

2) To find a procedure to establish the hierarchy of the alternatives according to each of the m criteria and overall.

We discuss now these issues.

1) The fuzzy choice function C has been generated by the fuzzy preference relation Q which was decided by the experts. In its turn, the fuzzy choice function C generates a fuzzy revealed preference relation R (Defini-

tion 4.2 (i))

$$R(x_i, x_j) = \bigvee_{S \in \mathcal{B}} [C(S)(x_i) \wedge S(x_j)]$$

for any alternatives $x_i, x_j \in X$.

The fuzzy revealed preference relation R reflects the preferences of the company displayed by the act of choosing and Q reflects the preferences of the company according to the expertise. Since in theory R is not always reflexive, we will replace the elements of the main diagonal with 1. In practice this assumption is normal.

The fuzzy choice function C reflects faithfully the expertise if Q and R are sufficiently close to each other. The closeness between Q and R can be measured with *the grade of similarity* . There are several modalities to define it. Recall such measures [63]:

$$M(Q,R) = \begin{cases} 1 & \text{if} \quad Q = R = \emptyset \\ \dfrac{\sum\limits_{x,y \in X} min(Q(x,y), R(x,y))}{\sum\limits_{x,y \in X} max(Q(x,y), R(x,y))} & \text{otherwise} \end{cases}$$

$$L(Q,R) = 1 - max_{x,y \in X} |Q(x,y) - R(x,y)|.$$

$$S(Q,R) = \begin{cases} 1 & \text{if} \quad Q = R = \emptyset \\ \dfrac{\sum\limits_{x,y \in X} |Q(x,y) - R(x,y)|}{\sum\limits_{x,y \in X} |Q(x,y) + R(x,y)|} & \text{otherwise} \end{cases}$$

For instance, choosing the first grade of similarity we say that "the choice is made with the grade of similarity $M(Q,R)$".

2) To establish the hierarchy of the alternatives with respect to each criterion we will apply the *degree of dominance* , concept studied in the previous chapter. As seen before, the degree of dominance ranks the alternatives with respect to the act of choice and not to the preference relation. Recall the definition of the degree of dominance $D_S(x)$ of an alternative x with respect to a criterion S (Definition 7.1):

$$D_S(x) = S(x) \wedge \bigwedge_{y \in X} [C(S)(y) \rightarrow C(S)(x)].$$

$D_S(x)$ reflects x's position among the other alternatives according to criterion S. The set of real numbers $\{D_S(x) | x \in X\}$ will be ordered such that we will obtain a hierarchy of alternatives according to S. For a global hierarchy of the offers we also apply the *aggregated degree of dominance* introduced in the previous chapter:

$D(x) = \frac{1}{card(\mathcal{B})} \sum_{S \in \mathcal{B}} D_S(x)$

By ordering the elements of the set $\{D(x) | x \in X\}$ one obtains an overall hierarchy of alternatives.

In the final act of making a choice, the company will decide which offer to choose.

These considerations provide an algorithm for solving the choice problem, that will be described in the next subsection.

8.2.3 An algorithm

In this subsection we propose an algorithm for solving the choice problem described previously.

The input data are the following:

n=the number of alternatives (offers);

m=the number of criteria;

m=the number of experts;

$Q_1, \ldots, Q_m$=Boolean matrices that describe the results of the expertise where $Q_k = (q_{ij}^k)$ for any $k = 1, \ldots, m$;

$S_1, \ldots, S_m$=the fuzzy subsets of the available set of alternatives $X = \{x_1, \ldots, x_n\}$, where

$S_j = a_{1j}\chi\{x_1\} + a_{2j}\chi\{x_2\} + \ldots + a_{nj}\chi\{x_n\}$ for $j = 1, \ldots, m$.

The steps of the algorithm are:

Step 1

The fuzzy preference relation Q obtained by the aggregation of individual preference relations $Q_1, \ldots, Q_m$ is calculated:

$Q = \frac{1}{m} \sum_{k=1}^{m} Q_k.$

Step 2

The fuzzy choice function C associated to the fuzzy preference relation Q is calculated:

$$\begin{aligned} C(S_j)(x_i) &= S_j(x_i) \wedge \bigwedge_{u=1}^{n} [S_j(x_u) \rightarrow Q(x_i, x_u)] \\ &= a_{ij} \wedge \bigwedge_{u=1}^{n} [a_{uj} \rightarrow q_{iu}] \end{aligned}$$

for any $j = 1, \ldots, m$ and $i = 1, \ldots, n$.

Step 3

The fuzzy revealed preference relation R associated to C is calculated:
$R(x_i, x_j) = \bigvee_{t=1}^{m} [C(S_t)(x_i) \wedge S_t(x_j)]$
for any $i, j = 1, \ldots, n$.

A minimal condition on preference relations is reflexivity. In case when R is not reflexive, we substitute R with its reflexive closure R', i.e. the fuzzy relation obtained by replacing the elements of R's main diagonal with 1.

Step 4

We compute the grade of similarity of the initial fuzzy preference relation Q and the fuzzy revealed preference relation R':

$$M(Q, R') = \frac{\sum_{i,j=1}^{n} (Q(x_i, x_j) \wedge R'(x_i, x_j))}{\sum_{i,j=1}^{n} (Q(x_i, x_j) \vee R'(x_i, x_j))}.$$

Step 5

We calculate the degree of dominance of alternative x_i with respect to the criterion S_j and the aggregated degree of dominance of x_i.
$D_{S_j}(x_i) = S_j(x_i) \wedge \bigwedge_{y \in X} [C(S_j)(y) \to C(S_j)(x_i)]$,
$i = 1, \ldots, n$, $j = 1, \ldots, m$.
$D(x_i) = \frac{1}{m} \sum_{j=1}^{m} D_{S_j}(x_i)$, $i = 1, \ldots, n$.

Step 6

Ordering the set $\{D_{S_j}(x_1), \ldots, D_{S_j}(x_n)\}$ one obtains a hierarchy of the set of alternatives with respect to S_j. Ordering the set $\{D(x_1), \ldots, D(x_n)\}$ one obtains a global hierarchy of alternatives.

8.2.4 A numerical illustration

In this subsection we shall illustrate the above algorithm with a simple example.

Take the particular case $n = 5$ alternatives, $m = 3$ criteria. The set of alternatives is $X = \{x_1, \ldots, x_5\}$.

The activity of the three experts is materialized in the following preference matrices:

$$Q_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

The three criteria are given by the following fuzzy subsets of S

$S_1 = 0.3\chi\{x_1\} + 0.6\chi\{x_2\} + 0.8\chi\{x_3\} + 0.5\chi\{x_4\} + 0.6\chi\{x_5\}$;
$S_2 = 0.4\chi\{x_1\} + 0.7\chi\{x_2\} + 0.8\chi\{x_3\} + 0.9\chi\{x_4\} + 0.2\chi\{x_5\}$;
$S_3 = 0.7\chi\{x_1\} + 0.6\chi\{x_2\} + 0.5\chi\{x_3\} + 0.1\chi\{x_4\} + 0.4\chi\{x_5\}$.

We follow the steps formulated in the previous subsection.

Step 1

The matrix of the fuzzy preferences Q is obtained from Boolean matrices Q_1, Q_2 and Q_3:

$$Q = \begin{pmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & 1 & 1 & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 1 \end{pmatrix}.$$

Step 2

The fuzzy choice function resulting from Q is computed by:

$C(S_j)(x_i) = S_j(x_i) \wedge \bigwedge_{u=1}^{5} (S_j(x_u) \to Q(x_i, x_u))$, $i = 1, \ldots, 5$, $j = 1, \ldots, 3$.

For example,

$C(S_1)(x_1) = S_1(x_1) \wedge [S_1(x_1) \to Q(x_1, x_1)] \wedge [S_1(x_2) \to Q(x_1, x_2)] \wedge [S_1(x_3) \to Q(x_1, x_3)] \wedge [S_1(x_4) \to Q(x_1, x_4)] \wedge [S_1(x_5) \to Q(x_1, x_5)] = 0.3 \wedge [0.3 \to 1] \wedge [0.6 \to \frac{1}{3}] \wedge [0.8 \to \frac{2}{3}] \wedge [0.5 \to \frac{1}{3}] \wedge [0.6 \to \frac{2}{3}] = 0.3$.

After all computations, we obtain the table:

$C(S_j)(x_i)$	x_1	x_2	x_3	x_4	x_5
S_1	0.3	$\frac{1}{3}$	0.8	0.5	$\frac{1}{3}$
S_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
S_3	$\frac{1}{3}$	$\frac{1}{3}$	0.5	0.1	$\frac{1}{3}$

Step 3

The elements of the fuzzy revealed preference matrix R associated to C are calculated by the formula:

$R(x_i, x_j) = [C(S_1)(x_i) \wedge S_1(x_j)] \vee [C(S_2)(x_i) \wedge S_2(x_j)] \vee [C(S_3)(x_i) \wedge S_3(x_j)]$, $i, j = 1, \ldots, 5$.

The matrix R is :

$$R = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0.5 & \frac{2}{3} & 0.8 & \frac{2}{3} & 0.6 \\ 0.4 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0.5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

We replace R by its reflexive closure R'.

$$R' = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0.5 & \frac{2}{3} & 1 & \frac{2}{3} & 0.6 \\ 0.4 & \frac{2}{3} & \frac{2}{3} & 1 & 0.5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}.$$

Step 4

Now we find out how similar matrices Q and R' are. For this, first we compute:

$$Q \wedge R' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0.5 & \frac{2}{3} & 1 & \frac{2}{3} & 0.6 \\ 0.4 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}, \quad Q \vee R' = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & 1 & 1 & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 1 & 0.5 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 1 \end{pmatrix}.$$

The grade of similarity of Q and R' is $M(Q, R') = \dfrac{\sum\limits_{i,j=1}^{5} (Q \wedge R')(x_i, x_j)}{\sum\limits_{i,j=1}^{5} (Q \vee R')(x_i, x_j)} = \frac{11.83}{16.5} = 0.71$.

Steps 5 and 6

The degrees of dominance of alternatives in X with respect to criteria S_1, S_2 and S_3 are represented in the table:

$D_{S_j}(x_i)$	x_1	x_2	x_3	x_4	x_5
S_1	0.3	$\frac{1}{3}$	0.8	0.5	$\frac{1}{3}$
S_2	$\frac{1}{3}$	$\frac{1}{3}$	0.8	0.9	0.2
S_3	$\frac{1}{3}$	$\frac{1}{3}$	0.5	0.1	$\frac{1}{3}$

The aggregated degrees of dominance of the alternatives are

$D(x_1) = 0.322$, $D(x_2) = 0.333$, $D(x_3) = 0.7$, $D(x_4) = 0.5$, $D(x_5) = 0.288$.

Therefore the order of the alternatives is: $D(x_5) < D(x_1) < D(x_2) < D(x_4) < D(x_3)$.

8.2.5 Discussion

In this subsection we will make an analysis of the results from the previous section in the context of our example.

We deal with a situation where a company wants to buy 3 types of products. The offers come from 5 producers. The set of alternatives (=offers) is given by $S = \{x_1, \ldots, x_5\}$.

The partial information on the product i (e.g. its quality) is represented in the vector $\mathbf{a_i}$:

$\mathbf{a_1} = (0.3, 0.6, 0.8, 0.5, 0.6)$;

$\mathbf{a_2} = (0.4, 0.7, 0.8, 0.9, 0.2)$;

$\mathbf{a_3} = (0.7, 0.6, 0.5, 0.1, 0.4)$.

It means that $a_{11} = 0.3$ represents the degree to which information on the quality of the first product given by the first producer exists.

The certain information on the products (e.g. cost, delivery times) is also taken into consideration, by the participation of 3 experts in the decision making. For example, $q_{12}^1 = 1$ means that the first expert considers that x_1 is preferred to x_2, etc. The overall expertise is reflected in the aggregated matrix Q of fuzzy preferences. The element $q_{12} = \frac{1}{3}$ can be interpreted as offer x_1 is preferred to offer x_2 with the degree of intensity $\frac{1}{3}$.

From the fuzzy choice functions obtained above the fuzzy revealed preference R' is derived. Offer 1 is revealed preferred to the other offers with the degree $\frac{1}{3}$ and the same applies for offer 2, etc.

One first conclusion shows that the result of the expertise coincides with the preferences of the company based on fuzzy choices with the grade of similarity 0.71.

Another conclusion is that according to criterion S_1, the company might choose offer 3, according to criterion S_2 offer 4 and according to criterion S_3 offer 3. If the company takes into account all criteria, it might choose offer 3.

This application assumed that there is no conflict on the experts'judgements. An open problem is what happens in case of a conflict on the experts'judgements.

8.3 Application 3

A producer manufactures m types of products $P_1, \ldots, P_m$. Every year he organizes an auction to sell his products. n companies $x_1, \ldots, x_n$ are interested in participating in this auction. The bidding concerns the right to sell the products. The sales obtained in year T are given in the following table:

	P_1	P_2	$\ldots$	P_m
x_1	a_{11}	a_{12}	$\ldots$	a_{1m}
x_2	a_{21}	a_{22}	$\ldots$	a_{2m}
$\ldots$				
x_n	a_{n1}	a_{n2}	$\ldots$	a_{nm}

where a_{ij} denotes the number of units of product P_j sold by company x_i in year T. For the year $T+1$ the producer would like to increase the number of sales with the n companies. The companies give an estimation of the sales for year $T+1$ contained in a matrix (c_{ij}) with n rows and m columns; c_{ij} denotes the number of units of product P_j that the company x_i estimates to sell in year $T+1$.

In making his choice, the producer will those companies that have an efficient sales market, good marketing, and a good image. The decision analysis will require two aspects:

(a) the sales a_{ij} for year T;

(b) the estimated sales c_{ij} for year $T+1$.

The sales for year T can be considered results of the act of choice, or more clearly, values of a choice function, and the preferences will be given by the revealed preference relation associated to these choice functions. With the resulting preference relation and the estimated sale for the year $T+1$, a fuzzy choice function can be defined. This choice function will be used to rank the companies with respect to each type of product. Dividing the values a_{ij} and c_{ij} respectively by a power of 10 conveniently chosen we may assume that $0 \leq a_{ij}, c_{ij} \leq 1$ for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

In establishing the mathematical model the following steps are needed:

–(A) To build a fuzzy choice function from the sales of year T.

The set of alternatives is $X = \{x_1, \ldots, x_n\}$.

For each $j = 1, \dots, m$ denote by S_j the subset of X whose elements are those companies that have had "good" sales for product P_j in year T. Only the companies whose sales are greater than a threshold e_j are considered.

If $\mathcal{H} = \{S_1, \dots, S_m\}$ then $\langle X, \mathcal{H} \rangle$ is a fuzzy choice space (we will identify S_j with its characteristic function). The sales (a_{ij}) of year T lead to a choice function $C' : \mathcal{H} \to \mathcal{F}(X)$ defined by:

(1) $C'(S_j)(x_i) = a_{ij}$

for each $j = 1, \dots, m$ and $x_i \in S_j$.

This context is similar to Banerjee [4]. There $\mathcal{H}$ contains all non-empty finite subsets of X.

–(B) The choice function C' gives a fuzzy revealed preference relation R on X defined cf. Remark 4.2:

(2) $R(x_i, x_j) = \bigvee \{C'(S_k)(x_i) | x_i, x_j \in S_k\} = \bigvee \{a_{ik} | x_i, x_j \in S_k\}$

for any $x_i, x_j \in X$.

$R(x_i, x_j)$ represents the degree to which alternative x_i is preferred to alternative x_j as a consequence of current sales.

Since in most cases R is not reflexive, we replace it by its reflexive closure R'.

–(C) From the fuzzy revealed preference matrix R' and the matrix c_{ij} of estimated sales one can define a fuzzy choice function C, whose values will estimate the potential sales for the year $T+1$. Starting from C one will rank the alternatives for each type of product.

The set of alternatives is $X = \{x_1, \dots, x_n\}$. For each $j = 1, \dots, m$ A_j will denote the fuzzy subset of X given by

(3) $A_j(x_i) = c_{ij}$ for any $i = 1, \dots, n$.

Take $\mathcal{A} = \{A_1, \dots, A_m\}$. One obtains the fuzzy choice space $\langle X, \mathcal{A} \rangle$. The choice function $C : \mathcal{A} \to \mathcal{F}(X)$ is defined by (see Remark 4.2.2)

(4) $C(A_j)(x_i) = A_j(x_i) \wedge \bigwedge_{k=1}^{n} [A_j(x_k) \to R'(x_i, x_k)]$

$= c_{ij} \wedge \bigwedge_{k=1}^{n} [c_{ij} \to R(x_i, x_k)]$

for any $i = 1, \dots, n$ and $j = 1, \dots, m$.

Applying the degree of dominance for the fuzzy choice function C one will obtain a ranking of the companies with respect to each product. This ranking gives the information that the mathematical model described above offers to the producer with respect to the sales activity for the following year.

We present next the algorithm of this problem.

The input data are:

m= the number of types of products

n=the number of companies

a_{ij}=the matrix of sales for year T

c_{ij}=the matrix of estimated sales for year $T+1$

$(e_1, \dots, e_m)$=the threshold vector

Assume $0 \leq a_{ij} \leq 1$, $0 \leq c_{ij} \leq 1$ for any $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

From the mathematical model we can derive the following steps:

Step 1 Determine the subsets $S_1, \ldots, S_m$ of $X = \{x_1, \ldots, x_n\}$ by
$S_k = \{x_i \in X | a_{ik} \geq e_k\}$, $k = 1, \ldots, m$.

Step 2 Compute the matrix of revealed preferences $R = (R(x_i, x_j))$ by
$R(x_i, x_j) = \bigvee\limits_{x_i, x_j \in S_k} a_{ik}$.

Replace R with its reflexive closure R'.

Step 3 Determine the fuzzy sets $A_1, \ldots, A_m$
$A_j = c_{ij}\chi\{x_1\} + \ldots + c_{nj}\chi\{x_n\}$ for $j = 1, \ldots, m$

Step 4 Obtain the choice function C applying (3)

Step 5 Determine the degrees of dominance $D_{A_j}(x_i)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Step 6 Rank the set of alternatives with respect to each product P_j by ranking the set $\{D_{A_j}(x_1), \ldots, D_{A_j}(x_n)\}$.

For a better understanding of this model we present a numerical illustration. Consider the initial data $m = 3$ products and $n = 5$ companies willing to sell these products.

The sales for year T are given in the following table:

	P_1	P_2	P_3
x_1	0.3	0.6	0.7
x_2	0.8	0.1	0.5
x_3	0.7	0.6	0.1
x_4	0.1	0.8	0.7
x_5	0.8	0.1	0.7

The estimated sales for year $T + 1$ are given in the following table:

	P_1	P_2	P_3
x_1	0.5	0.7	0.7
x_2	0.8	0.3	0.6
x_3	0.8	0.7	0.2
x_4	0.2	0.8	0.8
x_5	0.8	0.2	0.8

The thresholds are $e_1 = e_2 = e_3 = 0.2$.

We follow now the steps described above.

Step 1 The subsets S_1, S_2, S_3 of X are:
$S_1 = \{x_1, x_2, x_3, x_5\}$, $S_2 = \{x_1, x_3, x_4\}$, $S_3 = \{x_1, x_2, x_4, x_5\}$.

Step 2 We compute the matrix of revealed preferences R. Then we replace it by its reflexive closure R'.

$$R = \begin{pmatrix} 0.7 & 0.7 & 0.6 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.5 & 0.8 \\ 0.7 & 0.7 & 0.7 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.7 & 0.8 \end{pmatrix}. \; R' = \begin{pmatrix} 1 & 0.7 & 0.6 & 0.7 & 0.7 \\ 0.8 & 1 & 0.8 & 0.5 & 0.8 \\ 0.7 & 0.7 & 1 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.8 & 1 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.7 & 1 \end{pmatrix}.$$

For example, $R(x_1, x_2) = \bigvee_{x_1, x_2 \in S_k} a_{1k} = a_{11} \vee a_{13} = 0.3 \vee 0.7 = 0.7.$

Step 3 The fuzzy sets A_1, A_2, A_3 are:

$A_1 = 0.5\chi\{x_1\} + 0.8\chi\{x_2\} + 0.8\chi\{x_3\} + 0.2\chi\{x_4\} + 0.8\chi\{x_5\};$
$A_2 = 0.7\chi\{x_1\} + 0.3\chi\{x_2\} + 0.7\chi\{x_3\} + 0.8\chi\{x_4\} + 0.2\chi\{x_5\};$
$A_3 = 0.7\chi\{x_1\} + 0.6\chi\{x_2\} + 0.2\chi\{x_3\} + 0.8\chi\{x_4\} + 0.8\chi\{x_5\}.$

Step 4 The corresponding fuzzy choice functions are:

$C(A_1) = 0.5\chi\{x_1\} + 0.8\chi\{x_2\} + 0.7\chi\{x_3\} + 0.2\chi\{x_4\} + 0.8\chi\{x_5\}$
$C(A_2) = 0.6\chi\{x_1\} + 0.3\chi\{x_2\} + 0.6\chi\{x_3\} + 0.8\chi\{x_4\} + 0.2\chi\{x_5\}$
$C(A_3) = 0.7\chi\{x_1\} + 0.5\chi\{x_2\} + 0.2\chi\{x_3\} + 0.7\chi\{x_4\} + 0.7\chi\{x_5\}.$

Step 5 The corresponding degrees of dominance are represented in the table:

$D_{A_j}(x_i)$	x_1	x_2	x_3	x_4	x_5
A_1	0.5	0.8	0.7	0.2	0.8
A_2	0.6	0.3	0.6	0.8	0.2
A_3	0.7	0.5	0.2	0.8	0.7

The producer needs to decide one contractor according to the dominant values for each criterion. According to criterion A_1,
$D_{A_1}(x_4) < D_{A_1}(x_1) < D_{A_1}(x_3) < D_{A_1}(x_2) = D_{A_1}(x_5)$,
therefore companies 2 and 5 will be chosen.
According to criterion A_2,
$D_{A_2}(x_5) < D_{A_2}(x_2) < D_{A_2}(x_1) = D_{A_2}(x_3) < D_{A_2}(x_4)$,
therefore company 4 will be chosen.
According to criterion A_3,
$D_{A_3}(x_3) < D_{A_3}(x_2) < D_{A_3}(x_1) = D_{A_3}(x_5) < D_{A_3}(x_4)$,
therefore company 4 will be chosen.

In the situation described above the sales of year T are regarded as the result of the act of choice. As such we deal with a first problem of fuzzy choices (in the sense of Banerjee) where the choice function C' is defined by the values of the sales of year T.

The fuzzy revealed preference relation R associated with C' expresses the preferences of the buyers in year T. R gives only partial information on the manner in which the sales of year $T + 1$ will occur.

Estimative values of the sales of year $T+1$ will be taken into account. For instance, these estimative values can be obtained by statistical samplings. By combining the estimations of year $T + 1$ with the preferences of year T one obtains a new fuzzy choice problem (in the general sense of this thesis) that models the sales of year $T + 1$.

The multicriterial hierarchy of alternatives obtained from the second fuzzy choice problem will be useful to the company in organizing the sales of year $T + 1$.

Chapter 9

Summary and Conclusions

Most individual or group activities can be considered to be acts of choice between feasible alternatives. An agent must be able to choose some alternatives from an available set of alternatives. This is the intuitive idea behind the notion of choice function or consumer.

According to Uzawa [61], the behavior of a consumer is rational if "he has a definite preference over all conceivable commodity bundles and he chooses those commodity bundles that are optimum with respect to his preference subject to budgetary constraints".

Then the rationality is to find a "good binary relation" R such that the choice is determined by the R-greatest elements of any available sets of alternatives.

In the real world most preferences are vague, consequently they are more adequately modelled by fuzzy binary relations.

Some authors have studied the case when preferences are vague but the act of choice is exact [5, 6, 7, 16, 33, 38].

Banerjee's thesis [4] is that "If preferences are permitted to be fuzzy, it seems natural to permit the choice functions to be fuzzy as well. This also tallies with the experience." He studies the revealed preference theory for a class of fuzzy choice functions.

In [4, 43, 58, 62, 64] fuzzy choice functions in different forms have been studied.

Our notion of fuzzy choice function differs from that of Banerjee where the domain of the choice function is made only of finite (crisp) sets of alternatives. In this thesis it is assumed that both the domain and the range of a choice function are made of fuzzy sets. For a fuzzy set S in the domain of the choice function and for an alternative x the real number $S(x)$ can be considered as the availability degree of x.

Any continuous t-norm $*$ leads to a different set theory; the operations with fuzzy sets and fuzzy relations are expressed by means of the t-norm $*$, residuum $\rightarrow$ and negation $\neg$ associated to $*$. This is the main motivation of our attempt to develop a general theory of fuzzy revealed preference for an arbitrary continuous t-norm. Besides the generality of some definitions and propositions, another argument is the specificity of the results for different t-norms.

The contributions of this thesis can be grouped in five main themes.

I. Revealed preference and congruence axioms for fuzzy choice functions.

II. Rationality and normality of fuzzy choice functions.

III. Consistency conditions for fuzzy choice functions.

IV. Degree of dominance for fuzzy choice functions.

V. Applications.

In this thesis two notions of rationality for fuzzy choice functions have been introduced: M-rationality and G-rationality. Both assume the existence of a fuzzy preference relation Q. M-rationality (resp. G-rationality) extends to fuzzy sets the notion of the set of Q-maximal (resp.Q-greatest) elements of an available set of alternatives. When Q is the fuzzy preference relation defined by (3.2.5), M-normality and G-normality of fuzzy choice functions are obtained.

In connection with these notions axioms of revealed preference $WAFRP$, $SAFRP$ and congruence $WFCA$, $SFCA$ have been studied. This study has been done in the thesis in two contexts.

The first was in the line with the Uzawa-Arrow-Sen theory [60, 2, 49, 48, 50] in which the domain of the choice function includes all finite subsets of alternatives. At a fuzzy level, correspondingly, hypotheses ($H1$) and ($H2$) have been found.

The main result of Section 5.1 is Theorem 5.1 in which axioms $WAFRP$, $SAFRP$, $WFCA$, $SFCA$ and four other conditions of rationality are compared. Some equivalences or implications are true for an arbitrary continuous t-norm, others only for Lukasiewicz or Gödel t-norms. The Gödel t-norm is more effective for revealed preference axioms (conditions (i)-(iv) are equivalent) and the Lukasiewicz t-norm is more effective for congruence axioms (conditions (v) -(vi) are equivalent). The example in Section 5.5 establishes the limitations of this theorem. Reflecting on Theorem 5.1 and the associated example it can be said that the product t-norm has a reduced significance in fuzzy revealed preference theory.

A second setting for axioms of revealed preference and congruence follows the Richter-Hansson-Suzumura theory [41, 27, 54]. In this case hypotheses ($H1$) and ($H2$) are lifted. In [27, 54], $WARP$ and $SARP$ have been used in equivalent forms expressed in terms of C-connected sequences. For fuzzy choice functions these equivalent forms have led to axioms $WAFRP^\circ$, $SAFRP^\circ$ which are different from $WAFRP$ and $SAFRP$. Another axiom of revealed preference $HAFRP$, the fuzzy version of axiom $HARP$ introduced by Hansson in [27] has been considered. In Chapter 6 the equivalences between $WFCA$ and $WAFRP^\circ$, $SFCA$ and $HAFRP$ have been established.

Open problem 1: No result on $SAFRP^{\circ}$ has been obtained. Is there any relation between $SAFRP^{\circ}$ and the rest of the axioms?

In [4] Banerjee studies in a different context fuzzy forms of revealed preference and congruence axioms. The choice functions in Banerjee's axioms are defined in combinations of operations with Gödel and Lukasiewicz t-norms. In this thesis a different approach has been adopted, working only with a fixed t-norm.

Open problem 2: Could Banerjee's results [4] be formulated and proved for the fuzzy choice functions considered in this thesis?

The last section of Chapter 6 attempts to obtain the Richter theorem [41] in a fuzzy context. A choice function is totally $*$-rational if it is rationalizable by a fuzzy preference relation which is reflexive, $*$-transitive and total. This is the fuzzy version of a rational choice function in Richter's terminology [41, 42]. We prove that any totally $*$-rational fuzzy choice function is $*$-congruous (i.e. it satisfies $SAFRP$ for an arbitrary continuous t-norm $*$). One implication of Richter theorem is obtained henceforth.

Open problem 3: Is the converse true: any $*$-congruous choice function is totally $*$-rational?

Trying to follow Richter's proof in order to obtain this implication, the concept of $*$-semirational fuzzy choice function was necessary. In Section 6.3 a surprising result has been obtained: any fuzzy choice function is $*$-semirational. To prove this fact the fuzzy version of the Szpilrajn theorem, which was proved in Section 3.3, is needed.

As mentioned already, the third type of contributions refers to consistency conditions, which involve properties concerning the expansion or contraction of the feasible sets of alternatives. Under hypotheses $(H1)$ and $(H2)$ it was especially intended to study conditions $F\alpha$, $F\beta$, $F\gamma$ and $F\delta$, the fuzzy forms of properties α, β, γ and δ studied by Sen in [49, 48, 50].

For example, we prove that a fuzzy choice function verifies $F\alpha$ and $F\beta$ if and only if $WFCA$ holds (Theorem 5.2). Another result shows that the fuzzy preference relation R is quasi-transitive if and only if $F\delta$ is verified.

Open problem 4: In this thesis we have not obtained any result concerning condition $F\gamma$. Can Sen's results on condition γ [48] be extended in our context?

Open problem 5: The consistency conditions $F\alpha$, $F\beta$ and $F\gamma$ have been studied for the Gödel t-norm. Are they still true for an arbitrary continuous t-norm or for a particular t-norm?

Open problem 6: In this thesis specifically the concepts of G-rationality and G-normality have been analyzed. Can significant results be obtained concerning the M-rationality and M-normality of fuzzy choice functions?

In the literature on multiple criteria decision making (see e.g. [67]) it has been often emphasized how important the selection of alternatives is for the act of choice. Usually the alternatives are not clearly differentiated, therefore their ranking according to various criteria is needed. This facilitates the process of decision making.

In Chapter 7 we have introduced the concept of degree of dominance of an alternative, as a method of ranking the alternatives according to different criteria. These criteria can be taken as the available sets of alternatives.

The degree of dominance of an alternative x in an available set S of alternatives reflects x's position towards the other alternatives (with respect to S). This notion expresses the dominance of an alternative with regard to the act of choice, not to a preference relation. With the degree of dominance one can build a hierarchy of alternatives for each available set S. If one defines a concept of aggregated degree of dominance (that unifies the degrees of dominance with regard to various available sets) one obtains an overall hierarchy of alternatives.

In Chapter 7 we have considered the degree of dominance $D_S^Q(x)$ of x in S in terms of Q, where Q is an arbitrary fuzzy relation on X. If Q is a fuzzy revealed preference relation associated with a fuzzy choice function C, we have proved that $D_S(x) \leq D_S^Q(x)$, $x \in X$ and $S \in \mathcal{B}$.

Open problem 7: Characterize the fuzzy choice functions C on the fuzzy choice space $\langle X, \mathcal{B} \rangle$ such that $D_S(x) = D_S^Q(x)$ for any $x \in X$ and $S \in \mathcal{B}$.

Open problem 8: By replacing $D_S(x)$ with $D_S^Q(x)$ one can formulate axioms of congruence analogous to FC^*1, FC^*2 and FC^*3. What might be the dependencies between these new axioms and how are they connected to FC^*1, FC^*2 and FC^*3?

The analysis of the three applications of Chapter 8 leads to the following conclusions:

• all these applications describe concrete economic situations where partial information or human subjectivity appears;

• the mathematical modelling is done by formulating some fuzzy choice problems where criteria are represented by fuzzy available sets of alternatives;

• the degree of dominance is the mathematical instrument on which the algorithms of multicriterial hierarchy are based.

Bibliography

[1] G. A. Akerlof. The market for lemons: Quality uncertainty and the market mechanism. *Quarterly Journal of Economics*, 84:488–500, 1970.

[2] K. J. Arrow. Rational choice functions and orderings. *Economica*, 26:121–127, 1959.

[3] B. De Baets and H. De Meyer. On the existence and construction of T-transitive closures. *Information Sciences*, 152:167–179, 2003.

[4] A. Banerjee. Fuzzy choice functions, revealed preference and rationality. *Fuzzy Sets and Systems*, 70:31–43, 1995.

[5] C. R. Barrett, P. K. Pattanaik, and M. Salles. On the structure of fuzzy social welfare functions. *Fuzzy Sets and Systems*, 19:1–11, 1986.

[6] C. R. Barrett, P. K. Pattanaik, and M. Salles. On choosing rationally when preferences are fuzzy. *Fuzzy Sets and Systems*, 34:197–212, 1990.

[7] C. R. Barrett, P. K. Pattanaik, and M. Salles. Rationality and aggregation of preferences in an ordinal fuzzy framework. *Fuzzy Sets and Systems*, 49:9–13, 1992.

[8] U. Bodenhofer and F. Klawonn. A formal study of linearity axioms for fuzzy orderings. *Fuzzy Sets and Systems*, 145:323–496, 2004.

[9] G. Bordes. Consistency, rationality and collective choice. *Review of Economic Studies*, 43:451–457, 1976.

[10] M. Bunge. *Scientific Research*, volume 1. Springer, 1967.

[11] R. Bělohlávek. *Fuzzy relational systems. Foundations and principles.* Kluwer, 2002.

[12] C. Carlsson and R. Fullér. *Fuzzy reasoning in decision making and optimization*. Physica-Verlag, 2002.

[13] C. Cornelis, C. Van der Donck, and E. E. Kerre. Sinha-Dougherty approach to the fuzzification of set inclusion revisited. *Fuzzy Sets and Systems*, 134:283–295, 2003.

[14] B. A. Davey and M. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, 2002.

[15] B. Van de Walle, S. Heitsch, and P. Faratin. Coping with one-to-many multi-criteria negotiations in electronic markets. In *Proceedings of the Second International Workshop on Negotiations in Electronic Markets: beyond Price Discovery, Munich, Germany*, pages 747–751, 2001.

[16] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers. Dordrecht, 1994.

[17] I. Georgescu. A fuzzy analysis of a Richter theorem in fuzzy consumers. In *Proceedings of the 3rd International Conference in Fuzzy Logic and Technology EUSFLAT 2003, Zittau, Germany*, pages 423–428, 2003.

[18] I. Georgescu. Rational and congruous fuzzy consumers. In *Proceedings of the International Conference on Fuzzy Information Processing, Theory and Applications FIP 2003, Beijing, China*, pages 133–137, 2003.

[19] I. Georgescu. Consistency conditions in fuzzy consumers theory. *Fundamenta Informaticae*, 61:223–245, 2004.

[20] I. Georgescu. On the axioms of revealed preference in fuzzy consumer theory. *Journal of Systems Science and Systems Engineering*, 13:279–296, 2004.

[21] I. Georgescu. Revealed preference, congruence and rationality: a fuzzy approach. *Fundamenta Informaticae*, 65:307–328, 2005.

[22] I. Georgescu. Degree of dominance and congruence axioms for fuzzy choice functions. *Fuzzy Sets and Systems*, forthcoming.

[23] N. Georgescu-Roegen. Choice and revealed preference. *Southern Economic Journal*, 21:119–130, 1954.

[24] S. Gottwald. *Fuzzy sets and fuzzy logic*. Vieweg, Braunschweig, 1993.

[25] R. H. Guttman and P. Maes. Cooperative vs. competitive multi-agent negotiations in retail electronic commerce. In *Proceedings of the Second International Workshop on Cooperative Information Agents CIA'98 , Paris, France*, 1998.

[26] P. Hájek. *Metamathematics of fuzzy logic*. Kluwer, 1998.

[27] B. Hansson. Choice structures and preference relations. *Synthese*, 18:443–458, 1968.

[28] H. S. Houthakker. Revealed preference and utility functions. *Economica*, 17:159–174, 1950.

[29] J. S. Kelly. *Arrow impossibility theorems.* Academic Press, 1978.

[30] S. Klaue, K. Kurbel, and I. Loutschko. Automated negotiations on agent-based e-marketplaces: An overview. In *Proceedings of the 14th Bled Electronic Commerce Conference, Bled, Slovenia*, pages 508–519, 2001.

[31] E. P. Klement, R. Mesiar, and E. Pap. *Triangular norms.* Kluwer, 2000.

[32] T. S. Kuhn. *The structure of scientific revolutions.* University of Chicago Press, 1970.

[33] P. Kulshreshtha and B. Shekar. Interrelationship among fuzzy preference - based choice function and significance of rationality conditions: a taxonomic and intuitive perspective. *Fuzzy Sets and Systems*, 109:429–445, 2000.

[34] K. Kurbel and I. Loutschko. A framework for multi-agent electronic marketplaces: Analysis and classification of existing systems. In *Proceedings of International ICSC Congress on Information Science Innovations ISI 2001, Dubai*, 2001.

[35] K. Kurbel and I. Loutschko. Towards multi-agent electronic marketplaces: What is there and what is missing? *The Knowledge Engineering Review*, 18:33–46, 2003.

[36] K. Menger. Statistical metrics. *Proceedings of the National Academy of Science, USA*, 28:535–537, 1942.

[37] M. Norris, S. West, and K. Gaughan. *eBusiness essentials.* John Wiley and Sons, Ltd., 2000.

[38] S. A. Orlovsky. Decision-making with a fuzzy preference relation. *Fuzzy Sets and Systems*, 1:155–167, 1978.

[39] S. Ovchinnikov. Decision-making with a ternary binary relation. In *Proceedings of the Conference on Information Processing and Management of Uncertainty IPMU 2004, Perugia, Italy*, pages 511–516, 2004.

[40] H. Raiffa. *The art and science of negotiation.* Harvard University Press, Cambridge, USA, 1982.

[41] M. Richter. Revealed preference theory. *Econometrica*, 34:635–645, 1966.

[42] M. Richter. Rational choice. In J. S. Chipman and et al., editors, *Preference, utility, and demand*, pages 29–58. New-York: Harcourt Brace Jovanovich, 1971.

[43] M. Roubens. Some properties of choice functions based on valued binary relations. *European Journal of Operations Research*, 40:309–321, 1989.

[44] P. A. Samuelson. A note on the pure theory of consumers' behaviour. *Economica*, 5:61–71, 1938.

[45] P. A. Samuelson. Consumption theory in terms of revealed preference. *Economica*, 15:243–253, 1948.

[46] P. A. Samuelson. Some psychological aspects of mathematics and economics. *The Review of Economics and Statistics*, 36:380–386, 1954.

[47] B. Schweizer and A. Sklar. Statistical metric spaces. *Pacific Journal of Mathematics*, 10:312–334, 1960.

[48] A. K. Sen. Quasi-transitivity, rational choice and collective decisions. *Review of Economic Studies*, 36:381–393, 1969.

[49] A. K. Sen. Choice functions and revealed preference. *Review of Economic Studies*, 38:307–312, 1971.

[50] A. K. Sen. Social choice theory: A re-examination. *Econometrica*, 45:53–89, 1977.

[51] A. K. Sen. *Choice, welfare and measurement.* Cambridge MA: MIT Press and Oxford: Blackwell, 1982.

[52] A. K. Sen. Internal consistency of choice. *Econometrica*, 61:495–521, 1993.

[53] D. Sinha and E. R. Dougherty. Fuzzification of set inclusion: beyond applications. *Fuzzy Sets and Systems*, 55:15–42, 1993.

[54] K. Suzumura. Rational choice and revealed preference. *Review of Economic Studies*, 43:149–159, 1976.

[55] K. Suzumura. Remarks on the theory of collective choice. *Economica*, 43:381–390, 1976.

[56] K. Suzumura. Houthakker's axiom in the theory of rational choice. *Journal of Economic Theory*, 14:284–290, 1977.

[57] K. Suzumura. *Rational choice, collective decisions and social welfare.* Cambridge University Press, Cambridge, 1983.

[58] Z. Switalski. Choice functions associated with fuzzy preference relations. In J. Kacprzyk and M. Roubens, editors, *Nonconventional Preference Relations in Decision Making*, pages 106–118. Springer Verlag, Berlin, 1988.

[59] E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae*, 16:386–389, 1930.

[60] H. Uzawa. A note on preference and axioms of choice. *Annals of the Institute of Statistical Mathematics*, 8:35–40, 1956.

[61] H. Uzawa. Preference and rational choice in the theory of consumption. In K. J. Arrow, S. Karlin., and P.Suppes, editors, *Mathematical Methods in the Social Sciences*. Stanford, Stanford University Press, 1959.

[62] X. Wang. A note on congruence conditions of fuzzy choice functions. *Fuzzy Sets and Systems*, 145:355–358, 2004.

[63] X. Wang, B. De Baets, and E. E. Kerre. A comparative study of similarity measures. *Fuzzy Sets and Systems*, 73:259–268, 1995.

[64] Ph. De Wilde. Fuzzy utility and equilibria. *IEEE Transactions on Systems, Man and Cybernetics*, 34:1774–1785, 2004.

[65] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[66] L. A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.

[67] M. Zeleny. *Multiple criteria decision making*. McGraw-Hill, New York, 1982.

[68] H. J. Zimmermann. *Fuzzy set theory and its applications*. Kluwer, 1984.

Turku Centre for Computer Science
TUCS Dissertations

23. **Gábor Magyar**, On Solution Approaches for Some Industrially Motivated Combinatorial Optimization Problems
24. **Linas Laibinis**, Mechanised Formal Reasoning About Modular Programs
25. **Shuhua Liu**, Improving Executive Support in Strategic Scanning with Software Agent Systems
26. **Jaakko Järvi**, New Techniques in Generic Programming : C++ is More Intentional than Intended
27. **Jan-Christian Lehtinen**, Reproducing Kernel Splines in the Analysis of Medical Data
28. **Martin Büchi**, Safe Language Mechanisms for Modularization and Concurrency
29. **Elena Troubitsyna**, Stepwise Development of Dependable Systems
30. **Janne Näppi**, Computer-Assisted Diagnosis of Breast Calcifications
31. **Jianming Liang**, Dynamic Chest Images Analysis
32. **Tiberiu Seceleanu**, Systematic Design of Synchronous Digital Circuits
33. **Tero Aittokallio**, Characterization and Modelling of the Cardiorespiratory System in Sleep-disordered Breathing
34. **Ivan Porres**, Modeling and Analyzing Software Behavior in UML
35. **Mauno Rönkkö**, Stepwise Development of Hybrid Systems
36. **Jouni Smed**, Production Planning in Printed Circuit Board Assembly
37. **Vesa Halava**, The Post Correspondence Problem for Marked Morphisms
38. **Ion Petre**, Commutation Problems on Sets of Words and Formal Power Series
39. **Vladimir Kvassov**, Information Technology and the Productivity of Managerial Work
40. **Franck Tétard**, Managers, Fragmentation of Working Time, and Information Systems
41. **Jan Manuch**, Defect Theorems and Infinite Words
42. **Kalle Ranto**, Z_4-Goethals Codes, Decoding and Designs
43. **Arto Lepistö**, On Relations between Local and Global Periodicity
44. **Mika Hirvensalo**, Studies on Boolean Functions Related to Quantum Computing
45. **Pentti Virtanen**, Measuring and Improving Component-Based Software Development
46. **Adekunle Okunoye**, Knowledge Management and Global Diversity - A Framework to Support Organisations in Developing Countries
47. **Antonina Kloptchenko**, Text Mining Based on the Prototype Matching Method
48. **Juha Kivijärvi**, Optimization Methods for Clustering
49. **Rimvydas Rukšėnas**, Formal Development of Concurrent Components
50. **Dirk Nowotka**, Periodicity and Unbordered Factors of Words
51. **Attila Gyenesei**, Discovering Frequent Fuzzy Patterns in Relations of Quantitative Attributes
52. **Petteri Kaitovaara**, Packaging of IT Services – Conceptual and Empirical Studies
53. **Petri Rosendahl**, Niho Type Cross-Correlation Functions and Related Equations
54. **Péter Majlender**, A Normative Approach to Possibility Theory and Soft Decision Support
55. **Seppo Virtanen**, A Framework for Rapid Design and Evaluation of Protocol Processors
56. **Tomas Eklund**, The Self-Organizing Map in Financial Benchmarking
57. **Mikael Collan**, Giga-Investments: Modelling the Valuation of Very Large Industrial Real Investments
58. **Dag Björklund**, A Kernel Language for Unified Code Synthesis
59. **Shengnan Han**, Understanding User Adoption of Mobile Technology: Focusing on Physicians in Finland
60. **Irina Georgescu**, Rational Choice and Revealed Preference: A Fuzzy Approach

Turku Centre *for* Computer Science

Lemminkäisenkatu 14 A, 20520 Turku, Finland | www.tucs.fi

University of Turku
- Department of Information Technology
- Department of Mathematics

Åbo Akademi University
- Department of Computer Science
- Institute for Advanced Management Systems Research

Turku School of Economics and Business Administration
- Institute of Information Systems Sciences

ISBN 952-12-1528-3
ISSN 1239-1883