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Limit Cycles for Generalized Liénard-type and Lotka-Volterra Systems

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Ping Yan

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May 2005 Ping Yan

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List of original publications

This thesis consists of this paper and the following seven original research papers:

I Gyllenberg, M. and Yan, P., The generalized Liénard systems, *Discrete Contin. Dyn. Syst.* 8 (2002), 1043–1057.

II Yan, P. and Jiang, J.F., On global asymptotic stability of second order nonlinear differential systems, *Appl. Anal.* 81 (2002), 681–703.

III Gyllenberg, M., Yan, P. and Jiang, J.F., The qualitative behavior of a second-order system with zero diagonal coefficient, *J. Math. Anal. Appl.* 291 (2004), 322–340.

IV Yan, P. and Jiang, J.F., Periodic solutions of general autonomous systems of Liénard type, *Appl. Anal.* 83 (2004), 735–746.

V Gyllenberg, M., Yan, P. and Wang, Y., A 3D competitive Lotka-Volterra system with three limit cycles: A falsification of a conjecture by Hofbauer and So, *Appl. Math. Lett.* (to appear).

VI Gyllenberg, M. and Yan, P., Necessary and sufficient conditions for oscillations and centers of generalized Liénard systems, (submitted).

VII Gyllenberg, M., Yan, P. and Wang, Y., Limit cycles for the competitor-competitor-mutualist Lotka-Volterra systems, (revised).

Abstract

Since 1940s, many mathematical models from physics, engineering, chemistry, biology, economics, etc., have been displayed as autonomous planar systems. A wide class of autonomous planar systems can be transformed into Liénard-type systems. Also, due to the well-known paper of I. G. Petrovskii and E. M. Landis concerning the maximum number of limit cycles of all quadratic differential systems (the second part of Hilbert's 16th problem), the study of the qualitative behavior of the solutions of autonomous planar systems of Liénard-type has become more and more important and has attracted the attention of many pure and applied mathematicians.

The purpose of this thesis is to develop the qualitative theory of autonomous planar systems of Liénard-type. More explicitly, we give conditions for global asymptotic stability, existence of local centers and global centers, existence of oscillatory solutions, existence and nonexistence of periodic solutions, and also existence and uniqueness of limit cycles for some autonomous planar systems of generalized Liénard-type. Moreover, in case of having uniqueness of limit cycles, the hyperbolicity of the limit cycle is relevant. We apply different techniques for different types of systems. The main tools used are some nonlinear integral inequalities, methods of comparison and some transformation techniques (especially the generalization of the Filippov transformation). Furthermore, some powerful methods for Liénard systems, especially those developed by G. Villari and F. Zanolin, are applied in this thesis. We apply the criteria for existence, uniqueness and hyperbolicity of limit cycles, existence of centers, existence of oscillatory solutions, and global asymptotic stability of an uniqueness positive equilibrium to the Gause-type predator-prey systems and a class of second-order autonomous systems found in the literature. On the other hand, for three-dimensional competitive Lotka-Volterra systems, M. L. Zeeman identified 33 stable equivalent classes. Among these, only classes 26-31 may have limit cycles. J. Hofbauer and J. W.-H. So conjectured that the number of limit cycles is at most two for these systems. We construct three limit cycles for class 29 without a heteroclinic polycycle in Zeeman's classification and thus give a counterexample to Hofbauer and So's conjecture. For competitor-competitor-mutualist Lotka-Volterra systems, we show that the number of periodic orbits (and hence a fortiori of limit cycles) is finite, and furthermore, we construct an example with at least two limit cycles. It is also shown that, unlike in three-dimensional competitive Lotka-Volterra systems, the nontrivial periodic coexistence does happen even if none of the three species can resist invasion from either of the other species. In this case, new amenable conditions are given on the coefficients under which the system has no nontrivial periodic coexistence. These conditions imply that the positive equilibrium, if it exists, is globally asymptotically stable.

Part I:
Research summary

Chapter 1

Generalized Liénard systems

1.1 Global asymptotic stability

The development of a mathematical theory is often guided by practical problems. For differential equations, the situation is particularly clear. The driving force behind the research in autonomous planar systems of Liénard-type was furnished much more by practical problems then by great mathematicians. During the twentieth century, applied electronics advanced rapidly, physicists invented the triode vacuum tube which was able to produce stable self-excited oscillations of constant amplitude, thus making it possible to propagate sound and pictures through electronics. However, it was not possible to describe this oscillation phenomenon by linear differential equations. In 1926, van der Pol first obtained a differential equation, which was later named after him, to describe oscillations of constant amplitude of a triode vacuum tube:

$$\ddot{x}+\mu(x^2-1)\dot{x}+x=0 \quad (\mu>0). \tag{1.1}$$

After transforming this equation into an equivalent differential system in the phase plane

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-x+\mu(1-x^2)y, \end{cases} \tag{1.2}$$

he used graphical methods to prove the existence of an isolated closed orbit (limit cycle). In 1928, the French engineer A. Liénard first studied the problem of limit cycles of the equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \tag{1.3}$$

or its equivalent differential systems

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-f(x)y-g(x), \end{cases} \tag{1.4}$$

and

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{1.5}$$

where $F(x) = \int_0^x f(s)ds$.

It is worthwhile to mention that, owing to research development from other fields such as physics, engineering, chemistry, biology, economics, etc., research on the qualitative theory of autonomous planar systems of Liénard-type has become more important. The main problem in the study of such models consists of giving a complete description of the behavior of solutions as $t \to +\infty$. In general, this is not possible, due to the complexity of the equations and the phenomena involved. The aim of the qualitative theory is to give an approximate description of the behavior of the system by identifying suitable regions of the phase space where the solutions behave in a similar way.

In recent years, several authors [26, 46, 81] have considered the following second order differential equation:

$$\ddot{x} + (f(x) + k(x)\dot{x})\dot{x} + g(x) = 0, \tag{1.6}$$

where f, g and k are all continuous functions. Clearly, when $k(x) \equiv 0$, (1.6) reduces to the Liénard equation (1.3). Using the transformation $y = a_0(x)\dot{x} + F_0(x)$, one can change (1.6) into

$$\begin{cases} \frac{dx}{dt} = \frac{1}{a_0(x)}[y - F_0(x)], \\ \frac{dy}{dt} = -a_0(x)g(x), \end{cases} \tag{1.7}$$

where $a_0(x) = \exp(\int_0^x k(s)ds)$ and $F_0(x) = \int_0^x a_0(s)f(s)ds$.

Therefore, motivated by theoretical interest and plausible applications, Qian [81], Jiang [47] and Sugie [89] investigated a more general nonlinear system:

$$\begin{cases} \frac{dx}{dt} = \frac{1}{a(x)}[h(y) - F(x)], \\ \frac{dy}{dt} = -a(x)g(x). \end{cases} \tag{1.8}$$

The global asymptotic stability of the zero solution of a planar autonomous system is related to the Markus-Yamabe problem. The following conjecture was explicitly stated by Markus and Yamabe [72] in 1960: If the eigenvalues $\lambda_1(x)$, ... $\lambda_n(x)$ of the Jacobian matrix $Df_n(x)$ of a class C^1 vector field f_n: $\mathbf{R^n} \to \mathbf{R^n}$ in the n-dimensional space $\mathbf{R^n}$ all have negative real parts at every x in $\mathbf{R^n}$ and if $f_n(0) = 0$, then the origin is a globally asymptotically stable equilibrium point for the n-dimensional nonlinear autonomous system of ordinary differential equations $\dot{x} = f_n(x)$.

The Markus-Yamabe conjecture for the case $n = 2$ has been given an affirmative answer independently by several authors [17, 25, 27]. For $n \geq 3$, the Markus-Yamabe conjecture has been proved to be false [4, 5, 10]. Therefore, the Markus-Yamabe conjecture has been completely solved. However, it is still of interest to give necessary and sufficient conditions to guarantee the zero solution of a planar autonomous system to be globally asymptotically stable [35, 46, 47, 51, 81, 90, 89, 95, 98].

To study the global asymptotic stability of the zero solution of (1.8), the significant point is to find conditions for deciding whether all orbits intersect the isocline $h(y) = F(x)$. We also need to examine the behavior of orbits near the origin. If system (1.8) has a homoclinic orbit, then the zero solution of (1.8) is not even stable. Roughly speaking, if

(i) all positive semiorbits are bounded and cross the isocline $h(y) = F(x)$,

(ii) no nontrivial periodic orbit exists, and

(iii) no homoclinic orbit exists,

then the zero solution of (1.8) is globally asymptotically stable.

Qian [81] established necessary and sufficient conditions for the global asymptotic stability of the zero solution of (1.8). Under considerably weaker conditions, Jiang [47] generalized the results of [81]. Sugie [89] investigated the same topic and obtained an implicit necessary and sufficient condition under which the zero solution of (1.8) with $a(x) \equiv 1$ is globally asymptotically stable [89, Theorem 3.1]. Since, in general, it is not an easy matter to verify whether all orbits intersect the isocline $h(y) = F(x)$, even for the Liénard system (1.8) with $a(x) \equiv 1$ and $h(y) \equiv y$ (see, for example, [24, 34, 35, 36, 90, 93, 95, 98], and the references contained therein), the result of Sugie [89, Theorem 3.1] is of theoretical interest only. For an application, Sugie [89] considered the system:

$$\begin{cases} \frac{dx}{dt} = m|y|^p \operatorname{sgn} y - F(x), \\ \frac{dy}{dt} = -g(x), \end{cases} \tag{1.9}$$

with $m > 0$ and $p \geq 1$. But the problem of determining what happens when $0 < p < 1$ is left open in [89].

The aim of Paper I in this thesis is to extend and improve the results mentioned above and to derive necessary and sufficient conditions under which the zero solution of (1.8) is globally asymptotically stable. The main advantage of our global asymptotic stability criteria is that they are explicit, so it is not difficult to verify them. In addition, our results can be applied to system (1.9) even for $0 < p < 1$. We have the following theorems which can be applied to system (1.9) for $0 < p < +\infty$.

Theorem 1.1. *Suppose that the system* (1.8) *satisfies the following conditions:*

(A_0) $F(0) = 0$, $a(x) > 0$ *for* $x \in \mathbf{R}$ *and* $xg(x) > 0$ *for* $x \neq 0$*;*

(A_1) $yh(y) > 0$ *for* $y \neq 0$, $h(y)$ *is strictly increasing and* $h(\pm\infty) = \pm\infty$*;*

(A_2^*) $F(G_0^{-1}(-z)) \leq F(G_0^{-1}(z))$ *for any* $z \in (0, \min\{-G_0(-\infty), G_0(+\infty)\})$ *and* $F(G_0^{-1}(-z)) \not\equiv F(G_0^{-1}(z))$ *for* $0 < z \ll 1$, *where* $G_0(x) = \int_0^x a^2(s)|g(s)|ds$, *and the notation* $0 < z \ll 1$ *denotes* z *sufficiently small;*

(A_3^*) *there exist constants* $\alpha > \frac{1}{4}$ *and* $\delta > 0$ *such that* $|F(x)| > 0$ for $0 < |x| \le \delta$, *and, for any fixed real number* $k \ge 1$,

$$\int_0^x \frac{a^2(s)g(s)}{|F(s)|}ds \ge \frac{1}{k}h^{-1}(k\alpha|F(x)|) \text{ for } 0 < |x| \ll 1,$$

where $h^{-1}(u)$ *is the inverse function of* $u = h(y)$;

(A_4^*) $\limsup_{x\to+\infty} F(x) > -\infty$ *and* $\liminf_{x\to-\infty} F(x) < +\infty$.
Then the origin of (1.8) *is globally asymptotically stable if and only if*

$$\limsup_{x\to+\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_-(s)}ds + F(x)\right] = +\infty \tag{1.10}$$

and

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_+(s)}ds - F(x)\right] = +\infty, \tag{1.11}$$

where $F_-(x) = \max\{0, -F(x)\}$ *and* $F_+(x) = \max\{0, F(x)\}$.

We have a more general result in Paper I:

Theorem 1.2. *Assume that the system* (1.8) *satisfies the conditions* $(A_0) - (A_4)$. *Then the zero solution of* (1.8) *is globally asymptotically stable if and only if* (1.10) *and* (1.11) *hold.*

In the study of (1.8), it seems reasonable to assume (A_0). However, assumption (A_1) is strong. On the other hand, it is also important to find other explicit conditions to decide whether all orbits intersect the isoline $h(y) = F(x)$ and whether system (1.8) has a homoclinic orbit. For this reason, some attempts have been made to get some further results for system (1.8).

We state an additional condition:

(A_1^*) $yh(y) > 0$ for $y \neq 0$, $h(y)$ is strictly increasing and the curve $h(y) = F(x)$ is well defined and continuous on all $x \in \mathbf{R}$.

Theorem 1.3. *Assume that the system* (1.8) *satisfies the conditions* $(A_0), (A_1^*), (A_4)$, *and that all positive semi-orbit are bounded. Then the zero solution of* (1.8) *is globally asymptotically stable if and only if system* (1.8) *has no closed orbits.*

Here we give three theorems of nonexistence of closed orbits (see, for example, [123] and Paper I).

Theorem 1.4. *Suppose system* (1.8) *satisfies the conditions* $(A_0), (A_1^*)$, *and the simultaneous equations*

$$F(u) = F(x), G(u) = G(x)$$

do not have a solution (u, x) *with* $-\infty < u < 0$ *and* $0 < x < +\infty$, *where* $G(x) = \int_0^x a^2(s)g(s)ds$. *Then system* (1.8) *has no closed orbits.*

Theorem 1.5. *Suppose that system* (1.8) *satisfies the conditions* (A_0) *and* (A_1^*) *and that the inequality*

$$\frac{f(u)}{a^2(u)g(u)} \geq \frac{f(x)}{a^2(x)g(x)} \quad \left(\text{or} \quad \frac{\mathrm{f(u)}}{\mathrm{a^2(u)g(u)}} \leq \frac{\mathrm{f(x)}}{\mathrm{a^2(x)g(x)}}\right)$$

holds for any (u,x) *satisfying* $G(u) = G(x)$ *with* $-\infty < u < 0$ *and* $0 < x < +\infty$. *Then system* (1.8) *has no closed orbits.*

Theorem 1.6. *Suppose that system* (1.8) *satisfies the conditions* $(A_0), (A_1^*)$ *and* (A_2^*). *Then system* (1.8) *has no closed orbits.*

1.2 Oscillations and centers

In this section we study oscillations of all nontrivial solutions and centers of system (1.8). The system (1.8) has in recent years been the object of intensive studies with particular emphasis on the asymptotic behavior of solutions (see [47, 81, 89]). To study the oscillation of solutions of (1.8), as discussed in some recent papers (see [29, 34, 47, 64, 93, 89, 98, 108, 109, 114]), for the right half plane, a significant point is to find conditions ensuring that all positive orbits $\gamma^+(P)$ (where $P = (0, p)$ with $p > 0$) intersect the characteristic curve $h(y) = F(x)$ and then cross the negative y-axis; this property of $\gamma^+(P)$ plays an important role in the analysis of the center, oscillation, asymptotic stability and boundedness conditions of (1.8). There have been many studies in this direction in which sufficient conditions to obtain the above mentioned property of $\gamma^+(P)$ were given. For example (see [18, 24, 35, 36, 73, 75, 77, 101, 120]), no solution of (1.5) approaches the origin directly in the right half plane (i.e., in a nonoscillatory way) if one of the following conditions is satisfied (in the following, $f(x) := F'(x)$ if $F(x)$ is continuously differentiable and $G(x) := \int_0^x g(s)ds$):

(1) (McHarg [73]) $f(x) > 0$ for $x > 0$ and there exist $k > 0$ and $a > 0$ such that

$$f(x) < kg(x) \quad \text{for } 0 < x < a.$$

(2) (Wendel [101]) There exist $k > 0$ and $a > 0$ such that

$$0 < f(x) < kg(x) \quad \text{for } 0 < x < a.$$

(3) (Nemyckii and Stepanov [75]) There exist $\alpha > \frac{1}{4}$ and $a > 0$ such that

$$f(x) > 0 \quad \text{and } \alpha f(x)F(x) \leq g(x) \quad \text{for } 0 < x < a.$$

(4) (Filippov [18]) There exist $0 < \beta < 8$ and $a > 0$ such that

$$F^2(x) \leq \beta G(x) \quad \text{for } 0 < x < a.$$

(5) (Opial [77]) There exist $\alpha > \frac{1}{4}$ and $a > 0$ such that

$$\alpha|F(x)| \le \int_0^x \frac{g(u)}{|F(u)|}du \ \text{ for } 0 < x < a.$$

(6) (Hara and Yoneyama [35], Hara, Yoneyama and Sugie [36], Sugie [90]) If one of the following conditions holds:

(i) there exists a positive sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \le 0$ for $n \ge 1$;

(ii) There exist $\alpha > \frac{1}{4}$ and $a > 0$ such that

$$F(x) > 0 \ \text{ and } \ \frac{1}{F(x)}\int_0^x \frac{g(u)}{F(u)}du \ge \alpha \ \text{ for } 0 < x < a.$$

(7) (Yu [120]) There exist $a > 0$, $k_1 > 0$ and $k_2 < 0$ such that

$$k_2 \le \frac{f(x)}{g(x)} \le k_1 \ \text{ for } 0 < x < a.$$

Our investigation shows that condition (6) is much weaker than condition (4). The problem concerning the oscillation of solutions of (1.8) with $a(x) \equiv 1$ has been studied by some authors (see, for example, [64, 109] and the references cited therein). Li and Tang [64] discussed the oscillation of solutions of (1.8) with $a(x) \equiv 1$ requiring the existence of $h''(y)$ and $h'(0) > 0$. Yan and Jiang [109] proved that the solutions of (1.8) with $a(x) \equiv 1$ are oscillatory under the condition $h'(0) > 0$. But the problem of what happens when $h'(0) = 0$ or $h'(0) = +\infty$ remains unsolved. In the present paper, no restrictions on the differentiability of $h(y)$ are required. We give necessary and sufficient conditions for all nontrivial solutions of (1.8) being oscillatory. Our theorem can be applied to system (1.8) even for $h'(0) = 0$, $h'(0) = +\infty$ and $\lim_{|x|\to+\infty} F(x)\operatorname{sgn} x = -\infty$.

The problem of finding the center of the system (1.5) has been widely studied and continues to attract attention; see, for example, [77, 90, 95, 98, 101, 108, 115, 120, 127] and the references cited therein. Our purpose is to develop a center theory for the system (1.8). This work was motivated by the papers of Hara and Yoneyama [35] and Sugie [90], in which a detailed analysis of center properties was given for system (1.5). We will follow closely the presentation of Hara, Yoneyama and Sugie, and show that all of their results on this subject can be generalized to (1.8).

The technical tool is based on a nonlinear integral inequality and a phase plane analysis. Also the methods for Liénard-type systems, especially those developed by Villari and Zanolin [98], Hara and Sugie [34], and Sugie and Hara [93] are also applied.

In Paper VI, We have the following results:

Theorem 1.7. *Suppose that the conditions* (A_0), (A_1), (A_6) *and* (A_7) *are satisfied. Then all nontrivial solutions of* (1.8) *oscillate if and only if* (1.10) *and* (1.11) *hold.*

Theorem 1.8. *If conditions* (A_0), (A_1) *and* (A_4) *hold, and there exists* $K_0 > 0$ *such that*
(A_3*) $\quad F(G^{-1}(-w)) = F(G^{-1}(w))$ *for* $0 \le w < K_0$.
Then the origin is a local center of (1.8).

Theorem 1.9. *Suppose that the origin is a local center of* (1.8)*, and that the conditions* $(A_0) - (A_3)$ *and* (A_5) *are satisfied. Then the origin is a global center of* (1.8) *if and only if* (1.10) *and* (1.11) *hold.*

By a method similar to that of Section 1.2, we can relax condition (A_1). We have the following further results:

Theorem 1.10. *Assume that the system* (1.8) *satisfies the conditions* $(A_0), (A_1^*)$ *and* (A_6) *and that all positive semi-orbits are bounded. Then all nontrivial solutions of* (1.8) *oscillate.*

Theorem 1.11. *Assume that the system* (1.8) *satisfies the conditions* $(A_0), (A_1^*), (A_3*)$ *and* (A_4). *Then the origin is a local center of* (1.8).

1.3 Limit cycles for generalized Gause-type predator-prey systems

The first attempts to describe population cycles mathematically can be found in Lotka [67] and Volterra [99]. The classical Lotka-Volterra system is

$$\begin{cases} \dot{x} = ax - bxy, \\ \dot{y} = cxy - dy, \end{cases} \tag{1.12}$$

which admits no isolated periodic orbits, and the interior equilibrium is a center surrounded by neutrally stable orbits. The models for which of Lotka and Volterra were later generalized by Gause [21] into the following Gause-type predator-prey model for consumer-resource interaction:

$$\begin{cases} \dot{x} = h(x) - yp(x), \\ \dot{y} = cyp(x) - dy. \end{cases} \tag{1.13}$$

The number, x and y, denote the prey and predator densities, respectively. The function h is the growth function of the predator and this function is assumed to be continuously differentiable. In the absence of predators, the prey population should converge towards a positive limit with $h(x) > 0$ for $0 < x < K$, $h(x) < 0$ for $x > K$ or $x < 0$, and $h(x) = 0$ for $x = K$ or $x = 0$. The function p is called the functional response. This function is expected to be increasing and continuously differentiable. Moreover, it has a unique zero at the origin. The functional response denotes the number of prey eaten by a predator per unit of time as a function of prey density. The constants c and d denote the conversion factor and the death-rate of the predator, respectively. The conditions are such, that the solutions of the system (1.13) remain positive and bounded. In the models of Lotka and Volterra, the functions h and p were assumed to be linear.

Gause-type predator-prey systems can possess unique attractors also when it does not possess limit cycles. If the unique attractor is an equilibrium, the Gause-type predator-prey system possesses global stability since the solutions are bounded and hence the equilibrium must attract all initial conditions, except possibly initial conditions on the $x-$ and $y-$axis. Hence, the existence of limit cycles in Gause-type predator-prey systems is related to the existence and stability of the positive equilibrium. If there exists a unique positive equilibrium which is unstable, then there must exist at least one limit cycle. For a Gause-type predator-prey model under Kolmogorov conditions, May [71] claimed that there must occur a "unique" stable limit cycle. In response to this, Albrecht et al [1] constructed a Gause-type predator-prey model satisfying the Kolmogorov conditions for which there are uncountably many periodic solutions inside an annular region bounded by two limit cycles. This observation makes the problem of determining conditions which guarantee the uniqueness of limit cycles or the global stability of positive equilibrium in Gause-type predator-prey systems very challenging.

In this section we consider a general Gause-type predator-prey systems of the general form

$$\begin{cases} \dot{x} = \psi(x) - \xi(y)p(x), \ x(0) \geq 0, \\ \dot{y} = \eta(y)q(x), \ y(0) \geq 0, \end{cases} \tag{1.14}$$

where x and y are functions of t, which represent the prey and predator populations at a given time $t \geq 0$, respectively. Hence, we will restrict our attention to the first quadrant and make the following assumptions:

$(A1)$ $\psi(x) \in C^1[0,+\infty)$, $\psi(0) = 0$ and there exists $K > 0$ such that $\psi(K) = 0$ and $(x-K)\psi(x) < 0$ for $x > 0$ and $x \neq K$;

$(A2)$ $\xi(y), \eta(y) \in C^1[0,+\infty)$, $\xi(0) = 0 = \eta(0) = 0$ and $\xi'(y) > 0$ and $\eta'(y) > 0$ for $y \geq 0$;

$(A3)$ $p(x) \in C^1[0,+\infty)$, $p(0) = 0$ and $p(x) > 0$ for $x > 0$;

$(A4)$ $q(x) \in C^1[0,+\infty)$, there exists $x^* \in (0,K)$ such that $q(x^*) = 0$, and $(x-x^*)q(x) > 0$ for $x \in (0,x^*) \bigcup (x^*,K)$;

$(A5)$ System (1.14) has no equilibrium at infinity except at the infinity of the positive x-axis and y-axis.

We use some transformations of variables to reduce system (1.14) to the generalized Liénard system (1.8) with $a(x) \equiv 1$. We present explicit conditions guaranteeing the uniqueness of limit cycles (based on a theorem of Gasull and Guillamon [20]) and results on nonexistence of limit cycles, the global stability of a positive equilibrium of system and center problems. Further, in the case of having uniqueness, the limit cycle could bifurcate under small perturbations and the dynamics of the population could be qualitatively modified. If a limit cycle is hyperbolic then it will persist under small C^1-perturbations. Hence, the hyperbolicity of a limit cycle for (1.8) implies the non-appearance of new periodic solutions near to it and so a similar behavior for the C^1-close system, even if it is not of type (1.8).

We have the following results (see, for example, [103]):

Theorem 1.12. *Suppose that system* (1.14) *satisfies the conditions* $(A1)-(A5)$, *then the solutions of* (1.14) *in the interior of the first quadrant are positive and eventually bounded.*

Theorem 1.13. *Suppose system* (1.14) *satisfies the conditions* $(A1)-(A5)$. *Then the dynamics of* (1.14) *in the region* $\Omega_1 = \{(x,y) : 0 < x < K, 0 < y < +\infty\}$ *is equivalent to that of the generalized Liénard system*

$$\begin{cases} \dot{u} = \phi(v) - F(u), \\ \dot{v} = -g(u), \end{cases} \tag{1.15}$$

in the region $\Omega_2 = \{(u,v) : x^* - K < u < x^*, h^{-1}(-y^*) < v < h^{-1}(+\infty)\}$, *where* $F(u) = \psi(-u+x^*)/p(-u+x^*) - \xi(y^*)$, $\phi(v) = \xi(h(v)+y^*) - \xi(y^*)$, $g(u) = -q(-u+x^*)/p(-u+x^*)$, *and* $h(v)$ *is a solution of the initial-value problem* $dh(v)/dv = \eta(h(v)+y^*)$, $h(0) = 0$.

Therefore, we can establish conditions to ensure the uniqueness and hyperbolicity of limit cycles, the nonexistence of limit cycles, the global stability of a positive equilibrium of systems and oscillation and to solve center problems by utilizing a wealth of existing methods or the results for the generalized Liénard systems (1.8) given in the above sections ([20, 57, 58, 59, 103, 105, 106]).

Chapter 2

A class of second-order autonomous systems

2.1 Global asymptotic stability of second-order nonlinear differential systems

In a series papers [52, 53, 54, 55], Krechetov studied the following real system of two differential equations

$$\begin{cases} \dot{x} = f_1(x) + h_2(x)y, \\ \dot{y} = f_3(x) + h_4(x)y, \end{cases} \tag{2.1}$$

where $f_1(x),\ f_3(x),\ h_2(x)$ and $h_4(x)$ are continuous on $\mathbf{R}$. Using Liapunov functions, he investigated the question of stability, described the configurations of the domains of stability (when there is no global stability) and constructed estimates of the boundaries of these domains. Egorov and Kartuzova [16] studied the same problem and formulated necessary and sufficient conditions for the zero solution of (2.1) to be globally asymptotically stable under rather restrictive assumptions on the functions $h_i(x)$.

Theorem 2.1. *(Egorov and Kartuzova [16]). Suppose that* $f_1(x),\ f_3(x),\ h_2(x)$ *and* $h_4(x)$ *are continuous on* $\mathbf{R}$ *with* $f_1(0) = f_3(0) = 0$ *and that they satisfy the following conditions:*

(1) $h_1(x) + h_4(x) < 0$ *for* $x \neq 0$*;*

(2) $h_1(x)h_4(x) - h_2(x)h_3(x) := \delta(x) > 0$ *for* $x \neq 0$*, where* $h_i(x) = \frac{f_i(x)}{x}$ *for* $x \neq 0$ *and* $i = 1,\ 3$*;*

(3) $h_2(x) \neq 0$ *for all* x*;*

(4) $h_1(x) + \frac{h_2(x)H_{42}(x)}{x} < 0$ *for* $x \neq 0$*.*

Then the zero solution of (2.1) *is globally asymptotically stable if and only if*

$$\int_0^{\pm\infty} \delta(x)[h_2(x)]^{-2}x dx + \limsup_{x\to\pm\infty} |\Psi(x)| = +\infty. \tag{2.2}$$

Here $H_{42}(x) := \int_0^x h_4(s)[h_2(s)]^{-1}ds$ *and* $\Psi(x) := \left[h_1(x) + \frac{h_2(x)H_{42}(x)}{x}\right]\frac{x}{h_2(x)}$.

In Paper II, we investigate the global asymptotic stability of the system (2.1) without the assumption (4) and condition (2.2) in Theorem 2.1. The transformation technique plays an important role. Under suitable assumptions, we prove that the system (2.1) is equivalent to equations of the following type

$$\begin{cases} \dot{x} = \phi(z - F(x)), \\ \dot{z} = -g(x), \end{cases} \tag{2.3}$$

which is a generalized form of the Liénard system. The study of the system (2.3) has an independent interest and value.

2.2 Qualitative behavior of second-order systems with zero diagonal coefficient

In Paper III, we study the qualitative behavior of the solutions of the following autonomous system of two differential equations with zero diagonal coefficient

$$\begin{cases} \dot{x} = p_2(y)q_2(x)y, \\ \dot{y} = p_3(y)q_3(x)x + p_4(y)q_4(x)y, \end{cases} \tag{2.4}$$

where $p_i(y)$ and $q_i(x)$ ($i = 2,3,4$) are continuous real functions defined on $\mathbf{R} = (-\infty, +\infty)$.

Krechetov [56] studied the global asymptotic behavior of solutions of system (2.4), described the configurations of the domains of stability (when there is no global asymptotic stability) and constructed estimates of the boundaries of these domains. In the study of stability for (2.4), the most important condition given by Krechetov [56] is

$$q_2(x)q_4(x) > 0 \quad \text{for all} \quad x \in \mathbf{R}. \tag{2.5}$$

By using the Lyapunov function method, he gave necessary and sufficient conditions for the zero solution of (2.4) to be globally asymptotically stable under some additional assumptions.

In Paper II, we first introduce the transformation techniques to investigate the global asymptotic stability of the following system (2.6), the special case (i.e., $p_3(y) \equiv p_4(y)$) of system (2.4),

$$\begin{cases} \dot{x} = p_2(y)q_2(x)y, \\ \dot{y} = p_3(y)q_3(x)x + p_3(y)q_4(x)y. \end{cases} \tag{2.6}$$

Without the assumption (2.5), in paper [107], under the following conditions

$$\begin{aligned} &p_2(y) > 0,\ p_3(y) > 0 \ \text{ for all } \ y, \\ &q_2(x) > 0,\ q_3(x) < 0 \ \text{ for all } \ x, \end{aligned} \tag{2.7}$$

they transformed system (2.6) into the following Liénard-type system

$$\begin{cases} \dot{x} = \phi(z - F(x)), \\ \dot{z} = -g(x), \end{cases} \tag{2.8}$$

and obtained necessary and sufficient conditions for the zero solution of (2.6) (resp. (2.8)) to be globally asymptotically stable. Such a system (2.8) with $\phi(u) \equiv u$ arises in several different settings: modelling phenomena appearing in the study of physical, as well as biological, chemical, and economical systems It has naturally been studied by a number of authors [18, 29, 34, 35, 36, 90, 98, 116, 124]. In Paper III, we investigate the qualitative behavior of system (2.4) without the assumption (2.5). No restriction on the sign of $q_4(x)$ is required; we only assume that

$$\begin{aligned} &p_2(y) > 0,\ p_3(y) > 0,\ p_4(y) > 0 \ \text{ for all } \ y, \\ &q_2(x) < 0,\ q_3(x) > 0 \text{ or } q_2(x) > 0,\ q_3(x) < 0) \ \text{ for all } \ x, \\ &\rho(y) \in C^1(\mathbf{R}),\ \rho'(y) > 0 \ \text{ for all } \ y,\ \rho(\pm\infty) = \pm\infty, \\ &\text{where } \rho(y) := \tfrac{y p_4(y)}{p_3(y)}. \end{aligned} \tag{2.9}$$

If $p_3(y) \equiv p_4(y)$, one case of assumption (2.9) reduces to (2.7). Under assumption (2.9), we prove that system (2.4) is equivalent to a form of system (2.8) which is a Liénard-type system, and give some conditions for the existence of oscillatory solutions, the existence of local centers and global centers, and the existence, uniqueness and hyperbolicity of nontrivial periodic solutions for system (2.4) (resp. (2.8)).

2.3 Existence and nonexistence of periodic solutions of general autonomous systems of Liénard-type

In 1942, Levinson and Smith [63] first studied the existence of nonzero periodic solutions of the general autonomous equation of Liénard-type

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0 \tag{2.10}$$

or its equivalent system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -f(x,y)y - g(x). \end{cases} \tag{2.11}$$

Since then, many authors have made contributions to the theory of this system with regards to the existence of nonzero periodic solutions. The books by Sansone and

Conti [83], Zhang [124] and Ye [116] contain a summary of the results on this problem. On reviewing all of the known results, we find that in order to obtain a criterion for the existence of nonzero periodic solutions, almost every author required that the restoring force $g(x)$ and damping $f(x,y)$ should be not too small, that is, $f(x,y)$ should have a lower bound in a strip region $|x| \leq d$ and should be non-negative outside this strip region, and $\int^{\pm\infty} g(x)dx = +\infty$. Ponzo and Wax [79] gave a result on the existence of a nonzero periodic solution which does not require $f(x,y)$ to have a lower bound. Unfortunately, Zheng [126] gave an example

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(x^2-4)(y^2+1)y - x \end{cases} \tag{2.12}$$

to show that the conditions of Ponzo and Wax cannot guarantee the existence of a nonzero periodic solution if $f(x,y)$ does not have a lower bound. Yu and Huang [118] also dealt with the existence of nonzero periodic solutions of (2.11), and pointed out that system (2.12) has a nonzero periodic solution. Yan and Jiang [110] considered the system (2.11), and noted that system (2.12) has no nonzero periodic solution. Also, Wang, Jiang and Yan [100] gave a complete analysis of global bifurcation for the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(x^2-\delta)(y^2+1)y - x, \end{cases} \tag{2.13}$$

where δ is a parameter. In addition, it was shown by Lemma 5 in [100] that system (2.13) has no nonzero periodic solution when $\delta \geq \sqrt[3]{q\pi^2/16} \approx 1.7707$.

Yu and Huang [119] studied a more general system than (2.11), namely,

$$\begin{cases} \dot{x} = p(y), \\ \dot{y} = -f(x,y)p(y)q(y) - r(y)g(x), \end{cases} \tag{2.14}$$

under the assumptions $\int_0^{\pm\infty} g(s)ds = +\infty$ They obtained some sufficient conditions for the existence of one nonzero periodic solution of (2.14). Moreover, as a result of [119] they pointed out that system (2.12) has at least one nonzero periodic solution.

The purpose of Paper IV is to study the problem how small the extent for $f(x,y)$ should be to warrant the existence of nonzero periodic solutions of (2.14). Our investigation shows that whether (2.14) has a nonzero periodic solution strongly depends on the integral $\int^{\pm\infty} |f(x,y)q(y)|^{-1}dy$, where $|x|$ is sufficiently small. We find some sufficient conditions for the existence of nonzero periodic solutions of (2.14), roughly speaking, if $\int^{\pm\infty} |f(x,y)q(y)|^{-1}dy = \pm\infty$ for a small $|x|$ and some additional assumption hold, then (2.14) has at least one nonzero periodic solution. Our results allow us to avoid the classical assumptions:

$$\int_0^{\pm\infty} g(x)dx = +\infty, \tag{2.15}$$

$$f(x,y) > 0 \text{ (or } \geq 0) \text{ for } |x| \text{ sufficiently large.} \tag{2.16}$$

In Paper IV we also give some sufficient conditions for nonexistence of periodic solutions of (2.14), which state that if $\int^{+\infty} |f(x,y)q(y)|^{-1}dy$ (or $\int^{-\infty} |f(x,y)q(y)|^{-1}dy$) is finite for a small $|x|$ and some additional assumptions hold, then there does not exist a nonzero periodic solution of the system (2.14). Some examples illustrating our results are given in Paper IV.

Chapter 3

Limit cycles in three-dimensional Lotka-Volterra systems

3.1 A 3D Lotka-Volterra competitive system with three limit cycles: A falsification of a conjecture by Hofbauer and So

Lotka-Volterra (L-V) interaction of n biological species is modeled by a system of differential equations

$$\frac{dx_i}{dt} = x_i\left(r_i + \sum_{j=1}^{n} a_{ij}u_j\right), \qquad i = 1, 2, \cdots, n, \tag{3.1}$$

where x_i represents the number (or density) of individuals of species i, the a_{ij}'s are the interaction coefficients, and r_i is the per capita growth rate of species i in the absence of interaction. For example, $r_i > 0$ means that species i is able to grow with food from the environment, while $r_i < 0$ means that it cannot survive when left alone in the environment. One can also have $r_i = 0$ which means that the population stays constant if the species do not interact.

Although the L-V model is a model that originated from mathematical ecology, it plays an important role in many other research fields including optical maser [60], fluid mechanics [70] and neural networks [76]. Nevertheless, we would like to emphasize its close relationship with replicator dynamics, which is an important branch of deterministic dynamical systems motivated by evolutionary game theory (see [40, 41]).

The dynamics of the two-dimensional L-V systems is well understood. In particular, two-dimensional L-V systems cannot have limit cycles: if there is a periodic orbit, then the interior singular point is a center (i.e., surrounded by a continuum of periodic orbits). Hence a center is a codimension one phenomenon for two-dimensional L-V systems. Using numerical simulations, three-dimensional L-V systems have been seen to allow already complicated dynamics: the period doubling route to chaos and many other phenomena known from the interaction of the quadratic map have been observed (see [2, 19, 84]).

An L-V system with interaction matrix $A=(a_{ij})$ is called *competitive* if $r_i>0$ and $a_{ij}<0$ for all $1\le i,j\le n$. It describes the competition between two or more species that share and compete for the same resources, habitat or territory (interference competition). This is different from exploitative competition, where individuals do not directly interfere with one another, but compete indirectly through their consumption of a common resource [11]. From the viewpoint of evolutionary game theory, all replicator dynamics on the standard $(n-1)$ dimensional simplex S_n can be imbedded into a competitive L-V system on $\mathbf{R}^{\mathbf{n}}_{+}$ which has a global attractor S_n [41].

For three-dimensional competitive L-V systems, the dynamical possibilities are more restricted: Hirsch [38] has showed that all nontrivial orbits approach a "carrying simplex", a Lipchitz two-dimensional manifold-with-corner homeomorphic to the standard simplex in $\mathbf{R}^{\mathbf{3}}_{+}$. This then leads to the Poincaré-Bendixson theorem for three-dimensional systems, which states that three-dimensional competitive L-V systems behave like general planar systems. Based on the remarkable result of Hirsch, Zeeman [121] defined a combinatorial equivalence relation on the set of all three-dimensional L-V competitive systems and identified 33 stable equivalence classes. Of these, classes 1-25 and classes 32 and 33 exhibit convergence to equilibrium for all orbits, while limit cycles are possible for the remaining 6 classes, i.e., in classes 26 to 31 (see [15, 121]). Open problems remain concerning the number of periodic orbits in the later classes. Hofbauer and So [43] first give an example in class 27 (with heteroclinic polycycle) with two limit cycles surrounding the interior equilibrium. Recently, Lu and Luo [68] constructed two limit cycles in three cases without a heteroclinic polycycle (cases 26, 28 and 29).

Apparently, the main questions now are (i) whether or not there are at most finitely many limit cycles on the carrying simplex and (ii) whether there can be more than two limit cycles in three-dimensional competitive L-V systems. Regarding question (i) Xiao and Li [102] have proved that the number of limit cycles of the three-dimensional competitive L-V systems is finite if the system does not have any heteroclinic polycycle. It is a very interesting open problem to prove whether or not the number of limit cycles of system (1) is finite in the small neighborhood of the heteroclinic polycycles. Question (ii) is a very difficult problem. Hofbauer and So [43] conjectured that the number of limit cycles is at most two for system (1). Note that the existence of the limit cycles in the references [43, 102, 68] were all generated by local Hopf bifurcation. The discussion in [43] implies that the maximum order of a focus would be 2 and that one could not generate more than two limit cycles from local Hopf bifurcation. This motivated their belief that two is the maximum number of limit cycles in three-dimensional competitive L-V systems. However, it is worthy of note that Hofbauer-So Conjecture neglects the fact that the global dynamical behavior of system (1) might generate the third limit cycle by the Poincaré-Bendixson theorem in three-dimensional competitive L-V systems [88]. If we can construct by generic Hopf bifurcation a L-V competitive system which is strongly persistent [9] and has two limit cycles and if the first bifurcated (outer) cycle is unstable, then by the Poincaré-Bendixson theorem in three-dimensional competitive systems we can get the third limit cycle.

Recently, Lu and Luo [69] were the first to give an example in class 27 (with a heteroclinic polycycle) with three limit cycles. This gives a partial answer to Hofbauer and So's conjecture. In Paper V, we construct three limit cycles in case 29 without heteroclinic polycycle and thus give a counterexample to Hofbauer and So's conjecture which is qualitatively different from that of Lu and Luo. We conjecture that there also exist three limit cycles in case 26. We leave this as a future research problem.

3.2 Limit cycles for the competitor-competitor-mutualist Lotka-Volterra systems

The dynamics of an ecosystem with $n \geq 2$ interacting populations can be modelled by the general Lotka-Volterra system

$$\dot{x}_i = x_i(r_i + \sum_{j=1}^{n} a_{ij}x_j), \qquad i = 1,2,...,n, \tag{3.2}$$

where x_i is the density of the ith population, r_i is the intrinsic growth rate of the ith population and the coefficient a_{ij} describes the influence of the jth population upon the ith population (Hofbauer and Sigmund [40]). The signs of a_{ij} and a_{ji} determine the nature of the interaction between the populations i and j: the system (3.2) can describe all of the three basic types of interaction, viz., competition, collaboration (mutualism) and host-parasite (predator-prey) interactions.

The dynamics of two-dimensional Lotka-Volterra systems is well understood. Bomze [6] gave a complete classification of all possible phase portraits for this case. In particular, there are no limit cycles in two-dimensional Lotka-Volterra systems: if there is a periodic orbit, then the equilibrium in $\text{Int}\mathbf{R}^{\mathbf{2}}_{+}$ is a center, that is, it is surrounded by a continuum of periodic orbits. As is well known, this is the case in the classical Lotka-Volterra predator-prey system. It should, however, be noted that the phase portrait does not reveal the whole dynamics. For example, the solution may blow up in finite time (this is clear because the system (3.2) contains the system $\dot{x}_i = x_i^2$ as a special case).

As one steps from two to higher dimensions the situation becomes far more complicated and difficult. Using numerical simulations, three-dimensional Lotka-Volterra systems allow already complicated dynamics. The period doubling route to chaos and many other phenomena known from the interaction of the quadratic map have been observed (see [2, 19, 84]).

For three-dimensional competitive Lotka-Volterra systems, the dynamical possibilities are more restricted: Hirsch [38] has showed that all nontrivial orbits approach a "carrying simplex", a Lipshitz two-dimensional manifold-with-corner homeomorphic to the standard simplex in $\mathbf{R}_+^3$. Based on this, Zeeman [121] has given a classification of all possible stable phase portraits of three-dimensional competitive Lotka-Volterra systems and has shown that in some three-dimensional competitive Lotka-Volterra systems limit cycles can indeed occur. Recently, Liang and Jiang [65] did the same for Competitor-Competitor-Mutualist Lotka-Volterra systems. Hofbauer and So [43], Xiao and Li [102] and Lu and Luo [68] have also presented examples of three-dimensional competitive Lotka-Volterra systems with at least two limit cycles.

In Paper VII, we focus on the limit cycles for the Competitor-Competitor-Mutualist Lotka-Volterra systems. The specific system we shall consider models two competing populations that both collaborate with a third one. Such systems are of great biological relevance. The two competing populations may for instance represent two different types of the same species (a "resident" and "mutant" in the terminology of adaptive dynamics, Metz et al. [74]; Geritz et al. [22, 23]). More models of this type can be found in [28, 65] and the references therein. We shall prove that the number of nontrivial periodic orbits (and hence a fortiori of limit cycles) is finite in Competitor-Competitor-Mutualist Lotka-Volterra systems. We also construct an example of a system of this type with at least two limit cycles by using local Hopf bifurcation and analyse the scale of the parameters.

It also deserves to be noted that it is under the assumption $M_{12} = a_{11}a_{22} - a_{12}a_{21} < 0$ that Liang and Jiang [65] obtained the existence of the nontrivial limit cycle, generated by Hopf bifurcation, in Competitor-Competitor-Mutualist Lotka-Volterra systems [65, Theorem 5.5]. Hence in this case, in the competitive subcommunity of two species 1 and 2, at least one can resist invasion by the other. For the three-dimensional competitive Lotka-Volterra systems, van den Driessche and Zeeman [15] have shown that if none of the species can resist invasion by either of the others, then there is no periodic orbit and therefore limit cycles do not exist and global dynamics are known. For the system (0.38), it is obvious that none of the species can resist invasion by the other in the mutualistic subcommunity of two species 1 and 3, or 2 and 3. Therefore, it is a very interesting question whether there exist periodic orbits in Competitor-Competitor-Mutualist Lotka-Volterra systems if none of the species can resist invasion by the other in the competitive subcommunity of two species 1 and 2. We answer this question by providing an example which has a stable limit cycle. Meanwhile, new amenable conditions are also given on the coefficients r_i, a_{ij}, under which system (3.2) has no periodic orbits if none of the species can resist invasion from either of the others. Thus all trajectories converge to equilibria. Based on this, we also present an example of global stability for a positive equilibrium, but the Volterra multipliers method [65, Theorem 5.6] cannot be applied.

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Original research papers

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THE GENERALIZED LIÉNARD SYSTEMS

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Abstract. We consider the generalized Liénard system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{a(x)}[h(y) - F(x)],\\ \frac{dy}{dt} &= -a(x)g(x),\end{aligned} \tag{0.1}$$

where a is a positive and continuous function on $\mathbf{R} = (-\infty, \infty)$, and F, g and h are continuous functions on $\mathbf{R}$. Under the assumption that the origin is a unique equilibrium, we obtain necessary and sufficient conditions for the origin of system (0.1) to be globally asymptotically stable by using a nonlinear integral inequality. Our results substantially extend and improve several known results in the literature.

1. **Introduction.** It is well known that the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.1}$$

is of great importance in various applications. Hence, qualitative and asymptotic behavior of this equation and some of its extensions have been widely studied by a number of authors; results can be found in many books [14, 18, 19, 23, 31, 32, 33]. In recent years, several authors [8, 15, 22] have considered the following second order differential equation

$$\ddot{x} + (f(x) + k(x)\dot{x})\dot{x} + g(x) = 0, \tag{1.2}$$

where f, g and k are all continuous functions. Clearly, when $k(x) \equiv 0$, (1.2) is reduced to the Liénard equation (1.1). Using the transformation $y = a_0(x)\dot{x} + F_0(x)$, one can change (1.2) into

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{a_0(x)}[y - F_0(x)],\\ \frac{dy}{dt} &= -a_0(x)g(x),\end{aligned} \tag{1.3}$$

where $a_0(x) = \exp(\int_0^x k(s)ds)$ and $F_0(x) = \int_0^x a_0(s)f(s)ds$. Therefore, motivated by theoretical interest and plausible applications, Qian[22], Jiang[16] and Sugie[27] investigated a more general nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{a(x)}[h(y) - F(x)],\\ \frac{dy}{dt} &= -a(x)g(x).\end{aligned} \tag{1.4}$$

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In this paper, we give necessary and sufficient conditions for the global asymptotic stability of the zero solution of the system (1.4). we assume that $a > 0$, F, g and h are continuous functions which ensure the existence of a unique solution to the initial value problem.

The global asymptotic stability of the zero solution of a planar autonomous system is related to the Markus-Yamabe problem. The following conjecture was explicitly stated by Markus and Yamabe [20] in 1960: If the eigenvalues $\lambda_1(x)$, ... $\lambda_n(x)$ of the Jacobian matrix $Df_n(x)$ of a class C^1 vector field f_n: $\mathbf{R^n} \to \mathbf{R^n}$ in the n-dimensional space $\mathbf{R^n}$ all have negative real parts at every x in $\mathbf{R^n}$ and if $f_n(0) = 0$, then the origin is a globally asymptotically stable equilibrium point for the n-dimensional nonlinear autonomous system of ordinary differential equations $\dot{x} = f_n(x)$.

The Markus-Yamabe conjecture for the case $n = 2$ has been given an affirmative answer independently by several authors [5, 9, 10]. For $n \geq 3$, the Markus-Yamabe conjecture has been proved to be false [1, 2, 4]. Therefore, the Markus-Yamabe conjecture has been completely solved. However, it is still of interest to give necessary and sufficient conditions to guarantee the zero solution of a planar autonomous system to be globally asymptotically stable [12, 15, 16, 17, 22, 24, 27, 28, 29].

To study the global asymptotic stability of the zero solution of (1.4), the significant point is to find conditions for deciding whether all orbits intersect the isocline $h(y) = F(x)$, and we also need to examine the behavior of orbits near the origin. If system (1.4) has a homoclinic orbit, then the zero solution of (1.1) is not even stable. Roughly speaking, if

(i) all positive semiorbits are bounded and cross the isocline $h(y) = F(x)$,

(ii) no nontrivial periodic orbit exists,

(iii) no homoclinic orbit exists,

then the zero solution of (1.4) is globally asymptotically stable.

Recently, Qian [22] established necessary and sufficient conditions for the global asymptotic stability of the zero solution of (1.4). Under considerably weaker conditions, Jiang [16] generalized the results of [22], Sugie [27] investigated the same topic and obtained an implicit necessary and sufficient condition under which the zero solution of (1.4) with $a(x) \equiv 1$ is globally asymptotically stable [27, Theorem 3.1]. Since, in general, it is not an easy matter to verify whether all orbits intersect the isocline $h(y) = F(x)$ even for the Liénard system (1.4) with $a(x) \equiv 1$ and $h(y) = y$ (see, for example, [7, 11, 12, 13, 24, 26, 28, 29], and the references contained therein), the result of Sugie [27, Theorem 3.1] is of theoretical interest only. For an application, Sugie [27] considered the system

$$\begin{aligned} \frac{dx}{dt} &= m|y|^p \operatorname{sgn} y - F(x), \\ \frac{dy}{dt} &= -g(x), \end{aligned} \tag{1.5}$$

with $m > 0$ and $p \geq 1$. But the problem of what happens when $0 < p < 1$ is left open in [27].

The purpose of the present paper is to extend and improve the results mentioned above and to derive necessary and sufficient conditions under which the zero solution of (1.4) is globally asymptotically stable. The main advantage of our global asymptotic stability criteria is that they are explicit, so that it is not difficult to verify them. In addition, our results can be applied to system (1.5) even for $0 < p < 1$.

As a corollary of our main results, we have the following theorem which can be applied to system (1.5) for $0 < p < \infty$.

Theorem 1.1. *Suppose that the system* (1.4) *satisfies the following conditions:*

(A_0) $F(0) = 0$, *$a(x) > 0$ for $x \in \mathbf{R}$, $xg(x) > 0$ for $x \neq 0$;*

(A_1) *$yh(y) > 0$ for $y \neq 0$, $h(y)$ is strictly increasing and $h(\pm\infty) = \pm\infty$;*

(A_2^*) *$F(G_0^{-1}(-z)) \leq F(G_0^{-1}(z))$ for any $z \in (0, \min\{-G_0(-\infty), G_0(\infty)\})$ and $F(G_0^{-1}(-z)) \not\equiv F(G_0^{-1}(z))$ for $0 < z \ll 1$, where $G_0(x) = \int_0^x a^2(s)|g(s)|ds$, and the notation $0 < z \ll 1$ denotes z sufficiently small;*

(A_3^*) *there exist constants $\alpha > \frac{1}{4}$ and $\delta > 0$ such that $|F(x)| > 0$* for $0 < |x| \leq \delta$, *and for any fixed real number $k \geq 1$,*

$$\int_0^x \frac{a^2(s)g(s)}{|F(s)|}ds \geq \frac{1}{k}h^{-1}(k\alpha|F(x)|) \quad \text{for} \quad 0 < |x| \ll 1,$$

where $h^{-1}(u)$ is the inverse function of $u = h(y)$;

(A_4^*) $\limsup_{x\to\infty} F(x) > -\infty$ *and* $\liminf_{x\to-\infty} F(x) < \infty$.

Then the origin is globally asymptotically stable if and only if

$$\limsup_{x\to\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_-(s)}ds + F(x)\right] = \infty \tag{1.6}$$

and

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_+(s)}ds - F(x)\right] = \infty, \tag{1.7}$$

where $F_-(x) = \max\{0, -F(x)\}$ and $F_+(x) = \max\{0, F(x)\}$.

Our technique here is based on a nonlinear integral inequality and a transformation of (1.1) which is similar to the one used by Filippov [6]. Also the methods for Liénard systems, especially those developed by Villari and Zanolin [29], Hara and Sugie [11], will be applied in this paper.

The orgnization of this paper is as follows. In section 2 we adapt Filippov's transformation to equation (1.4) and prove some auxiliary Lemmas which will be essential to our proofs. In section 3 we study the problem of the intersection of semiorbits for (1.4) with the characteristic curve $h(x) = F(x)$. In section 4 we establish necessary and sufficient conditions for the zero solution of (1.4) to be globally asymptotically stable, an example illustrating our main result is also given in this section.

2. **Filippov Transformation and Auxiliary Lemmas.** We consider the generalized Liénard system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{a(x)}[h(y) - F(x)],\\ \frac{dy}{dt} &= -a(x)g(x),\end{aligned} \tag{2.1}$$

where $F(x)$, $g(x)$, $a(x)$ and $h(y)$ are continuous real functions defined on $\mathbf{R}$. Throughout this paper, we always assume the conditions (A_0) and (A_1) presented in Theorem 1.1 hold. These assumptions guarantee that the origin is the only critical point of (2.1). We also assume that the corresponding initial value problem has a unique solution.

We call the curve $h(y) = F(x)$ the characteristic curve of system (2.1). We write $\gamma^+(P)$ (resp. $\gamma^-(P)$) the positive (resp. negative) semiorbit of (2.1) starting at a point $P \in \mathbf{R^2}$. For the sake of convenience, we denote

$$C^+ = \{(x, y) : x > 0, h(y) = F(x)\},\ C^- = \{(x, y) : x < 0, h(y) = F(x)\}.$$

The curves C^+, C^- and the y-axis divide the planar domain $\mathbf{R^2}$ into four parts:

$$\begin{array}{ll} D_1 = \{(x, y) : x \geq 0,\ h(y) > F(x)\}, & D_2 = \{(x, y) : x \geq 0,\ h(y) < F(x)\}. \\ D_3 = \{(x, y) : x \leq 0,\ h(y) < F(x)\}, & D_4 = \{(x, y) : x \leq 0,\ h(y) > F(x)\}. \end{array}$$

Let $G(x) = \int_0^x a^2(s)g(s)ds$. Imitating Filippov's transformation [6], we now transform the system (2.1) as follows. For $x > 0$ we set

$$z = z_1(x) = G(x), \tag{2.2}$$

the inverse function of which is denoted by $x = x_1(z)$. We then define

$$F_1(z) = F(x_1(z)),\ z \in (0, G(\infty)). \tag{2.3}$$

Similarly, for $x < 0$ we write

$$z = z_2(x) = G(x), \tag{2.4}$$

denote the inverse function of (2.4) by $x = x_2(z)$, and define

$$F_2(z) = F(x_2(z)),\ z \in (0, G(-\infty)). \tag{2.5}$$

The transformation (2.2) & (2.4) turns the system (2.1) into the equivalent equations

$$\frac{dz}{dy} = F_1(z) - h(y),\ z \in (0, G(\infty)), \tag{2.6}$$

$$\frac{dz}{dy} = F_2(z) - h(y),\ z \in (0, G(-\infty)). \tag{2.7}$$

for $x > 0$ and $x < 0$, respectively. The transformation (2.2) & (2.4) is just Filippov's transformation of the generalized Liénard system

$$\begin{aligned} \frac{dx}{dt} &= h(y) - F(x), \\ \frac{dy}{dt} &= -g^*(x), \end{aligned} \tag{2.8}$$

where $g^*(x) = a^2(x)g(x)$. Thus, we obtain the following proposition.

Proposition 2.1. *Assume that* (A_0) *and* (A_1) *hold. then the qualitative behavior of* (2.1) *is the same as that of* (2.8).

Throughout this paper we shall suppose that the following condition holds:

(A_2) $F_2(z) \leq F_1(z)$ for $z \in (0, \min\{G(-\infty), G(\infty)\})$ and $F_1(z) \not\equiv F_2(z)$ for all sufficiently small $z > 0$.

Lemma 2.2. *(see* [16, Proposition 2.3] *or* [30, Lemma 2.1]*). Suppose that the conditions* (A_0)*,* (A_1) *and* (A_2) *are satisfied. If the positive semiorbit of* (2.1) *starting from* $P_1 = (0, y_1)$ *with* $y_1 > 0$ *intersects the positive* y*-axis once more at* $P_2 = (0, y_2)$*, then* $y_2 < y_1$*. In particular, the system* (2.1) *(or* (2.8)*) has no nontrivial periodic solution.*

Lemma 2.3. *Let $Y(x)$, $\psi(x)$ be positive continuous functions defined on $0 < a \leq x \leq b$ and let $\omega(u)$ be a positive increasing continuous function for $u > 0$, and let*

$$\Omega(u) = \int_{0^+}^{u} \frac{dt}{\omega(t)} \tag{2.9}$$

exist for $u > 0$ with $\Omega(0) = 0$. Then for $\lambda > 0$ the inequality

$$Y(x) \geq \lambda \int_a^x \psi(t)\omega(Y(t))dt \quad \text{for } a \leq x \leq b \tag{2.10}$$

implies the inequality

$$\Omega(Y(x)) \geq \lambda \int_a^x \psi(t)dt \quad \text{for } a \leq x \leq b. \tag{2.11}$$

Proof: Define

$$V(x) = \lambda \int_a^x \psi(t)\omega(Y(t))dt \quad \text{for } a \leq x \leq b. \tag{2.12}$$

Then (2.10) can be restated as $Y(x) \geq V(x)$. Because $\omega(u)$ is increasing, this may be rewritten as

$$\omega(Y(x)) \geq \omega(V(x)),$$

$$\tfrac{V'(x)}{\omega(V(x))} \geq \lambda\psi(x)$$

for $a < x \leq b$. Using the notation $\Omega(u)$ introduced in (2.9), we have

$$\frac{d\Omega(V(x))}{dx} \geq \lambda\psi(x) \quad \text{for } a < x \leq b. \tag{2.13}$$

Now, integrating (2.13) from a to x, we get

$$\Omega(V(x)) - \Omega(V(a)) \geq \lambda \int_a^x \psi(t)dt.$$

Since $V(a) = 0$, it follows that

$$\Omega(V(x)) \geq \lambda \int_a^x \psi(t)dt \quad \text{for } a \leq x \leq b. \tag{2.14}$$

Because $Y(x) \geq V(x)$ for $a \leq x \leq b$, and $\Omega(u)$ is increasing, we obtain by (2.14),

$$\Omega(Y(x)) \geq \lambda \int_a^x \psi(t)dt \quad \text{for } a \leq x \leq b.$$

This completes the proof.

3. **Intersection of Orbits with the Characteristic Curve.** In this section, we are concerned with conditions ensuring the intersection of $\gamma^+(P)$ (resp. $\gamma^-(P)$) with the characteristic curve of (2.1), for any given $P \in \mathbf{R}^2$.

The system (2.1) is said to satisfy the assumption (A_3^+) if one of the following conditions holds:

$(A_3^+)_1$ $\limsup_{x\to\infty} F(x) \neq -\infty$;

$(A_3^+)_2$ $\limsup_{x\to\infty} F(x) = -\infty$, and there exist $\beta > \frac{1}{4}$ and $N_1 > 0$ such that $F(x) < 0$ for $x \geq N_1$, and for any fixed $k \geq 1$ and $b \geq N_1$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{a^2(s)g(s)}{F(s)}ds \leq \frac{1}{k}h^{-1}(k\beta F(x)) \quad \text{for } x \geq \bar{b}.$$

The system (2.1) is said to satisfy (A_3^-) if one of the following conditions holds:

$(A_3^-)_1$ $\liminf_{x\to-\infty} F(x) \neq \infty$;

$(A_3^-)_2$ $\liminf_{x\to-\infty} F(x)=\infty$, and there exist $\beta>\frac{1}{4}$ and $N_1>0$ such that $F(x)>0$ for $x\le -N_1$, and for any fixed $k\ge 1$ and $b\ge N_1$, there exists $\bar b>b$ satisfying

$$\int_{-b}^{x}\frac{a^2(s)g(s)}{F(s)}ds\ge \frac{1}{k}h^{-1}(k\beta F(x))\quad \text{for } x\le -\bar b.$$

We say that system (2.1) satisfies the assumption (A_3) if both (A_3^+) and (A_3^-) hold.

Theorem 3.1. *Suppose that the conditions* (A_0), (A_1) *and* (A_3^+) *hold. Then every positive semiorbit of* (2.1) *departing from* D_1 *intersects the characteristic curve* C^+ *if and only if*

$$\limsup_{x\to\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_-(s)}ds+F(x)\right]=\infty, \tag{3.1}$$

where $F_-(x)=\max\{0,-F(x)\}$.

Proof: We first prove sufficiency. Suppose the conclusion does not hold. Then there is a point $P=(x_0,y_0)\in D_1$ such that $\gamma^+(P)$ does not intersect C^+. Let $(x(t)y(t))(t\ge 0)$ be the solution of (2.1) passing through a point P whose maximal existence interval is $[0,\ \omega_+)$. Note that $x'(t)>0$ and $y'(t)<0$ in the region D_1; hence $x(t)$ is increasing and $y(t)$ is decreasing as t is increasing. Suppose that $x(t)$ is bounded, then $(x(t),y(t))$ stays in the region $\{(x,y): 0<x<K_1,\ h(y)>F(x)\}$ for some $K_1>0$. Hence it must intersect the characteristic curve, which is a contradiction. Therefore $x(t)\to\infty$ as $t\to\omega_+$.

Case 1: Suppose $\limsup_{x\to\infty}F(x)=\infty$, that is, there exists a sequence $\{x_n\}$ such that $x_n\to\infty$ as $n\to\infty$ and $\lim_{n\to\infty}F(x_n)=\infty$, then $(x(t),y(t))$ must intersect the characteristic curve, which is a contradiction.

Case 2: Suppose $\int_0^\infty \frac{a^2(x)g(x)}{1+F_-(x)}dx=\infty$, then

$$\begin{aligned}
y(t)-y_0 &= -\int_0^t a(x(s))g(x(s))ds\\
&= -\int_0^t \frac{a^2(x(s))g(x(s))}{h(y(s))-F(x(s))}\dot x(s)ds\\
&= -\int_{x_0}^{x(t)} \frac{a^2(\xi)g(\xi)}{h(y(s))-F(\xi)}d\xi\\
&\le -\int_{x_0}^{x(t)} \frac{a^2(\xi)g(\xi)}{h(y_0)+F_-(\xi)}d\xi\to-\infty
\end{aligned}$$

as $t\to\omega_+$. Therefore the orbit of the above solution can be considered as a function $y(x)$ which is a solution of the equation

$$\frac{dy}{dx}=-\frac{a^2(x)g(x)}{h(y)-F(x)} \tag{3.2}$$

and $y(x)\to-\infty$ as $x\to\infty$.

Case $(A_3^+)_1$: There exist $c>0$ and a sequence $\{x_n\}$ such that $x_n\to\infty$ $(n\to\infty)$, and $F(x_n)\ge -c$, hence $(x(t),\ y(t))$ must intersect the characteristic curve, which is a contradiction.

Case $(A_3^+)_2$**:** There exists $b > N_1$ such that $F(x) < 0$ and $y(x) < 0$ for $x \geq b$. Since $y(x)$ is a solution of (3.2), putting $H_1(u) = \int_0^u h(y)dy$ for $u \leq 0$, we have

$$\begin{aligned} H_1(y(x)) - H_1(y(b)) &= \int_b^x H_1'(y(s)) \frac{a^2(s)g(s)}{F(s) - h(y(s))} ds \\ &\geq \int_b^x h(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \\ &= \int_b^x (h \circ H_1^{-1})(H_1(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \end{aligned}$$

for $x \geq b$. Hence

$$H_1(y(x)) \geq \int_b^x (-h \circ H_1^{-1})(H_1(y(s)) \frac{a^2(s)g(s)}{-F(s)} ds$$

for $x \geq b$. It follows from Lemma 2.3 that

$$H_2(H_1(y(x)) \geq \int_b^x \frac{a^2(s)g(s)}{-F(s)} ds \quad \text{for } x \geq b, \tag{3.3}$$

where $H_2(u) = \int_{0+}^u \frac{dt}{(-h \circ H_1^{-1})(t)}$. Changing variables $H_1^{-1}(t) = \tau$, then $H_2(u) = -H_1^{-1}(u)$, by (3.3), it is easy to see that

$$y(x) \leq \int_b^x \frac{a^2(s)g(s)}{F(s)} ds \quad \text{for } x \geq b. \tag{3.4}$$

From the assumption $(A_3^+)_2$, there exist $\beta > \frac{1}{4}$ and $b_1 > b$ such that

$$\int_b^x \frac{a^2(s)g(s)}{F(s)} ds \leq h^{-1}(\beta F(x)) \quad \text{for } x \geq b_1. \tag{3.5}$$

By virtue of (3.4) and (3.5), we have $y(x) \leq h^{-1}(\beta F(x))$ for $x \geq b_1$. Because $h(y)$ is strictly increasing, we obtain $h(y(x)) \leq \beta F(x)$ for $x \geq b_1$. Hence $F(x) - h(y(x)) \geq \beta_1 F(x)$ for $x \geq b_1$, where $\beta_1 = 1 - \beta$. By a similar argument, we have

$$\begin{aligned} H_1(y(x)) - H_1(y(b_1)) &= \int_{b_1}^x h(y(s)) \frac{a^2(s)g(s)}{F(s) - h(y(s))} ds \\ &\geq \frac{1}{\beta_1} \int_{b_1}^x h(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \\ &= \frac{1}{\beta_1} \int_{b_1}^x (h \circ H_1^{-1})(H_1(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \end{aligned}$$

for $x \geq b_1$. Hence

$$H_1(y(x)) \geq \frac{1}{\beta_1} \int_{b_1}^x (-h \circ H_1^{-1})(H_1(y(s)) \frac{a^2(s)g(s)}{-F(s)} ds$$

for $x \geq b_1$. By Lemma 2.3 one has

$$\begin{aligned} H_2(H_1(y(x))) &\geq \frac{1}{\beta_1} \int_{b_1}^x \frac{a^2(s)g(s)}{-F(s)} ds, \\ y(x) &\leq \frac{1}{\beta_1} \int_{b_1}^x \frac{a^2(s)g(s)}{F(s)} ds \end{aligned} \tag{3.6}$$

for $x \geq b_1$. From the assumption $(A_3^+)_2$, there exists $b_2 > b_1$ such that

$$\int_{b_1}^x \frac{a^2(s)g(s)}{F(s)} ds \leq \beta_1 h^{-1} \left(\frac{\beta}{\beta_1} F(x) \right) \quad \text{for } x \geq b_2. \tag{3.7}$$

By virtue of (3.6) and (3.7), we have $y(x) \leq h^{-1}(\frac{\beta}{\beta_1}F(x))$ for $x \geq b_2$. Thus $F(x) - h(y(x)) \geq \beta_2 F(x)$ for $x \geq b_2$, where $\beta_2 = 1 - \frac{\beta}{\beta_1}$. Repeating this procedure, we obtain two sequences $\{b_n\}$ and $\{\beta_n\}$ such that $\beta_n = 1 - \frac{\beta}{\beta_{n-1}}$ and $F(x) - h(y(x)) \geq \beta_n F(x)$ for $x \geq b_n$. If $\beta_n > 0$ $(n = 1, 2, \ldots)$, then $\{\beta_n\}$ is decreasing, and $\{\beta_n\}$ converges to some real number λ. On the other hand, $\lambda = 1 - \frac{\beta}{\lambda}$ and $\beta > \frac{1}{4}$ which shows that λ is a complex number, which is a contradiction. Hence, $\beta_n \leq 0$ for some n, that is $F(x) \geq h(y(x))$ for all $x \geq b_n$, a contradiction. This completes the proof of sufficiency.

Necessity. Suppose (3.1) does not hold. Then there exist $M_1 > 0$ and $L > 0$ such that $F(x) < M_1$ for $x \geq 0$ and $\int_L^\infty \frac{a^2(x)g(x)}{1+F_-(x)}dx < 1$. Suppose $(x(t), y(t))$ is a solution of (2.1), and $(x(0), y(0)) = (L,\ M_1 + M_0 + 1) = P$, where $M_0 > 0$ satisfies $h(M_1 + M_0) \geq M_1 + 1$.

We will show that $y(t) > M_1 + M_0$ for $t > 0$. Suppose this is not the case. There exists $t_1 > 0$ such that $y(t_1) = M_1 + M_0$ and $M_1 + M_0 < y(t) \leq M_1 + M_0 + 1$ for all $t \in [0,\ t_1)$, and we have

$$\begin{aligned} y(t_1) &= M_1 + M_0 + 1 - \int_0^{t_1} \frac{a^2(x(s))g(x(s))}{h(y(s)) - F(x(s))}\dot{x}(s)ds \\ &\geq M_1 + M_0 + 1 - \int_L^{x(t_1)} \frac{a^2(\xi)g(\xi)}{1 + F_-(\xi)}d\xi > M_1 + M_0. \end{aligned}$$

This is a contradiction. Hence, $\dot{x}(t) = h(y(t)) - F(x(t)) > M_1 + 1 - F(x(t)) > 1$ for all $t \geq 0$. Thus the solution $(x(t),\ y(t))$ is unbounded and $\gamma^+(P)$ is above the characteristic curve $h(y) = F(x)$. This completes the proof.

In quite the same manner, we can prove the following result.

Theorem 3.2. *Suppose that the conditions* (A_0), (A_1) *and* (A_3^-) *hold. Then every positive semiorbit of* (2.1) *departing from* D_3 *intersects the characteristic curve* C^- *if and only if*

$$\limsup_{x\to-\infty} \left[\int_0^x \frac{a^2(s)g(s)}{1 + F_+(s)}ds - F(x)\right] = \infty, \tag{3.8}$$

where $F_+(x) = \max\{0, F(x)\}$.

The system (2.1) is said to satisfy the condition $(A_3^+)'$ (resp. $(A_3^-)'$) if $-F(x)$, $-h(-y)$, $a(x)$, $g(x)$ satisfy the condition (A_3^+) (resp. (A_3^-)).

By the transformations $t \to -t$ and $y \to -y$ in (2.1), we have the following results with respect to the negative semiorbits of (2.1).

Theorem 3.3. *Suppose that the conditions* (A_0), (A_1) *and* $(A_3^+)'$ *hold. Then every negative semiorbit of* (2.1) *departing from* D_2 *intersects the characteristic curve* C^+ *if and only if*

$$\limsup_{x\to\infty} \left[\int_0^x \frac{a^2(s)g(s)}{1 + F_+(s)}ds - F(x)\right] = \infty, \tag{3.9}$$

Theorem 3.4. *Suppose that the conditions* (A_0), (A_1) *and* $(A_3^-)'$ *hold. Then every negative semiorbit of* (2.1) *departing from* D_4 *intersects the characteristic curve* C^- *if and only if*

$$\limsup_{x\to-\infty} \left[\int_0^x \frac{a^2(s)g(s)}{1 + F_-(s)}ds + F(x)\right] = \infty. \tag{3.10}$$

Remark 3.5. If $\liminf_{x\to\infty} F(x) > -\infty$, then (3.1) is equivalent to

$$\limsup_{x\to\infty}\left[\int_0^x a^2(s)g(s)ds + F(x)\right] = \infty, \tag{3.11}$$

and if $\limsup_{x\to-\infty} F(x) < -\infty$, then (3.8) is equivalent to

$$\limsup_{x\to-\infty}\left[\int_0^x a^2(s)g(s)ds - F(x)\right] = \infty. \tag{3.12}$$

Remark 3.6. From the necessity proof of Theorem 3.1 we know that if

$$\limsup_{x\to\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_+(s)}ds - F(x)\right] < \infty, \tag{3.13}$$

then there exists a point $P \in D_2$ such that $\gamma^-(P)$ does not intersect C^+. Similarly, if

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{a^2(s)g(s)}{1+F_-(s)}ds + F(x)\right] < \infty, \tag{3.14}$$

then there exists a point $P \in D_4$ such that $\gamma^-(P)$ does not intersect C^-.

Remark 3.7. If $h(y) \equiv y$, $a(x) \equiv 1$, then Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 give the corresponding results of Hara, Yoneyama and Sugie [13], and Sugie [24].

Remark 3.8. The condition $(A_3^+)_2$ is a generalization of the following condition:

$(A_3^+)_{2^*}$ $\limsup_{x\to\infty} F(x) = -\infty$, and there exist $N_1 > 0$, $\beta_0 > \frac{1}{4}$ and $\bar\beta_0 > 0$ such that $h(y)$ is continuously differentiable on $(-\infty,\ -N_1]$, $h'(-y) \geq \frac{\beta_0}{\bar\beta_0}$ for $y \geq N_1$, $F(x) < 0$ for $x \geq N_1$ and for any $b \geq N_1$, there exists $\bar b > b$ satisfying

$$\int_b^x \frac{a^2(s)g(s)}{F(s)}ds \leq \bar\beta_0 F(x) \quad \text{for } x \geq \bar b.$$

In fact, if the condition $(A_3^+)_{2^*}$ is satisfied, then there exist $N_1 > 0$, $\beta_0 > \frac{1}{4}$ and $\bar\beta_0 > 0$ such that $h'(-y) \geq \frac{\beta_0}{\bar\beta_0}$ for $y \geq N_1$, and for any fixed real number $k \geq 1$, there exists $N_2 > N_1$ satisfying $k\bar\beta_0 F(x) \leq -N_1$ for $x \geq N_2$, and

$$\begin{aligned} h(k\bar\beta_0 F(x)) &< h(k\bar\beta_0 F(x)) - h(k\bar\beta_0 F(N_2)) \\ &= k\bar\beta_0 h'(\xi)\frac{F(x)-F(N_2)}{F(x)}F(x) \end{aligned}$$

for $x > N_2$, where ξ is between $k\bar\beta_0 F(x)$ and $k\bar\beta_0 F(N_2)$. Since $\lim_{x\to\infty} F(x) = -\infty$, for any $b \geq N_1$, it can be shown that there exist $\frac{1}{4} < \beta < \beta_0$ and $b^* > b$ such that $h(k\bar\beta_0 F(x)) < k\beta F(x)$ for $x \geq b^*$. Because $h(y)$ is strictly increasing, we have $\bar\beta_0 F(x) \leq \frac{1}{k}h^{-1}(k\beta F(x))$ for $x \geq b^*$. Hence the condition $(A_3^+)_{2^*}$ implies $(A_3^+)_2$.

By the same argument, it can be seen that condition $(A_3^-)_2$ is a generalization of the following condition:

$(A_3^-)_{2^*}$ $\lim_{x\to-\infty} F(x) = \infty$, and there exist $N_1 > 0$, $\beta_0 > \frac{1}{4}$ and $\bar{\beta}_0 > 0$ such that $h(y)$ is continuously differentiable on $[N_1,\ \infty)$, $h'(y) \geq \frac{\beta_0}{\bar{\beta}_0}$ for $y \geq N_1$, $F(x) > 0$ for $x \leq -N_1$ and for any $b \geq N_1$, there exists $\bar{b} > b$ satisfying

$$\int_{-b}^{x} \frac{a^2(s)g(s)}{F(s)} ds \geq \bar{\beta}_0 F(x) \quad \text{for } x \leq -\bar{b}.$$

It follows immediately from Remark 3.8 that the condition (A_3) is a generalization of condition (A_4) in [16]. Thus, Theorem 3.1 and Theorem 3.2 contain the Proposition 3.2 in [16]. Moreover, the condition (A_3^+) with $a(x) \equiv 1$ is a generalization of condition (A_4) in [30].

4. **Global Asymptotic Stability.** In order to give a criterion for the zero solution of (2.1) to be globally asymptotically stable, we must provide a criterion excluding homoclinic orbits.

The system (2.1) is said to satisfy (A_4^+) if one of the following three conditions holds:

$(A_4^+)_1$ There exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and $F(x_n) \geq 0$ for $n \geq 1$;

$(A_4^+)_2$ There exist constants $m > 0$, $p > 0$ and $\delta_1 > 0$ such that

$$|h(y)| \geq m|y|^p \quad \text{for} \quad 0 < -y < -\delta_1$$

and

$$F_1(z) \geq -az^{\frac{p}{p+1}} \quad \text{for} \quad 0 < z < \delta_1,$$

where $0 < a < m(1+p)(\frac{1+p}{mp})^{\frac{p}{(1+p)}}$;

$(A_4^+)_3$ There exist constants $\alpha > \frac{1}{4}$ and $\delta_2 > 0$ such that

$$F(x) < 0 \quad \text{for} \quad 0 < x \leq \delta_2,$$

and for any fixed real number $k \geq 1$,

$$\int_{0+}^{x} \frac{a^2(s)g(s)}{F(s)} ds \leq \frac{1}{k} h^{-1}(k\alpha F(x)) \quad \text{for} \quad 0 < x \ll 1.$$

The system (2.1) is said to satisfy (A_4^-) if one of the following three conditions holds:

$(A_4^-)_1$ There exists a negative increasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and $F(x_n) \leq 0$ for $n \geq 1$;

$(A_4^-)_2$ There exist constants $m > 0$, $p > 0$ and $\delta_3 > 0$ such that

$$|h(y)| \geq m|y|^p \quad \text{for} \quad 0 < y < \delta_3,$$

and

$$F_2(z) \leq az^{\frac{p}{p+1}} \quad \text{for } 0 < z < \delta_3,$$

where $0 < a < m(1+p)(\frac{1+p}{mp})^{\frac{p}{(1+p)}}$;

$(A_4^-)_3$ There exist constants $\alpha > \frac{1}{4}$ and $\delta_4 > 0$ such that

$$F(x) > 0 \quad \text{for} \quad 0 < -x \leq \delta_4,$$

and for any fixed real number $k \geq 1$,

$$\int_{0-}^{x} \frac{a^2(s)g(s)}{F(s)} ds \geq \frac{1}{k} h^{-1}(k\alpha F(x)) \quad \text{for } 0 < -x \ll 1.$$

The system (2.1) is said to satisfy the condition (A_4) if both (A_4^+) and (A_4^-) hold.

Lemma 4.1. *Assume that the conditions (A_0) and (A_1) are true. Then*

(i) *if (A_4^+) holds, then for any $P=(x_0,y_0)\in C^+$, the negative semiorbit $\gamma^-(P)$ must intersect the positive y-axis at $(0,y_1)$ with $y_1>0$;*

(ii) *if (A_4^-) holds, then for any $P=(x_0,y_0)\in C^-$, the negative semiorbit $\gamma^-(P)$ must intersect the negative y-axis at $(0,y_2)$ with $y_2<0$;*

Proof: We only prove (i); (ii) can be proved in a similar way.

Let $P=(x_0,\ y_0)\in C^+$ and $(x(t),\ y(t))$ be the solution of (2.1) with $x(0)=x_0,\ y(0)=y_0$. By the uniqueness of the solutions of (2.1), we only have to show that every orbit $\gamma^-(P)$ of (2.1) passing through a point $P=(x_0,y_0)$ with $x_0>0$ sufficiently small, intersects the positive y-axis at $(0,\ y_1)$ with $y_1>0$. Since $\lim_{y\to\infty}h(y)=\infty$, the system (2.1) has no vertical asymptote in the first quadrant. Therefore, $\gamma^-(P)$ must intersect the y-axis at $B(0,\ y_1)$ with $y_1\geq 0$. We only have to show that $y_1\neq 0$. We do this separately for the different cases of (A_4^+).

Case $(A_4^+)_1$**:** It is obvious in this case.

Case $(A_4^+)_2$**:** In this case the proof is completely analogous to the proof of [16, Lemma 3.1] or [30, Theorem 2.4].

Case $(A_4^+)_3$**:** It follows from (A_0) that the orbit $\gamma^-(P)$ of (2.1) does not touch the characteristic curve at any point $(x,\ h^{-1}(F(x)))$ with $0\leq x<x_0$. Thus, we consider only the region $\{(x,y):x>0,\ h(y)>F(x)\}$, and $F(x)<0$ for $0<x<\delta_2$. Suppose that the conclusion does not hold. Then there exists a point $P\in C^+$ such that $\gamma^-(P)$ does not intersect the positive y-axis. Let $(x(t),\ y(t))$ $(0\leq t<\infty)$ denote the solution of (2.1) which passes through such a point P. Then $\gamma^-(P)$ must be contained in the fourth quadrant, and $x(t)$ decreases and $y(t)$ increases as t is increasing. Since the origin is the unique equilibrium of (2.1), $\lim_{t\to-\infty}x(t)=\lim_{t\to-\infty}y(t)=0$. The solution $(x(t),\ y(t))$ defines a function $y=y(x)$ on $0\leq x\leq\delta_2$, which is a solution on $0<x<\delta_2$ of equation (3.2).

It follows from $\lim_{x\to 0^+}y(x)=0$ that $y(x)<0$ for $0<x\leq\delta_2$. By assumption $(A_4^+)_3$, there exist $\alpha>\frac{1}{4}$ and $x_1\in(0,\delta_2)$ such that $F(x)<0$ for $0<x\leq x_1$, and

$$\int_{0^+}^{x}\frac{a^2(s)g(s)}{F(s)}ds\leq h^{-1}(\alpha F(x))\quad\text{for } 0<x\leq x_1. \tag{4.1}$$

Now, we restrict our attention to the interval $(0,\ x_1]$. Putting $H_1(u)=\int_0^u h(y)dy$ for $u\leq 0$, we have by (3.2), for any sufficiently small $\varepsilon>0$,

$$\begin{aligned} H_1(y(x))-H_1(y(\varepsilon)) &= \int_\varepsilon^x H_1'(y(s))\frac{a^2(s)g(s)}{F(s)-h(y(s))}ds\\ &\geq \int_\varepsilon^x h(y(s))\frac{a^2(s)g(s)}{F(s)}ds\\ &= \int_\varepsilon^x (h\circ H_1^{-1})(H_1(y(s))\frac{a^2(s)g(s)}{F(s)}ds \end{aligned}$$

for $\varepsilon\leq x\leq x_1$. Hence

$$H_1(y(x))\geq\int_\varepsilon^x(-h\circ H_1^{-1})(H_1(y(s))\frac{a^2(s)g(s)}{-F(s)}ds$$

for $\varepsilon\leq x\leq x_1$. It follows from Lemma 2.3 that

$$H_2(H_1(y(x))\geq\int_\varepsilon^x\frac{a^2(s)g(s)}{-F(s)}ds\quad\text{for } \varepsilon\leq x\leq x_1, \tag{4.2}$$

where $H_2(u) = \int_{0^+}^{u} \frac{dt}{(-h\circ H_1^{-1})(t)}$. Changing variables $H_1^{-1}(t) = \tau$, it is easy to see that $H_2(u) = -H_1^{-1}(u)$. By (4.2), we have

$$y(x) \leq \int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{F(s)} ds \quad \text{for } \varepsilon \leq x \leq x_1. \tag{4.3}$$

(i) If $\int_{0^+}^{x_1} \frac{a^2(s)g(s)}{F(s)} ds = -\infty$, we reach a contradiction by (4.3).

(ii) If $\int_{0^+}^{x_1} \frac{a^2(s)g(s)}{F(s)} ds > -\infty$, we see from (4.3) that

$$y(x) \leq \int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)} ds \quad \text{for } 0 < x \leq x_1. \tag{4.4}$$

By virtue of (4.1) and (4.4), we have $y(x) \leq h^{-1}(\alpha F(x))$ for $0 < x \leq x_1$. Because $h(y)$ is strictly increasing, we obtain $h(y(x)) \leq \alpha F(x)$ for $0 < x \leq x_1$. Since $y = y(x)$ is above the characteristic curve $h(y) = F(x)$, we have $\frac{1}{4} < \alpha < 1$. Let $\alpha_1 = 1 - \alpha$, then we get that $F(x) - h(y(x)) \geq \alpha_1 F(x)$ for $0 < x \leq x_1$. In a similar way, for any sufficiently small $\varepsilon > 0$, we have by (3.2)

$$\begin{aligned} H_1(y(x)) - H_1(y(\varepsilon)) &= \int_{\varepsilon}^{x} h(y(s)) \frac{a^2(s)g(s)}{F(s) - h(y(s))} ds \\ &\geq \frac{1}{\alpha_1} \int_{\varepsilon}^{x} h(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \end{aligned}$$

for $\varepsilon \leq x \leq x_1$. Therefore

$$\begin{aligned} H_1(y(x)) &\geq \frac{1}{\alpha_1} \int_{\varepsilon}^{x} h(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \\ &= \frac{1}{\alpha_1} \int_{\varepsilon}^{x} (-h \circ H_1^{-1})(H_1(y(s)) \frac{a^2(s)g(s)}{-F(s)} ds \end{aligned}$$

for $\varepsilon \leq x \leq x_1$. By Lemma 2.3 we have

$$\begin{aligned} H_2(H_1(y(x)) &\geq \frac{1}{\alpha_1} \int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{-F(s)} ds, \\ y(x) &\leq \frac{1}{\alpha_1} \int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{F(s)} ds \end{aligned}$$

for $\varepsilon \leq x \leq x_1$. Hence

$$y(x) \leq \frac{1}{\alpha_1} \int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)} ds \tag{4.5}$$

for $0 < x \leq x_1$. By assumption $(A_4^+)_3$, there exists $x_2 \in (0,\ x_1)$ such that

$$\int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)} ds \leq \alpha_1 h^{-1}(\frac{\alpha}{\alpha_1} F(x)) \tag{4.6}$$

for $0 < x \leq x_2$. By virtue of (4.5) and (4.6), we have $y(x) \leq h^{-1}(\frac{\alpha}{\alpha_1} F(x))$ for $0 < x \leq x_2$. Because $h(y)$ is strictly increasing, we get $h(y(x)) \leq \frac{\alpha}{\alpha_1} F(x)$ for $0 < x \leq x_2$. Thus, $F(x) - h(y(x)) \geq \alpha_2 F(x)$ with $\alpha_2 = 1 - \frac{\alpha}{\alpha_1}$. Repeating this procedure, we obtain two sequences $\{x_n\}$ and $\{\alpha_n\}$ such that $\alpha_n = 1 - \frac{\alpha}{\alpha_{n-1}}$ and $F(x) - h(y(x)) \geq \alpha_n F(x)$ for $0 < x \leq x_n$. If $\alpha_n \leq 0$, we have a contradiction. Suppose $\alpha_n > 0$ $(n = 1, 2, \ldots)$, then $\{\alpha_n\}$ is decreasing, and hence $\{\alpha_n\}$ converges to some real number λ. On the other hand, $\lambda = 1 - \frac{\alpha}{\lambda}$ and $\alpha > \frac{1}{4}$ show that λ is a complex number, which is a contradiction. This completes the proof.

We now state our main result.

Theorem 4.2. *Assume that the system* (2.1) *satisfies the conditions* $(A_0) - (A_4)$. *Then the zero solution of* (2.1) *is globally asymptotically stable if and only if* (3.1) *and* (3.8) *hold.*

Proof: Necessity. If either (3.1) or (3.8) is false, then Theorem 3.1 and Theorem 3.2 imply that (2.1) has at least one unbounded solution lying in D_1 or D_3. Thus, the origin is not globally asymptotically stable.

Sufficiency. The proof is similar to that of Theorem 3.3 in [16] and Theorem 3.1 in [27], so we omit it.

Remark 4.3. If $h(y) \equiv y$, $a(x) \equiv 1$, then Lemma 4.1 gives the corresponding results of Hara, Yoneyama and Sugie [13] and Sugie [24].

Remark 4.4. The condition $(A_4^+)_3$ is a generalization of the following condition:

$(A_4^+)_3*$ there exist constants $\alpha_0 > 0$ and $\delta > 0$ such that $h(y)$ is continuously differentiable on $[-\delta, 0]$,

$$F(x) < 0 \quad \text{for} \quad 0 < x \le \delta$$

and

$$\int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)} ds \le \alpha_0 F(x) \quad \text{for } 0 < x \le \delta,$$

where $\alpha = h'(0)\alpha_0 > \frac{1}{4}$.

In fact, if the condition $(A_4^+)_3*$ is satisfied, then there exist constants $0 < \bar{\delta} < \delta$ and $\frac{1}{4} < \bar{\alpha} < \alpha$ such that $h'(y) > \frac{\bar{\alpha}}{\alpha_0}$ for $-\bar{\delta} \le y \le 0$, and for any fixed real number $k \ge 1$, we have

$$\begin{aligned} \frac{1}{k} h^{-1}(k\bar{\alpha}F(x)) &= \frac{1}{k} h^{-1}(k\bar{\alpha}F(x)) - \frac{1}{k} h^{-1}(0) \\ &= \frac{1}{k} \frac{dh^{-1}(u)}{du} \,|_{u=\xi}\, k\bar{\alpha}F(x), \quad k\bar{\alpha}F(x) < \xi < 0 \\ &= \frac{\bar{\alpha}F(x)}{h'(h^{-1}(\xi))} > \alpha_0 F(x) \quad \text{for } 0 < x \ll 1. \end{aligned}$$

Thus the condition $(A_4^+)_3*$ implies the condition $(A_4^+)_3$.

By the same argument, it can be seen that the condition $(A_4^-)_3$ is a generalization of the following condition:

$(A_4^-)_3*$ there exist constants $\alpha_0 > 0$ and $\delta > 0$ such that $h(y)$ is continuously differentiable on $[0, \delta]$,

$$F(x) > 0 \quad \text{for} - \delta \le x < 0$$

and

$$\int_{0^-}^{x} \frac{a^2(s)g(s)}{F(s)} ds \ge \alpha_0 F(x) \quad \text{for} \quad -\delta \le x < 0,$$

where $\alpha = h'(0)\alpha_0 > \frac{1}{4}$.

Therefore, the condition (A_4) is a generalization of condition (A_3) in [16] and condition (A_{10}) in [30]. Hence, Lemma 4.1 contains Lemma 3.1 in [16].

Remark 4.5. If $a(x) \equiv 1$, then by condition $(A_4)_2$, Lemma 4.1 is seen to be a generalization of Theorem 4.6 and Theorem 4.12 of Sugie [27]. But our results hold also for $0 < p < 1$.

Remark 4.6. Theorem 4.2 is a generalization of [16, Theorem 3.3]. This follows from Remark 3.8 and Remark 4.4. Our results do not need the differentiability condition of $h(y)$. Moreover, Theorem 4.2 can be applied to system (1.3) even for $0 < p < 1$.

Example 4.7. In system (2.1), we take $a(x) \equiv 1$, $h(y) = |y|^{\frac{1}{2}}\mathrm{sgn}y$, $g(x) = 4x^3$, and

$$F(x) = \begin{cases} -x & \text{for } x \geq 1, \\ -x^3 & \text{for } 0 \leq x < 1, \\ 2x^3 & \text{for } -1 < x < 0, \\ 2x & \text{for } x \leq -1. \end{cases}$$

Then $\int_0^{-\infty} g(x)dx = \int_0^{\infty} g(x)dx = \infty$, and for $0 < z < \infty$,

$$F_2(z) = F(-z^{\frac{1}{4}}) < F(z^{\frac{1}{4}}) = F_1(z).$$

Hence (A_0), (A_1) and (A_2) are satisfied. Because $\int_{0+}^{x} \frac{g(s)}{F(s)}ds = -4x$ for $0 < x < 1$, and because for any fixed real number $k \geq 1$,, one has

$$\int_{0+}^{x} \frac{g(s)}{F(s)}ds = -4x \leq -kx^6 = \frac{1}{k}h^{-1}(kF(x))$$

for $0 < x \ll 1$, the condition $(A_4^+)_3$ is satisfied. For any $b > 1$ and fixed real number $k \geq 1$, we have

$$\int_{b}^{x} \frac{g(s)}{F(s)}ds = -\frac{4}{3}x^3 + \frac{4}{3}b^3,$$

$$\frac{1}{k}h^{-1}(kF(x)) = -kx^2,$$

it is clear that $(A_3^+)_2$ is satisfied. It is obvious that $(A_3^-)_1$, $(A_4^-)_1$, (3.1) and (3.8) are also satisfied. Thus, by Theorem 4.2, the zero solution is globally asymptotically stable. However, $h(y) = |y|^{\frac{1}{2}}\mathrm{sgn}y$, $p = \frac{1}{2}$, and $\lim_{y\to\pm\infty} h'(y) = 0$, hence, the condition (A_4^+) in [16] is not satisfied. Therefore, the result of [16] cannot be applied to this example.

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Paper II

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On Global Asymptotic Stability of Second Order Nonlinear Differential Systems

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This paper investigates the global asymptotic stability of the autonomous planar systems $\dot{x} = p_2(y)q_2(x)y$, $\dot{y} = p_3(y)q_3(x)x + p_3(y)q_4(x)y$ and $\dot{x} = f_1(x) + h_2(x)y$, $\dot{y} = f_3(x) + h_4(x)y$, under the assumption that all functions involved in the equations are continuous and that the origin is a unique equilibrium. We present necessary and sufficient conditions for the origin to be globally asymptotically stable.

Keywords: Second order differential system; Global asymptotic stability; Filippov transformation

AMS: 34D05; 34C05

1 INTRODUCTION

In a series papers [1–4], Krechetov studied the following real system of two differential equations

$$\begin{aligned} \dot{x} &= f_1(x) + h_2(x)y, \\ \dot{y} &= f_3(x) + h_4(x)y, \end{aligned} \tag{1}$$

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where $f_1(x)$, $f_3(x)$, $h_2(x)$ and $h_4(x)$ are continuous on $\mathbf{R}$. Using Liapunov functions, he investigated the question of stability in the large, described the configurations of the domains of stability (when there is no global stability) and constructed estimates of the boundaries of these domains. Egorov and Kartuzova [6] studied the same problem and formulated necessary and sufficient conditions for the zero solution of (1) to be globally asymptotically stable under rather restrictive assumptions on the functions $h_i(x)$. We quote this result here.

THEOREM 1.1 (*Egorov and Kartuzova*) *Suppose that* $f_1(x)$, $f_3(x)$, $h_2(x)$ *and* $h_4(x)$ *are continuous on* $\mathbf{R}$ *with* $f_1(0) = f_3(0) = 0$ *and that they satisfy the following conditions*:

(1) $h_1(x) + h_4(x) < 0$ *for* $x \neq 0$;
(2) $h_1(x)h_4(x) - h_2(x)h_3(x) := \delta(x) > 0$ *for* $x \neq 0$, *where* $h_i(x) = f_i(x)/x$ *for* $x \neq 0$ *and* $i = 1, 3$;
(3) $h_2(x) \neq 0$ *for all* x;
(4) $h_1(x) + h_2(x)H_{42}(x)/x < 0$ *for* $x \neq 0$.

Then the zero solution of (1) *is globally asymptotically stable if and only if*

$$\int_0^{\pm\infty} \delta(x)[h_2(x)]^{-2} x\, dx + \limsup_{x \to \pm\infty} |\Psi(x)| = +\infty \tag{2}$$

Here $H_{42}(x) := \int_0^x h_4(s)[h_2(s)]^{-1}\, ds$, $\Psi(x) := [h_1(x) + h_2(x)H_{42}(x)/x]x/h_2(x)$.

In the paper [5], Krechetov considered the following autonomous system of two differential equations with zero diagonal coefficient

$$\begin{aligned} \dot{x} &= p_2(y)q_2(x)y \\ \dot{y} &= p_3(y)q_3(x)x + p_4(y)q_4(x)y \end{aligned} \tag{3}$$

and studied the same problems as in the previous papers [1–4]. In the study of stability for (1), the most crucial condition added by Egorov and Kartuzova [6] is (4) in Theorem 1.1, while for (3), the most important condition given by Krechetov [5] is

$$q_2(x)q_4(x) > 0 \quad \text{for all} \quad x. \tag{4}$$

The purpose of the present article is to investigate the global asymptotic stability of the systems (1) and (3) without the assumption (4) in Theorem 1.1 and inequality (4) above. The transformation technique plays an important role in this paper. Under suitable assumptions, we shall prove that the systems (1) and (3) are equivalent to the equations of the following type

$$\begin{aligned} \dot{x} &= \phi(z - F(x)), \\ \dot{z} &= -g(x), \end{aligned} \tag{5}$$

which is a generalization of the Liénard system. Study the system (5) has an independent interest.

The organization of this article is as follows. In Section 2, we give suitable transformations which change the systems (1) and (3) into the form of (5). In Section 3, we study the problem of the intersection of positive semiorbits for (5) with the characteristic curve $z = F(x)$. In Section 4, we give necessary and sufficient conditions for the origin of (1) and (3) to be globally asymptotically stable. Some examples illustrating the results are given in this paper.

2 TRANSFORMATIONS FOR (1) AND (3)

First, we transform the system (1). Suppose that

$$h_2(x) \neq 0 \quad \text{for all} \quad x. \tag{6}$$

Without loss of generality, we may assume that $h_2(x) < 0$ for all x. If $h_2(x) > 0$, then we replace y by $-y$. Using the substitution

$$y = H_{42}(x) - z \tag{7}$$

where $H_{42}(x)$ is given in the Theorem 1.1 of Egorov and Kartuzova, we obtain that

$$\begin{aligned} \dot{x} &= g_1(x) - h_2(x)z \\ \dot{z} &= -g_3(x). \end{aligned} \tag{8}$$

Here $g_1(x) := f_1(x) + h_2(x)H_{42}(x)$, $g_3(x) := f_3(x) - h_4(x)f_1(x)[h_2(x)]^{-1} = -x\delta(x)[h_2(x)]^{-1}$. Obviously, the qualitative behavior of (8) is the same as that of the system

$$\begin{aligned} \dot{x} &= z - \frac{g_1(x)}{h_2(x)}, \\ \dot{z} &= \frac{g_3(x)}{h_2(x)}, \end{aligned} \tag{9}$$

which is a Liénard system. Therefore, we obtain the following proposition.

Proposition 2.1 *If* $h_2(x) \neq 0$, *then the qualitative behavior of* (1) *is the same as that of* (9).

Next, we restrict our attention to the system (3). Here we only consider the case $p_3(y) \equiv p_4(y)$. Under this restriction, the system (3) becomes

$$\begin{aligned} \dot{x} &= p_2(y)q_2(x)y, \\ \dot{y} &= p_3(y)q_3(x)x + p_3(y)q_4(x)y. \end{aligned} \tag{10}$$

The basic assumption given in [5] is

$$\begin{aligned} &p_i(y) > 0 \quad \text{for all } y \quad \text{and} \quad i = 2,\ 3, \\ &q_2(x) > 0,\ q_3(x) < 0,\ q_4(x) > 0 \quad \text{for all } x. \end{aligned} \tag{11}$$

In the following, we only assume that

$$\begin{aligned} &p_2(y) > 0,\ p_3(y) > 0 \quad \text{for all } y \\ &q_2(x) < 0,\ q_3(x) > 0 \quad \text{for all } x. \end{aligned} \tag{12}$$

In this situation, $p_2(y)q_2(x)y$ and $-y$ have the same sign. Thus, the qualitative behavior of (10) is identical to that of the system

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= -\frac{p_3(y)q_3(x)}{p_2(y)q_2(x)}x - \frac{p_3(y)q_4(x)}{p_2(y)q_2(x)}y. \end{aligned} \tag{13}$$

From (13), we obtain the second order equation

$$\ddot{x} = \frac{p_3(-\dot{x})q_3(x)}{p_2(-\dot{x})q_2(x)}x - \frac{p_3(-\dot{x})q_4(x)}{p_2(-\dot{x})q_2(x)}\dot{x}. \tag{14}$$

It follows from (14) that

$$\frac{d}{dt}\left[-\int_0^{-\dot{x}} \frac{p_2(s)}{p_3(s)}\,ds + \int_0^x \frac{q_4(s)}{q_2(s)}\,ds\right] = \frac{q_3(x)}{q_2(x)}x.$$

Letting $\psi(y) = \int_0^y p_2(s)/p_3(s)\,ds$ and introducing the substitution

$$z = -\psi(-\dot{x}) + \int_0^x \frac{q_4(s)}{q_2(s)}\,ds.$$

We change the system (13) into

$$\begin{aligned}\dot{x} &= -\psi^{-1}\left(\int_0^x \frac{q_4(s)}{q_2(s)}\,ds - z\right),\\ \dot{z} &= \frac{q_3(x)}{q_2(x)}x.\end{aligned}$$

If we let ϕ denote ψ^{-1} and replace x and z by $-x$ and $-z$ respectively, then we obtain

$$\begin{aligned}\dot{x} &= \phi(z - F(x)),\\ \dot{z} &= -g(x),\end{aligned} \tag{15}$$

where $F(x) = -\int_0^{-x} (q_4(s)/q_2(s))\,ds$ and $g(x) = -(q_3(-x)/q_2(-x))x$.

PROPOSITION 2.2. *Under the assumption* (12), *the qualitative behavior of* (10) *is just the same as that of* (15).

Remark 2.1. For the case $p_3(y) \not\equiv p_4(y)$, we assume that $\rho'(y) > 0$ for all y, where $\rho(y) := yp_4(y)/p_3(y)$. We transform the system (3), using the substitution $z = \rho(y)$, we have

$$\begin{aligned}\dot{y} &= p_3(y)q_3(x)x + p_3(y)q_4(x)z,\\ \frac{d}{dz}\left[\rho^{-1}(z)\right]\dot{z} &= p_3(\rho^{-1}(z))q_3(x)x + p_3(\rho^{-1}(z))q_4(x)z,\end{aligned}$$

we change the system (3) into

$$\dot{x} = p_2(\rho^{-1}(z))\frac{\rho^{-1}(z)}{z}q_2(x)z,$$
$$\dot{z} = \rho'(\rho^{-1}(z))p_3(\rho^{-1}(z))q_3(x)x + \rho'(\rho^{-1}(z))p_3(\rho^{-1}(z))q_4(x)z,$$

the following discussion is similar to that of the case $p_3(y) \equiv p_4(y)$, we leave this to another paper.

3 INTERSECTION WITH THE CHARACTERISTIC CURVE OF (15)

The curve $L: z = F(x)$ is called the characteristic curve of (15). Let

$$L^+ = \{(x, F(x)): x \geq 0\} \quad \text{and} \quad L^- = \{(x, F(x)): x < 0\}.$$

Then $L = L^+ \cup L^-$. In (15), if $\phi(u) \equiv u$, then it is a Liénard system. Villari and Zanolin [7] and Hara, Yoneyama and Sugie [8] have given necessary and sufficient conditions for all positive semiorbits to intersect the characteristic curve. Employing the techniques in [7,8], we shall study the more general system (15).

First of all, we present the basic conditions. We assume that

(C_1) $F(x)$ and $g(x)$ are continuous on $\mathbf{R}$ with $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$ and $\phi(u)$ is continuous differentiable and strictly increasing with $\phi(0) = 0$ and $\phi(\pm\infty) = \pm\infty$.

(C_2) For any fixed number $k > 0$, there exists $M(k) > 0$ with $M(k) \equiv k$ for $0 < k \leq 1$ such that

$$|\phi(ku)| \leq M(k)\phi(|u|) \quad \text{for all} \quad u. \tag{16}$$

Sometimes, we only need the condition

(C_2') For any fixed $k \in (0, 1]$ and $u \in \mathbf{R}$,

$$|\phi(ku)| \leq k\phi(|u|). \tag{17}$$

For example, if $\phi(u) = u^3$, then (C_2) and (C_2') are satisfied.

Proposition 3.1. *If (C_1) is satisfied, then for any initial point $p(x_0, z_0)$, (15) has a unique trajectory passing through p.*

Proof By Peano's Theorem (see [9, p. 10]), (15) has at least one solution $(x(t),\ z(t))$ satisfying $x(0) = x_0$ and $z(0) = z_0$. Along such a solution, we have

$$\frac{dz}{dx} = -\frac{g(x)}{\phi(z - F(x))}, \tag{18}$$

$$z(x_0) = z_0. \tag{19}$$

In order to prove this proposition, we only have to prove that if $p \neq O = (0,\ 0)$, then the initial value problem (18) and (19) has a unique solution.

(i) Suppose $p \notin L$, that is, $z_0 \neq F(x_0)$. Then there exists a rectangle $E : |x - x_0| \leq a$ and $|z - z_0| \leq b$ such that E does not intersect L. Therefore, (C_1) implies that $\partial/\partial z[g(x)(\phi(z - F(x)))^{-1}]$ is continuous on E. Applying the Picard–Lindelöf Theorem, we know that the initial value problem (18) and (19) has a unique solution on E.

(ii) Suppose $p \in L$, that is, $z_0 = F(x_0)$, for example, $x_0 > 0$. If the conclusion is not true in this case, then (18) has two solutions $z = z_i(x)$ with $z_i(x_0) = z_0$ for $i = 1, 2$ and $z_1(x) \not\equiv z_2(x)$ for $x_1 \leq x < x_0$ without loss of generality, we may assume that $z = z_i(x)$ $(x \in [x_1, x_0])$ is under the characteristic curve L for $i = 1,\ 2$. Thus, there is an $x^* \in [x_1,\ x_0)$ with $z_1(x^*) > z_2(x^*)$. Set

$$\bar{x} = \sup\{x\colon x \in [x^*,\ x_0) \ \text{ such that } \ z_1(s) > z_2(s) \ \text{ for any } \ s \in [x^*,\ x]\}.$$

Then, $z_1(x) > z_2(x)$ for $x \in [x^*,\ \bar{x})$ and $z_1(\bar{x}) = z_2(\bar{x})$. This shows that (18) has two solutions passing through the point $(\bar{x},\ z_1(\bar{x}))$. The first step (i) implies that $(\bar{x},\ z_1(\bar{x})) \in L$.

Hence, $\bar{x} = x_0$. Using (18), we obtain that

$$\frac{d(z_1(x) - z_2(x))}{dx} = \frac{g(x)(\phi(z_1(x) - F(x)) - \phi(z_2(x)) - F(x))}{\phi(z_1(x) - F(x))\phi(z_2(x) - F(x))} \tag{20}$$

It follows from (C_1) that $\phi(u)$ is strictly increasing with $u\phi(u) > 0$ for $u \neq 0$. Therefore, from $z_i(x) - F(x) < 0$ for $x \in [x^*,\ x_0)$ and

$i = 1, 2$, $z_2(x) < z_1(x)$ and (20), we can conclude that

$$\frac{d(z_1(x) - z_2(x))}{dx} > 0 \quad \text{for} \quad x \in [x^*, \ x_0).$$

This implies that $z_1(x) - z_2(x)$ is strictly increasing on $[x^*, x_0]$. Thus, $z_1(x) - z_2(x) < z_1(x_0) - z_2(x_0) = 0$, that is, $z_1(x) < z_2(x)$ for $x \in [x^*, x_0)$, a contradiction. This completes the proof.

The system (15) is said to satisfy (C_3^+) if one of the following two conditions holds:

$(C_3^+)_1$ $\limsup_{x \to +\infty} F(x) > -\infty$;
$(C_3^+)_2$ there exist constants $N > 0$ and $\beta > 1/4$ such that $F(x) < 0$ for all $x \geq N$ and for any $b \geq N$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{g(s)}{\phi(-F(s))} ds \geq -\beta F(x) \quad \text{for all} \quad x \geq \bar{b}.$$

The system (15) is said to satisfy (C_3^-) if one of the following two conditions holds:

$(C_3^-)_1$ $\liminf_{x \to -\infty} F(x) < +\infty$;
$(C_3^-)_2$ There exist constants $N > 0$ and $\beta > 1/4$ such that $F(x) > 0$ for $x \leq -N$ and for any $b > N$, there exists $\bar{b} > b$ satisfying

$$\int_{-b}^x \frac{g(s)}{\phi(F(s))} ds \geq \beta F(x) \text{ for all } x \leq -\bar{b}. \tag{21}$$

The system (15) is said to satisfy (C_3) if both (C_3^+) and (C_3^-) hold.

For example, if $\phi(u) = u^3$, $g(x) = x^3$ and $F(x) = -|x|^{1/3}$. Then $\limsup_{x \to +\infty} F(x) = -\infty$, for any $b > 0$, we have

$$\int_b^x \frac{g(s)}{\phi(-F(s))} ds = \frac{x^3}{3} - \frac{b^3}{3} \quad \text{for all} \quad x \geq b,$$

it is obvious that $(C_3^+)_2$ and $(C_3^-)_1$ are satisfied. Thus (C_3) is satisfied.

THEOREM 3.1 *Suppose that the conditions* (C_1), (C_2) *and* (C_3^+) *hold. Then every positive semiorbit of* (15) *departing from* $D_1 = \{(x, z) = x \geq 0, z > F(x)\}$ *intersects the characteristic curve* L^+ *if and* *only if*

$$\limsup_{x\to+\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_-(s))}\,ds + F(x)\right] = +\infty, \tag{22}$$

where $F_-(x) = \max\{0, -F(x)\}$.

Proof: Sufficiency Suppose this is not the case. Then there is a point $p = (x_0, z_0) \in D_1$ such that the positive semiorbit $O^+(p)$ of (15) does not intersect L^+. Let $x = x(t)$ and $z = z(t)$ for $t \in [0, \omega_+)$ be the solution of (15) passing through p. Then we claim that

$$\lim_{t\to\omega_+} x(t) = +\infty. \tag{23}$$

If (23) is not true, then $\lim_{t\to\omega_+} x(t) = x^* < +\infty$. Let $p^* = (x^*, F(x^*)) \in L^+$ and $\overline{op}^*$ be the characteristic curve arc from O to p^*. Then $O^+(p)$ is contained in the bounded domain surrounded by z-axis, $z = z_0$, $x = x^*$ and $\overline{op}^*$. Thus, $\lim_{t\to\omega_+}(x(t), z(t))$ must exist and is an equilibrium of (15). But from (C_1), the origin is the unique equilibrium of (15). This implies that $x^* = 0$. However, $x(t) > x_0$ for $t > 0$ and hence $x^* > x_0 \geq 0$ which is a contradiction. This shows that (23) is true. Therefore, the solution $(x(t), z(t))$ determines a function $z = z(x)$ defined on $[x_0, +\infty)$ which lies above L^+ and is strictly decreasing. Clearly,

$$z_0 > \lim_{x\to+\infty} z(x) \geq \limsup_{x\to+\infty} F(x), \tag{24}$$

now, (24) shows that all positive semiorbits of (15) starting from D_1 intersect L^+ as long as $\limsup_{x\to+\infty} F(x) = +\infty$. Here we do not require condition (C_2). Therefore, it suffices to consider the case $\limsup_{x\to+\infty} F(x) < +\infty$.

Suppose that $(C_3^+)_1$ holds. Then it follows from (24) that $z(x)$ is bounded on $[x_0, +\infty)$. From the definition of $F_-(x)$, we have

$$z(x) - F(x) \leq z_0 + F_-(x) \leq k(1 + F_-(x)) \tag{25}$$

where $k = \max\{1, |z_0|\}$. From (C_2), $\phi(u)$ is strictly increasing. This fact, together with (25) and (16), yields

$$\phi(z(x) - F(x)) \leq \phi(k(1 + F_-(x))) \leq M(k)\phi(1 + F_-(x)) \quad \text{for all} \quad x \geq x_0.$$

Thus,

$$
\begin{aligned}
z(x) - z_0 &= -\int_{x_0}^{x} \frac{g(s)}{\phi(z(s) - F(s))}\, ds \\
&\leq -\frac{1}{M(k)} \int_{x_0}^{x} \frac{g(s)}{\phi(1 + F_{-}(s))}\, ds \ (x \geq x_0).
\end{aligned} \tag{26}
$$

Letting $x \to +\infty$ in (26) and applying (22), we conclude that $\lim_{x\to+\infty} z(x) = -\infty$, contradicting $(C_3^+)_1$ and (24). We note that if $\liminf_{x\to+\infty} F(x) > -\infty$ then we can prove that $z(x) \to -\infty$ as $x \to +\infty$ without the assumption (C_2).

Assume that $(C_3^+)_2$ is true. From the proof in the last paragraph, we only have to consider the case $\lim_{x\to+\infty} z(x) = -\infty$. Let $\beta_1 = 1 - \beta$ and define $\beta_{n+1} = 1 - \beta/\beta_n$. If $1/4 < \beta < 1$, then it is easy to prove that $0 < \beta_n < 1$ and $\{\beta_n\}$ is a strictly decreasing sequence. Now, we choose a sufficiently large number N such that $z(x) < 0$ and $F(x) < 0$ for all $x \geq N$. Applying $(C_3^+)_2$, we can find $b_1 \geq N$ such that for all $x \geq b_1$,

$$
z(x) - z(N) = -\int_{N}^{x} \frac{g(s)}{\phi(z(s) - F(s))}\, ds \leq -\int_{N}^{x} \frac{g(s)}{\phi(-F(s))}\, ds \leq \beta F(s). \tag{27}
$$

From (27) it immediately follows that for $x \geq b_1$,

$$
0 < z(x) - F(x) \leq (\beta - 1)F(x). \tag{28}
$$

Thus, we have $1/4 < \beta < 1$. Using (C_1) and (C_2) again, we obtain

$$
\phi(z(x) - F(x)) \leq \phi((\beta - 1)F(x)) \leq (1 - \beta)\phi(-F(x)) = \beta_1 \phi(-F(x)) \ (x \geq b_1).
$$

By $(C_3^+)_2$, there exists $b_2 > b_1$, such that for all $x \geq b_2$,

$$
\begin{aligned}
z(x) - z(b_1) = -\int_{b_1}^{x} \frac{g(s)}{\phi(z(s) - F(s))}\, ds &\leq -\frac{1}{\beta_1} \int_{b_1}^{x} \frac{g(s)}{\phi(-F(s))}\, ds \\
&\leq \frac{\beta}{\beta_1} F(x),
\end{aligned}
$$

that is, when $x \geq b_2$, we have

$$0 < z(x) - F(x) < \left(\frac{\beta}{\beta_1} - 1\right)F(x) = -\beta_2 F(x).$$

Continuing this procedure, we can prove that there is a sequence $\{b_n\}$ such that $0 < z(x) - F(x) \leq -\beta_n F(x)$ for all $x \geq b_n$. Since $1/4 < \beta < 1$, β_n is a positive decreasing sequence, hence, $\lim_{n\to+\infty} \beta_n = \lambda$ exists. Obviously, $\lambda \in [0, 1]$ is a real number. But from the definition of β_n, we have $\lambda = 1 - \beta/\lambda$, $\beta > 1/4$, which implies that λ is a complex number, which is a contradiction. This proves the sufficiency.

Necessity Suppose that (22) is not true. Then there exist numbers $M, L > 0$ such that

$$F(x) \leq M \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad \int_L^{+\infty} \frac{g(x)}{\phi(1 + F_-(x))}\, dx < 1.$$

Let $(x(t),\ z(t))$ be the solution of (15) with $(x(0),\ z(0)) = (L, M + 2)$. Then we assert that $z(t) > M + 1$ for all $t \geq 0$. Otherwise, there is a $\tau > 0$ such that $z(t) > M + 1$ for $t \in [0,\ \tau)$ and $z(\tau) = M + 1$. Thus

$$z(t) - F(x(t)) \geq M + 1 - (M - F_-(x(t))) = 1 + F_-(x(t))$$

for all $t \in [0, \tau]$.

Integrating, we have

$$\begin{aligned} M + 1 = z(\tau) &= M + 2 - \int_0^\tau \frac{g(x(s))}{\phi(z(s) - F(s))} x'(s)\, ds \\ &\geq M + 2 - \int_0^\tau g(x(s))(\phi(1 + F_-(x(s))))^{-1} x'(s)\, ds \\ &\geq M + 2 - \int_L^{+\infty} \frac{g(\zeta)}{\phi(1 + F_-(\zeta))}\, d\zeta > M + 1. \end{aligned}$$

This causes a contradiction and proves the necessity.

In quite the same manner, we can prove the following result.

THEOREM 3.2 *Suppose that the conditions* (C_1), (C_2) *and* (C_3^-) *hold. Then every positive semiorbit of* (15) *starting from* $D_3 = \{(x, z) : x \leq 0,\ z < F(x)\}$ *intersects the characteristic curve* L^-

if and only if

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_+(s))}\,ds - F(x)\right] = +\infty, \tag{29}$$

where $F_+(x) = \max\{0,\ F(x)\}$.

The system (15) is said to be satisfying the condition $(C_3^+)'$ $((C_3^-)')$ if $-F(x)$, $g(x)$ and $\phi(u)$ satisfy the condition (C_3^+) $((C_3^-))$.

By the transformations $t \to -t$ and $z \to -z$ in (15), we have the following results with respect to the negative semiorbits of (15).

THEOREM 3.3 *Let* ϕ, F *and* g *satisfy the conditions* (C_1), (C_2) *and* $(C_3^+)'$. *If* $\phi(u)$ *is an odd function, then every negative semiorbit of* (15) *starting from a point in* $D_2 = \{(x,\ z) : x \geq 0,\ z < F(x)\}$ *intersects the characteristic curve* L^+ *at a point* $B(x_1,\ F(x_1))$ *with* $x_1 > 0$ *if and only if*

$$\limsup_{x\to+\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_+(s))}\,ds - F(x)\right] = +\infty. \tag{30}$$

THEOREM 3.4 *Let* ϕ, F *and* g *satisfy the conditions* (C_1), (C_2) *and* $(C_3^-)'$. *If* $\phi(u)$ *is an odd function, then every negative semiorbit of* (15) *starting from a point in* $D_4 = \{(x,\ z) : x \leq 0,\ z > F(x)\}$ *intersects the characteristic curve* L^- *at a point* $B(x_2,\ F(x_2))$ *with* $x_2 < 0$ *if and only if*

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_-(s))}\,ds + F(x)\right] = +\infty. \tag{31}$$

Remark 3.1 If $\liminf_{x\to+\infty} F(x) > -\infty$, then (22) is equivalent to

$$\limsup_{x\to+\infty}\left[\int_0^x g(s)\,ds + F(x)\right] = +\infty, \tag{32}$$

and if $\limsup_{x\to-\infty} F(x) < +\infty$, then (29) is equivalent to

$$\limsup_{x\to-\infty}\left[\int_0^x g(s)\,ds - F(x)\right] = +\infty. \tag{33}$$

From the proof of Theorem 3.1, we know that the conclusion of the Theorem 3.1 is also true if $\liminf_{x\to+\infty} F(x) > -\infty$ and (C_2) and (22) are replaced by (C_2') and (32) respectively. Similarly, suppose that $\limsup_{x\to-\infty} F(x) < +\infty$. Then the result of Theorem 3.2 also holds when (C_2) and (29) are replaced by (C_2') and (33) respectively.

Remark 3.2 From the necessity proof of Theorem 3.1, we know that if

$$\limsup_{x\to+\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_+(s))}\,ds - F(x)\right] < +\infty, \tag{34}$$

then there exists a point $p \in D_2$ such that $O^-(p)$ does not intersect L^+. Similarly, if

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_-(s))}\,ds + F(x)\right] < +\infty, \tag{35}$$

then there exists a point $p \in D_4$ such that $O^-(p)$ does not meet L^-.

4 THE GLOBAL STABILITY OF THE ZERO SOLUTION OF (15)

In order to give a criterion for the zero solution of (15) to be globally asymptotically stable, we must provide a criterion for non-existence of nontrivial periodic solution for (15). To this end, we introduce transformations similar to those of Filippov's [10].

Let $G(x) = \int_0^x g(s)\,ds$. If $x > 0$, then we set

$$u = u_1(x) = G(x), \quad u \in (0,\ G(+\infty)), \tag{36}$$

the inverse function of which is denoted by $x = x_1(u)$. Replacing $x(>0)$ in $F(x)$ by $x = x_1(u)$, we have

$$F_1(u) = F(x_1(u)), \quad u \in (0,\ G(+\infty)). \tag{37}$$

Similarly, if $x < 0$, then we write

$$u = u_2(x) = G(x), \quad u \in (0,\ G(-\infty)), \tag{38}$$

whose inverse function is given by $x = x_2(u)$. Thus, substituting $x = x_2(u)$ in $F(x)$ if $x < 0$, we obtain

$$F_2(u) = F(x_2(u)), \quad u \in (0,\ G(-\infty)). \tag{39}$$

Therefore, the Eq. (15) in the cases $x > 0$ and $x < 0$ are equivalent to the following two equations, respectively:

$$\frac{du}{dz} = -\phi(z - F_1(u)),\ u \in (0,\ G(+\infty)) \tag{40}$$

$$\frac{du}{dz} = -\phi(z - F_2(u)),\ u \in (0,\ G(-\infty)). \tag{41}$$

Now we introduce the condition (C_4) :

$$F_2(u) \le F_1(u) \text{ if } u \in (0, \min\{G(-\infty), G(+\infty)\}) \text{ and}$$
$$F_1(u) \not\equiv F_2(u) \text{ if } 0 < u \ll 1$$

where the notation $0 < u \ll 1$ denotes u sufficiently small.

Proposition 4.1. *Suppose that the conditions* (C_1), (C_2) *and* (C_4) *are satisfied. If the positive semiorbit of* (15) *starting from* $A_0 = (0, z_0) \times (z_0 > 0)$ *again intersects the positive* z*-axis at* $A_1 = (0, z_1)$, *then* $z_1 < z_0$. *In particular*, (15) *has no nontrivial periodic solutions.*

Proof Let $O^+(A_0)$ denote the positive semiorbit passing through A_0. Assume that the conclusion is false. Then $z_1 \ge z_0$. Therefore, the orbit arc $\overline{A_0A_1} \subset O^+(A_0)$ must intersect the negative z-axis at $B = (0,\ z_{-1})$ with $z_{-1} < 0$. Let $u = u_1(z)$ and $u = u_2(z)$ be the solutions of (40) and (41) with the initial condition $u_i(z_{-1}) = 0$ for $i = 1, 2$. Then $u = u_i(z)$ is defined on $[z_{-1}, z_0]$ for $i = 1, 2$. Since $z_1 \ge z_0$, we have

$$u_2(z_0) \ge u_1(z_0) = 0. \tag{42}$$

The condition (C_4) implies that $F_2(u) \le F_1(u)$ for all $u \in (0,\ \min\{G(-\infty), G(+\infty)\})$. Therefore, it follows from (C_2) that

$$-\phi(z - F_2(u)) \le -\phi(z - F_1(u)),\ \text{for all } z \text{ and}$$
$$u \in (0,\ \min\{G(-\infty),\ G(+\infty)\}). \tag{43}$$

Applying Kamke's Theorem (see [11, p. 29]), we know that if (43) holds and $u_2(z_{-1}) \le u_1(z_{-1})$ then $u_2(z) \le u_1(z)$ for all $z \in [z_{-1}, z_0]$. On the other hand, Coppel discussed in [11, p. 30] that $u_2(z_0) = u_1(z_0)$ if and only if $u_1(z)$ coincides with $u_2(z)$ on $[z_{-1}, z_0]$, that is, $u_1(z) \equiv u_2(z)$ for any $z \in [z_{-1}, z_0]$. We claim that $u_2(z) < u_1(z)$ for all $z \in (z_{-1}, z_0]$. Suppose this is not the case, then there exists a point $z^* \in (z_{-1}, z_0]$ such that $u_2(z^*) = u_1(z^*)$. This implies that $u_1(z) \equiv u_2(z)$ for all $z \in [z_{-1}, z^*]$. Hence $F_1(u_1(z)) \equiv F_2(u_1(z))$ for all $z \in [z_{-1}, z^*]$, that is, $F_1(u) \equiv F_2(u)$ for $u \in (0, u_1(z^*)]$, contradicting (C_4). This proves that our claim is true. In particular, $u_2(z_0) < u_1(z_0)$, contradicting (42). The proof is complete.

The system (15) is said to satisfy (C_5^+) if one of the following conditions holds:

$(C_5^+)_1$ there exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \ge 0$ for each n;

$(C_5^+)_2$ there exist constants $a > 0$ and $\beta > 1/4$ such that $F(x) < 0$ for $0 < x \le a$ and

$$\int_0^x \frac{g(s)}{\phi(-F(s))}\, ds \ge -\beta F(x) \quad \text{for } 0 < x \le a.$$

The system (15) is said to satisfy (C_5^-) if one of the following conditions holds:

$(C_5^-)_1$ there is a negative increasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \le 0$ for each n ;

$(C_5^-)_2$ there are constants $b < 0$ and $\beta > 1/4$ such that $F(x) > 0$ for $b \le x < 0$ and

$$\int_0^x \frac{g(s)}{\phi(F(s))}\, ds \le \beta F(x) \quad \text{for } b \le x < 0.$$

The system (15) is said to satisfy the condition (C_5) if both (C_5^+) and (C_5^-) hold.

Lemma 4.1 *Assume that the conditions* (C_1) *and* (C_2) *are true. Then*

(i) *if* (C_5^+) *holds, then for any* $p = (x_0, z_0) \in L^+$, *the negative semiorbit* $O^-(p)$ *must intersect the positive z-axis at* $(0, z_p)$ *with* $z_p > 0$;

(ii) *if* (C_5^-) *holds, then for any* $p = (x_0, z_0) \in L^-$, *the negative semiorbit* $O^-(p)$ *must intersect the negative z-axis at* $(0, z_p)$ *with* $z_p < 0$.

Proof We only prove (i); (ii) can be proved in a similar way.

Let $p = (x_0, z_0) \in L^+$ and $(x(t), z(t))$ be the solution of (15) with $x(0) = x_0, z(0) = z_0$ whose maximal existence interval is (ω_-, ω_+). By (C_1), $\phi(+\infty) = +\infty$. Therefore, (15) has no vertical asymptote in the first quadrant. This implies that every negative semiorbit of (15) passing the point $p \in D_1 \cup L^+$ in the first quadrant must intersect the positive z-axis. By Proposition 3.1, there exists a unique orbit of (15) passing through a given initial point. Hence if $O^-(p)$ intersects the positive z-axis, then for any $q = (x, F(x)) \in L^+$ with $x \geq x_0$ $O^-(q)$ also intersects the positive z-axis.

First, assume that $(C_5^+)_1$ holds. Then there exists a decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \geq 0$ for each n. Let $p_n = (x_n, z_n)$ with $z_n = F(x_n)$. The facts presented in the above paragraph show that $O^-(p_n)$ intersects the positive z-axis for each n. Therefore, all negative semiorbits $O^-(p)$ passing through $p \in L^+$ must intersect the positive z-axis.

Next, assume that $(C_5^+)_2$ holds. We assert that all negative semiorbits passing through $p = (x_0, z_0) \in L^+$ with $0 < x_0 \leq a$ intersect the positive x-axis and therefore intersect the positive z-axis. Otherwise, then there exists $p \in L^+$ with $0 < x_0 \leq a$ such that $O^-(p)$ is contained in the fourth quadrant. Let $(x(t), z(t))$ $(-\infty < t \leq 0)$ denote the solution of (15) passing through such a point $p = (x_0, F(x_0))$. Then $x(t)$ decreases and $z(t)$ increases as t is decreasing. Since the origin is the unique equilibrium of (15), $x(t) \to 0$ and $z(t) \to 0$ as $t \to +\infty$. Because the solution $(x(t), z(t))$ defines a function $z = z(x) \leq 0$ on $0 < x \leq x_0$, which is a solution of the Eq. (18) from 0 to x, we obtain that

$$\begin{aligned} z(x) &= -\int_0^x \frac{g(s)}{\phi(z(s) - F(s))}\, ds \\ &\leq -\int_0^x \frac{g(s)}{\phi(-F(s))}\, ds \leq \beta F(x) \quad \text{for } 0 < x \leq x_0, \end{aligned} \tag{44}$$

where the second inequality follows from $(C_5^+)_2$. Define $\beta_1 = 1 - \beta$ and $\beta_n = 1 - \beta/\beta_{n-1}$. Then, we shall prove that

$$\beta_n F(x) \leq F(x) - z(x) < 0 \tag{45}$$

for all $x \in (0,\ x_0)$ and each n.

From (44), it follows immediately that (45) is true for $n = 1$. Suppose (45) holds for $n = m$. Then we have $0 < \beta_m < 1$ and $\phi(z(x) - F(x)) \leq \phi(-\beta_m F(x)) \leq \beta_m \phi(-F(x))$. The same procedure as used in proof of (44) shows that

$$z(x) \leq -\frac{1}{\beta_m} \int_0^x \frac{g(s)}{\phi(-F(s))}\, ds \leq \frac{\beta}{\beta_m} F(x) \quad \text{for } 0 < x \leq x_0,$$

and this implies that $\beta_{m+1} F(x) \leq F(x) - z(x) < 0$ for all $x \in (0,\ x_0)$. By induction, (45) is true for every n. It follows from $F(x) < 0$ for $0 < x \leq x_0$ and (45) that $\beta_n > 0$ for each n. Therefore, $0 < \beta_n < 1$ for $n \geq 1$ and β_n is decreasing. Suppose that $\beta_n \to \lambda$ as $n \to +\infty$. Then λ is a real number which satisfies the algebraic equation $\lambda = 1 - \beta/\lambda$. However, from $\beta > 1/4$, it follows that λ is a complex number. This desired contradiction shows that our conclusion is true and completes the proof.

THEOREM 4.1 *Suppose that the system* (15) *satisfies the conditions* (C_1) *through* (C_5). *Then the origin is globally asymptotically stable if and only if* (22) *and* (29) *hold.*

Proof: Necessity If either (22) or (29) is false, then Theorem 3.1 and 3.2 implies that (15) has at least one unbounded solution lying in D_1 or D_3. Thus, the origin is not globally asymptotically stable.

Sufficiency Suppose that (22) and (29) hold. We shall prove that the origin is globally asymptotically stable.

By (C_1), $\phi(\pm\infty) = \pm\infty$, which implies that (15) has no vertical asymptote. Hence, all positive semiorbits departing from $D_2 \cup D_4$ enter $D_1 \cup D_3$ or converge to the origin and all positive semiorbits starting from D_1 and D_3 will intersect L^+ and L^- respectively if (22) and (29) hold. Thus, all points in the plane $\mathbf{R}^2$ can be divided into two classes:

$$S_1 = \Big\{ p \in \mathbf{R}^2 \colon \text{there is } t_0 \geq 0 \text{ such that } \varphi_t(p) \in D_2 \text{ or } \\ \varphi_t(p) \in D_4 \text{ for } t \geq t_0 \Big\}$$

and

$$S_2 = \big\{ p \in \mathbf{R}^2 \colon O^+(p) \text{ spirals around the origin} \big\},$$

where $\varphi_t(p)$ denotes the solution of (15) passing through p. The above arguments show that $\mathbf{R}^2 = S_1 \cup S_2$.

Suppose that $p \in S_1$. Then $\varphi_t(p) \in D_2 \cup D_4$ for $t \geq t_0$. Without loss of generality, we may assume that $\varphi_t(p) = (x(t),\ z(t)) \in D_2$ for $t \geq t_0$. Therefore, $\varphi_t(p)$ is bounded and $x(t),\ z(t)$ is decreasing as t increases. Thus, the ω-limit set of such an orbit $O^+(p)$ must be a singleton. Since the origin is a unique equilibrium of (15), this singleton point is certainly the origin.

Suppose that $p \in S_2$. Then $O^+(p)$ spirals around the origin. From the condition (C_4) and Proposition 4.1, $O^+(p)$ is bounded and its ω-limit set $\omega(p)$ cannot have a homoclinic orbit if the conditions (C_1) through (C_5) hold, that is, there cannot exist $q \in \mathbf{R}^2$ with $q \neq O$ such that $\lim_{t\to+\infty} \varphi_t(q) = O$ and $\lim_{t\to-\infty} \varphi_t(q) = O$. Suppose, to reach a contradiction, that such a point q exists. Proposition 4.1 implies that $O^-(q)$ cannot spiral around the origin. Hence, there exists $t_1 < 0$ such that for $t \leq t_1$ either $\varphi_t(q) \in D_1$, in this case, $F(x) < 0$ for $0 < x \ll 1$, or $\varphi_t(q) \in D_3$, in which case, $F(x) > 0$ for $0 < -x \ll 1$. Without loss of generality, we may assume that $\varphi_t(q) \in D_1$ for $t \leq t_1$ and $F(x) < 0$ for $0 < x \ll 1$. Applying Theorem 3.1, we obtain that $O^+(\varphi_t(q))$ intersects L^+, that is, there is $t_2 > t_1$ such that $\varphi_{t_2}(q) \in L^+$. This implies that $\varphi_t(q) \in D_1$ for all $t \leq t_2$. However, applying Lemma 4.2 to (15), we know that $O^-(\varphi_{t_2}(q))$ will intersect the positive z-axis, which is a contradiction. This shows that the system (15) cannot have a homoclinic orbit. This fact and the Poincare-Bendixson theorem now allow us to conclude that all positive semiorbits converge to the origin.

It remains to be shown that the origin is locally Liapunov stable. Suppose the origin is unstable. Then, by definition, for some $\varepsilon_0 > 0$

there are sequences $p_n \to O$ and $t_n \geq 0$ such that

$$\varphi_t(p_n) \in B(\varepsilon_0) \quad \text{for} \quad 0 \leq t \leq t_n \tag{46}$$

and $\varphi_{t_n}(p_n) \in \partial B(\varepsilon_0)$ where $B(\varepsilon_0) = \{(x,\ z) : x^2 + z^2 \leq \varepsilon_0^2\}$ and $\partial B(\varepsilon_0)$ denotes the boundary of $B(\varepsilon_0)$. Since $\partial B(\varepsilon_0)$ is compact, we can assume $q_n = \varphi_{t_n}(p_n) \to q$ as $n \to +\infty$. We claim that $t_n \to +\infty$ as $n \to +\infty$. If not, then there is a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to \mu < +\infty$ as $k \to +\infty$. Then $p_{n_k} = \varphi_{-t_{n_k}}(q_{n_k}) \to \varphi_{-\mu}(q)$ as $k \to +\infty$, that is, $\varphi_{-\mu}(q) = O$, and hence $q = O$. However, $q \in \partial B(\varepsilon_0)$. This produces the desired contradiction and proves our claim. We shall prove that $O^-(q)$ is bounded. For any $\tau > 0$, there is an N such that $t_n \geq \tau$ for $n \geq N$. Hence, from (46), we have

$$\varphi_{-\tau}(q_n) = \varphi_{t_n-\tau}(p_n) \in B(\varepsilon_0) \quad \text{for} \quad n \geq N. \tag{47}$$

Letting $n \to +\infty$ in (47), we have $\varphi_{-\tau}(q) = \lim_{n\to\infty} \varphi_{t_n-\tau}(p_n) \in B(\varepsilon_0)$. Since τ is arbitrary, we obtain that $\varphi_{-\tau}(q) \in B(\varepsilon_0)$ for $\tau > 0$, that is, $O^-(q) \subset B(\varepsilon_0)$. The global attractivity property proved in the above paragraph implies that $\alpha(q)$ cannot be a limit cycle, and hence $O \in \alpha(q)$. If $\alpha(q) = \{0\}$, then the orbit $O(q)$ passing through q is homoclinic. If $\alpha(q) \neq \{O\}$, then the Poincare-Bendixon theorem implies that $\alpha(q)$ contains a homoclinic orbit. But we have proved that under the conditions (C_1) through (C_5) the system (15) does not have a homoclinic orbit. This contradiction proves that the origin is globally asymptotically stable and completes the proof of the theorem.

Remark 4.1 If $\varphi(u) = u$, then Theorem 4.1 generalized the corresponding results of Hara and Yoneyama [13,14] who gave only the sufficient conditions.

Finally, we shall apply Theorem 4.3 to the systems (1) and (10). Let $\phi(y) \equiv y, F(x) = x/h_2(x)[h_1(x) + h_2(x)H_{42}(x)/x] = \Psi(x), g(x) = -g_3(x)/h_2(x) = \delta(x)x[h_2(x)]^{-2}$ and $u = G(x) = \int_0^x \delta(s)/h_2^2(s)s ds$. With these notations, we can obtain the function $F_1(u)$ and $F_2(u)$ in (37) and (39).

COROLLARY 4.1 *Let the functions $f_1(x)$, $f_3(x)$, $h_2(x)$ and $h_4(x)$ be continuous on* **R**. *Suppose that $h_2(x) \neq 0$ for all x and $\delta(x) > 0$ for all $x \neq 0$. Assume that the conditions (C_3) through (C_5) hold. Then the* *origin of* (1) *is globally asymptotically stable if and only if*

$$\limsup_{x\to+\infty}\left[\int_0^x \frac{g(s)}{1+F_-(s)}\,ds + F(x)\right] = +\infty$$

and

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{g(s)}{1+F_+(s)}\,ds - F(x)\right] = +\infty.$$

We note that in the system (15) if there exists a constant $a < \sqrt{8}$ such that $F_1(u) \geq -a\sqrt{u}$ and $F_2(u) \leq a\sqrt{u}$ for $0 < u \ll 1$, then (C_5) is satisfied. Thus, the results of [15] are corollaries of Theorem 4.1 with $\varphi(u) = u$.

Example 4.1 Consider the system of differential equations

$$\begin{aligned} \dot{x} &= -x^2(x^3+x+1) - y, \\ \dot{y} &= 5x^6(x^3+x+1) + 2x + 5x^4 y. \end{aligned} \tag{48}$$

In (48), $f_1(x) = -x^2(x^3+x+1)$, $f_3(x) = 5x^6(x^3+x+1) + 2x$, $h_2(x) \equiv -1$ and $h_4(x) = 5x^4$. By calculation, we have

$$\begin{aligned} &h_1(x) = -x(x^3+x+1), \quad h_3(x) = 2 + 5x^6(x^3+x+1) \\ &\delta(x) = h_1h_4 - h_2h_3 \equiv 2, \quad H_{42}(x) = -x^5 \\ &h_1(x) + h_4(x) = 4x^4 - x^2 - x > 0 \text{ for } |x| \text{ sufficiently large,} \\ &h_1(x) + \frac{h_2(x)H_{42}(x)}{x} = -x(x+1) > 0 \text{ for } -1 < x < 0. \end{aligned}$$

Therefore, the conditions (1) and (4) in Egorov and Kartuzova's Theorem are not satisfied, that is, Egorov and Kartuzova's Theorem cannot be applied to (48). However, in our notation,

$$F(x) = x^3 + x \quad \text{and} \quad g(x) = 2x.$$

It is easy to prove that for such $F(x)$ and $g(x)$ the conditions (C_3) through (C_5) hold and $\lim_{|x|\to+\infty} \operatorname{sgn} x F(x) = +\infty$. Therefore, applying Corollary 4.1, we conclude that the origin is globally asymptotically stable.

COROLLARY 4.2 *Let* $p_i(y)$ $(i = 2,\ 3)$ *and* $q_j(x)$ $(j = 2,\ 3,\ 4)$ *be continuous on* $\mathbf{R}$ *and* (12) *be satisfied. Assume that the inverse function* $\phi(u)$ *of the function* $u = \int_0^y (p_2(s)/p_3(s))\,ds$ *satisfies* (C_1) *and* (C_2'), *and* $F(x) = -\int_0^{-x}(q_4(s)/q_2(s))\,ds$ *and* $g(x) = -(q_3(-x)/q_2(-x))x$ *satisfy* (C_4) *and* (C_5). *If* $\limsup_{x\to-\infty} \int_0^x (q_4(s)/q_2(s))\,ds < +\infty$ *and* $\liminf_{x\to+\infty} \int_0^x \times (q_4(s)/q_2(s))\,ds > -\infty$, *then the zero solution of* (10) *is globally asymptotically stable if and only if*

$$\liminf_{|x|\to+\infty}\left[\int_0^x \frac{sq_3(s)}{q_2(s)}\,ds - \operatorname{sgn}x \int_0^x \frac{q_4(s)}{q_2(s)}\,ds\right] = -\infty. \tag{49}$$

Corollary 4.2 follows from Remark 3.1.

Example 4.2 Consider the system with zero diagonal coefficient

$$\begin{aligned}\dot{x} &= -y(1 + x^4)[1 + (1 + y^2)^{3/2}]\\ \dot{y} &= (1 + y^2)^{3/2}(1 + x^4)x + (1 + y^2)^{3/2}(1 + x^4)(x - x^2)y.\end{aligned} \tag{50}$$

By calculation, we have

$$\begin{aligned}u = \psi(y) &= \int_0^y \frac{1 + (1 + s^2)^{3/2}}{(1 + s^2)^{3/2}}\,ds = y + \frac{y}{\sqrt{1 + y^2}}\\ F(x) &= \int_0^{-x} \frac{(1 + s^4)(s - s^2)}{1 + s^4} = \frac{x^3}{3} + \frac{x^2}{2}\\ g(x) &= x\end{aligned}$$

It is easy to check that $F(x)$ and $g(x)$ satisfy the conditions (C_1), (C_3), (C_4) and (C_5). Moreover, $\lim_{|x|\to+\infty} F(x)\operatorname{sgn}x = +\infty$ and therefore, (49) holds. Let $y = \psi^{-1}(u) := \phi(u)$. Then $\phi(u)$ is an odd function on $\mathbf{R}$ with $\phi(\pm\infty) = \pm\infty$. For any $u \in [0,\ \infty)$ and $k \in (0,\ 1]$, we have

$$ku = ky + \frac{ky}{\sqrt{1 + y^2}} \leq \psi(ky).$$

Therefore, $\phi(ku) \leq ky = k\phi(u)$, that is (C_2') is true. From Corollary 4.2, it follows that the zero solution of (50) is global asymptotically stable. However, $g_4(x) = (1 + x^4)(x - x^2)$ changes sign on $\mathbf{R}$. Hence, the result of Krechetov [5] cannot be applied to (50).

Remark 4.2 In the papers [16–19], the second author gave other conditions to guarantee that every solution of a second-order differential equation converges. In the forthcoming paper, we will discuss the existence of homoclinic orbits, existence of oscillatory solutions, and existence of centers of system (15).

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The qualitative behavior of a second-order system with zero diagonal coefficient

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Abstract

This paper investigates the qualitative behavior of solutions of the autonomous planar system with zero diagonal coefficient $\dot{x} = p_2(y)q_2(x)y$, $\dot{y} = p_3(y)q_3(x)x + p_4(y)q_4(x)y$. Under suitable assumptions, the necessary and sufficient conditions for all solutions to be oscillatory, and for the origin to be a global center are established. The theorems on the existence and uniqueness of nontrivial periodic solutions are also proved.

1. Introduction

This paper is devoted to the investigation of the qualitative behavior of the solutions of the autonomous system of two differential equations with zero diagonal coefficient

$$\dot{x} = p_2(y)q_2(x)y, \qquad \dot{y} = p_3(y)q_3(x)x + p_4(y)q_4(x)y, \tag{1.1}$$

where $p_i(y)$ and $q_i(x)$ $(i = 2, 3, 4)$ are continuous real functions defined on $\mathbf{R} = (-\infty, +\infty)$.

Krechetov [8] studied the global asymptotic behavior of solutions of system (1.1), described the configurations of the domains of stability (when there is no global asymptotic

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stability) and constructed estimates of the boundaries of these domains. In the study of stability for (1.1), the most important condition given by Krechetov [8] is

$$q_2(x)q_4(x) > 0 \quad \text{for all } x \in \mathbf{R}, \tag{1.2}$$

by using the Lyapunov function method, he gave necessary and sufficient conditions for the zero solution of (1.1) to be globally asymptotically stable under some additional assumptions.

Recently, Yan and Jiang [12] first introduced the transformation techniques to investigate the global asymptotic stability of the following system (1.3), the special case (i.e., $p_3(y) \equiv p_4(y)$) of system (1.1),

$$\dot{x} = p_2(y)q_2(x)y, \qquad \dot{y} = p_3(y)q_3(x)x + p_3(y)q_4(x)y \tag{1.3}$$

without the assumption (1.2). In paper [12], under the following conditions

$$\begin{aligned} &p_2(y) > 0, \qquad p_3(y) > 0 \quad \text{for all } y, \\ &q_2(x) > 0, \qquad q_3(x) < 0 \quad \text{for all } x, \end{aligned} \tag{1.4}$$

they transformed system (1.3) into the following generalized Liénard system

$$\dot{x} = \phi\bigl(z - F(x)\bigr), \qquad \dot{z} = -g(x), \tag{1.5}$$

and obtained necessary and sufficient conditions for the zero solution of (1.3) (respectively (1.5)) to be globally asymptotically stable. Such system (1.5) with $\phi(u) \equiv u$ arises in several different settings, modelling phenomena appearing in the study of physical, as well as biological, chemical, and economical systems, it naturally has been studied by a number of authors [1–5,10,11,13,14]. The main problem connected to the study of such models consists of giving a complete description of the behavior of solutions as $t \to +\infty$. In general, this is not possible, due to the complexity of the equations and the phenomena involved. The aim of the qualitative theory is to give an approximate description of the behavior of the system, by identifying suitable regions of the phase space, where the solutions behave in a similar way.

In the present paper, we shall investigate the qualitative behavior of system (1.1) without the assumption (1.2). Especially, we shall pay our attention to the oscillation, center, existence and uniqueness of nontrivial periodic solutions of system (1.1) (respectively (1.5)). In this paper, no restriction on the sign of $q_4(x)$ is required, we only assume that

$$\begin{aligned} &p_2(y) > 0, \qquad p_3(y) > 0, \qquad p_4(y) > 0 \quad \text{for all } y, \\ &q_2(x) < 0, \qquad q_3(x) > 0 \quad (\text{or } q_2(x) > 0,\ q_3(x) < 0) \quad \text{for all } x, \\ &\rho(y) \in C^1(\mathbf{R}), \qquad \rho'(y) > 0 \quad \text{for all } y,\ \rho(\pm\infty) = \pm\infty,\ \text{where } \rho(y) := \frac{yp_4(y)}{p_3(y)}. \end{aligned} \tag{1.6}$$

If $p_3(y) \equiv p_4(y)$, one case of assumption (1.6) reduces to (1.4). Under assumption (1.6), we shall prove that system (1.1) is equivalent to a form of system (1.5) which is a Liénard-like system, the investigation of the qualitative behavior of solutions of system (1.5) has independent interest and value. For example, applying the results in this paper, the following system and equation have a unique nontrivial periodic solution,

$$\begin{aligned}\dot{x} &= x^3 - 3x^5 + 3x^7 - x^9 + 3\bigl(x^6 - 2x^4 + x^2\bigr)y + 3\bigl(x - x^3\bigr)yy^2 + y^3,\\ \dot{y} &= -x,\end{aligned} \tag{1.7}$$

and

$$\ddot{x} + 3\bigl(x^2 - 1\bigr)\dot{x}^{5/3} + 3x\dot{x}^{2/3} = 0. \tag{1.8}$$

The technique tool of this paper is based on some transformations (including Filippov's transformation [1]), and the methods for Liénard systems, especially those developed by Villari and Zanolin [11], Hara and Sugie [3].

The organization of this paper follows. In Section 2 we introduce suitable transformations which change (1.1) into the form of (1.5), present assumptions and some auxiliary lemmas which will be essential to our proofs. In Section 3 we give the necessary and sufficient conditions for all solutions of (1.5) to be oscillatory and for the origin to be a global center. In Section 4 we give the theorems of existence and uniqueness of nontrivial periodic solutions of (1.5). A brief discussion is given in Section 5.

2. Transformation for (1.1) and auxiliary lemmas

In this section, we first transform system (1.1) into a Liénard-like system, and then state some results which will be useful in subsequent sections.

We transform system (1.1), suppose that the assumption (1.6) is satisfied, we only discuss the case $q_2(x) < 0$, $q_3(x) > 0$ for all x, the other case (i.e., $q_2(x) > 0$, $q_3(x) < 0$ for all x) can be considered in a similar way. By using the substitution $u = \rho(y)$, where $\rho(y)$ is given in (1.6), from (1.1), we have

$$\begin{aligned}\dot{y} &= p_3(y)q_3(x)x + p_3(y)q_4(x)u,\\ \frac{d}{du}\bigl[\rho^{-1}(u)\bigr]\dot{u} &= p_3\bigl(\rho^{-1}(u)\bigr)q_3(x)x + p_3\bigl(\rho^{-1}(u)\bigr)q_4(x)u,\end{aligned}$$

we change system (1.1) into

$$\begin{aligned}\dot{x} &= \rho^{-1}(u)p_2\bigl(\rho^{-1}(u)\bigr)q_2(x),\\ \dot{u} &= \rho'\bigl(\rho^{-1}(u)\bigr)p_3\bigl(\rho^{-1}(u)\bigr)q_3(x)x + \rho'\bigl(\rho^{-1}(u)\bigr)p_3\bigl(\rho^{-1}(u)\bigr)q_4(x)u,\end{aligned} \tag{2.1}$$

by assumption (1.6), $\rho^{-1}(u)p_2(\rho^{-1}(u))q_2(x)$ and $-u$ have the same sign, it is easy to see that the qualitative behavior of (1.1) is identical to that of the system

$$\begin{aligned}\dot{x} &= -u,\\ \dot{u} &= -\frac{\rho'(\rho^{-1}(u))p_3(\rho^{-1}(u))q_3(x)}{\rho_1(u)p_2(\rho^{-1}(u))q_2(x)}x - \frac{\rho'(\rho^{-1}(u))p_3(\rho^{-1}(u))q_4(x)}{\rho_1(u)p_2(\rho^{-1}(u))q_2(x)}u,\end{aligned} \tag{2.2}$$

where $\rho_1(u) = \rho^{-1}(u)/u$ for $u \neq 0$, $\rho_1(0) = \lim_{u\to 0}\rho^{-1}(u)/u$. From (2.2), we get

$$\ddot{x} + \frac{\rho'(\rho^{-1}(-\dot{x}))p_3(\rho^{-1}(-\dot{x}))q_4(x)}{\rho_1(-\dot{x})p_2(\rho^{-1}(-\dot{x}))q_2(x)}\dot{x} - \frac{\rho'(\rho^{-1}(-\dot{x}))p_3(\rho^{-1}(-\dot{x}))q_3(x)}{\rho_1(-\dot{x})p_2(\rho^{-1}(-\dot{x}))q_2(x)}x = 0. \tag{2.3}$$

It follows from (2.3) that

$$\frac{d}{dt}\left[\int_0^x \frac{q_4(s)}{q_2(s)}\,ds - \int_0^{-\dot x} \frac{\rho_1(s)p_2(\rho^{-1}(s))}{\rho'(\rho^{-1}(s))p_3(\rho^{-1}(s))}\,ds\right] - \frac{q_3(x)}{q_2(x)}x = 0.$$

Letting

$$\psi(y) = \int_0^y \frac{\rho_1(s)p_2(\rho^{-1}(s))}{\rho'(\rho^{-1}(s))p_3(\rho^{-1}(s))}\,ds$$

and introducing the substitution

$$z = -\psi(-\dot x) + \int_0^x \frac{q_4(s)}{q_2(s)}\,ds.$$

We change system (2.3) into

$$\dot x = -\psi^{-1}\left(\int_0^x \frac{q_4(s)}{q_2(s)}\,ds - z\right), \qquad \dot z = \frac{q_3(x)}{q_2(x)}x. \tag{2.4}$$

If we let ϕ denote ψ^{-1} and replace x and z by $-x$ and $-z$, respectively, then we obtain

$$\dot x = \phi\big(z - F(x)\big), \qquad \dot z = -g(x), \tag{2.5}$$

where

$$F(x) = -\int_0^{-x} \frac{q_4(s)}{q_2(s)}\,ds \quad \text{and} \quad g(x) = -\frac{q_3(-x)}{q_2(-x)}x.$$

Lemma 2.1. *Under the assumption* (1.6), *the qualitative behavior of* (1.1) *is the same as that of* (2.5).

In the following, we shall present the basic assumptions and auxiliary lemmas. We assume that

(C_1) $F(x)$ and $g(x)$ are continuous on $\mathbf{R}$ with $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$ and $\phi(u)$ is continuous differentiable and strictly increasing with $\phi(0) = 0$ and $\phi(\pm\infty) = \pm\infty$.

(C_2) For any fixed number $k > 0$, there exists $M(k) > 0$ with $M(k) \equiv k$ for $0 < k \leqslant 1$ such that

$$\big|\phi(ku)\big| \leqslant M(k)\phi\big(|u|\big) \quad \text{for all } u.$$

Sometimes, we only need the condition

(C_2') For any fixed $k \in (0,\ 1]$ and $u \in \mathbf{R}$,

$$\bigl|\phi(ku)\bigr| \leqslant k\phi\bigl(|u|\bigr).$$

Lemma 2.2 (see [12, Proposition 3.1]). *If (C_1) is satisfied, then for any initial point $p(x_0,\ z_0)$, (2.5) has a unique orbit passing through p.*

We call the curve L: $z = F(x)$ the characteristic curve of (2.5), we denote

$$L^+ = \bigl\{\bigl(x, F(x)\bigr):\ x > 0\bigr\} \quad \text{and} \quad L^- = \bigl\{\bigl(x, F(x)\bigr):\ x < 0\bigr\}.$$

Let $G(x) = \int_0^x g(s)\,ds$. If $x > 0$, then we set

$$u = u_1(x) = G(x), \quad u \in \bigl(0, G(+\infty)\bigr), \tag{2.6}$$

the inverse function of which is denoted by $x = x_1(u)$. Replacing $x(> 0)$ in $F(x)$ by $x = x_1(u)$, we have

$$F_1(u) = F\bigl(x_1(u)\bigr), \quad u \in \bigl(0, G(+\infty)\bigr). \tag{2.7}$$

Similarly, if $x < 0$, then we write

$$u = u_2(x) = G(x), \quad u \in \bigl(0, G(-\infty)\bigr), \tag{2.8}$$

whose inverse function is given by $x = x_2(u)$. Thus, substituting $x = x_2(u)$ in $F(x)$ if $x < 0$, we obtain

$$F_2(u) = F\bigl(x_2(u)\bigr), \quad u \in \bigl(0, G(-\infty)\bigr). \tag{2.9}$$

Therefore, Eqs. (2.5) in the cases $x > 0$ and $x < 0$ are equivalent to the following two equations, respectively:

$$\frac{du}{dz} = -\phi\bigl(z - F_1(u)\bigr), \quad u \in \bigl(0, G(+\infty)\bigr), \tag{2.10}$$

$$\frac{du}{dz} = -\phi\bigl(z - F_2(u)\bigr), \quad u \in \bigl(0, G(-\infty)\bigr). \tag{2.11}$$

Now we introduce the condition (C_3). The system (2.5) is called to satisfy the condition (C_3) if the following condition hold:

$$F_1(u) \equiv F_2(u) \quad \text{for } u \in \bigl(0, \min\bigl\{G(+\infty), G(-\infty)\bigr\}\bigr),$$

where $F_1(u)$ and $F_2(u)$ are given in (2.10) and (2.11).

If the condition (C_3) is true, then Eqs. (2.10) and (2.11) are identical in $(0, \min\{G(+\infty), G(-\infty)\})$, employing an argument similar to that in [4,10], we have the following lemma which shows that the orbit of (2.5) have deformed mirror symmetry about the z-axis.

Lemma 2.3. *Suppose that the conditions (C_1) and (C_3) are satisfied, $G(+\infty) = G(-\infty)$. If an orbit of (2.5) starting from $A = (0, z_A)$ $(z_A > 0)$ passes through a point $B = (0, z_B)$ $(z_B < 0)$, then it reaches the point A again.*

3. The oscillation and the global center for system (2.5)

First, we give the result on the oscillation of all solutions for (2.5). A solution $(x(t), z(t))$ of (2.5) is oscillatory if there are two sequences $\{t_n\}$ and $\{\tau_n\}$ tending monotonically to $+\infty$ such that $x(t_n) = 0$ and $z(\tau_n) = 0$ for every $n \geqslant 1$. As is usual in the investigation of oscillation properties, by solution, we mean those which are defined in the future. Some attempts have been made to find necessary as well as sufficient conditions on F, ϕ and g for solutions of (2.5) to be continued in the future [7].

The system (2.5) is said to satisfy (C_4^+) if one of the following conditions holds:

$(C_4^+)_1$ there exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \leqslant 0$ for each n;

$(C_4^+)_2$ there exist constants $a > 0$ and $\beta > \frac{1}{4}$ such that $F(x) > 0$ for $0 < x \leqslant a$ and

$$\int_0^x \frac{g(s)}{\phi(F(s))}\,ds \geqslant \beta F(x) \quad \text{for } 0 < x \leqslant a.$$

The system (2.5) is said to satisfy (C_4^-) if one of the following conditions holds:

$(C_4^-)_1$ there is a negative increasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \geqslant 0$ for each n;

$(C_4^-)_2$ there are constants $b < 0$ and $\beta > \frac{1}{4}$ such that $F(x) < 0$ for $b \leqslant x < 0$ and

$$\int_0^x \frac{g(s)}{\phi(-F(s))}\,ds \geqslant -\beta F(x) \quad \text{for } b \leqslant x < 0.$$

The system (2.5) is said to satisfy the condition (C_4) if both (C_4^+) and (C_4^-) hold.

Lemma 3.1. *Assume that the conditions* (C_1) *and* (C_2') *hold. If the condition* (C_4^+) $((C_4^-))$ *is satisfied, then every positive semiorbit of* (2.5) *passing through* $(x_0, F(x_0))$ *with* $x_0 > 0$ $(x_0 < 0)$ *will intersect the negative* z*-axis* (*the positive* z*-axis*).

Proof. Suppose that (C_4^+) holds. Then it is easy to see that if the positive semiorbit $O^+(p)$ with $p_0 = (x_0, F(x_0)) \in L^+$ intersects the negative z-axis, then $O^+(p)$ also intersects the negative z-axis for every $p = (x, F(x)) \in L^+$ with $x > x_0$. Thus, in order to prove the conclusion, we only have to prove that there exists a sequence $\{p_n\} \subset L^+$ such that $p_n \to 0$ and $O^+(p_n)$ intersects the negative z-axis for every n.

If $(C_4^+)_1$ is true, then there is a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and $F(x_n) \leqslant 0$ for $n \geqslant 1$. Since (2.5) has no vertical asymptote, $O^+(p_n)$ must intersect the negative z-axis. So the conclusion in this situation is proved.

Suppose that $F(x) > 0$ for $0 < x \leqslant a$ and $(C_4^+)_2$ is satisfied. If the conclusion is false, then there exists a point $p_0(x_0, F(x_0))$ with $x_0 > 0$ such that $O^+(p_0)$ does not intersect the negative z-axis. Then $O^+(p_0)$ must be contained in the first quadrant and its ω-limit set is the origin. Let $(x(t), z(t))$ be the solution of (2.5) passing through p_0. Then such a solution defines a function $z = z(x)$ on $0 \leqslant x \leqslant x_0$ which is a solution on $0 < x \leqslant x_0$ of the following equation

$$\frac{dz}{dx} = -\frac{g(x)}{\phi(z - F(x))}. \tag{3.1}$$

Clearly, $0 < z(x) < F(x)$ for $0 < x < x_0$. Without loss of generality, we may assume that $x_0 \leqslant a$. Since $z = z(x)$ is under L^+ and $\phi(u)$ is strictly increasing, we have

$$0 > \phi\big(z(x) - F(x)\big) \geqslant \phi\big(-F(x)\big) \geqslant -\phi\big(F(x)\big),$$

where the last inequality follows from (C_2') with $k = 1$. Therefore, integrating (3.1) from 0 to x, we obtain that

$$z(x) = -\int_0^x \frac{g(s)}{\phi(z(s) - F(s))}\, ds \geqslant \int_0^x \frac{g(s)}{\phi(F(s))}\, ds \geqslant \beta F(x) \quad \text{for } 0 < x \leqslant x_0. \tag{3.2}$$

Here the last inequality follows from condition $(C_4^+)_2$. From (3.2), we have $\beta < 1$.

Let $\beta_1 = 1$ and $\beta_{n+1} = 1 - \beta/\beta_n$. Then from (3.2), we get that

$$0 < F(x) - z(x) < \beta_1 F(x) \quad \text{for } 0 < x \leqslant x_0.$$

By induction and the same method as the proof of (3.2), we can prove that

$$0 < F(x) - z(x) \leqslant \beta_n F(x) \quad \text{for } 0 < x \leqslant x_0 \text{ and each } n.$$

Thus, $\{\beta_n\}$ is a positive decreasing sequence which must converge to a real number λ. From $\beta_{n+1} = 1 - \beta/\beta_n$ and $\frac{1}{4} < \beta < 1$, we obtain that $\lambda = 1 - \beta/\lambda$. Therefore, λ is a complex number, a contradiction. This proves the lemma. □

The system (2.5) is said to satisfy (C_5^+) if one of the following two conditions holds:

$(C_5^+)_1$ $\limsup_{x\to+\infty} F(x) > -\infty$;

$(C_5^+)_2$ there exist constants $N > 0$ and $\beta > \frac{1}{4}$ such that $F(x) < 0$ for all $x \geqslant N$ and for any $b \geqslant N$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{g(s)}{\phi(-F(s))}\, ds \geqslant -\beta F(x) \quad \text{for all } x \geqslant \bar{b}.$$

The system (2.5) is said to satisfy (C_5^-) if one of the following two conditions holds:

$(C_5^-)_1$ $\liminf_{x\to-\infty} F(x) < +\infty$;

$(C_5^-)_2$ There exist constants $N > 0$ and $\beta > \frac{1}{4}$ such that $F(x) > 0$ for $x \leqslant -N$ and for any $b > N$, there exists $\bar{b} > b$ satisfying

$$\int_{-b}^x \frac{g(s)}{\phi(F(s))}\, ds \geqslant \beta F(x) \quad \text{for all } x \leqslant -\bar{b}.$$

The system (2.5) is said to satisfy (C_5) if both (C_5^+) and (C_5^-) hold.

Lemma 3.2 (see [12, Theorem 3.1]). *Suppose that the conditions* (C_1), (C_2), *and* (C_5^+) *hold. Then every positive semiorbit of* (2.5) *departing from* $D_1 = \{(x,z)\colon x \geqslant 0,\ z > F(x)\}$ *intersects the characteristic curve* L^+ *if and only if*

$$\limsup_{x\to+\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_-(s))}\,ds + F(x)\right] = +\infty, \tag{3.3}$$

where $F_-(x) = \max\{0, -F(x)\}$.

Lemma 3.3 (see [12, Theorem 3.2]). *Suppose that the conditions* (C_1), (C_2), *and* (C_5^-) *hold. Then every positive semiorbit of* (2.5) *starting from* $D_3 = \{(x,z)\colon x \leqslant 0,\ z < F(x)\}$ *intersects the characteristic curve* L^- *if and only if*

$$\limsup_{x\to-\infty}\left[\int_0^x \frac{g(s)}{\phi(1+F_+(s))}\,ds - F(x)\right] = +\infty, \tag{3.4}$$

where $F_+(x) = \max\{0, F(x)\}$.

Remark 3.1. If $\liminf_{x\to+\infty} F(x) > -\infty$, then (3.3) is equivalent to

$$\limsup_{x\to+\infty}\left[\int_0^x g(s)\,ds + F(x)\right] = +\infty, \tag{3.5}$$

and if $\limsup_{x\to-\infty} F(x) < +\infty$, then (3.4) is equivalent to

$$\limsup_{x\to-\infty}\left[\int_0^x g(s)\,ds - F(x)\right] = +\infty. \tag{3.6}$$

It follows from the proof of sufficiency of Theorem 3.1 in [12] that the conclusion of Lemma 3.2 is also true if $\liminf_{x\to+\infty} F(x) > -\infty$, (3.3) is replaced by (3.5) and the condition (C_2) is removed. Similarly, suppose that $\limsup_{x\to-\infty} F(x) < +\infty$. Then the result of Lemma 3.3 also holds when (3.4) is replaced by (3.6) and the condition (C_2) is removed. From the proof of necessity of Theorem 3.1 in [12], we know that if

$$\lim_{x\to+\infty}\sup\left[\int_0^x \frac{g(s)}{\phi(1+F_+(s))}\,ds - F(x)\right] < +\infty, \tag{3.7}$$

then there exists a point $p \in D_2 = \{(x,z)\colon x \geqslant 0,\ z < F(x)\}$ such that $O^-(p)$ does not intersect L^+. Similarly, if

$$\lim_{x\to-\infty}\sup\left[\int_0^x \frac{g(s)}{\phi(1+F_-s))}\,ds + F(x)\right] < +\infty, \tag{3.8}$$

then there exists a point $p \in D_4 = \{(x, z)\colon x \leqslant 0, z > F(x)\}$ such that $O^-(p)$ does not intersect L^-.

Theorem 3.4. *Assume that the conditions* (C_1), (C_2), (C_4), *and* (C_5) *are satisfied. Then all nontrivial solutions of* (2.5) *are oscillatory if and only if* (3.3) *and* (3.4) *hold.*

Proof. From Lemmas 3.2 and 3.3, the necessity is obvious. Now we give the proof of sufficiency.

Let $p \in \mathbf{R}^2$ with $p \neq 0$. Then it follows from Lemmas 3.2 and 3.3 that $O^+(p)$ must intersect the characteristic curve L, where we have used the fact that (2.5) has no any asymptote. Therefore, in order to prove the conclusion, we only have to prove that if $p \in L^+(L^-)$ then $O^+(p)$ must intersect $L^+(L^-)$ again. Lemma 3.1 implies that $O^+(p)$ must intersect the negative z-axis (the positive z-axis). Applying Lemma 3.4 (Lemma 3.3), we know that $O^+(p)$ will intersect $L^-(L^+)$. Using the fact that (2.5) has no any asymptote once again, we obtain that $O^+(p)$ intersects $L^+(L^-)$ again. This implies all positive semiorbits spiral around the origin. This completes the proof. □

Remark 3.2. If $\phi(u) \equiv u$, then Theorem 3.4 gives the results of [5].

Next, we give the result on a global center for (2.5). The origin is called to be a global center for (2.5) if all orbits of (2.5) are closed curves surrounding it.

If the condition (C_3) is true and $G(+\infty) = G(-\infty)$, then Eqs. (2.10) and (2.11) are identical. Therefore, $O^+(p)$ is closed as long as $O^+(p)$ spirals around the origin. The global center result can be immediately obtained from Theorem 3.4 and Lemma 2.3.

Theorem 3.5. *Suppose that the conditions* (C_1), (C_2), (C_3), (C_4), *and* (C_5) *are satisfied,* $G(+\infty) = G(-\infty)$. *Then the origin of* (2.5) *is a global center if and only if* (3.3) *and* (3.4) *hold.*

Remark 3.3. If $\phi(u) \equiv u$, then Theorem 3.5 reduces to the results of [10].

Remark 3.4. If $\lim_{x\to+\infty} \inf F(x) > -\infty$ and $\lim_{x\to-\infty} \sup F(x) < +\infty$ and we replace (C_2), (3.3), and (3.4) by (C_2'), (3.5), and (3.6), respectively, then the conclusion of Theorem 3.5 is also true.

Example 3.1. Consider the system with zero diagonal coefficient

$$\begin{aligned} \dot{x} &= -\left(1+x^2\right)\left[1+\left(1+y^2\right)^{3/2}\right]y, \\ \dot{y} &= \left(1+x^2\right)\left(1+y^2\right)^{3/2}x + 2\left(1+x^2\right)x\sin x^2\left(1+y^2\right)^{3/2}, \end{aligned} \tag{3.9}$$

in which $p_2(y) = 1 + (1+y^2)^{3/2}$, $p_3(y) = p_4(y) = (1+y^2)^{3/2}$, $q_2(x) = -(1+x^2)$, $q_3(x) = (1+x^2)$, and $q_4(x) = 2(1+x^2)x\sin x^2$. Thus,

$$u = \Psi(y) = \int_0^y \frac{p_2(s)}{p_3(s)}\,ds = y + \frac{y}{\sqrt{1+y^2}},$$

$$F(x) = -\int_0^{-x} \frac{q_4(s)}{q_2(s)}\,ds = \int_0^{-x} 2s\sin s^2\,ds = 1-\cos x^2,$$
$$g(x) = -\frac{q_3(-x)}{q_2(-x)}x = x.$$

It is easy to see that $y = \Psi^{-1}(u) := \phi(u)$ satisfied (C_2'). Obviously, $\phi(u)$, $F(x)$, and $g(x)$ satisfy the conditions (C_1), (C_3), $(C_4^-)_1$, and (C_5). Moreover, (3.3) and (3.4) are immediately satisfied. In order to prove that the origin of (3.9) is a global center, it remains to check $(C_4^+)_2$

$$\lim_{x\to 0^+} \frac{\int_0^x \frac{s}{\phi(F(s))}\,ds}{F(x)} = \lim_{x\to 0^+} \frac{1}{2\sin x^2 \phi(F(x))} = +\infty,$$

which shows that there exist numbers $a > 0$ and $\beta > \frac{1}{4}$ such that

$$\int_0^x \frac{s^3}{\phi(F(s))}\,ds \geqslant \beta F(x) \quad \text{for } 0 < x \leqslant a.$$

In other words, $(C_4^+)_2$ is true. By Remark 3.4, the origin of (3.9) is a global center.

4. Existence and uniqueness of nontrivial periodic solutions

Throughout this section, we assume that $\phi(u)$ is an odd function. The system (2.5) is said to satisfy (C_6^+) if one of the following conditions holds:

$(C_6^+)_1$ there exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \geqslant 0$ for each n;

$(C_6^+)_2$ there exist constants $a > 0$ and $\beta > \frac{1}{4}$ such that $F(x) < 0$ for $0 < x \leqslant a$ and

$$\int_0^x \frac{g(s)}{\phi(-F(s))}\,ds \geqslant -\beta F(x) \quad \text{for } 0 < x \leqslant a.$$

The system (2.5) is said to satisfy (C_6^-) if one of the following conditions holds:

$(C_6^-)_1$ there is a negative increasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) \leqslant 0$ for each n;

$(C_6^-)_2$ there are constants $b < 0$ and $\beta > \frac{1}{4}$ such that $F(x) > 0$ for $b \leqslant x < 0$ and

$$\int_0^x \frac{g(s)}{\phi(F(s))}\,ds \geqslant \beta F(x) \quad \text{for } b \leqslant x < 0.$$

The system (2.5) is said to satisfy the condition (C_6) if both (C_6^+) and (C_6^-) hold.

Lemma 4.1 (see [12, Lemma 4.1]). *Assume that the conditions* (C_1) *and* (C_2) *are true. Then*

(i) *if* (C_6^+) *holds, then for any* $p = (x_0, z_0) \in L^+$, *the negative semiorbit* $O^-(p)$ *must intersect the positive* z*-axis at* $(0, z_p)$ *with* $z_p > 0$;
(ii) *if* (C_6^-) *holds, then for any* $p = (x_0, z_0) \in L^-$, *the negative semiorbit* $O^-(p)$ *must intersect the negative* z*-axis at* $(0, z_p)$ *with* $z_p < 0$.

Remark 4.1. The condition $(C_5^-)_2$ of [12] should be $(C_6^-)_2$ in this paper (i.e., the inequality sign $\leqslant$ in the condition $(C_5^-)_2$ of [12] should be $\geqslant$).

Theorem 4.2. *Suppose that* (C_1), (C_2), (C_4^+), (C_5^-), (C_6^+), (3.4), (3.5), *and* (3.7) *hold. If* $F_1(u) \leqslant F_2(u)$ *for* $0 < u \ll 1$, *then* (2.5) *has at least one nontrivial periodic solution.*

Proof. Since (C_1), (C_2), and (C_4^+) are satisfied, it follows from Lemma 3.1 that every positive semiorbit of (2.5) passing through $(x_0, F(x_0))$ with $x_0 > 0$ will intersect the negative z-axis. Thus, for $-z_0 > 0$ sufficiently small, by Lemma 4.1, (2.10) has a solution $u = u_1(z)$ with $u_1(z_0) = 0$ which is defined on $[z_0, z_1]$ with $z_1 > 0$ and $u(z_1) = 0$. Let $p_0 = (0, z_0)$. Then, by Lemma 3.3, the positive semiorbit $O^+(p_0)$ will meet L^-. We assert that for sufficiently small $-z_0$, $O^+(p_0)$ must intersect the positive z-axis at $(0, z_2)$ with $z_2 \geqslant z_1$. Let $u = u_2(z)$ be the solution of (2.11) with $u_2(z_0) = 0$. It follows from the condition $F_1(u) \leqslant F_2(u)$ $(0 < u \ll 1)$ and the proof of Proposition 4.1 of [12] that $u_1(z) \leqslant u_2(z)$ if $z \geqslant z_0$ and z is in the common existence interval of $u_1(z)$ and $u_2(z)$. Since $u_2(0) \geqslant u_1(0) > 0$, thus, $O^+(p)$ meets the negative x-axis and hence intersects the positive z-axis at $p_2(0, z_2)$. Moreover, $z_2 \geqslant z_1$. Let $p_1 = (0, z_1)$ and $\widehat{p_1 p_2}$ be the orbit arc of $O^+(p_1)$ and $\overline{p_2 p_1}$ be the closed segment from p_2 to p_1. Then $C = \widehat{p_1 p_2} \cup \overline{p_2 p_1}$ is a Jordan curve and the exterior of C is positively invariant. By (3.7) and Remark 3.1, there exists a point p in the negative z-axis such that $O^-(p)$ does not intersect L^+. Such a point p must be in the exterior of C. Applying Lemma 3.3, we conclude that $O^+(p)$ meets L^- and therefore enters into D_1. From (3.7), it is easy to see that $F(x) \geqslant -M$ for some $M > 0$. Thus, by (3.5) and Remark 3.1, we know that $O^+(p)$ must intersect L^+ and hence meets the negative z-axis at q which lies above p. Therefore, $O^+(p)$ is bounded and spirals around the origin as in Fig. 1. The Poincare–Bendixson theorem implies that $\omega(p)$ is a nontrivial periodic solution. The proof is complete. □

In a similar way, we can prove the following

Theorem 4.3. *Suppose that* (C_1), (C_2), (C_4^+), (C_5^+), (C_6^+), (3.3), (3.6), *and* (3.8) *hold. If* $F_1(u) \leqslant F_2(u)$ *for* $0 < u \ll 1$, *then* (2.5) *has at least one nontrivial periodic solution.*

Theorem 4.4. *Let* (C_1) *and* (C_2') *hold. Assume that* F *is continuously differentiable with* $F'(x) := f(x) > 0$ *for* $x \notin [a, b]$, $f(0) < 0$ *and* $F(b) - F(a) > 0$, *where* $a < 0 < b$. *If*

$$\lim_{|x| \to \infty} \sup\bigl[G(x) + F(x)\operatorname{sgn} x\bigr] = +\infty, \tag{4.1}$$

then (2.5) *has at least one nontrivial periodic solution.*

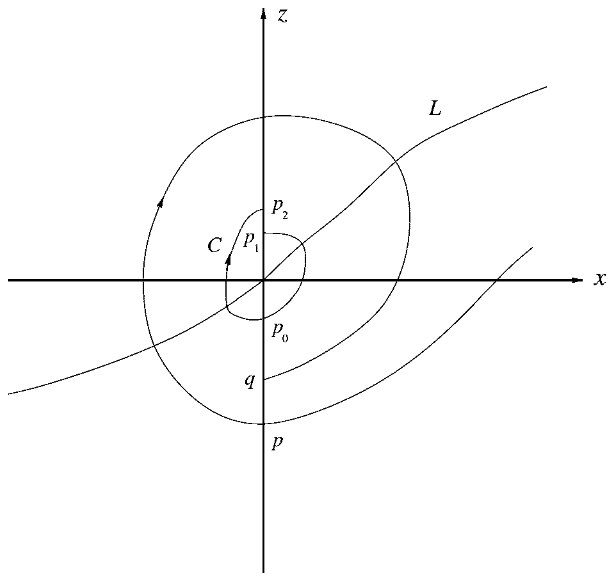

Fig. 1.

Proof. Under the assumptions of this theorem, it is not difficult to prove that all nontrivial solutions of (2.5) are oscillatory. Since $\lim_{|z|\to\infty} \frac{g(x)}{\phi(z-F(x))} = 0$ uniformly for all $x \in [a,b]$, for any $\varepsilon > 0$, there exists $M_0 > 0$ such that

$$\left|\frac{g(x)}{\phi(z - F(x))}\right| < \frac{\varepsilon}{b-a} \quad \text{for all } x \in [a,b] \text{ and } |z| \geqslant M_0. \tag{4.2}$$

Furthermore, there is $M_1 > M_0$ such that for $|y_0| > M_1$ the solution $z = z(x)$ of (3.1) with $z(0) = y_0$ is defined on $[a,b]$ and $|z(x)| \geqslant M_0$ for all $x \in [a,b]$. Thus, for any $x_1, x_2 \in [a,b]$, it follows from (4.2) that

$$\left|z(x_1) - z(x_2)\right| < \varepsilon. \tag{4.3}$$

Let $\Phi(y) = \int_0^y \phi(\sigma)\,d\sigma$. Then $\Phi(y)$ is an even function. Define

$$V(x,z) = \Phi\bigl(z - F(x)\bigr) + G(x). \tag{4.4}$$

Thus, along a solution of (2.5), we have

$$\frac{dV}{dt} = -f(x)\phi^2\bigl(z - F(x)\bigr). \tag{4.5}$$

Let $p_0 = (0, y_0)$ with $y_0 > 0$. Then $O^+(p_0)$ crosses the negative z-axis at $q_0(0, y_0^*)$. If $\lim_{y_0\to+\infty} y_0^* = y^* > -\infty$, then $O^-(q^*)$ does not intersect L^+ where $q^* = (0, y^*)$. Therefore, $O^+(q^*)$ is bounded. From $f(0) < 0$ and (4.5), we obtained that the origin is a repeller. The Poincare–Bendixson theorem implies that $\omega(q^*)$ is a nontrivial periodic solution. Suppose that $y^* = -\infty$ and choose $\varepsilon = \frac{1}{2}(F(b) - F(a))$ in (4.2) and (4.3). Then, there is $M_2 > M_1 + F(b) - F(a)$ such that $y_0^* < -(M_1 + F(b) - F(a))$ as long as $y_0 > M_2$. Now, we fix the point $p_0 = (0, y_0)$ with $y_0 > M_2$. The remain proof makes use of the following Fig. 2.

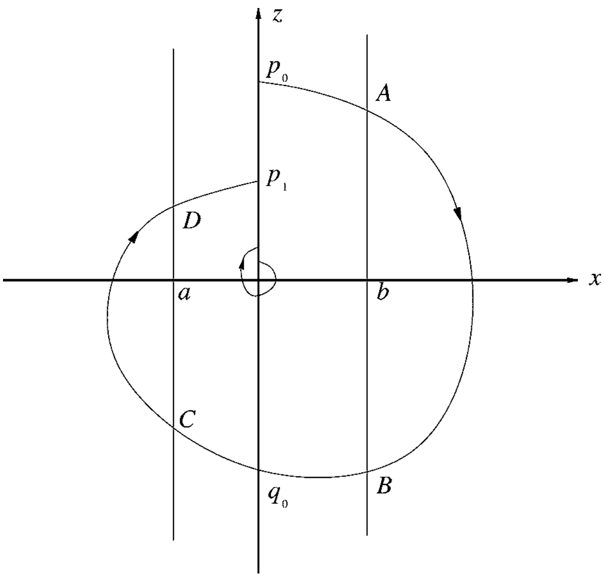

Fig. 2.

The points A, B, C, D, and P_1 have the coordinates (b, y_b^+), (b, y_b^-), (a, z_a^-), (a, z_a^+), and $(0, y_1)$, respectively. we shall prove that $y_1 < y_0$. Since $f(x) > 0$ for $x \notin [a, b]$, it follows from (4.5) that

$$\Phi\big(y_b^+ - F(b)\big) > \Phi\big(y_b^- - F(b)\big), \tag{4.6}$$

$$\Phi\big(z_a^- - F(a)\big) > \Phi\big(z_a^+ - F(a)\big). \tag{4.7}$$

We note that Φ is an even function and is strictly increasing on $[0, \infty)$. Therefore, (4.6) and (4.7) imply that

$$y_b^+ - F(b) > F(b) - y_b^-, \qquad F(a) - z_a^- > z_a^+ - F(a).$$

That is,

$$y_b^+ > -y_b^- + 2F(b), \tag{4.8}$$

$$z_a^+ < -z_a^- + 2F(a). \tag{4.9}$$

By (4.3), we have

$$y_1 - z_a^+, \qquad y_b^- - z_a^-, \qquad y_b^+ - y_0 < \varepsilon \tag{4.10}$$

where $\varepsilon = \frac{1}{2}(F(b) - F(a))$. Combining (4.8), (4.9) with (4.10), we conclude that $y_1 < y_0$. Hence $O^+(p_0)$ is bounded and $\omega(p_0)$ is a nontrivial periodic solution. The proof is complete. □

In the following, we consider a special form of (2.5)

$$\dot{x} = \big(z - F(x)\big)^{2n+1}, \qquad \dot{z} = -g(x). \tag{4.11}$$

Theorem 4.5. *Let $g(x)$ and $f(x) := F'(x)$ be continuous for all x with the properties*:

(i) $xg(x) > 0$ *for* $x \neq 0$;
(ii) $F'(x) := f(x) > 0$ *for* $x \notin [\alpha, \beta]$, $f(x) < 0$ *for* $x \in [\alpha, \beta]$ *where* $\alpha < 0 < \beta$ *and* $F(b) - F(a) > 0$ *for some* $a < \alpha$ *and* $b > \beta$;
(iii) $G(\alpha) = G(\beta)$ *where* $G(x) = \int_0^x g(\sigma)\, d\sigma$;
(iv) $\lim_{|x| \to +\infty} [G(x) + F(x) \operatorname{sgn} x] = +\infty$.

Then (4.11) *has exactly one nontrivial periodic solution which is exponentially asymptotically, orbitally stable.*

Proof. The existence of a nontrivial periodic solution has been given in Theorem 4.4. It remains to prove the uniqueness.

Suppose that Γ: $x = x(t)$, $z = z(t)$ for $0 \leqslant t \leqslant T$ is any nontrivial periodic solution of (4.11) whose characteristic multiplier is

$$\gamma = -\int_0^T (2n+1)\bigl(z(t) - F\bigl(x(t)\bigr)\bigr)^{2n} f\bigl(x(t)\bigr)\, dt.$$

Let $\gamma_0 = \int_0^T (z(t) - F(x(t)))^{2n} f(x(t))\, dt$. Then, it suffices to prove that $\gamma_0 > 0$.

Defining the function

$$V(x, z) = \frac{(z - F(x))^{2n+2}}{2(n+1)} + G(x)$$

and restricting V on the periodic orbit Γ, we have

$$V(t) = \frac{(z(t) - F(x(t)))^{2n+2}}{2(n+1)} + G\bigl(x(t)\bigr).$$

Therefore,

$$\frac{dV}{dt} = -f\bigl(x(t)\bigr)\bigl(z(t) - F\bigl(x(t)\bigr)\bigr)^{4n+2}. \tag{4.12}$$

Choose a positive number $h < \min_{0 \leqslant t \leqslant T} V(t)$. Then we change the form of (4.12) into

$$\frac{1}{V - h}\frac{dV}{dt} + 2(n+1)\bigl(z - F(x)\bigr)^{2n} f(x) = \frac{2(n+1)(G(x) - h)(z - F(x))^{2n} f(x)}{V - h}. \tag{4.13}$$

Integrating (4.13) from 0 to T, we obtain that

$$\gamma_0 = \int_0^T \frac{(G(x(t)) - h)(z(t) - F(x(t)))^{2n} f(x(t))}{V(t) - h}\, dt. \tag{4.14}$$

In order to prove $\gamma_0 > 0$, we only have to show that we can choose a suitable number h such that

$$\bigl(G\bigl(x(t)\bigr) - h\bigr) f\bigl(x(t)\bigr) \geqslant 0 \quad \text{for all } t \in [0, T].$$

Assume that $V(t)$ attains the minimum value at $t = t_0 \in [0, T]$, then it follows from (4.12) that $f(x(t_0)) = 0$. We claim that $z(t_0) \neq F(x(t_0))$. In the following, we will prove our claim. Suppose the contrary, that is, $z(t_0) = F(x(t_0))$. Since the orbit Γ does not pass through the origin, $\dot{z}(t_0) = -g(x(t_0)) \neq 0$. Thus, $z(t) - F(x(t)) = z(t_0) + \dot{z}(t_0) \times (t - t_0) - F(x(t_0)) - f(x(t_0))\dot{x}(t_0)(t - t_0) + o(t - t_0) = -g(x(t_0))(t - t_0) + o(t - t_0)$. From the first equation of (4.11), we have

$$x(t) - x(t_0) = -\frac{(g(x(t_0)))^{2n+1}}{2(n+1)}(t - t_0)^{2n+2} + o\big((t - t_0)^{2n+2}\big).$$

Therefore, for $|t - t_0|$ sufficiently small, either $x(t) \geqslant x(t_0)$ or $x(t) \leqslant x(t_0)$. This implies that either $f(x(t)) \leqslant 0$ or $f(x(t)) \geqslant 0$ as long as $|t - t_0|$ is sufficiently small. From (4.12), we can conclude that in a neighborhood of t_0, $V(t)$ is strictly monotone. In other words, $V(t)$ cannot attains the minimum value at $t = t_0$, a contradiction arises. This proves our claim that $z(t_0) \neq F(x(t_0))$.

Let $h = G(x(t_0))$, by the above claim, we have $(z(t_0) - F(x(t_0)))^{2n+2} > 0$ and $h < V(t)$ for all t. Since $f(x(t_0)) = 0$, $x(t_0) = \alpha$ or β and $h = G(\alpha) = G(\beta)$. By the assumption (ii) and (iii) $f(x)(G(x) - h) > 0$ for any $x \neq \alpha, \beta$. It follows from (4.14) that $\gamma_0 > 0$. By Theorem 11.3 of [6, p. 256], the nontrivial periodic solution is exponentially, asymptotically, orbitally stable. This completes the proof. □

Theorem 4.6. *Let $F'(x) := f(x)$ and $g(x)$ be continuous for all x with the properties*:

(i) $xg(x) > 0$ *for* $x \neq 0$;
(ii) $f(x) > 0$ *for* $|x| > \Delta$, $xF(x) < 0$ *for* $|x| < \Delta$ *and* $F(\Delta) = F(-\Delta) = 0$;
(iii) $\lim_{|x|\to+\infty}[G(x) + F(x)\operatorname{sgn} x] = +\infty$.

Then (4.11) *has exactly one nontrivial periodic solution which is exponentially asymptotically, orbitally stable.*

Proof. The method of the following proof is due to Sansone (see [9] or [14]). The existence of periodic solution follows from Theorem 4.4. We only give the proof of the uniqueness.

Let

$$\lambda(x, z) = \frac{z^{2(n+1)}}{2(n+1)} + G(x).$$

Then, along a solution of (4.11), we have

$$\frac{d\lambda}{dt} = -g(x)\big[z^{2n+1} - \big(z - F(x)\big)^{2n+1}\big].$$

From the assumption (ii), it follows that

$$\frac{d\lambda}{dt} = -g(x)\big[z^{2n+1} - \big(z - F(x)\big)^{2n+1}\big] > 0 \quad \text{for all } |x| < \Delta. \tag{4.15}$$

Using (4.15), we can prove that the points $(-\Delta, 0)$ and $(\Delta, 0)$ must be in the interior of any periodic orbit. Suppose that (4.11) has two periodic orbits Γ_1 and Γ_2. Without loss of

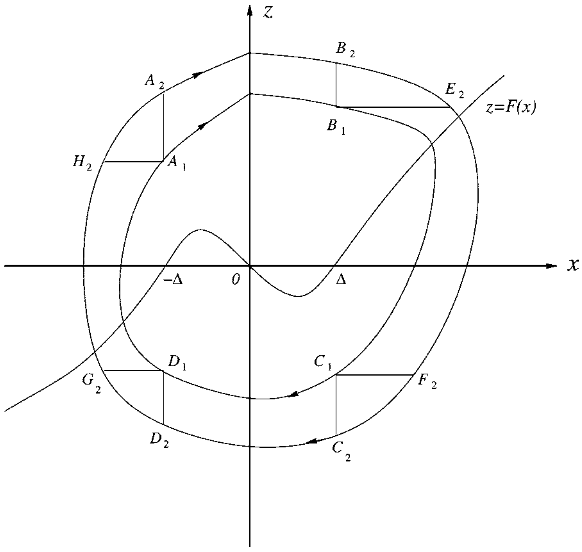

Fig. 3.

generality, we may assume that Γ_1 is in the interior Γ_2. It is easy to see that

$$0 = \oint\limits_{\Gamma_i} d\lambda = \oint\limits_{\Gamma_i} \left[z^{2n+1} - \left(z - F(x)\right)^{2n+1}\right] dz \quad (i = 1, 2). \tag{4.16}$$

In the following, we shall prove that

$$\oint\limits_{\Gamma_2} d\lambda < \oint\limits_{\Gamma_1} d\lambda. \tag{4.17}$$

(4.16) and (4.17) imply that the nontrivial periodic orbit of (4.11) is unique. The following proof will make use of the Fig. 3.

On the orbit arc $A_i B_i$, we have $z = z_i(x)$ for $|x| < \Delta$ which is the solution of (3.1). By (4.16), we can calculate that

$$\int\limits_{\widehat{A_i B_i}} d\lambda = \int\limits_{-\Delta}^{\Delta} \frac{-g(x)[(z_i(x))^{2n+1} - (z_i(x) - F(x))^{2n+1}]}{(z_i(x) - F(X))^{2n+1}} \, dx \quad \text{for } i = 1, 2.$$

Thus,

$$\int\limits_{\widehat{A_1 B_1}} d\lambda - \int\limits_{\widehat{A_2 B_2}} d\lambda$$

$$= -\int\limits_{-\Delta}^{\Delta} \frac{g(x)[(z_1(x)(z_2(x) - F(x)))^{2n+1} - (z_2(x)(z_1(x) - F(x)))^{2n+1}]}{[(z_1(x) - F(x))(z_2(x) - F(x))]^{2n+1}} \, dx. \tag{4.18}$$

Since $z_1(x)(z_2(x) - F(x)) - z_2(x)(z_1(x) - F(x)) = F(x)(z_2(x) - z_1(x))$, we have $[z_1(x)(z_2(x) - F(x)) - z_2(x)(z_1(x) - F(x))]\operatorname{sgn} x > 0$ for $0 < |x| < \Delta$. Together this inequality and (4.18), we have

$$\int_{\widehat{A_1B_1}} d\lambda > \int_{\widehat{A_2B_2}} d\lambda. \tag{4.19}$$

Similarly, we can prove that

$$\int_{\widehat{C_1D_1}} d\lambda > \int_{\widehat{C_2D_2}} d\lambda. \tag{4.20}$$

On the orbit arc $\widehat{B_1C_1}$ we have $x = x_1(z)$ and on the orbit arc $\widehat{E_2F_2}$ we have $x = x_2(z)$. From (4.16), we obtain that

$$\int_{\widehat{B_1C_1}} d\lambda = \int_{z_{B_1}}^{z_{C_1}} \bigl[z^{2n+1} - \bigl(z - F\bigl(x_1(z)\bigr)\bigr)^{2n+1}\bigr]\,dz \quad \text{and}$$

$$\int_{\widehat{E_2F_2}} d\lambda = \int_{z_{E_2}}^{z_{F_2}} \bigl[z^{2n+1} - \bigl(z - F\bigl(x_2(z)\bigr)\bigr)^{2n+1}\bigr]\,dz,$$

where $z_{C_1} = z_{F_2} := z_1$ and $z_{B_1} = z_{E_2} := z_2$. Obviously, $x_2(z) > x_1(z)$ for $z_1 \leqslant z \leqslant z_2$. The assumption (ii) implies that $F(x)$ is strictly increasing if $|x| > \Delta$. Hence $F(x_2(z)) > F(x_1(z))$ for all $z \in (z_1, z_2)$.

$$\int_{\widehat{B_1C_1}} d\lambda - \int_{\widehat{E_2F_2}} d\lambda = \int_{z_2}^{z_1} \bigl[\bigl(z - F\bigl(x_2(z)\bigr)\bigr)^{2n+1} - \bigl(z - F\bigl(x_1(z)\bigr)\bigr)^{2n+1}\bigr]\,dz > 0,$$

that is,

$$\int_{\widehat{B_1C_1}} d\lambda > \int_{\widehat{E_2F_2}} d\lambda. \tag{4.21}$$

In a similar way, we can show that

$$\int_{\widehat{D_1A_1}} d\lambda > \int_{\widehat{G_2H_2}} d\lambda. \tag{4.22}$$

Since $x[z^{2n+1} - (z - F(x))^{2n+1}] > 0$ for all $|x| > \Delta$, from (4.15), we conclude that

$$\int_L d\lambda < 0, \tag{4.23}$$

where $L = \widehat{B_2E_2} \cup \widehat{F_2C_2} \cup \widehat{D_2G_2} \cup \widehat{H_2A_2}$. From the inequalities (4.19) to (4.23), we deduce that (4.17). The proof is complete. $\square$

In the following, we apply the results in this section to the systems (1.7) and (1.8).

Rewriting (1.7), we have

$$\dot{x} = \left[y - \left(x^3 - x\right)\right]^3, \qquad \dot{y} = -x. \tag{4.24}$$

It is not difficult to show that (4.24) satisfies all properties of Theorem 4.4 or Theorem 4.5. Therefore, (4.24) has a unique limit cycle.

If the two sides of (1.8) are divided by $\dot{x}^{-2/3}$, we have

$$\dot{x}^{-2/3}\ddot{x} + 3\left(x^2 - 1\right)\dot{x} + 3x = 0,$$

that is,

$$\frac{d}{dt}\left[\dot{x}^{1/3} + \left(\frac{x^3}{3} - x\right)\right] + x = 0.$$

Let

$$z = \dot{x}^{1/3} + \left(\frac{x^3}{3} - x\right).$$

Then

$$\dot{x} = \left[z - \left(\frac{x^3}{3} - x\right)\right]^3, \qquad \dot{z} = -x. \tag{4.25}$$

Applying Theorem 4.4 or Theorem 4.5 to (4.25), we immediately obtain that (4.25) has a unique limit cycle.

5. Brief discussion

Krechetov [8] studied the global asymptotic behavior of solutions of system (1.1) by using the Lyapunov function method, and he gave necessary and sufficient conditions for the zero solution of (1.1) to be globally asymptotically stable under the main condition (1.2) and some other assumptions. In paper [12] the authors first introduced the transformation techniques to investigate the global asymptotic stability of system (1.3), the special case of system (1.1) with $p_3(y) \equiv p_4(y)$. Under the condition (1.4), they transformed system (1.3) into the generalized Liénard system (1.5) without the assumption (1.2) and obtained necessary and sufficient conditions for the zero solution of (1.3) (respectively (1.5)) to be globally asymptotically stable. Moreover, they also found conditions for deciding whether all positive (respectively negative) nontrivial orbit of (1.5) intersect the characteristic curve $z = F(x)$ and obtained sufficient conditions under which there is no homoclinic orbit for (1.5).

Motivated by paper [12], we find that no restriction on the sign of $q_4(x)$ is required for (1.1) under assumption (1.6) (it should be noticed that if $p_3(y) \equiv p_4(y)$ one case of assumption (1.6) reduces to (1.4)). By introducing suitable transformations we prove that system (1.1) is equivalent to a form of system (1.5) under assumption (1.6). In this paper we have investigated the qualitative behavior of systems (1.1) and (1.5). Especially, we give the necessary and sufficient conditions for all nontrivial solutions of (1.1) (respectively (1.5)) to be oscillatory and for the origin to be a global center, and we also study the existence and uniqueness of nontrivial periodic solutions of system (1.1) (respectively (1.5)). Furthermore, we establish the sufficient conditions under which no solution of (1.5) approaches the origin directly in the right (or left) half plane (i.e., in a nonoscillatory way), which plays an important role in the analysis of oscillation and center conditions of (1.5).

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Paper IV

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Periodic Solutions of General Autonomous System of Liénard Type

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This article gives some sufficient conditions for the existence and nonexistence of nonzero periodic solutions of the general autonomous system of Liénard type

$$\begin{cases} \dot{x} = p(y), \\ \dot{y} = -f(x,\ y)p(y)q(y) - r(y)g(x). \end{cases}$$

The main purpose of this article is to study the problem of how small the extent for the function $f(x, y)$ should be to guarantee the existence of nonzero periodic solutions of this system. With some standard additional assumptions, we prove that if

$$\int^{\pm\infty} |f(x,\ y)q(y)|^{-1} dy = \pm\infty \quad \text{for a small } |x|,$$

then the system has at least one nonzero periodic solution, otherwise, the system has no nonzero periodic solution.

Keywords: Generalized Liénard system; Periodic solution

AMS: 34C25

1. INTRODUCTION

In 1942, Levinson and Smith [2] first studied the existence of nonzero periodic solutions of the general autonomous equation of Liénard type

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \tag{1}$$

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or its equivalent system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -f(x,y)y - g(x). \end{cases} \tag{2}$$

Since then, many authors have made contributions to the theory of this system regarding the existence of nonzero periodic solutions. The books by Sansone and Conti [4], Zhang *et al.* [13] and Ye *et al.* [10] contain a summary of the results on this problem. On reviewing all the known results, we find that in order to obtain a criterion for the existence of nonzero periodic solutions almost every author required that the restoring force $g(x)$ and damping $f(x,y)$ should be not too small, that is, $f(x,y)$ should have a lower bound in a strip region $|x| \leq d$ and should be nonnegative outside this strip region, and $\int^{\pm\infty} g(x)\,dx = +\infty$. Ponzo and Wax [3] gave a result on the existence of a nonzero periodic solution which does not require $f(x,y)$ to have a lower bound. Unfortunately, Zheng [14] gave an example

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(x^2 - 4)(y^2 + 1)y - x \end{cases} \tag{3}$$

to show that the conditions of Ponzo and Wax cannot guarantee the existence of a nonzero periodic solution if $f(x,y)$ has not a lower bound. Yu and Huang [11] also dealt with the existence of nonzero periodic solutions of (2), and pointed out that system (3) has a nonzero periodic solution. Yan and Jiang [8] considered the system (2), they noted that system (3) has no nonzero periodic solution. Also, Wang *et al.* [7] gave a complete analysis of global bifurcation for the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(x^2 - \delta)(y^2 + 1)y - x, \end{cases} \tag{4}$$

where δ is a parameter. In addition, it was shown by Lemma 5 in [7] that system (4) has no nonzero periodic solution when $\delta \geq \sqrt[3]{q\pi^2/16} \approx 1.7707$.

Yu and Huang [12] studied a more general system than (2), namely,

$$\begin{cases} \dot{x} = p(y), \\ \dot{y} = -f(x,y)p(y)q(y) - r(y)g(x), \end{cases} \tag{5}$$

under the assumptions $\int_0^{\pm\infty} g(s)\,ds = +\infty$, they obtained some sufficient conditions for the existence of one nonzero periodic solution of (5). Moreover, as a result of [12] they pointed out that system (3) has at least one nonzero periodic solution.

In the present article we study the generalized Liénard system (5), where the functions $p(y), q(y), r(y), g(x)$ and $f(x,y)$ are continuous for all values of their arguments, and are subject to the conditions which ensure that the existence of unique solution to the initial value problem.

The purpose of this article is to study the problem of how small the extent for $f(x,y)$ should be to warrant the existence of nonzero periodic solutions of (5). Our investigation shows that whether (5) has a nonzero periodic solution strongly depends on the integral $\int^{\pm\infty} |f(x,y)q(y)|^{-1}dy$, where $|x|$ is sufficiently small. The article is organized as follows: In Section 2 we find some sufficient conditions for the existence of nonzero periodic solutions of (5), roughly speaking, if $\int^{\pm\infty} |f(x,y)q(y)|^{-1}\,dy = \pm\infty$ for a small $|x|$ and some additional assumptions hold, then (5) has at least one nonzero periodic solution. Our results allow to avoid the classical assumptions:

$$\int_0^{\pm\infty} g(x)\,dx = +\infty, \tag{6}$$

$$f(x,y) > 0 \text{ (or } \geq 0) \quad \text{for } |x| \text{ sufficiently large.} \tag{7}$$

In Section 3 we give some sufficient conditions about nonexistence of periodic solutions of (5), the main results in this section are Theorems 3.1 and 3.2 which state that if $\int^{+\infty} |f(x,y)q(y)|^{-1}dy$ (or $\int^{-\infty} |f(x,y)q(y)|^{-1}dy$) is finite for a small $|x|$ and some additional assumptions hold, there does not exist a nonzero periodic solution of the system (5). Some examples illustrating our results are given in this article.

2. EXISTENCE OF NONZERO PERIODIC SOLUTIONS

In this section, we give sufficient conditions for the existence of nonzero periodic solutions of (5).

At first we assume:

(E_1) $xg(x) > 0$ for all $x \neq 0$;
(E_2) $f(0,0) < 0$;
(E_3) $q(y)$ and $r(y)$ are positive on $\mathbf{R}$, $yp(y) > 0$ for all $y \neq 0$, and $\liminf_{y\to\pm\infty} |p(y)|/r(y) > 0$;
(E_4) There exist constants $a < 0, b > 0$ and a function $f_0(x) \in C((-\infty, a] \cup [b, +\infty))$ such that $f(x,y)q(y) \geq f_0(x)$ for $x \in [-\infty, a] \cup [b, +\infty), y \in \mathbf{R}$. $G(x) = \int_0^x g(s)\,ds$, $F^+(x) = \int_b^x f_0(s)\,ds$ $(x \geq b)$ and $F^-(x) = \int_a^x f_0(s)\,ds$ $(x \leq a)$ satisfy the following conditions $(E_4^+)_i$ and $(E_4^-)_i$ respectively,

$(E_4^+)_1$ $\liminf_{x\to+\infty} F^+(x) > -\infty$,
$(E_4^+)_2$ $\limsup_{x\to+\infty}(G(x) + F^+(x)) = +\infty$;
$(E_4^-)_1$ $\limsup_{x\to-\infty} F^-(x) < +\infty$,
$(E_4^-)_2$ $\limsup_{x\to-\infty}(G(x) - F^-(x)) = +\infty$.

(E_5) There exist functions $\varphi(x) \in C[a,b]$ and $h(y) \in C(\mathbf{R})$ such that

$(E_5)_1$ $f(x,y) \geq \varphi(x)h(y)$ for $a \leq x \leq b, y \in \mathbf{R}$,
$(E_5)_2$ For every constant $m > 0$, there is a positive number $c(m)$ such that

$$\frac{|p(y)|h(y)q(y)}{r(y)} \geq c(m) \quad \text{for } |y| \geq m, \text{ and}$$
$$\int_1^{+\infty} \frac{dy}{h(\pm y)q(\pm y)} = +\infty.$$

THEOREM 2.1 *Assume that the conditions* (E_1)–(E_5) *hold, and that the following condition is satisfied.*

(E_6) There exist a constant $a_0 < a$ and a function $A(x) \in C^1((-\infty, a_0], (0, +\infty))$ such that

$$p(A(x))A'(x) + p(A(x))f(x, A(x))q(A(x)) + r(A(x))g(x) \geq 0,$$

for $x \in (-\infty, a_0]$.

Then (5) has at least one nonzero periodic solution.

Before proving Theorem 2.1, we introduce some notations. Let $L^+ = (0, b] \times \{0\}, L^- = [a, 0) \times \{0\}$. $\Omega^+_{a_0} = \{(x, y)\colon x \leq a_0, y > 0\}$, $\Omega^+_b = \{(x, y)\colon x \geq b, y > 0\}$. The positive and negative x-axis are denoted by X^- and X^- respectively. Y^+ and Y^- denote the positive and negative y-axis respectively. If $x_0 \in \mathbf{R}$, we write $L^+_{x_0} = \{(x_0, y)\colon y > 0\}, L^-_{x_0} = \{(x_0, y)\colon y < 0\}$. Furthermore, we denote throughout this article by $\gamma^+(p)$ and $\gamma^-(p)$, respectively, the positive and negative semi-trajectory of (5) passing through an arbitrary point p at time 0, $\gamma(p) = \gamma^+(p) \cup \gamma^-(p)$, and let $[0, T_p)$ denote the right maximal interval of $\gamma^+(p)$, where $T_p \leq +\infty$.

First of all, we establish several lemmas which will play an important role in the proof of Theorem 2.1 in the sequel.

LEMMA 2.1 *If the conditions* (E_1), (E_2) *and* (E_3) *are satisfied then the origin is the only critical point of* (5), *and it is locally repulsive.*

Proof In view of conditions (E_1) and (E_3), it is easy to see that the origin is a unique critical point of (5). Next, we will show that the origin is repulsive. Set

$$G(x) = \int_0^x g(s)\, ds,$$
$$\lambda(x, y) = G(x) + \int_0^y \frac{p(s)}{r(s)}\, ds.$$

From (E_1) and (E_3), it is clear that for $0 < c \ll 1$ the curve $\lambda(x, y) = c$ is a closed curve surrounding the origin. By (E_2) and (E_3), for $0 < c \ll 1$, along the closed curve $\lambda(x, y) = c$ we have

$$\frac{d\lambda(x, y)}{dt}\Big|_{(5)} = -\frac{f(x, y)p^2(y)q(y)}{r(y)} > 0 \quad \text{for all } y \neq 0. \tag{8}$$

Hence the origin is locally repulsive. This completes the proof of Lemma 2.1.

LEMMA 2.2 *Assume the conditions* (E_1)–(E_4) *hold,* $p \in \Omega^+_{a_0}$. *Then* $\gamma^+(p)$ *must intersect* L^+_a.

Proof Suppose the conclusion is false. Since $dx/dt = p(y) > 0$ for $y > 0$ and $dy/dt = -r(y)g(x) > 0$ on X^-, it follows that $\gamma^+(p)$ will stay in the region $\Omega^+_a = \{(x, y)\colon x \leq a, y > 0\}$. Let $(x(t), y(t))$ be the coordinates of $\gamma^+(p), (x(0), y(0)) = p$. By Lemma 2.1, we know that no solution can approach the origin except itself. Since Ω^+_a does not contain any equilibrium, the Poincaré–Bendixson Theorem tells us that

$\gamma^+(p)$ must be unbounded. It is easy to see that there exist a increasing sequence $\{\sigma_k\} \subset (0, T_p)$ and a constant $\bar{a} \in (a_0, a]$ satisfying

$$\lim_{k\to+\infty} x(\sigma_k) = \bar{a}, \qquad \lim_{k\to+\infty} y(\sigma_k) = +\infty. \tag{9}$$

Because the origin is locally repulsive and $dy/dt > 0$ on X^-, it is obvious that there exists a constant $\bar{y} > 0$ such that $y(t) \geq \bar{y}$ for $0 \leq t < T_p$. Therefore, from the condition (E_3) it is sure that there exists a constant $M > 0$ such that

$$\frac{p(y(t))}{r(y(t))} \geq M \quad \text{for } 0 \leq t < T_p.$$

By (E_3), (E_4) and (5), we have

$$\begin{aligned}\frac{dy}{dt} &= -f(x,y)q(y)p(y) - \gamma(y)g(x)\\ &\leq -f_0(x)p(y) - r(y)g(x),\end{aligned} \tag{10}$$

for $0 \leq t < T_p$. Now, integrating (10) from 0 to σ_k along $\gamma^+(p)$, we get

$$\begin{aligned} y(\sigma_k) - y(0) &\leq -\int_0^{\sigma_k} \left(f_0(x(s)) + \frac{r(y(s))g(x(s))}{p(y(s))} \right) p(y(s))\, ds \\ &\leq -\int_0^{\sigma_k} \left(f_0(x(s)) + \frac{g(x(s))}{M} \right) p(y(s))\, ds \\ &= -\int_{x(0)}^{(\sigma_k)} \left(f_0(z) + \frac{g(z)}{M} \right) dz. \end{aligned} \tag{11}$$

By (9), the right of (11) is bounded, the left of (11) is positive infinite, which is a contradiction. Hence $\gamma^+(p)$ must intersect L_a^+. The proof of Lemma 2.2 is complete.

LEMMA 2.3 *If the conditions* (E_1), (E_2), (E_3) *and* (E_5) *are satisfied*, $p \in L_a^+$. *Then* $\gamma^+(p)$ *must intersect* Y^+.

Proof Suppose this is not the case. Let $(x(t), y(y))$ be the coordinates of $\gamma^+(p)$, $(x(0), y(0)) = p$. Then by an argument similar to the Lemma 2.2, there must exist an increasing sequence $\{t_k\} \subset (0, T_p)$ and a constant $\bar{a}_1 \in (a, 0]$ satisfying

$$\lim_{k\to+\infty} x(t_k) = \bar{a}_1, \qquad \lim_{k\to+\infty} y(t_k) = +\infty. \tag{12}$$

By (E_3), (E_5) and (5), we obtain

$$\begin{aligned} \frac{dy}{dt} &= -f(x,y)p(y)q(y) - r(y)g(x)\\ &\leq -\varphi(x)h(y)p(y)q(y) - r(y)g(x)\\ \frac{1}{h(y)q(y)}\frac{dy}{dt} &\leq -\varphi(x)p(y) - \frac{r(y)g(x)}{h(y)q(y)}, \end{aligned} \tag{13}$$

for $0 \le t < T_p$. Integrating (13) from 0 to t_k along $\gamma^+(p)$, we have

$$\int_{y(0)}^{y(t_k)} \frac{dy}{h(y)q(y)} \le -\int_0^{t_k} \left(\varphi(x(s)) + \frac{r(y(s))g(x(s))}{p(y(s))h(y(s))q(y(s))}\right) p(y(s))\, ds$$
$$= -\int_a^{x(t_k)} \left(\varphi(z) + \frac{r(y(s))g(z)}{p(y(s))h(y(s))}\right) dz. \tag{14}$$

Since $dy/dt > 0$ on the negative x-axis and the origin is locally repulsive, it is easy to see that there is a constant $\bar{y}_1 > 0$ such that $y(t) \ge \bar{y}$, for $0 \le t < T_p$. From $(E_5)_2$ we know that there exists a constant $m_0 > 0$ such that $(p(y(t))h(y(t))q(y(t))/r(y(t))) \ge m_0$ for $t \in [0, T_p)$. Therefore, by (E_5) and (12), the right of (14) is bounded, the left of (14) is positive infinite, which is a contradiction. Thus, $\gamma^+(p)$ must intersect the positive y-axis. This completes the proof of Lemma 2.3.

LEMMA 2.4 *Assume the conditions* (E_1), (E_2), (E_3) *and* (E_5) *hold*, $p \in Y^+$. *Then* $\gamma^+(p)$ *must intersect* $L^+ \cup L_b^+$.

Proof Since $dx/dt = p(y) > 0$ for $y > 0$, if $\gamma^+(p)$ does not intersect L^+, by using an exactly similar argument as in the proof of Lemma 2.3, then $\gamma^+(p)$ must intersect L_b^+. This completes the proof of Lemma 2.4.

LEMMA 2.5 *Assume that the conditions* (E_1)–(E_5) *hold and that* $p \in L_b^+$. *Then* $\gamma^+(p)$ *must intersect* X^+.

Proof Suppose it is not the case. Since $dx/dt = p(y) > 0$ for $y > 0$, it follows that $\gamma^+(p)$ will stay in the region Ω_b^+. Let $(x(t), y(t))$ be the coordinates of $\gamma^+(p)$, $(x(0), y(0)) = p$. Because there is no critical point of (5) in Ω_b^+ under the conditions of Lemma 2.5, according to the theory of limit sets, it is certain that $\gamma^+(p)$ is unbounded.

On the other hand, it follows from (5) that

$$\frac{dy}{dt} = -f(x, y)p(y)q(y) - r(y)g(x). \tag{15}$$

Integrating (15) from 0 to $t \in [0, T_p)$ along $\gamma^+(p)$, by (E_4) we have

$$\int_{y(0)}^{y(t)} dy = -\int_0^t (f(x, y)p(y)q(y) + r(y)g(x))\, ds$$
$$\le -\int_0^t f(x, y)p(y)q(y)\, ds$$
$$\le -\int_0^t f_0(x)p(y)\, ds$$
$$= -\int_{x(0)}^{x(t)} f_0(z)\, dz$$
$$= -F(x(t)) + F(x(0)). \tag{16}$$

Then (16) implies

$$y(t) - y(0) \leq -F(x(t)) + F(x(0)),$$

and condition (E_4) imply that $y(t)$ is bounded from above for $t \in [0, T_p)$. Thus, it is sure that there exists a constant $M_1 > 0$ such that $0 < y(t) \leq M_1$ for all $t \in [0, T_p)$.

Notice that $\gamma^+(p)$ is unbounded, it is clear that $\lim_{t \to T_p^-} x(t) = +\infty$. In the following, we are going to obtain a contradiction. Let

$$M_2 = 1 + \max_{0 \leq y \leq M_1} \frac{p(y)}{r(y)}.$$

Integrating (15) from 0 to t along $\gamma^+(p)$, we have

$$\begin{aligned}
\int_{y(0)}^{y(t)} dy &= -\int_0^t (f(x(s), y(s))p(y(s))q(y(s)) + r(y(s))g(x(s)))\, ds \\
&\leq -\int_0^t \left(f_0(x(s)) + \frac{r(y(s))g(x(s))}{p(y(s))} \right) p(y(s))\, ds \\
&\leq -\int_0^t \left(f_0(x(s)) + \frac{g(x(s))}{M_2} \right) \frac{dx(s)}{ds}\, ds \\
&= -\int_{x(0)}^{x(t)} f(z)\, dz - \frac{1}{M_2} \int_{x(0)}^{x(t)} g(z)\, dz \\
&= -F(x(t)) + F(x(0)) - \frac{1}{M_2} G(x(t)) + \frac{1}{M_2} G(x(0)).
\end{aligned}$$

This implies

$$y(t) \leq y(0) - \frac{1}{M_2}(F(x(t)) + G(x(t))) - \frac{M_2 - 1}{M_2} F(x(t)) + F(x(0)) + \frac{1}{M_2} G(x(0)). \quad (17)$$

In view of $(E_4^+)_1$, $(E_4^+)_2$ and $\lim_{t \to T_p^-} x(t) = +\infty$, it is easy to see that inequality (17) implies $\liminf_{t \to T_p^-} y(t) = -\infty$. But this contradicts the fact that $y(t) > 0$ for all $t \in [0, T_p)$, and hence $\gamma^+(p)$ must intersect X^+. This completes the proof of Lemma 2.5.

By an argument similar to Lemmas 2.3–2.5 above, we can prove the following Lemma 2.6.

LEMMA 2.6 *If the conditions* (E_1)–(E_5) *hold,* $p \in X^+$. *Then* $\gamma^+(p)$ *must intersect* Y^- *and* X^-.

Proof of Theorem 2.1 According to Lemma 2.1, the origin is a unique critical point of (5) and is repulsive. For $0 < c \ll 1$, the oval $\lambda(x, y) \equiv c$ can serve as an inner bound for the annulus, and the trajectories of (5) cross this closed curve from its interior to its exterior.

Now let us complete the outer bound.

Let $P_0 = (a_0, A(a_0))$. In view of Lemmas 2.2 and 2.3, we see that $\gamma^+(P_0)$ will intersect Y^+ at some point P_1. Then, it follows from Lemmas 2.4–2.6 that $\gamma^+(P_0)$ through point P_1 will intersect X^- at some point P_2.

Along the curve $y = A(x)$, we have

$$\begin{aligned}\frac{dy}{dx}\Big|_{(5)} - \frac{dA(x)}{dx} &= -f(x,y)q(y) - \frac{r(y)g(x)}{p(y)} - \frac{dA(x)}{dx} \\ &= -\frac{p(A(x))f(x,A(x))q(A(x)) + r(A(x))g(x) + p(A(x))(dA(x)/dx)}{p(A(x))} \\ &\leq 0\end{aligned}$$

for $x \in (-\infty, a_0]$, which shows that $\gamma^+(P_0)$ through point P_2 cannot cross the curve $y = A(x)$ $(-\infty < x \leq a_0)$. Therefore, by Lemmas 2.3 and 2.4, $\gamma^+(P_0)$ through P_2 will intersect Y^+ again at some point P_3, the above proof and the uniqueness of solutions of (5) show that P_3 is under P_2. Then the trajectories of (5) cross the closed curve $\overline{P_3P_2}P_2P_3$ from its exterior into its interior. This completes the outer bound for the annulus. Hence, by the Poincaré–Bendixson Theorem, system (5) has at least one nonzero periodic solution. The proof of Theorem 2.1 is complete.

A slight modification in the proof of Theorem 2.1 leads to the following Theorem 2.2.

THEOREM 2.2 *Assume that the conditions* (E_1)–(E_5) *hold, and the following condition is satisfied.*

(E_6') There exist a constant $b_0 > b$ and a function $A(x) \in C^1((b_0, +\infty), (-\infty, 0))$ such that

$$p(A(x))A'(x) + p(A(x))f(x, A(x))q(A(x)) + r(A(x))g(x) \leq 0,$$

for all $x \in [b_0, +\infty)$.

Then (5) has at least one nonzero periodic solution.

COROLLARY 2.1 *Suppose that the conditions* (E_1)–(E_5) *hold, and that one of the following two conditions is satisfied.*

(E_7^+) There exist a constant $a_0 < a$ and a function $A'(x) \in C^1((-\infty, a_0], (0 + \infty))$ such that

$$p(A(x))A'(x) + p(A(x))f_0(x) + r(A(x))g(x) \geq 0,$$

for all $x \in (-\infty, a_0]$;

(E_7^-) There exist a constant $b_0 > b$ and a function $A(x) \in C^1([b_0, +\infty), (-\infty, 0))$ such that

$$p(A(x))A'(x) + p(A(x))f_0(x) + r(A(x))g(x) \leq 0,$$

for all $x \in [b_0, +\infty)$.

Then (5) has at least one nonzero periodic solution.

Obviously, when the conditions (E_3) and (E_4) hold, the conditions (E_6) and (E_6') imply (E_7^+) and (E_7^-) respectively. Hence, Corollary 2.1 is a direct corollary of Theorems 2.1 and 2.2.

Remark 2.1 For the system (2) the condition (E_3) holds naturally, therefore, our Theorems 2.1 and 2.2 are available for (2).

Remark 2.2 If $p(y) \equiv y, q(y) \equiv 1, r(y) \equiv 1$, then system (5) reduces to system (2), and the condition (E_5) is the condition (E_4) of [8]. Our theorem includes the cases where $f(x,y)$ has no lower bound in the strip region $|x| \leq d$, $f(x,y)$ isn't nonnegative in $|x| \geq K$ where K is an arbitrary positive number, and $G(\pm\infty) < +\infty$. Hence Theorems 2.1 and 2.2 extend and improve the corresponding results of [1–3, 5, 6, 8, 9, 11–13].

Example 2.1 Let $\alpha_1, \alpha_2, \alpha_3$ and β be the positive numbers, and $\alpha_2 \leq (1/3), \alpha_1 \geq \alpha_3$. Then the following system

$$\begin{cases} \dot{x} = |y|^{\alpha_1} \operatorname{sgn} y, \\ \dot{y} = -(1+x^2+y^2)^{1/3}(\sqrt{x^2+x^2|y|} - (\beta + \sin^2 xy)^{-1/2})(1+|y|^{\alpha_2})|y|^{\alpha_1} \operatorname{sgn} y \\ \qquad - (1+|y|^{\alpha_3})x(1+x^2)^{-3/2} \end{cases} \tag{18}$$

has at least one nonzero periodic solution.

Proof For this system, we have $p(y) = |y|^{\alpha_1} \operatorname{sgn} y, q(y) = 1 + |y|^{\alpha_2}, r(y) = 1 + |y|^{\alpha_3}$, $f(x,y) = (1+x^2+y^2)^{1/3}(\sqrt{x^2+x^2|y|} - (\beta+\sin^2 xy)^{-1/2})$ and $g(x) = x(1+x^2)^{-3/2}$. Obviously, $G(\pm\infty) < +\infty$, it is easy to know that the conditions (E_1), (E_2) and (E_3) hold. Choose $b = -a = 1$, $f_0(x) = |x| - \beta^{-1/2}$, then $f(x,y)q(y) \geq f_0(x)$ for $x \in (-\infty, -1] \cup (1, +\infty)$, $y \in \mathbf{R}$, on the other hand, by some computation, we have

$$F^+(x) = \int_1^x f_0(s)\,ds = \frac{x^2}{2} - \beta^{-1/2}x - \frac{1}{2} + \beta^{-1/2},$$

$$F^-(x) = \int_{-1}^x f_0(s)\,ds = -\frac{x^2}{2} - \beta^{-1/2}x + \frac{1}{2} - \beta^{-1/2}.$$

Thus, the conditions $(E_4^+)_i$ and $(E_4^-)_i$ hold $(i = 1, 2)$. Let $\varphi(x) = |x| - \beta^{-1/2}$ and $h(y) = (1+y^2)^{1/3}$, then $f(x,y) \geq (1+y^2)^{1/3}(|x| - \beta^{-1/2})$ for $x \in [-1, 1], y \in \mathbf{R}$, and

$$\int_1^{+\infty} \frac{dy}{h(\pm y)q(\pm y)} = +\infty,$$

it is clear that the condition (E_5) holds. We next prove that condition (E_6) holds.

Set $A(x) = 6 - e^x$. Then we get

$$\lim_{x\to-\infty} [p(A(x))A'(x) + p(A(x))f(x, A(x))q(A(x)) + r(A(x))g(x)] = +\infty,$$

which implies that there exists a constant $a_0 < -1$ such that

$$p(A(x))A'(x) + p(A(x))f(x, A(x))q(A(x)) + r(A(x))g(x) > 0$$

for all $x \le a_0$, and so (E_6) holds. Therefore, it follows from Theorem 2.1 that the conclusion of Example 2.1 is true.

3. NONEXISTENCE OF NONZERO PERIODIC SOLUTIONS

In Section 2, we have proved that if $\int^{\pm\infty} |f(x,y)q(y)|^{-1}dy = \pm\infty$ for a $|x|$ sufficiently small and some additional assumptions hold, then (5) has at least a nonzero periodic solution. In this section, we shall consider the converse case, i.e., $\int^{\pm\infty} |f(x,y)q(y)|^{-1}dy$ is finite for every small $|x|$, and prove the following Theorem 3.1.

THEOREM 3.1 *Assume* (5) *satisfies the following conditions:*

(H_1) $xg(x) > 0$ *for all* $x \neq 0$;
(H_2) $q(y)$ *and* $r(y)$ *are positive on* $\mathbf{R}$, $yp(y) > 0$ *for all* $y \neq 0$, *and* $\liminf_{y\to\pm\infty} |p(y)|/r(y) > 0$;
(H_3) *There exist constant* $a < 0, b > 0$ *a negative integrable function* $\varphi(x)$ *defined on* $[a, b]$ *and a continuous function* $h(y) > 0$ *for* $y \neq 0$ *such that*

$(H_3)_1$ $f(x,y) \le \varphi(x)h(y)$ *for* $a < x < b, y \in \mathbf{R}$;
$(H_3)_2$ $\int_0^{+\infty} dy/h(y)q(y) \le \int_0^{a_1} \varphi(x)\,dx$, *where* $a_1 < 0$, $G(a_1) = \min\{G(a), G(b)\}$.

Then (5) *has no nonzero periodic solution.*

Proof Set

$$G(x) = \int_0^x g(s)\,ds,$$
$$\lambda(x,y) = G(x) + \int_0^y \frac{p(s)}{r(s)}\,ds,$$

it is obvious that the curve of constant energy $\Gamma : \lambda(x,y) = G(x) + \int_0^y (p(s)/r(s))ds = G(a_1)$ is closed in the strip region $a \le x \le b,\ -\infty < y < +\infty$. Because,

$$\frac{d\lambda}{dt}\Big|_{(5)} = -\frac{f(x,y)p^2(y)q(y)}{r(y)} > 0, \quad \text{for } y \neq 0, \tag{19}$$

then the trajectories of (5) cross this closed curve $\lambda(x,y) = G(a_1)$ from its interior to its exterior. This proves that the region surrounded by Γ is negatively invariant and $\alpha(p) = 0$ for each $p \in \Gamma$.

Now let $A = (a_1, 0), (x(t), y(t))$ be the coordinates of $\gamma^+(A), (x(0), y(0)) = A$. If $\gamma^+(A)$ intersects Y^+ at $B(0, \bar{y})(\bar{y} > 0)$, then, along the solution arc AB, we have

$$\frac{dy}{dx} = -f(x,y)q(y) - \frac{r(y)g(x)}{p(y)}$$
$$\ge -\varphi(x)h(y)q(y)$$
$$\frac{1}{h(y)q(y)}\frac{dy}{dx} \ge -\varphi(x). \tag{20}$$

Integrating (20) from a_1 to 0, we get

$$\int_0^{\bar{y}} \frac{dy}{h(y)q(y)} \geq -\int_{a_1}^0 \varphi(x)\,dx = \int_0^{a_1} \varphi(x)\,dx.$$

Because $h(y) > 0, q(y) > 0$ for $y > 0$, then

$$\int_0^{+\infty} \frac{dy}{h(y)q(y)} > \int_0^{\bar{y}} \frac{dy}{h(y)q(y)} \geq \int_0^{a_1} \varphi(x)\,dx,$$

which contradicts the condition $(F_3)_2$. This implies that $\gamma^+(A)$ is located in the strip region: $a_1 \leq x < 0, y \geq 0$. Since

$$\begin{aligned}\frac{dy(t)}{dt} &= -f(x,y)p(y)q(y) - r(y)g(x)\\ &\geq -\varphi(x)h(y)p(y)q(y) - r(y)g(x) \geq 0,\end{aligned}$$

for $t \in [0, T_A), y(t)$ is increasing on $[0, T_A)$, and thus $\lim_{t\to T_A^-} y(t) = +\infty$ and $\lim_{t\to-\infty}(x(t), y(t)) = (0,0)$. We have proved that there exists a trajectory $\gamma(A)$ of (5) such that one side of it tends to the origin and the other side approaches to infinity. Therefore, (5) has no nonzero periodic solutions. This proves Theorem 3.1.

Similarly, we can prove the following Theorem 3.2.

THEOREM 3.2 *Assume* (5) *satisfies the following conditions*:

(F_1) $xg(x) > 0$ *for all* $x \neq 0$;
(F_2) $q(y)$ *and* $r(y)$ *are positive on* $\mathbf{R}$, $yp(y) > 0$ *for all* $y \neq 0$, *and* $\liminf_{y\to\pm\infty} (|p(y)|/r(y)) > 0$;
(F_3) *There exist constants* $a < 0, b > 0$, *a negative integrable function* $\varphi(x)$ *defined on* $[a,b]$ *and a continuous function* $h(y) > 0$ *for* $y \neq 0$ *such that*

$(F_3)_1$ $f(x,y) \leq \varphi(x)h(y)$ *for* $a < x < b, y \in \mathbf{R}$;
$(F_3)_2$ $\int_0^{-\infty} (dy/h(y)q(y)) \geq \int_0^{b_1} \varphi(x)\,dx$, *where* $b_1 > 0$, $G(b_1) = \min\{G(a), G(b)\}$.

Then (5) *has no nonzero periodic solutions.*

Remark 3.1 Let $p \neq 0$, under the conditions of Theorems 3.1 and 3.2, we can prove that $\gamma^+(p)$ is unbounded. In fact, if it is not the case, then by (19), $\omega(p)$ does not contain the origin O. The Poincaré–Bendixson Theorem implies that $\omega(p)$ is a limit cycle. Since $\omega(p)$ is a Jordan curve and the trajectory $\gamma(A)$ passing through $A(a_1, 0)$ connects the origin and infinity, $\omega(p)$ must intersect $\gamma(A)$, this contradicts the uniqueness of initial value problems. Hence $\gamma^+(p)$ is unbounded.

Remark 3.2 Choose $a_1 = -\sqrt[3]{3\pi/4}$, since $\int_0^{+\infty} dy/(1+y^2) = \pi/2$, Theorem 3.1 implies that if $\delta > \sqrt[3]{9\pi^2/16} \approx 1.77$, (4) has no nonzero periodic solution. The result of Zheng [14] regarding (3) is included in our results.

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Paper V

Gyllenberg, M., Yan, P., and Wang, Y., A 3D competitive Lotka-Volterra system with three limit cycles: A falsification of a conjecture by Hofbauer and So, *Appl. Math. Lett.* (to appear).

A 3D competitive Lotka-Volterra system with three limit cycles: A falsification of a conjecture by Hofbauer and So*

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Abstract For 3-dimensional competitive Lotka-Volterra systems, Zeeman [1] identified 33 stable nullcline equivalence classes. Among these, only classes 26-31 may have limit cycles. Hofbauer and So [2] conjectured that the number of limit cycles is at most two for these systems. In this paper, we construct three limit cycles for class 29 without a heteroclinic polycycle in Zeeman's classification.

Key words and phrases: Competitive, Lotka-Volterra system, limit cycles, carrying simplex, Hopf bifurcation.

2000 Mathematics Subject Classification: 34C12, 34C23, 92D25.

Short title for page headings: 3D Competitive L-V Systems.

1 Introduction

Lotka-Volterra (L-V) competition is modelled by a system of differential equations describing the competition between two or more species that share

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and compete for the same resources, habitat or territory (interference competition). The n-dimensional competitive L-V model is

$$\frac{dx_i}{dt} = x_i\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right), \qquad i = 1, 2, \cdots, n, \tag{1}$$

where x_i is the number or density of species i, b_i is the intrinsic growth rate of species i, a_{ij}'s are the interaction coefficients. The parameters b_i and a_{ij} are strictly positive.

The dynamics of the 2-dimensional L-V competition model is well understood. If two species compete, there are no periodic solutions and all bounded orbits converge to an equilibrium point (see [3]). For 3-dimensional competitive L-V systems, the dynamical possibilities are more restricted than for general L-V systems: Hirsch [4] has shown that all nontrivial orbits approach a "carrying simplex", a Lipchitz 2-dimensional manifold-with-corner homeomorphic to the standard simplex in $\mathbb{R}^3_+$ via radial projection. This then leads to a Poincaré-Bendixson theorem for 3-dimensional competitive systems (see [5]). Recently, the existence and global attractivity of the carrying simplex have also been verified in time-periodic competitive L-V systems ([6]). Based on the remarkable result of Hirsch, Zeeman [1] defined a combinatorial equivalence relation on the set of all 3-dimensional L-V competitive systems and identifies 33 stable equivalence classes. Of these, only classes 26-31 may have limit cycles (see [1, 7]). Open problems remain concerning the number of limit cycles. Hofbauer and So [2] were the first to give an example in class 27 (with a heteroclinic polycycle) with two limit cycles surrounding the interior equilibrium. Lu and Luo [8] have constructed two limit cycles in three cases without a heteroclinic polycycle (classes 26, 28 and 29).

Apparently, the main questions now are (i) whether or not there are at most finitely many limit cycles on the carrying simplex; (ii) whether there can be more than two limit cycles in classes 26-31. For question (i), Xiao and Li [9] have proved that the number of limit cycles of the 3-dimensional competitive L-V systems is finite if the system does not have a heteroclinic polycycle. Question (ii), as pointed out by Hofbauer and So [2], is a very difficult problem and they conjectured that the number of limit cycles is at most two for 3-dimensional L-V competitive systems.

Recently, Lu and Luo [10] were the first to give an example in class 27 (with a heteroclinic polycycle) with three limit cycles. This gives a partial answer to Hofbauer's and So's conjecture. In this paper, we will construct three limit cycles in class 29 without a heteroclinic polycycle (see Figure

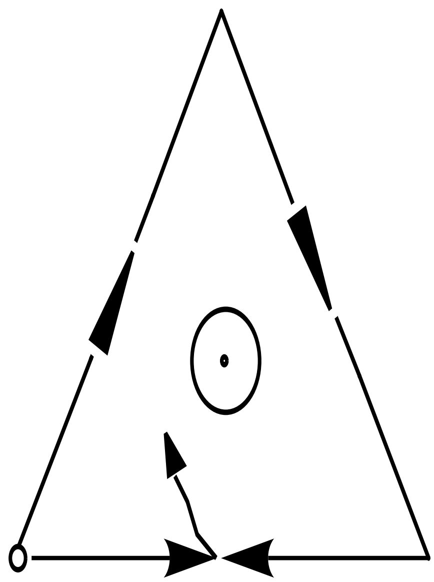

Figure 1: The phase portraits on Σ of class 29 with interior fixed point. The fixed point notation is as in [1].

1) and thus give a counterexample to Hofbauer's and So's conjecture which is qualitatively different from that of Lu and Luo [10]. We conjecture that there also exist three limit cycles in class 26.

2 An example with three limit cycles

In this section, we present an example of a 3-dimensional competitive L-V system with at least three limit cycles in class 29 without a heteroclinic polycycle.

The idea for constructing such an example with three limit cycles is as follows: We consider a 3-dimensional competitive L-V system of class 29 in Zeeman's classification, which is indeed uniformly persistent [11], and where the unique interior fixed point E has the following properties: (a) There is a pair of purely imaginary eigenvalues at E; (b) The first focal value vanishes, and (c) the second local value is positive. Thus E is a weak focus of multiplicity 2 repelling on its center manifold. This implies the existence of an asymptotically stable limit cycle Γ_1 by the Poincaré-Bendixson theorem on the carrying simplex Σ. If we now change some of the parameters slightly, the equilibrium will undergo a generic Hopf bifurcation, that is, the interior equilibrium E first becomes a weak focus of multiplicity 1 attracting on its center manifold and will be surrounded by another, smaller, unstable limit cycle Γ_2; then one of the parameters is changed slightly so that a supercritical Hopf bifurcation occurs, and hence E becomes a focus repelling on Σ and be surrounded by the smallest stable limit cycle Γ_3 of the three existing limit cycles.

The remaining work is to check to which class in Zeeman's classification the constructed system belongs. Using Zeeman's notation, we have $R_{ij} = \mathrm{sgn}(\alpha_{ij})$ and $Q_{kk} = \mathrm{sgn}(\beta_{kk})$, with $\alpha_{ij} = b_i a_{ji}/a_{ii} - b_j = (AR_i)_j - b_j$ and $\beta_{kk} = (AQ_k)_k - b_k$. Here R_i is the equilibrium on the x_i-axis, and Q_k is the positive equilibrium on the plane of $x_k = 0$.

If $Q_{33} = -1, R_{12} = -1, R_{13} = -1, R_{21} = -1, R_{23} = 1, R_{31} = 1, R_{32} = -1$, then the system (1) belongs to class 29 in Zeeman's classification.

Consider the 3-dimensional competitive L-V system

$$\dot{x}_i = x_i[A(E - x)]_i, \qquad i = 1, 2, 3, \tag{2}$$

where

$$A = (a_{ij}) = \begin{pmatrix} 2 & \frac{129}{26} & \lambda \\ \frac{27}{136} & 1 & \mu \\ \frac{99}{100} & \frac{79}{6} & \frac{181}{36} \end{pmatrix}$$

with two positive parameters μ and λ. A necessary condition (see [9]) that A has a positive real eigenvalue and a pair of purely imaginary eigenvalues is

$$\det(A) = (A_{11} + A_{22} + A_{33}) \cdot \mathrm{tr}A,$$

where $\mathrm{tr}(A) = \sum_{i=1}^{3} a_{ii}$, $A_{11} = a_{22}a_{33} - a_{23}a_{32}$, $A_{22} = a_{11}a_{33} - a_{13}a_{31}$ and $A_{33} = a_{11}a_{22} - a_{12}a_{21}$. Then a simple calculation yields that $\mu = \frac{148137475}{100576964} - \frac{11422593}{100576964}\lambda$. Let $y_i = 1 - x_i$, $i = 1, 2, 3$, and set $z = Ty$, then system (2) is transformed to a new one whose linear part is in the block diagonal form

$$\text{linear part} = \begin{pmatrix} -\frac{632955}{846118} & \frac{156919212655475}{255299938877256} - \frac{11422593}{100576964}\lambda & 0 \\ \frac{94122741}{21152950} & \frac{632955}{846118} & 0 \\ 0 & 0 & \frac{289}{36} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

where $y = \mathrm{col}(y_1, y_2, y_3)$, $z = \mathrm{col}(z_1, z_2, z_3)$, and

$$T = \begin{pmatrix} \frac{27}{136} & -\frac{253}{36} & \frac{148137475}{100576964} - \frac{11422593}{100576964}\lambda \\ \frac{99}{100} & \frac{79}{6} & -3 \\ \frac{340041807}{201153928} + \frac{300794949}{201153928}\lambda & \frac{387}{26} + \frac{79}{6}\lambda & \frac{19109734275}{2615001064} + \frac{11702860525}{1810385352}\lambda \end{pmatrix}.$$

This can be reduced to the 2-dimensional case by computing the center manifold

$$z_3 = F(z_1, z_2) = f_2(z_1, z_2) + f_3(z_1, z_2) + f_4(z_1, z_2) + \text{h.o.t.},$$

where $f_i = \sum_{j=0}^{i} c_{ij} {z_1}^{i-j} {z_2}^{j}$, and h.o.t. denotes the terms with order greater than or equal to five. Solving for the ${c_{ij}}'$s and substituting by appealing to the method in [8] one obtains a rather complicated and lengthy expression of the first focal value LV_1 and the second focal value LV_2:

$$LV_1 = \frac{f_1(\lambda)}{f_2(\lambda)}, \; LV_2 = \frac{g_1(\lambda)}{g_2(\lambda)},$$

where

$$\begin{aligned} f_1(\lambda) = \; & -7139120625(58876160308378538412703666150849813688344903 2\lambda^4 \\ & -10645591432228919681423121174963424577878777716\lambda^3 \\ & +58800212557536279681971948183832470586407248122\lambda^2 \\ & -110138947820001095130109579416340544376845179475\lambda \\ & +105100051109054286534963651334843321925555528125), \\ f_2(\lambda) = \; & 3937373499268036(153302801402250836060884765780160734488\lambda^3 \\ & +409896035832534100855188376976364755072 25\lambda^2 \\ & +318575289044259403377613633311781629262 5000\lambda \\ & +56932577068027366681632696744684683338250000), \end{aligned}$$

and $g_1(\lambda)$ is a polynomial of 14 terms with degree 13 and $g_2(\lambda)$ is a polynomial of 13 terms with degree 12.

We computed LV_1 and LV_2 as a rational number using the computer algebraic system Maple. In the following, we choose $a = 9.229462$, $b = 9.229464$. A straightforward calculation yields that LV_1 has a unique root $\lambda_0 \in (a, b)$ and $LV_1 > 0$ for $\lambda \in (a, \lambda_0)$, $LV_1 < 0$ for $\lambda \in (\lambda_0, b)$.

Moreover, $LV_2 > 0$ for $\lambda \in (a, b)$, $\det(A) < 0$ for $\lambda \in (a, b)$, and $\mu = \frac{148137475}{100576964} - \frac{11422593}{100576964}\lambda > 0$ for $\lambda \in (a, b)$. It follows that for $\lambda \in (a, b)$ system (2) is a competitive system that satisfies the condition of the eigenvalues, that is, for $\lambda \in (a, b)$ the equilibrium E of system (2) has a negative real eigenvalue and a pair of purely imaginary eigenvalues.

Since for any $\lambda \in (a, b)$, $Q_{33} = -1$, $R_{12} = -1$, $R_{13} = -1$, $R_{21} = -1$, $R_{23} = 1$, $R_{31} = 1$, $R_{32} = -1$, the system (2) belongs to class 29 in Zeeman's classification.

Now, we can construct three limit cycles for system (2). We have already shown that there exists $\lambda_0 \in (a, b)$ such that $LV_1 = 0$ and $LV_2 > 0$. This implies that E is repelling on its center manifold (which is on the carrying simplex Σ). On the other hand, it is easy to see that the system (2) is uniformly persistent. Then it follows from the Poincaré-Bendixson theorem of the 3-dimensional competitive system that there exists an asymptotically stable limit cycle on the carrying simplex Σ. To obtain the second limit cycle, perturb λ_0 to become slightly larger so that $LV_1 < 0$ and adjust μ such that $\mu = \frac{148137475}{100576964} - \frac{11422593}{100576964}\lambda$ which keeps the linear part of the system (2) in a center-focus form, then the second limit cycles bifurcates. In order to obtain the third limit cycle, we need the following lemma.

Lemma 2.1 *Consider the following matrix*

$$A_{\lambda,\varepsilon} = \begin{pmatrix} -2 & -\frac{129}{26} & -\lambda \\ -\frac{27}{136} & -1 & -\mu-\varepsilon \\ -\frac{99}{100} & -\frac{79}{6} & -\frac{181}{36} \end{pmatrix}$$

with two real parameters λ, ε, *where* $\mu = \frac{148137475}{100576964} - \frac{11422593}{100576964}\lambda$, $\varepsilon > 0$.

Then there exists an $\varepsilon_0 > 0$, *such that the real part of the conjugate complex roots of* $A_{\lambda,\varepsilon}$ *is positive for each* $\lambda \in (a,b)$ *and* $0 < \varepsilon < \varepsilon_0$.

proof. The characteristic equation of matrix $A_{\lambda,\varepsilon}$ is

$$\begin{vmatrix} -2-\bar{\lambda} & -\frac{129}{26} & -\lambda \\ -\frac{27}{136} & -1-\bar{\lambda} & -\mu-\varepsilon \\ -\frac{99}{100} & -\frac{79}{6} & -\frac{181}{36}-\bar{\lambda} \end{vmatrix} = 0,$$

which implies

$$(\bar{\lambda} + \frac{289}{36})(\bar{\lambda}^2 + \bar{\lambda}_0) + c_0\varepsilon = 0,$$

where $\bar{\lambda}_0 = \frac{170771}{10608} - \frac{99}{100}\lambda - \frac{79}{6}\mu - \frac{79}{6}\varepsilon$, $\mu = \frac{148137475}{100576964} - \frac{11422593}{100576964}\lambda$, $c_0 = \frac{4461929}{35100}$.

We take $\varepsilon_0 = \frac{1}{10000}$. Then $\bar{\lambda}_0 \in (1.2, 1.5)$ for all $\lambda \in (a,b)$, $\varepsilon \in (0,\varepsilon_0)$. We claim that there exists a positive constant $\alpha \in (0, 2\varepsilon)$ (where $\varepsilon \in (0,\varepsilon_0)$) such that

$$(\bar{\lambda} + \frac{289}{36})(\bar{\lambda}^2 + \bar{\lambda}_0) + c_0\varepsilon = (\bar{\lambda} + \frac{289}{36} + \alpha)(\bar{\lambda}^2 - \alpha\bar{\lambda} + \bar{\lambda}_0 + \alpha(\frac{289}{36} + \alpha)). \quad (3)$$

To prove the claim, we only have to prove that there exists $\alpha \in (0, 2\varepsilon)$ such that

$$(\frac{289}{36} + \alpha)(\bar{\lambda}_0 + \alpha(\frac{289}{36} + \alpha)) = \frac{289}{36}\bar{\lambda}_0 + c_0\varepsilon.$$

In fact, set

$$\varphi(x) = (\frac{289}{36} + x)(\bar{\lambda}_0 + x(\frac{289}{36} + x)).$$

Then

$$\varphi(0) = \frac{289}{36}\bar{\lambda_0} < c_0\varepsilon + \frac{289}{36}\bar{\lambda_0},$$
$$\varphi(2\varepsilon) = (\frac{289}{36} + 2\varepsilon)(\bar{\lambda_0} + 2\varepsilon(\frac{289}{36} + 2\varepsilon)) > c_0\varepsilon + \frac{289}{36}\bar{\lambda_0},$$

where $\varepsilon \in (0, \varepsilon_0)$. By the Intermediate Value theorem, there exists $\alpha \in (0, 2\varepsilon)$ such that $\varphi(\alpha) = c_0\varepsilon + \frac{289}{36}\bar{\lambda_0}$, that is, $(\frac{289}{36} + \alpha)(\bar{\lambda_0} + \alpha(\frac{289}{36} + \alpha)) = \frac{289}{36}\bar{\lambda_0} + c_0\varepsilon$, hence the claim is true. The lemma follows directly from (3) and $\alpha > 0$. ■

Now we return to the existence of the third limit cycle. Changing $A_{\lambda,0}$ to $A_{\lambda,\varepsilon}$ ($\varepsilon \in (0, \varepsilon_0)$) so slightly that the former two limit cycles are kept intact, it follows from Lemma 2.1 that E undergoes a supercritical Hopf bifurcation which implies that E will be surrounded by a new limit cycle which is the smallest of the three existing limit cycles.

Remark 2.1. The process of our construction is very artificial and technical. We should admit that the parameter in our example where three limit cycles coexist is extremely small (just in an interval with length 2×10^{-6}). They would be impossible to find by numerical integration. Since the second focal value seems to be closely related to the center problem (see [2]), we conjecture that the maximum order of a focus would be 2 and the maximum number of limit cycles in the 3-dimensional L-V competitive systems is 3.

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Paper VI

Gyllenberg, M. and Yan, P., Necessary and sufficient conditions for oscillations and centers of generalized Liénard systems, (submitted).

Necessary and Sufficient Conditions for Centers and Oscillations of Generalized Liénard Systems

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Abstract: In this paper we study the second-order nonlinear differential systems of Liénard-type $\dot{x} = \frac{1}{a(x)}[h(y) - F(x)]$, $\dot{y} = -a(x)g(x)$. We establish necessary and sufficient conditions to insure that all nontrivial solutions are oscillatory and the origin is a center by using a nonlinear integral inequality. Our results substantially extend and improve previous results known in the literature.

Key words: Generalized Liénard system, nonlinear integral inequality, oscillation, center.

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1 Introduction

This paper is concerned with the centers and oscillations of solutions of a generalized Liénard system of the type

$$\frac{dx}{dt} = \frac{1}{a(x)}[h(y) - F(x)],$$
$$\frac{dy}{dt} = -a(x)g(x). \tag{1}$$

The system (1) has in recent years been the object of intensive studies with particular emphasis on the asymptotic behavior of solutions (see [13, 20, 25]), because it can be considered as a natural generalization of the Liénard system

$$\frac{dx}{dt} = y - F(x),$$
$$\frac{dy}{dt} = -g(x). \tag{2}$$

As the system (2) appears in many mathematical models in physics, engineering, chemistry, biology, economics, etc., it naturally has been studied by a number of authors; many results can be found in the books [2, 11, 14, 21, 34, 35, 36].

It is well known that system (1) is of great importance in various applications, many other systems can be transformed into this form. Hence, qualitative and asymptotic behavior of this system and some of its extensions have been widely studied by many authors. To study the oscillation of solutions of (1), as discussed in some recent papers (see [5, 8, 13, 15, 24, 25, 27, 29, 30, 31]) with $a(x) \equiv 1$, for the right half plane, a significant point is to find conditions ensuring that all positive orbits $\gamma^+(P)$ (where $P = (0,\ p)$ with $p > 0$) intersect the characteristic curve $h(y) = F(x)$ and then cross the negative y-axis; this property of $\gamma^+(P)$ plays an important role in the analysis of the center, oscillation, asymptotic stability and boundedness conditions of (1). There have been many works in this direction in which sufficient conditions to obtain the above mentioned property of $\gamma^+(P)$ were given. For example (see [3, 4, 9, 10, 18, 17, 19, 28, 35]), no solution of (1) with $h(y) \equiv y$ and $a(x) \equiv 1$ approaches the origin directly in the right half plane (i.e., in a nonoscillatory way) if one of the following conditions is satisfied (in the following, $f(x) := F^{'}(x)$ if $F(x)$ is continuously differentiable and $G(x) := \int_0^x g(s)ds$):

(1) (McHarg [18]) $f(x) > 0$ for $x > 0$ and there exist $k > 0$ and $a > 0$ such that

$$f(x) < kg(x) \quad \text{for } 0 < x < a.$$

(2) (Wendel [28]) There exist $k > 0$ and $a > 0$ such that

$$0 < f(x) < kg(x) \quad \text{for } 0 < x < a.$$

(3) (Nemyckii and Stepanov [17]) There exist $\alpha > \frac{1}{4}$ and $a > 0$ such that

$$f(x) > 0, \alpha f(x)F(x) \leq g(x) \quad \text{for } 0 < x < a.$$

(4) (Filippov [3]) There exist $0 < \beta < 8$ and $a > 0$ such that

$$F^2(x) \leq \beta G(x) \quad \text{for } 0 < x < a.$$

(5) (Opial [19]) There exist $\alpha > \frac{1}{4}$ and $a > 0$ such that

$$\alpha |F(x)| \leq \int_0^x \frac{g(u)}{|F(u)|} du \quad \text{for } 0 < x < a.$$

(6) (Hara and Yoneyama [9], Hara, Yoneyama and Sugie [10], Sugie [22]) If one of the following conditions holds:

(i) there exists a positive sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and $F(x_n) \leq 0$ for $n \geq 1$;
(ii) There exist $\alpha > \frac{1}{4}$ and $a > 0$ such that

$$F(x) > 0, \; \frac{1}{F(x)} \int_0^x \frac{g(u)}{F(u)} du \geq \alpha \quad \text{for } 0 < x < a.$$

(7) (Yu [35]) There exist $a > 0$, $k_1 > 0$ and $k_2 < 0$ such that

$$k_2 \leq \frac{f(x)}{g(x)} \leq k_1 \quad \text{for } 0 < x < a.$$

Our investigation in this paper shows that condition (6) is much weaker than condition (4) (see Remark 5.5 in this paper). The problem concerning the oscillation of solutions of (1) with $a(x) \equiv 1$ has been studied by some authors (see, for example, [15, 30] and the references cited therein). Li and Tang [15] discussed the oscillation of solutions of (1) with $a(x) \equiv 1$ requiring the existence of $h''(y)$ and $h'(0) > 0$. Yan and Jiang [30] proved that the solutions of (1) with $a(x) \equiv 1$ are oscillatory under the condition $h'(0) > 0$. But the problem of what happens when $h'(0) = 0$ or $h'(0) = \infty$ remains unsolved. In the present paper, no restrictions on the differentiability of $h(y)$ are required. We give necessary and sufficient conditions for all nontrivial solutions of (1) being oscillatory. Our theorem can be applied to system (1) even for $h'(0) = 0$, $h'(0) = \infty$ and $\lim_{|x|\to\infty} F(x)$ sgn $x = -\infty$. Our results substantially extend and improve some results known in the literature.

Another purpose here is to develop a center theory for the system (1). This work was motivated by the papers of Hara and Yoneyama [9] and Sugie [25], in which a detailed analysis of center properties was given for system (2). We will follow closely the presentation of Hara, Yoneyama and Sugie, and show that all of their results on this subject can be generalized to (1).

The technical tool of this paper is based on a nonlinear integral inequality and a phase plane analysis. Also the methods for Liénard-type systems, especially those developed by Villari and Zanolin [27], Hara and Sugie [8], and Sugie and Hara [23] will be applied in our paper

The organization of this paper is as follows. In Section 2 we agree on some notation, present assumptions and some lemmas which will be essential to our proofs. In Section 3, we give some sufficient conditions of a local center for (1). In particular, we point out that the results of [37] in Section 3 are indeed corollaries of Hara and Yoneyama [9] and Sugie [25]. In Section 4, we give some sufficient and necessary conditions of a global center for (1). Moreover, we also point out that the results of [37] in Section 4 are corollaries of Sugie [25]. In Section 5 we give sufficient and necessary conditions for the oscillation of all solutions of (1). Some examples illustrating the results are also given in this paper.

2 Notation and Preliminaries

We consider the generalized Liénard system

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{a(x)}[h(y) - F(x)], \\ \frac{dy}{dt} &= -a(x)g(x), \end{aligned} \tag{3}$$

where $F(x)$, $g(x)$, $a(x)$ and $h(y)$ are continuous real functions defined on $\mathbf{R}$ satisfying:

(A_0) $F(0) = 0$, $a(x) > 0$ for $x \in \mathbf{R}$, $xg(x) > 0$ for $x \neq 0$;

(A_1) $yh(y) > 0$ for $y \neq 0$, $h(y)$ is strictly increasing and $h(\pm\infty) = \pm\infty$.

These assumptions guarantee that the origin is the only critical point of (3). We also assume that the initial value problem always has a unique solution.

We call the curve $h(y) = F(x)$ the characteristic curve of system (3). We write $\gamma^+(P)$ (resp., $\gamma^-(P)$) the positive (resp., negative) semiorbit of (3) starting at a point $P \in \mathbf{R^2}$. For the sake of convenience, we denote

$$D_1 = \{(x,y) : x \geq 0,\ h(y) > F(x)\},\quad D_2 = \{(x,y) : x > 0,\ h(y) \leq F(x)\},$$
$$D_3 = \{(x,y) : x \leq 0,\ h(y) < F(x)\},\quad D_4 = \{(x,y) : x < 0,\ h(y) \geq F(x)\}.$$

$$F_+(x) = \max\{0,\ F(x)\},\qquad F_-(x) = \max\{0,\ -F(x)\},$$
$$\Gamma_+(x) = \int_0^x a^2(s)g(s)(1+F_+(s))^{-1}ds,\quad \Gamma_-(x) = \int_0^x a^2(s)g(s)(1+F_-(s))^{-1}ds.$$

$$Y^+ = \{(0,y) : y > 0\},\qquad Y^- = \{(0,y) : y < 0\},$$
$$C^+ = \{(x,y) : x > 0,\ h(y) = F(x)\},\quad G(x) = \int_0^x a^2(s)|g(s)|ds.$$

Then by (A_0), $G(x)$ is strictly increasing, and therefore, the inverse function $G^{-1}(w)$ of $w = G(x)$ exists.

Throughout this paper we shall suppose that the following conditions hold:

(A_2) $\int_0^{-\infty} a^2(s)g(s)ds = \int_0^{\infty} a^2(s)g(s)ds$;

(A_3) $F(G^{-1}(-w)) = F(G^{-1}(w))$ for $0 < w < M$,
where $M = \min\{G(\infty),\ G(-\infty)\}$ (M may be ∞).

If $F(x)$ and $a^2(x)g(x)$ are even and odd functions, respectively, then it is obvious that (A_2) and (A_3) are satisfied and that all the orbits of (3) have mirror symmetry about the y-axis in the phase space. Moreover, for example, if $F(x) = 3x$, $a(x) \equiv 1$ and $g(x) = 2x$ for $x \geq 0$, and $F(x) = -3\sqrt{2}x$, $a(x) \equiv 1$ and $g(x) = 4x$ for $x \leq 0$, then (A_2) and (A_3) are also satisfied.

Firstly, employing an argument similar to that in [9, 22], we show that under the conditions (A_2) and (A_3), the orbits of (3) have deformed mirror symmetry about the y-axis.

Lemma 2.1 *Suppose that the conditions (A_2) and (A_3) are satisfied. If an orbit of (3) starting from a point $A(0, y_A)$ with $y_A > 0$ passes through a point $B(0, y_B)$ with $y_B < 0$, then it reaches the point A again.*

proof. Consider an orbit of (3) which starts from a point $A(0, y_A)$ with $y_A > 0$ and passes through a point $B(0, y_B)$ with $y_B < 0$. We denote this orbit by $T(x,y)$ and write $T_1(x,y)=\{(x,y)\in T : x\geq 0\}$ and $T_2(x,y)=\{(x,y)\in T : x<0\}$. Let $K=\int_0^\infty a^2(x)g(x)dx$ (K may be ∞) and let the mapping $\varphi:(x,y)\to(u,v)$ defined by

$$\begin{aligned} u &= \begin{cases} \sqrt{2G(x)} & \text{for } x\geq 0 \\ -\sqrt{-2G(x)} & \text{for } x<0, \end{cases} \\ v &= y. \end{aligned}$$

Then we can see that the image $\varphi_{T_1}(u,v)$ of $T_1(x,y)$ is an orbit of the system

$$\begin{aligned} u' &= h(v)-F^*(u), \\ v' &= -u, \end{aligned}$$

defined on $(-\sqrt{2K},\ \sqrt{2K})\times\mathbf{R}$, where

$$F^*(u)=\begin{cases} F(G^{-1}(\frac{1}{2}u^2)) & \text{for } 0\leq u<\sqrt{2K} \\ F(G^{-1}(-\frac{1}{2}u^2)) & \text{for } -\sqrt{2K}<u<0. \end{cases}$$

In fact, for any point $(u,v)\in\varphi_{T_1}$,

$$\begin{aligned} \frac{du}{dv} &= \frac{a^2(x)g(x)}{\sqrt{2G(x)}}\cdot\frac{h(y)-F(x)}{-a^2(x)g(x)} \\ &= \frac{h(v)-F^*(u)}{-u}. \end{aligned}$$

Note that the curve $\varphi_{T_1}(u,v)$ contains the points A and B. It follows from (A_2) and (A_3) that $F^*(u)$ is an even function on $(-\sqrt{2K},\ \sqrt{2K})$. Hence, the curve $\varphi_{T_1}(-u,v)$ is also an orbit of (3) which contains the points A and B. Let $T_3(x,y)$ be the inverse image of $\varphi_{T_1}(-u,v)$ under the mapping φ. Then for any point $(x,y)\in T_3$,

$$\begin{aligned} \frac{dx}{dy} &= \frac{\sqrt{-2G(x)}}{a^2(x)g(x)}\cdot\frac{h(v)-F(G^{-1}(\frac{u^2}{2}))}{-u} \\ &= \frac{h(v)-F(x)}{-a^2(x)g(x)}. \end{aligned}$$

Thus, $T_3(x,y)$ is an orbit of (3) which starts from the point B and arrives at the point A. Since the solutions of (3) are unique, $T_2(x,y)$ and $T_3(x,y)$ coincide, and hence the orbit $T(x,y)$ reaches the point A again. This completes the proof.

Remark 2.1. If the condition (A_3) holds for $w>0$ sufficiently small, then all the orbits of (3) near the origin have deformed mirror symmetry with respect to the y-axis.

Lemma 2.2 *Let $Y(x)$, $\psi(x)$ be positive continuous functions in $0 < a \leq x \leq b$ and let $\omega(u)$ be a positive increasing continuous function for $u > 0$, and let $\Omega(u) = \int_{0+}^{u} \frac{dt}{\omega(t)}$ exists for $u > 0$ with $\Omega(0) = 0$. Then for $\lambda > 0$ the inequality*

$$Y(x) \geq \lambda \int_a^x \psi(t)\omega(Y(t))dt \quad \text{for } a \leq x \leq b \tag{4}$$

implies the inequality

$$\Omega(Y(x) \geq \lambda \int_a^x \psi(t)dt \quad \text{for } a \leq x \leq b. \tag{5}$$

proof. Define

$$V(x) = \lambda \int_a^x \psi(t)\omega(Y(t)dt \quad \text{for } a \leq x \leq b. \tag{6}$$

Then (4) can be restated as $Y(x) \geq V(x)$. Because $\omega(u)$ is increasing, this may be rewritten as follows

$$\omega(Y(x)) \geq \omega(V(x)),$$

$$\tfrac{V'(x)}{\omega(V(x))} \geq \lambda\psi(x)$$

for $a < x \leq b$. By making use of the notation $\Omega(u)$, we have

$$\frac{d\Omega(V(x))}{dx} \geq \lambda\psi(x) \quad \text{for } a < x \leq b. \tag{7}$$

Now, integrating from a to x, we get by (7),

$$\Omega(V(x)) - \Omega(V(a)) \geq \lambda \int_a^x \psi(t)dt.$$

Since $V(a) = 0$, it follows that

$$\Omega(V(x)) \geq \lambda \int_a^x \psi(t)dt \quad \text{for } a \leq x \leq b. \tag{8}$$

Because $Y(x) \geq V(x)$ for $a \leq x \leq b$, and $\Omega(u)$ is increasing, we obtain by (8),

$$\Omega(Y(x)) \geq \lambda \int_a^x \psi(t)dt \quad \text{for } a \leq x \leq b.$$

This completes the proof.

3 Conditions of a Local Center

Definition 3.1. The origin is called a local center for (3) if all the orbits of (3) in some neighbourhood of it are closed curves surrounding it.

Now, we state the assumptions on (3). The assumption (A_0) through (A_3) have been presented in Section 2.

The assumption which guarantees that the origin is a local center of system (3) is given by (A_4). The system (3) is said to satisfy (A_4) if one of the following conditions holds:

$(A_4)_1$ there exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and $F(x_n) = 0$ for $n \geq 1$;

$(A_4)_2$ there exist constants $m > 0$, $p > 0$ and $\delta_1 > 0$ such that $|h(y)| \geq m|y|^p$ for $0 < |y| < \delta_1$, and

$$|F(G^{-1}(w))| \leq aw^{\frac{p}{p+1}} \quad \text{for } 0 < w \ll 1,$$

where $0 < a < m(1+p)(\frac{1+p}{mp})^{\frac{p}{(1+p)}}$, and the notation $0 < w \ll 1$ denotes w sufficiently small;

$(A_4)_3$ there exist constants $\alpha > \frac{1}{4}$ and $\delta_2 > 0$ such that $|F(x)| > 0$ for $0 < x \leq \delta_2$, and for any fixed real number $k \geq 1$,

$$\int_{0^+}^{x} \frac{a^2(s)g(s)}{|F(s)|} ds \geq \frac{1}{k} h^{-1}(k\alpha|F(x)|) \quad \text{for } 0 < x \ll 1,$$

where $h^{-1}(u)$ is the inverse function of $u = h(y)$.

Lemma 3.1 *If conditions (A_0), (A_1) and (A_4) hold, then for any $P = (x_0,\ y_0) \in C^+$,*
(i) *$\gamma^-(P)$ must intersect Y^+ at $A(0,\ y_A)$ with $y_A > 0$;*
(ii) *$\gamma^+(P)$ must intersect Y^- at $B(0,\ y_B)$ with $y_B < 0$.*

proof. We only prove (ii); (i) can be proved in a similar way.

Let $P = (x_0,\ y_0) \in C^+$ and $(x(t),\ y(t))$ be the solution of (3) with $x(0) = x_0,\ y(0) = y_0$. By the uniqueness of the solutions of (3), we only have to show that every orbit $\gamma^+(P)$ of (3) passing through $P = (x_0,\ y_0)$ $(0 < x_0 \ll 1)$ intersect Y^- at $B(0,\ y_B)$ with $y_B < 0$. Since $\lim_{y \to -\infty} h(y) = -\infty$, the system (3) has no vertical asymptote in the fourth quadrant. Therefore,

$\gamma^+(P)$ must intersect the y-axis at $B(0,\ y_B)$ with $y_B \leq 0$. We still have to show that $y_B \neq 0$. We do this separately for the different cases of (A_4).

Case $(A_4)_1$**:** It is obvious in this case.

Case $(A_4)_2$**:** In this case the proof is completely analogous to the proof of [13, Lemma 3.1] or [29, Theorem 2.4].

Case $(A_4)_3$**:** It follows from (A_0) that the orbit $\gamma^+(P)$ of (3) does not touch the characteristic curve at any point $(x,\ h^{-1}(F(x)))$ with $0 \leq x < x_0$. Thus, we consider only the region $\{(x,y) : x > 0,\ h(y) < F(x)\}$.

If $F(x) < 0$ for $0 < x \leq x_0$, it is clear that $y_B < 0$. Suppose that $F(x) > 0$ for $0 < x \leq x_0$ and that the conclusion does not hold. Then there exists a point $P \in C^+$ such that $\gamma^+(P)$ does not intersect Y^-. Let $(x(t),\ y(t))$ $(0 \leq t < \infty)$ denote the solution of (3) which passes through such a point P. Then $\gamma^+(P)$ must be contained in the first quadrant, and $x(t)$ decreases and $y(t)$ decreases as t is increasing. Since the origin is the unique equilibrium of (3), $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$. The solution $(x(t),\ y(t))$ defines a function $y = y(x)$ on $0 \leq x \leq x_0$, which is a solution on $0 < x < x_0$ of the following equation

$$\frac{dy}{dx} = -\frac{a^2(x)g(x)}{h(y) - F(x)}. \tag{9}$$

It follows from $\lim_{x\to 0^+} y(x) = 0$ that $y(x) > 0$ for $0 < x \leq x_0$. By assumption $(A_4)_3$, there exist $\alpha > \frac{1}{4}$ and $x_1 \in (0,\ x_0)$ such that $F(x) > 0$ for $0 < x \leq x_1$, and

$$\int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)} ds \geq h^{-1}(\alpha F(x)) \quad \text{for } 0 < x \leq x_1. \tag{10}$$

Now, we restrict our attention to the interval $(0,\ x_1]$. Putting $H_1(u) = \int_0^u h(y)dy$, we have by (9), for any $0 < \varepsilon \ll 1$,

$$\begin{aligned} H_1(y(x)) - H_1(y(\varepsilon)) &= \int_\varepsilon^x H_1'(y(s)) \frac{a^2(s)g(s)}{F(s) - h(y(s))} ds \\ &\geq \int_\varepsilon^x h(y(s)) \frac{a^2(s)g(s)}{F(s)} ds \\ &= \int_\varepsilon^x (h \circ H_1^{-1})(H_1(y(s))) \frac{a^2(s)g(s)}{F(s)} ds \end{aligned}$$

for $\varepsilon \le x \le x_1$. Hence

$$H_1(y(x)) \ge \int_{\varepsilon}^{x} (h \circ H_1^{-1})(H_1(y(s))\frac{a^2(s)g(s)}{F(s)}ds$$

for $\varepsilon \le x \le x_1$. It follows from Lemma 2.2 that

$$H_2(H_1(y(x)) \ge \int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{F(s)}ds \quad \text{for } \varepsilon \le x \le x_1, \tag{11}$$

where $H_2(u) = \int_{0^+}^{u} \frac{dt}{(h\circ H_1^{-1})(t)}$. Changing variables $H_1^{-1}(t) = \tau$, it is easy to see that $H_2(u) = H_1^{-1}(u)$. By (11), we have

$$y(x) \ge \int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{F(s)}ds \quad \text{for } \varepsilon \le x \le x_1. \tag{12}$$

(i) If $\int_{0^+}^{x_1} \frac{a^2(s)g(s)}{F(s)}ds = \infty$, we reach a contradiction by (12).

(ii) If $\int_{0^+}^{x_1} \frac{a^2(s)g(s)}{F(s)}ds < \infty$, we see from (12) that

$$y(x) \ge \int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)}ds \quad \text{for } 0 < x \le x_1. \tag{13}$$

By virtue of (10) and (13), we have $y(x) \ge h^{-1}(\alpha F(x))$ for $0 < x \le x_1$. Because $h(y)$ is strictly increasing, we obtain $h(y(x)) \ge \alpha F(x)$ for $0 < x \le x_1$. Since $y = y(x)$ is under the characteristic curve $h(y) = F(x)$, we have $\frac{1}{4} < \alpha < 1$. Let $\alpha_1 = 1 - \alpha$, then we get that $F(x) - h(y(x)) \le \alpha_1 F(x)$ for $0 < x \le x_1$. In a similar way, for any $0 < \varepsilon \ll 1$, we have

$$\begin{aligned} H_1(y(x)) - H_1(y(\varepsilon)) &= \int_{\varepsilon}^{x} h(y(s))\frac{a^2(s)g(s)}{F(s) - h(y(s))}ds \\ &\ge \frac{1}{\alpha_1}\int_{\varepsilon}^{x} h(y(s))\frac{a^2(s)g(s)}{F(s)}ds \end{aligned}$$

for $\varepsilon \le x \le x_1$. Therefore

$$\begin{aligned} H_1(y(x)) &\ge \frac{1}{\alpha_1}\int_{\varepsilon}^{x} h(y(s))\frac{a^2(s)g(s)}{F(s)}ds \\ &= \frac{1}{\alpha_1}\int_{\varepsilon}^{x} (h \circ H_1^{-1})(H_1(y(s))\frac{a^2(s)g(s)}{F(s)}ds \end{aligned}$$

for $\varepsilon \le x \le x_1$. By Lemma 2.2, we have

$$\begin{aligned} H_2(H_1(y(x)) &\ge \frac{1}{\alpha_1}\int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{F(s)}ds, \\ y(x) &\ge \frac{1}{\alpha_1}\int_{\varepsilon}^{x} \frac{a^2(s)g(s)}{F(s)}ds \end{aligned}$$

for $\varepsilon \le x \le x_1$. Hence

$$y(x) \ge \frac{1}{\alpha_1}\int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)}ds \tag{14}$$

for $\varepsilon \le x \le x_1$. By assumption $(A_4)_3$, there exists $x_2 \in (0,\ x_1)$ such that

$$\int_{0^+}^{x} \frac{a^2(s)g(s)}{F(s)}ds \ge \alpha_1 h^{-1}(\frac{\alpha}{\alpha_1}F(x)) \tag{15}$$

for $0 < x \le x_2$. By virtue of (14) and (15), we have $y(x) \ge h^{-1}(\frac{\alpha}{\alpha_1}F(x))$ for $0 < x \le x_2$. Because $h(y)$ is strictly increasing, we get $h(y(x)) \ge \frac{\alpha}{\alpha_1}F(x)$ for $0 < x \le x_2$. Thus, $F(x) - h(y(x)) \le \alpha_2 F(x)$ with $\alpha_2 = 1 - \frac{\alpha}{\alpha_1}$. Repeating this procedure, we obtain two sequences $\{x_n\}$ and $\{\alpha_n\}$ such that $\alpha_n = 1 - \frac{\alpha}{\alpha_{n-1}}$ and $F(x) - h(y(x)) \le \alpha_n F(x)$ for $0 < x \le x_n$. If $\alpha_n \le 0$, we have a contradiction. Suppose $\alpha_n > 0$ $(n = 1, 2, \ldots)$, then $(\alpha_n - \alpha_{n-1})(1-\alpha_n) = -\alpha_n + \alpha_n - \alpha < -(\alpha_n - \frac{1}{2})^2 \le 0$, $\{\alpha_n\}$ is decreasing, and hence $\{\alpha_n\}$ converges to some real number λ. On the other hand, $\lambda = 1 - \frac{\alpha}{\lambda}$ and $\alpha > \frac{1}{4}$ show that λ is a complex number, which is a contradiction. This completes the proof.

From Lemma 2.1 and Lemma 3.1, we deduce the following theorem.

Theorem 3.1 *If conditions (A_0), (A_1), (A_3) and (A_4) hold, then the origin is a local center of system (3).*

From Theorem 3.1 and Remark 2.1, we obtain the following result.

Corollary 3.1 *If conditions (A_0), (A_1) and (A_4) hold, and there exists $K_0 > 0$ such that*

(A_3*) $\quad F(G^{-1}(-w)) = F(G^{-1}(w))$ *for* $0 \le w < K_0$,

then the origin is a local center of system (3).

If $h(y) \equiv y$, $a(x) \equiv 1$, then Corollary 3.1 gives the results of Opial [19], Hara and Yoneyama [9] and Sugie [22] as follows.

Corollary 3.2 *Let $h(y) \equiv y$, $a(x) \equiv 1$, and conditions (A_0) and (A_3) (or (A_3*)) hold, and let either of the following two conditions hold.*

$(A_4*)_1$ *there exists a positive sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and $F(x_n) = 0$ for $n \geq 1$;*

$(A_4*)_2$ *there exist constants $\alpha > \frac{1}{4}$ and $\delta_3 > 0$ such that $|F(x)| > 0$ for $0 < x \leq \delta_3$, and*

$$\int_{0+}^{x} \frac{g(s)}{|F(s)|} ds \geq \alpha |F(x)| \quad \text{for } 0 < x \leq \delta_3.$$

Then the origin is a local center of system (3).

Remark 3.1. If there exist $\delta_4 > 0$ and a continuous function $r(x)$ such that for $0 < x < \delta_4$

(1) $r(x) \geq |F(x)| > 0$,

(2) $\frac{1}{r(x)} \int_{0+}^{x} \frac{g(s)}{r(s)} ds \geq \alpha > \frac{1}{4}$,

then

$$\int_{0+}^{x} \frac{g(s)}{|F(s)|} ds \geq \int_{0+}^{x} \frac{g(s)}{r(s)} ds \geq \alpha r(x) \geq \alpha |F(x)|$$

for $0 < x < \delta_4$. Obviously, the condition $(A_4*)_2$ is satisfied. Hence, the results of [37] in Section 3 are corollaries of Corollary 3.2. Thus, Theorem 2.1 and Theorem 2.2 of [35] are corollaries of Corollary 3.2. Moreover, the results of McHarg [18] and Wendel [28] in Section 1 are all results of Corollary 3.2.

Remark 3.2. The condition $(A_4)_3$ is a generalization of the following condition $(A_4*)_3$.

$(A_4*)_3$ there exist constants $\alpha_0 > 0$ and $\delta_5 > 0$ such that $h(y)$ is continuously differential on $[0,\ \delta_5]$, $|F(x)| > 0$ for $0 < x \leq \delta_5$, and

$$\int_{0+}^{x} \frac{a^2(s)g(s)}{|F(s)|} ds \geq \alpha_0 |F(x)| \quad \text{for } 0 < x \ll 1,$$

where $\alpha = h'(0)\alpha_0 > \frac{1}{4}$.

In fact, if the condition $(A_4*)_3$ is satisfied, then there exist constants $0 \le \bar{\delta} < \delta_5$ and $\frac{1}{4} < \bar{\alpha} < \alpha$ such that $h'(y) > \frac{\bar{\alpha}}{\alpha_0}$ for $0 \le y \le \bar{\delta}$, and for any fixed real number $k \ge 1$, we have

$$\begin{aligned}
\frac{1}{k}h^{-1}(k\bar{\alpha}|F(x)|) &= \frac{1}{k}h^{-1}(k\bar{\alpha}|F(x)|) - \frac{1}{k}h^{-1}(0) \\
&= \frac{1}{k}\frac{dh^{-1}(u)}{du}\ |_{u=\xi}\ k\bar{\alpha}|F(x)|, \quad 0 < \xi < k\bar{\alpha}|F(x)| \\
&= \frac{\bar{\alpha}|F(x)|}{h'(h^{-1}(\xi))} < \alpha_0|F(x)| \text{ for } 0 < x \ll 1.
\end{aligned}$$

Thus the condition $(A_4*)_3$ implies the condition $(A_4)_3$.

From Theorem 3.1 and Remark 3.2, we obtain the following result.

Corollary 3.3 *If conditions (A_0), (A_1) and (A_3), and either of the conditions $(A_4*)_1$ and $(A_4*)_3$ hold, then the origin is a local center of system (3).*

Remark 3.3. If $a(x) \equiv 1$, Corollary 3.3 reduces to Theorem 2.3 of [29].

Moreover, we also have

Corollary 3.4 *If conditions (A_0), (A_1) and (A_{3^*}) hold, and suppose*

(C_1) *there exists $K > 0$ such that $yh(y) \ge Ky^2$,*

(C_2) *there exist $a > 0$ and $r(x) \in C^2(\mathbf{R})$, for $0 < x < a$, we have*

(1) $r(x) \ge |F(x)| > 0$,

(2) $\frac{K}{r(x)}\int_0^x \frac{a^2(s)g(s)}{r(s)}ds \ge \alpha > \frac{1}{4}$.

Then the origin is a local center of system (3).

proof. For any fixed real number k, we have

$$\frac{1}{|F(x)|}\int_0^x \frac{a^2(s)g(s)}{|F(s)|} \ge \frac{1}{r(x)}\int_0^x \frac{a^2(s)g(s)}{r(s)}ds \ge \frac{\alpha}{K},$$

that is

$$\int_0^x \frac{a^2(s)g(s)}{|F(s)|}ds \ge \frac{\alpha}{K}|F(x)| = \frac{k\alpha|F(x)|}{kK} \ge \frac{1}{k}h^{-1}(k\alpha|F(x)|),\ 0 < x \ll 1.$$

Hence the condition $(A_4)_3$ in our paper is satisfied. The origin is a local center of (3) by Corollary 3.1.

Remark 3.4. If $a(x) \equiv 1$, then the Theorem 2 of [32] is a result of Corollary 3.4.

Remark 3.5. If $a(x) \equiv 1$, then by condition $(A_4)_2$, Lemma 3.1 is seen to be a generalization of Theorem 4.6 and Theorem 4.12 of Sugie [22]. But our result holds also for $0 < p < 1$.

Example 1. In system (3), we take $a(x) \equiv 1$, $h(y) = y^2 \text{sgn } y$, and

$$F(x) = \begin{cases} x\sqrt{x} & \text{for } x \geq 0 \\ -x\sqrt[4]{8x^2} & \text{for } x < 0, \end{cases} \quad g(x) = \begin{cases} x & \text{for } x \geq 0 \\ 2x & \text{for } x < 0. \end{cases}$$

Then it is easy to prove that $\int_0^{-\infty} g(x)dx = \int_0^{\infty} g(x)dx = \infty$, and for $0 \leq w < \infty$,

$$F(G^{-1}(-w)) = F(-\sqrt{w}) = \sqrt[4]{8w^3} = F(\sqrt{2w}) = F(G^{-1}(w)).$$

It is also clear that $\int_{0+}^{x} \frac{g(s)}{F(s)}ds = 2\sqrt{x}$ for $x > 0$, and for any fixed real number $k \geq 1$,

$$\int_{0+}^{x} \frac{g(s)}{F(s)}ds \geq \frac{1}{k}h^{-1}(kF(x)) \quad \text{for } 0 < x \ll 1,$$

thus $(A_4)_2$ is satisfied. Then the origin is a local center for (3) by Theorem 3.1.

Example 2. In system (3), we take $a(x) \equiv 1$, $h(y) = \sqrt{|y|}\text{sgn } y$, and

$$F(x) = \begin{cases} 3x & \text{for } x \geq 0 \\ -3\sqrt{2x} & \text{for } x < 0, \end{cases} \quad g(x) = \begin{cases} 2x & \text{for } x \geq 0 \\ 4x & \text{for } x < 0. \end{cases}$$

Then $\int_0^{-\infty} g(x)dx = \int_0^{\infty} g(x)dx = \infty$, and for $0 \leq w < \infty$,

$$F(G^{-1}(-w)) = F(-\sqrt{\frac{w}{2}}) = 3\sqrt{w} = F(\sqrt{w}) = F(G^{-1}(w)).$$

Hence (A_0), (A_1) and (A_3) are satisfied. It is easy to prove that $\int_{0+}^{x} \frac{g(s)}{F(s)}ds = \frac{2}{3}x$ for $x > 0$, and for any fixed real number $k \geq 1$,

$$\int_{0+}^{x} \frac{g(s)}{F(s)}ds \geq \frac{1}{k}h^{-1}(kF(x)) \quad \text{for } 0 < x \ll 1,$$

therefore $(A_4)_2$ is satisfied. Then the origin is a local center for (3) by Theorem 3.1.

4 Conditions of a Global Center

Definition 4.1. The origin is called a global center for (3) if all orbits of (3) are closed curves surrounding it.

The final assumption presented here is to guarantee that all positive (resp., negative) semiorbits $\gamma^+(P)$ (resp., $\gamma^-(P)$) for $P \in D_1$, (resp., $P \in D_2$) intersect C^+.

We say (3) satisfies the assumption (A_5) if both (A_5^+) and (A_5^-) hold.

The system (3) is said to satisfy (A_5^+) if one of the following conditions holds:

$(A_5^+)_1$ $\quad \overline{\lim}_{x\to\infty} F(x) \neq -\infty$;

$(A_5^+)_2$ $\quad \overline{\lim}_{x\to\infty} F(x) = -\infty$, and there exist $\beta > \frac{1}{4}$ and $N_1 > 0$ such that $F(x) < 0$ for $x \geq N_1$, and for any fixed $k \geq 1$ and $b \geq N_1$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{a^2(s)g(s)}{F(s)} ds \leq \frac{1}{k} h^{-1}(k\beta F(x)) \quad \text{for } x \geq \bar{b}.$$

The system (3) is said to satisfy (A_5^-) if one of the following conditions hold:

$(A_5^-)_1$ $\quad \underline{\lim}_{x\to\infty} F(x) \neq \infty$;

$(A_5^-)_2$ $\quad \underline{\lim}_{x\to\infty} F(x) = \infty$, and there exist $\beta > \frac{1}{4}$ and $N_1 > 0$ such that $F(x) > 0$ for $x \geq N_1$, and for any fixed $k \geq 1$ and $b \geq N_1$, there exist $\bar{b} > b$ satisfying

$$\int_b^x \frac{a^2(s)g(s)}{F(s)} ds \geq \frac{1}{k} h^{-1}(k\beta F(x)) \quad \text{for } x \geq \bar{b}.$$

Lemma 4.1 *Suppose that system* (3) *satisfies* (A_0) *and* (A_1). *Then*

(i) *If* (A_5^+) *holds, then for any* $P \in D_1$, $\gamma^+(P)$ *intersects* C^+ *if and only if*

$$\overline{\lim}_{x\to\infty}(\Gamma_-(x) + F(x)) = \infty; \tag{16}$$

(ii) *If* (A_5^-) *holds, then for any* $P \in D_2$, $\gamma^-(P)$ *intersects* C^+ *if and only if*

$$\overline{\lim}_{x\to\infty}(\Gamma_+(x) - F(x)) = \infty. \tag{17}$$

proof. We prove only (i); (ii) can be proved in a similar way.

Sufficiency. Suppose the conclusion is false. Then there is a point $P = (x_0,\ y_0) \in D_1$ such that $\gamma^+(P)$ does not intersect C^+. Let $(x(t),\ y(t))(t \geq 0)$ be the solution of (3) passing through such a point P whose maximal existence interval is $[0,\ \omega_+)$. Note that $x'(t) > 0$ and $y'(t) < 0$ in the region D_1, hence $x(t)$ is increasing and $y(t)$ is decreasing as t is increasing. Suppose that $x(t)$ is bounded, then $(x(t),\ y(t))$ stays in the region $\{(x, y) : 0 < x < K_1,$ and $h(y) > F(x)\}$ for some $K_1 > 0$. Hence it must intersect the characteristic curve, which is a contradiction. Therefore $x(t) \to \infty$ as $t \to \omega_+$.

Case 1: Suppose $\overline{\lim}_{x\to\infty} F(x) = \infty$, that is, there exists a sequence $\{x_n\}$ such that $x_n \to \infty (n \to \infty)$ and $\lim_{n\to\infty} F(x_n) = \infty$, then $(x(t),\ y(t))$ must intersect the characteristic curve, which is a contradiction.

Case 2: Suppose $\int_0^\infty \frac{a^2(x)g(x)}{1+F_-(x)} dx = \infty$, then

$$\begin{aligned} y(t) - y_0 &= -\int_0^t a(x(s))g(x(s))ds \\ &= -\int_0^t \frac{a^2(x(s))g(x(s))}{h(y(s)) - F(x(s))} \dot{x}(s)ds \\ &= -\int_{x_0}^{x(t)} \frac{a^2(\xi)g(\xi)}{h(y(s)) - F(\xi)} d\xi \\ &\leq -\int_{x_0}^{x(t)} \frac{a^2(\xi)g(\xi)}{h(y_0) + F_-(\xi)} d\xi \to -\infty \end{aligned}$$

as $t \to \omega_+$. Then the orbit of the above solution can be considered as a function $y(x)$ which is a solution of the equation (9), and $y(x) \to -\infty$ as $x \to \infty$.

Case $(A_5)_1$**:** There exist $c > 0$ and a sequence $\{x_n\}$ such that $x_n \to \infty\ (n \to \infty)$, and $F(x_n) \geq -c$, hence $(x(t),\ y(t))$ must intersect the characteristic curve, which is a contradiction.

Case $(A_5)_2$**:** There exists $b > N_1$ such that $F(x) < 0$ and $y(x) < 0$ for $x \geq b$. Since $y(x)$ is a solution of (9), putting $H_3(u) = \int_0^u h(y)dy$ for $u \leq 0$, we have

$$\begin{aligned} H_3(y(x)) - H_3(y(b)) &= \int_b^x H_3'(y(s))\frac{a^2(s)g(s)}{F(s)-h(y(s))}ds \\ &\geq \int_b^x h(y(s))\frac{a^2(s)g(s)}{F(s)}ds \\ &= \int_b^x (h\circ H_3^{-1})(H_3(y(s))\frac{a^2(s)g(s)}{F(s)}ds \end{aligned}$$

for $x \geq b$. Hence

$$H_3(y(x)) \geq \int_b^x (-h\circ H_3^{-1})(H_3(y(s))\frac{a^2(s)g(s)}{-F(s)}ds$$

for $x \geq b$. It follows from Lemma 2.2 that

$$H_4(H_3(y(x)) \geq \int_b^x \frac{a^2(s)g(s)}{-F(s)}ds \quad \text{for } x \geq b, \tag{18}$$

where $H_4(u) = \int_{0^+}^u \frac{dt}{(-h\circ H_3^{-1})(t)}$. Changing variable $H_3^{-1}(t) = \tau$, then $H_4(u) = -H_3^{-1}(u)$, by (18), it is easy to see that

$$y(x) \leq \int_b^x \frac{a^2(s)g(s)}{F(s)}ds \quad \text{for } x \geq b. \tag{19}$$

From the assumption $(A_5^+)_2$, there exist $\beta > \frac{1}{4}$ and $b_1 > b$ such that

$$\int_b^x \frac{a^2(s)g(s)}{F(s)}ds \leq h^{-1}(\beta F(x)) \quad \text{for } x \geq b_1. \tag{20}$$

By virtue of (19) and (20), we have $y(x) \leq h^{-1}(\beta F(x))$ for $x \geq b_1$. Because $h(y)$ is strictly increasing, we obtain $h(y(x)) \leq \beta F(x)$ for $x \geq b_1$. Hence $F(x) - h(y(x)) \geq \beta_1 F(x)$ for $x \geq b_1$, where $\beta_1 = 1 - \beta$. By a similar argument, we have

$$\begin{aligned} H_3(y(x)) - H_3(y(b_1)) &= \int_{b_1}^x h(y(s))\frac{a^2(s)g(s)}{F(s)-h(y(s))}ds \\ &\geq \frac{1}{\beta_1}\int_{b_1}^x h(y(s))\frac{a^2(s)g(s)}{F(s)}ds \\ &= \frac{1}{\beta_1}\int_{b_1}^x (h\circ H_3^{-1})(H_3(y(s))\frac{a^2(s)g(s)}{F(s)}ds \end{aligned}$$

for $x \geq b_1$. Hence

$$H_3(y(x)) \geq \frac{1}{\beta_1}\int_{b_1}^{x}(-h\circ H_3^{-1})(H_3(y(s))\frac{a^2(s)g(s)}{-F(s)}ds$$

for $x \geq b_1$. By Lemma 2.2, it can be shown that

$$\begin{aligned} H_4(H_3(y(x)) &\geq \frac{1}{\beta_1}\int_{b_1}^{x}\frac{a^2(s)g(s)}{-F(s)}ds, \\ y(x) &\leq \frac{1}{\beta_1}\int_{b_1}^{x}\frac{a^2(s)g(s)}{F(s)}ds \end{aligned} \tag{21}$$

for $x \geq b_1$. From the assumption $(A_5^+)_2$, there exists $b_2 > b_1$ such that

$$\int_{b_1}^{x}\frac{a^2(s)g(s)}{F(s)}ds \leq \beta_1 h^{-1}(\frac{\beta}{\beta_1}F(x)) \text{ for } x \geq b_2. \tag{22}$$

By virtue of (21) and (22), we have $y(x) \leq h^{-1}(\frac{\beta}{\beta_1}F(x))$ for $x \geq b_2$. Thus $F(x) - h(y(x)) \geq \beta_2 F(x)$ for $x \geq b_2$, where $\beta_2 = 1 - \frac{\beta}{\beta_1}$. Repeating this procedure, we obtain two sequences $\{b_n\}$ and $\{\beta_n\}$ such that $\beta_n = 1 - \frac{\beta}{\beta_{n-1}}$ and $F(x) - h(y(x)) \geq \beta_n F(x)$ for $x \geq b_n$. If $\beta_n > 0$ $(n = 1, 2, \ldots)$, then $\{\beta_n\}$ is decreasing, and $\{\beta_n\}$ converges to some real number λ, on the other hand $\lambda = 1 - \frac{\beta}{\lambda}$ and $\beta > \frac{1}{4}$ show that λ is a complex number, which is a contradiction. Hence, $\beta_n \leq 0$ for some n, that is $F(x) \geq h(y(x))$ for all $x \geq b_n$, a contradiction. This completes the proof of sufficiency.

Necessity. Suppose (16) does not hold. Then there exist $M_1 > 0$ and $L > 0$ such that $F(x) < M_1$ for $x \geq 0$ and $\int_L^\infty \frac{a^2(x)g(x)}{1+F_-(x)}dx < 1$. Suppose $(x(t),\ y(t))$ is a solution of (3), and $(x(0);\ y(0)) = (L,\ M_1 + M_0 + 1) = P$ where $M_0 > 0$ satisfying $h(M_1 + M_0) \geq M_1 + 1$.

We will show that $y(t) > M_1 + M_0$ for $t > 0$. Suppose not. There exists $t_1 > 0$ such that $y(t_1) = M_1 + M_0$ and $M_1 + M_0 < y(t) \leq M_1 + M_0 + 1$ for all $t \in [0,\ t_1)$, and we have

$$\begin{aligned} y(t_1) &= M_1 + M_0 + 1 - \int_0^{t_1}\frac{a^2(x(s))g(x(s))}{h(y(s)) - F(x(s))}\dot{x}(s)ds \\ &\geq M_1 + M_0 + 1 - \int_L^{x(t_1)}\frac{a^2(\xi)g(\xi)}{1 + F_-(\xi)}d\xi > M_1 + M_0. \end{aligned}$$

This is a contradiction. Hence, $\dot{x}(t) = h(y(t)) - F(x(t)) > M + 1 - F(x(t)) > 1$ for all $t \geq 0$. Thus the solution $(x(t),\ y(t))$ is unbounded and $\gamma^+(P)$ is above the characteristic curve $h(y) = F(x)$. Thus the necessity is proved. This completes the proof.

We now state our main result.

Theorem 4.1 *Suppose that the origin is a local center of* (3), *and that the conditions* $(A_0) - (A_3)$ *and* (A_5) *are satisfied. Then the origin is a global center of* (3) *if and only if* (16) *and* (17) *hold.*

proof. Sufficiency. By the uniqueness of the solutions of (3) and the fact that the origin is a local center of (3), no orbit of (3) tends to the origin. To prove the theorem, we must show that the orbits of (3) starting from all the points in D_i $(i = 1,\ 2,\ 3,\ 4)$ are closed curves surrounding the origin.

Consider the orbit of (3) starting from a point $(x_0,\ y_0) \in D_1$. Then by Lemma 4.1, this orbit intersects the characteristic curve at some point $(x_1,\ h^{-1}(F(x_1))$ with $x_1 > 0$. It follows from (A_0) that the orbit does not touch again the characteristic curve at any point $(x,\ h^{-1}(F(x))$ with $0 \le x < x_1$. Since $\lim_{y\to-\infty} h(y) = -\infty$, the system (3) has no vertical asymptote in the fourth quadrant. Therefore, the orbit must cross the y-axis at a point $B(0,\ y_B)$ with $y_B < 0$.

By replacing t by $-t$, we can also see that the orbit crosses the y-axis at a point $A(0,\ y_A)$ with $y_A > 0$. Thus by Lemma 2.1, the orbit reaches the point A again, and so the orbit is a closed curve surrounding the origin.

By a similar argument, the orbit of (3) starting from a point in D_2, D_3 or D_4 is also closed, thus the origin is a global center of (3).

Necessity. Suppose that the condition (16) or (17) does not hold. Then it follows immediately from Lemma 4.1 that the origin is not a global center of (3). The proof of Theorem 4.1 is now complete.

If $h(y) \equiv y$, $a(x) \equiv 1$, then Theorem 4.1 reduces to the result of Sugie [22] as follows.

Corollary 4.1 *Let* $h(y) \equiv y$, $a(x) \equiv 1$ *and suppose that the origin is a local center of* (3) *and that the conditions* (A_0), (A_2), (A_3) *and* (A_5) *are satisfied. Then the origin is a global center of* (3) *if and only if* (16) *and* (17) *hold.*

Corollary 4.2 *If conditions* (A_0), (A_1) *and* (A_3) *hold, and suppose*

(C_1) $\overline{\lim}_{x\to\infty} F(x) > -\infty$, $\lim_{x\to\infty} \inf F(x) < \infty$,

(C_2) *there exists* $K > 0$ *such that* $yh(y) \ge Ky^2$ *and one of the following conditions holds*

(i) *there exist* $a > 0$, $r(x) \in C^2(\mathbf{R})$, *for* $0 < x < a$,

(1) $r(x) \geq |F(x)| > 0$,

(2) $\frac{K}{r(x)} \int_0^x \frac{g(s)}{r(s)} ds \geq \alpha > \frac{1}{4}$,

(ii) *there exists* $a > 0$ *such that for* $0 < x < a$

$$\frac{K}{F(x)} \int_0^x \frac{g(s)}{F(s)} ds \geq \alpha > \frac{1}{4},$$

(iii) *there exist* $\alpha > 0$, $\gamma > 0$, $a > 0$ *such that for* $0 < x < a$

$$|F(x)| \leq \alpha |G(x)|^\gamma$$

where $\frac{1}{2} < \gamma < 1$ *or* $\alpha < \sqrt{8K}$, $\gamma = \frac{1}{2}$.

Then the origin is a global center of (3) if and only if (16) *and* (17) *hold.*

proof. From Remark 3.1, it is easy to see that the condition (i) is equivalent to condition (ii), by Corollary 3.4, we know the condition $(A_4)_3$ is a generalization of condition (i) and (ii).

Suppose the condition (iii) is satisfied and $F(x)$ does not satisfy the condition $(A_4)_1$ in our paper, then $|F(x)| > 0$ for $0 < x < a_1 < a$.

If $\frac{1}{2} < \gamma < 1$, $\alpha > 0$, then we have

$$\begin{aligned} \frac{K}{F(x)} \int_0^x \frac{g(s)}{F(s)} ds &\geq \frac{K}{\alpha (G(x))^\gamma} \int_0^x \frac{g(s)}{\alpha (G(s))^\gamma} ds \\ &= \frac{K}{(1-\gamma)\alpha^2} (G(x))^{-\gamma} (G(x))^{1-\gamma}. \end{aligned}$$

From $G(0) = 0$, if a_1 is sufficiently small, then $\frac{K}{F(x)} \int_0^x \frac{g(s)}{F(s)} ds \geq 1 > \frac{1}{4}$ for $0 < x < a_1$, hence the condition (ii) is satisfied.

If $\gamma = \frac{1}{2}$, $\alpha < \sqrt{8K}$, then from above,

$$\frac{K}{F(x)} \int_0^x \frac{g(s)}{F(s)} ds \geq \frac{K}{(1-\gamma)\alpha^2} > \frac{1}{4},$$

thus the condition (ii) is also satisfied. Therefore, the condition (C_2) implies the condition (A_4) in our paper, by Theorem 3.1 and Theorem 4.1, we know the origin is a global center of (3) if and only if (16) and (17) hold.

Remark 4.1. If $a(x) \equiv 1$, then Theorem 3 of [32] is a result of Corollary 4.2.

Remark 4.2. It is easy to see that Theorem 3.1 of [35], and Theorem 4.1, Theorem 4.2, Theorem 4.4 and Theorem 4.5 of [37] are all corollaries of Corollary 4.1, and that our results cover the corresponding results of [9].

Remark 4.3. The condition $(A_5^+)_2$ is a generalization of the following condition $(A_5^+)_{2^*}$.

$(A_5^+)_{2^*}$ $\overline{\lim}_{x\to\infty} F(x) = -\infty$, and there exist $N_1 > 0$, $\beta_0 > \frac{1}{4}$ and $\bar{\beta}_0 > 0$ such that $h(y)$ is continuously differentiable on $(-\infty,\ -N_1]$, $h'(-y) \geq \frac{\beta_0}{\bar{\beta}_0}$ for $y \geq N_1$, $F(x) < 0$ for $x \geq N_1$ and for any $b \geq N_1$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{a^2(s)g(s)}{F(s)}ds \leq \bar{\beta}_0 F(x) \quad \text{for } x \geq \bar{b}.$$

In fact, if the condition $(A_5^+)_{2^*}$ is satisfied, then there exist $N_1 > 0$, $\beta_0 > \frac{1}{4}$ and $\bar{\beta}_0 > 0$ such that $h'(-y) \geq \frac{\beta_0}{\bar{\beta}_0}$ for $y \geq N_1$, and for any fixed real number $k \geq 1$, there exists $N_2 > N_1$ satisfying $k\bar{\beta}_0 F(x) \leq -N_1$ for $x \geq N_2$, and

$$\begin{aligned} h(k\bar{\beta}_0 F(x)) &< h(k\bar{\beta}_0 F(x)) - h(k\bar{\beta}_0 F(N_2)) \\ &= k\bar{\beta}_0 h'(\xi)\frac{F(x) - F(N_2)}{F(x)}F(x) \end{aligned}$$

for $x > N_2$, where ξ is between $k\bar{\beta}_0 F(x)$ and $k\bar{\beta}_0 F(N_2)$. Since $\lim_{x\to\infty} F(x) = -\infty$, for any $b \geq N_1$, it can be shown that there exist $\frac{1}{4} < \beta < \beta_0$ and $b^* > b$ such that $h(k\bar{\beta}_0 F(x)) < k\beta F(x)$ for $x \geq b^*$. Because $h(y)$ is strictly increasing, we have $\bar{\beta}_0 F(x) \leq \frac{1}{k}h^{-1}(k\beta F(x))$ for $x \geq b^*$. Hence the condition $(A_5^+)_{2^*}$ implies $(A_5^+)_2$.

By the same argument, it can be seen that condition $(A_5^-)_2$ is a generalization of the following condition $(A_5^-)_{2^*}$

$(A_5^-)_{2^*}$ $\lim_{x\to\infty} F(x) = \infty$, and there exist $N_1 > 0$, $\beta_0 > \frac{1}{4}$ and $\bar{\beta}_0 > 0$ such that $h(y)$ is continuously differentiable on $[N_1,\ \infty)$, $h'(y) \geq \frac{\beta_0}{\bar{\beta}_0}$ for $y \geq N_1$, $F(x) > 0$ for $x \geq N_1$ and for any $b \geq N_1$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{a^2(s)g(s)}{F(s)}ds \geq \bar{\beta}_0 F(x) \quad \text{for } x \geq \bar{b}.$$

Remark 4.4. If $a(x) \equiv 1$, the Theorem 4.1 is a generalization of [29, Theorem 2.1]. This follows from Remark 4.3.

Example 3. In system (3), we take $a(x) \equiv 1$, $h(y) \equiv |y|^{\frac{3}{2}} \mathrm{sgn}\, y$ and

$$F(x) = \begin{cases} 4x \sin \frac{\pi}{2}x & \text{for } \ x \geq 1 \\ 4x & \text{for } \ 0 \leq x < 1 \\ -2x & \text{for } \ -2 \leq x < 0 \\ 2x \sin \frac{\pi}{4}x & \text{for } \ x < -2, \end{cases}$$

$$g(x) = \begin{cases} \frac{3}{2}x^{-2} & \text{for } \ x \geq 1 \\ \frac{3}{2}x^{\frac{1}{2}} & \text{for } \ 0 \leq x < 1 \\ -\frac{3}{4}(-\frac{x}{2})^{\frac{1}{2}} & \text{for } \ -2 < x < 0 \\ -3x^{-2} & \text{for } \ x \leq -2. \end{cases}$$

Then it is easy to show that

$$\int_0^{-\infty} g(x)dx = \int_0^{\infty} g(x)dx = \frac{5}{2}$$

and

$$G^{-1}(w) = \begin{cases} \frac{3}{5-2w} & \text{for } \ 1 \leq w < \frac{5}{2} \\ w^{\frac{2}{3}} & \text{for } \ 0 \leq w < 1 \\ -2w^{\frac{2}{3}} & \text{for } \ -1 \leq w < 0 \\ -\frac{6}{5+2w} & \text{for } \ -\frac{5}{2} < w < -1. \end{cases}$$

For $0 \leq w < 1$,

$$F(G^{-1}(-w)) = F(-2w^{\frac{2}{3}}) = 4w^{\frac{2}{3}} = F(w^{\frac{2}{3}}) = F(G^{-1}(w))$$

and for $1 \leq w < \frac{5}{2}$,

$$F(G^{-1}(-w)) = F(-\frac{6}{5-2w}) = \frac{12}{5-2w} = F(\frac{3}{5-2w}) = F(G^{-1}(w)).$$

Hence (A_2) and (A_3) are satisfied. It is also clear that $\int_{0+}^{x} \frac{g(s)}{F(s)}ds = \frac{3}{4}x^{\frac{1}{2}}$ for $0 < x \ll 1$, and that for any fixed real number $k \geq 1$,

$$\int_{0+}^{x} \frac{g(s)}{F(s)}ds \geq \frac{1}{k}h^{-1}(kF(x)) \quad \text{for } 0 < x \ll 1.$$

Thus $(A_4)_2$ is satisfied. It is obvious that (A_0), (A_1) and (A_5) are also satisfied. Then the origin is a local center for (3) by Theorem 3.1. It follows from Theorem 4.1 that the origin is a global center for (3).

5 Conditions of Oscillation

In this section, we give our main result about necessary and sufficient conditions for the oscillation of solutions of (3). We assume that all solutions of (3) can be continued in the forward direction up to $t = \infty$. A solution $(x(t),\ y(t))$ of (3) is oscillatory if there are two sequences $\{t_n\}$ and $\{\tau_n\}$ tending monotonically to ∞ such that $x(t_n) = 0$ and $y(\tau_n) = 0$ for every $n \geq 1$.

We say that (3) satisfies the assumption (A_6) if both (A_6^+) and (A_6^-) hold.

The system (3) is said to satisfy (A_6^+) if one of the following conditions holds:

$(A_6^+)_1$ There exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$, and $F(x_n) \leq 0$ for $n \geq 1$;

$(A_6^+)_2$ There exist constants $\alpha > \frac{1}{4}$ and $\delta_1 > 0$ such that

$$F(x) > 0 \quad \text{for} \quad 0 < x \leq \delta_1,$$

and for any fixed real number $k \geq 1$,

$$\int_{0+}^{x} \frac{a^2(x)g(s)}{F(s)} ds \geq \frac{1}{k} h^{-1}(k\alpha F(x)) \quad \text{for} \quad 0 < x \ll 1,$$

where $h^{-1}(u)$ is the inverse function of $u = h(y)$, and the notation $0 < x \ll 1$ denotes x sufficiently small.

The system (3) is said to satisfy (A_6^-) if one of the following conditions holds:

$(A_6^-)_1$ There exists a negative decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$, and $F(x_n) \geq 0$ for $n \geq 1$;

$(A_6^-)_2$ There exist constants $\alpha > \frac{1}{4}$ and $\delta_2 > 0$ such that

$$F(x) < 0 \quad \text{for} \quad 0 < -x \leq \delta_2,$$

and for any fixed real number $k \geq 1$,

$$\int_{0-}^{x} \frac{a^2(x)g(s)}{F(s)} ds \leq \frac{1}{k} h^{-1}(k\alpha F(x)) \quad \text{for} \quad 0 < -x \ll 1.$$

Lemma 5.1 *Suppose that the conditions* (A_0)*,* (A_1)*, and* (A_6^+) *hold. Then for any* $P = (x_0,\ y_0) \in C^+$*, the positive semiorbit* $\gamma^+(P)$ *intersects the negative* y*-axis.*

By a similar argument, we have the following lemma in the left half plane.

Lemma 5.2 *Suppose that the conditions* (A_0)*,* (A_1)*, and* (A_6^-) *hold. Then for any* $P = (x_0,\ y_0) \in C^-$*, the positive semiorbit* $\gamma^+(P)$ *intersects the positive* y*-axis.*

Remark 5.1. If $h(y) \equiv y$ and $a(x) \equiv 1$, then condition $(A_6^+)_2$ is the condition (ii) of (6) in Section 1 (cf. [9, 10, 22]).

By the above discussion, the condition (A_6) is a generalization of condition (A_3) in [13], condition (A_{10}) in [30], condition (A_2) in [30], and condition (C) in [15].

The final assumptions presented here are to guarantee that all positive orbits $\gamma^+(P)$ for $\in D_1$ (resp., $P \in D_3$) intersect C^+ (resp., C^-).

We say (3) satisfies the assumption (A_7) if both (A_7^+) and (A_7^-) hold.

The system (3) is said to satisfy (A_7^+) if one of the following conditions holds:

$(A_7^+)_1$ $\limsup_{x\to\infty} F(x) \neq -\infty$;

$(A_7^+)_2$ $\limsup_{x\to\infty} F(x) = -\infty$, and there exist $\beta > \frac{1}{4}$ and $N_1 > 0$ such that $F(x) < 0$ for $x \geq N_1$, and for any fixed $k \geq 1$ and $b \geq N_1$, there exists $\bar{b} > b$ satisfying

$$\int_b^x \frac{a^2(x)g(s)}{F(s)}ds \leq \frac{1}{k}h^{-1}(k\beta F(x)) \quad \text{for } x \geq \bar{b}.$$

The system (3) is said to satisfy (A_7^-) if one of the following conditions hold:

$(A_7^-)_1$ $\liminf_{x\to-\infty} F(x) \neq \infty$;

$(A_7^-)_2$ $\liminf_{x\to-\infty} F(x) = \infty$, and there exist $\beta > \frac{1}{4}$ and $N_1 > 0$ such that $F(x) > 0$ for $x \geq -N_1$, and for any fixed $k \geq 1$ and $b \geq N_1$, there exist $\bar{b} > b$ satisfying

$$\int_{-b}^{x} \frac{a^2(x)g(s)}{F(s)} ds \geq \frac{1}{k} h^{-1}(k\beta F(x)) \quad \text{for } x \geq -\bar{b}.$$

Lemma 5.3 *Suppose that the conditions* (A_0), (A_1), *and* (A_7^+) *hold. Then every positive semiorbit of* (3) *departing from* D_1 *intersects the characteristic curve* C^+ *if and only if*

$$\limsup_{x\to\infty} \left[\int_0^x \frac{a^2(x)g(s)}{1+F_-(s)} ds + F(x) \right] = \infty, \tag{23}$$

where $F_-(x) = \max\{0,\ -F(x)\}$.

In a similar way, we can prove the following lemma in the left half plane.

Lemma 5.4 *Suppose that the conditions* (A_0), (A_1), *and* (A_7^-) *hold. Then every positive semiorbit of* (3) *departing from* D_3 *intersects the characteristic curve* C^- *if and only if*

$$\limsup_{x\to-\infty} \left[\int_0^x \frac{a^2(x)g(s)}{1+F_+(s)} ds - F(x) \right] = \infty, \tag{24}$$

where $F_+(x) = \max\{0,\ F(x)\}$.

Remark 5.2. If $h(y) \equiv y$ and $a(x) \equiv 1$, then the conditions $(A_7^+)_2$ and $(A_7^-)_2$ are the conditions (A_2^+) and (A_2^-) in [10] respectively, the condition $(A_7^+)_2$ is the condition $(C_3)_2$ in [22].

Remark 5.3. By the above discussion, the condition (A_7) is a generalization of condition (A_4) (with $a(x) \equiv 1$) in [13] and condition (A_3) (with $a(x) \equiv 1$) in [30]. Moreover, the condition (A_7^+) is a generalization of condition (A_4) (with $a(x) \equiv 1$) in [29].

We are now in the position to give our main result about necessary and sufficient conditions for the oscillation of solutions of system (3).

Theorem 5.1 *Suppose that the conditions* (A_0), (A_1), (A_6) *and* (A_7) *are satisfied. Then all nontrivial solutions of (3) oscillate if and only if (23) and (24) hold.*

Proof. Necessary. If either (23) or (24) is false, then Lemma 5.3 and Lemma 5.4 imply that (3) has at least one unbounded solution lying in D_1 or D_3. Thus the necessity is proved.

Sufficiency. We prove the sufficiency by contradiction. Suppose that there exist a solution $(x(t),\ y(t))$ of (3) and $T_0 > 0$ such that $x(t) \neq 0$ for all $t \geq T_0$. We consider the case $x(t) > 0$ for all $t \geq T_0$. The Lemma 5.1 implies that $(x(t),\ y(t))$ does not tend to $(0,\ 0)$ as $t \to \infty$.

(i) suppose $(x(T_0),\ y(T_0)) \in D_1$, the Lemma 5.3 shows that there exist $T_1 > T_0$ such that $(x(t),\ y(t))$ intersects the characteristic curve $h(y) = F(x)$ at $t = T_1$. Then $x(t)$ and $y(t)$ are decreasing for all $t \geq T_1$. Thus there exists $K_0 \geq 0$ such that

$$\begin{aligned} x(t) &\to K_0 \quad as\ t \to \infty \qquad (25)\\ y(t) &\to -\infty \quad as\ t \to \infty \\ K_0 \leq x(t) &\leq x(T_1) \quad for\ all\ t \geq T_1. \end{aligned}$$

For $t \geq T_1$, we have

$$\begin{aligned} x(t) - x(T_1) &= \int_{T_1}^{t} \frac{1}{a(x)}(h(y(s)) - F(x(s))ds \\ &\leq \int_{T_1}^{t} \frac{1}{a(x)}(h(y(s)) - min_{K_0 \leq x \leq x(T_1)}\{F(x)\})ds \\ &\to -\infty \quad as\ t \to \infty, \end{aligned}$$

which contradicts (25).

(ii) Suppose $(x(T_0),\ y(T_0)) \in D_2$, by a similar method used in the case (i), we can reach a contradiction. In case $x(t) < 0$ for all $t \geq T_0$, we have also a contradiction by an argument similar to the one above. Hence all solution of (3) are oscillatory. Thus the proof of Theorem 5.1 is now complete.

Remark 5.4. Theorem 5.1 is a generalization of Theorem 1 in [30] and Theorem 1 in [15], this follows from Remark 2.2 and 2.4. Our results do not need the differentiability condition of $h(y)$, our Theorem 3.1 can be applied to system (3) even for $h'(0) = 0$, $h'(0) = \infty$, $h'(\pm\infty) = 0$, and $\lim_{|x| \to \infty} F(x)\ \mathrm{sgn}\ x = -\infty$.

If $h(y) \equiv y$ and $a(x) \equiv 1$, by Theorem 5.1 and Remarks 5.1 and 5.2, we have the following corollary which is the result of Hara, Yoneyama and Sugie [10].

Corollary 5.1 *Suppose that (3) with $h(y) \equiv y$ and $a(x) \equiv 1$ has a unique solution, and that the conditions (A_0), (A_1), (A_6) and (A_7) are satisfied. Then all nontrivial solutions of (3) with $h(y) \equiv y$ and $a(x) \equiv 1$ oscillate if and only if (23) and (24) hold.*

Remark 5.5. In system (3), we take $h(y) \equiv y$, $g(x) \equiv x$, $a(x) \equiv 1$, and

$$F(x) = \begin{cases} 2x & \text{for } x \geq \frac{1}{3}, \\ \frac{2x_{3n}-x_{6n}^3}{x_{3n}-x_{3n+1}}(x - x_{3n}) + 2x_{3n} & \text{for } x_{3n+1} \leq x \leq x_{3n}, \\ x_{6n}^3 & \text{for } x_{3n+2} \leq x \leq x_{3n+1}, \\ \frac{2x_{3n+3}-x_{6n}^3}{x_{3n+3}-x_{3n+2}}(x - x_{3n+3}) + 2x_{3n+3} & \text{for } x_{3n+3} \leq x \leq x_{3n+2}, \\ 0 & \text{for } x = 0, \\ -4x - x^2 & \text{for } x \leq 0, \end{cases}$$

where $x_n = \frac{1}{n}$, $n = 1, 2, ...$
Then $G(x) = \frac{x^2}{2}$, $F(0) = 0$, $F(x) \leq 2x$ for $0 < x < \frac{1}{3}$, $F(x) = 2x$ for $x \geq \frac{1}{3}$, and $F(x) = -4x - x^2$ for $x \leq 0$. Thus (A_0), (A_1), $(A_6^-)_1$, $(A_7^+)_1$, $(A_7^-)_1$, (23) and (24) are satisfied. For any $x \in (0, \frac{1}{3})$, we can choose n (sufficiently large) such that $0 < x_{3n+2} < x_{3n+1} < x$, and

$$\frac{1}{F(x)} \int_{0^+}^{x} \frac{g(s)}{F(s)} ds \geq \frac{3}{2} \int_{x_{3n+2}}^{x_{3n+1}} \frac{g(s)}{F(s)} ds = \frac{3}{2} \int_{x_{3n+2}}^{x_{3n+1}} \frac{s}{x_{6n}^3} ds = \frac{3(6n)^3(6n+3)}{4(3n+1)^2(3n+2)^2} > \frac{3}{4}$$

for $0 < x < \frac{1}{3}$. The condition $(A_6^+)_2$ is satisfied. Therefore, by Corollary 5.1, all nontrivial solutions oscillate.

Because $F^2(z_n) = 8G(z_n)$ when $z_n = \frac{1}{3n}$ $(n = 1, 2, ...)$, it follows that condition (4) in Section 1 is not satisfied. By the above discussion, condition $(A_6^+)_2$ is satisfied, hence, the condition (6) in Section 1 is really weaker than condition (4). The condition (6) is similar to (5) of Opial [19], but condition (6) is more precise. Moreover, condition (6) is a generalization of conditions (1), (2), (3), (4), and (7) in Section 1.

Example 4. In system (3), we take $a(x) = 1$, $h(y) = y^{\frac{1}{3}}$, $g(x) = x^5$, and $F(x) = -x|x|^{\beta}$, where β is a real number such that $0 \leq \beta < \frac{1}{2}$.
Then (A_0), (A_1), $(A_6^+)_1$, $(A_6^-)_1$, (23) and (24) are satisfied. Since $h^{-1}(u) = u^3$, for any $b > 1$ and fixed real number $k \geq 1$, we have

$$\lim_{x\to\infty} \frac{k}{h^{-1}(kF(x))} \int_b^x \frac{g(s)}{F(s)} ds = \lim_{x\to\infty} \frac{1}{(5-\beta)k^2 x^{3(1+\beta)}} (x^{(5-\beta)} - b^{(5-\beta)}) = \infty,$$

therefore $(A_7^+)_2$ is satisfied. Similarly, $(A_7^-)_2$ is also satisfied. Then all nontrivial solutions oscillate by Theorem 5.1. However $h'(0) = \infty$ and $h'(\pm\infty) = 0$, the previous results of [15, 30] cannot be applied to this example. It is easy to see from Remark 5.4 and Example 4 that our Theorem 5.1 can find more extensive applications.

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Paper VII

Gyllenberg, M., Yan, P., and Wang, Y., Limit cycles for the competitor-competitor-mutualist Lotka-Volterra systems, (revised).

Limit Cycles for the Competitor-Competitor-Mutualist Lotka-Volterra Systems*

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Abstract It is known that the limit cycle (or periodic coexistence) can occur in the Competitor-Competitor-Mutualist Lotka-Volterra systems

$$\begin{cases} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2 + a_{13}x_3), \\ \dot{x}_2 = x_2(r_2 - a_{21}x_1 - a_{22}x_2 + a_{23}x_3), \\ \dot{x_3} = x_3(r_3 + a_{31}x_1 + a_{32}x_2 - a_{33}x_3), \end{cases}$$

where r_i, a_{ij} are positive real constants. (c.f., [16]). In this paper, we shall construct an example with at least two limit cycles, and furthermore, we will show that the number of periodic orbits (and hence a fortiori of limit cycles) is finite. It is also showed that, contrary to three-dimensional competitive Lotka-Volterra systems, the nontrivial periodic coexistence does happen even if none of the three species can resist invasion from either of the others. In this case, new amenable conditions are given on the coefficients under which the system has no nontrivial periodic coexistence. These conditions imply that the positive equilibrium, if it exists, is globally asymptotically stable.

Key words and phrases: Competitor-Competitor-Mutualist, Lotka-Volterra systems, Limit cycles, Hopf Bifurcation.

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1 Introduction

The dynamics of an ecosystem with $n \geq 2$ interacting populations can be modelled by the general Lotka-Volterra system

$$\dot{x}_i = x_i(r_i + \sum_{j=1}^{n} a_{ij} x_j), \qquad i = 1, 2, ..., n, \tag{1}$$

where x_i is the density of the ith population, r_i is the intrinsic growth rate of the ith population and the coefficient a_{ij} describes the influence of the jth population upon the ith population (Hofbauer and Sigmund [12]). The signs of a_{ij} and a_{ji} determine the nature of the interaction between the populations i and j: the system (1) can describe all of the three basic types of interaction, viz., competition, collaboration (mutualism) and host-parasite (predator-prey) interactions.

The dynamics of two-dimensional Lotka-Volterra systems is well understood. Bomze [3] gave a complete classification of all possible phase portraits for this case. In particular, there are no limit cycles in two-dimensional Lotka-Volterra systems: if there is a periodic orbit, then the equilibrium in Int$\mathbb{R}^2_+$ is a center (that is, it is surrounded by a continuum of periodic orbits). As is well known, this is the case in the classical Lotka-Volterra predator-prey system. It should, however, be noted that the phase portrait does not reveal the whole dynamics. For example, the solution may blow up in finite time (this is clear because the system (1) contains the system $\dot{x}_i = x_i^2$ as a special case).

As one steps from two to higher dimensions the situation becomes far more complicated and difficult. By using numerical simulations, three-dimensional Lotka-Volterra systems allow already complicated dynamics. The period doubling route to chaos and many other phenomena known from the interaction of the quadratic map have been observed (see [1, 7, 21]).

For three-dimensional competitive Lotka-Volterra systems, the dynamical possibilities are more restricted: Hirsch [11] has showed that all nontrivial orbits approach a "carrying simplex", a Lipshitz two-dimensional manifold-with-corner homeomorphic to the standard simplex in $\mathbb{R}^3_+$. Based on this, Zeeman [26] has given a classification of all possible stable phase portraits of three-dimensional competitive Lotka-Volterra systems and has shown that in some three-dimensional competitive Lotka-Volterra systems limit cycles can indeed occur. Recently, Liang and Jiang [16] did the same for Competitor-Competitor-Mutualist Lotka-Volterra systems. Hofbauer and So [13], Xiao and Li [25] and Lu and Luo [17] have also presented examples of three-dimensional competitive Lotka-Volterra systems with at least two limit cycles.

In this paper, we focus on the limit cycles for the Competitor-Competitor-Mutualist Lotka-Volterra systems. The specific system we shall consider models two competing populations that both collaborate with a third one. Such systems are of great biological relevance. The two competing populations may, for instance, represent two different types of the same species (a "resident" and "mutant" in the terminology of adaptive dynamics, Metz et al. [18]; Geritz et al. [8, 9]). More models of this type can be found in [10, 16] and the references therein.

In the following sections, we shall prove that the number of nontrivial periodic orbits (and hence a fortiori of limit cycles) is finite in Competitor-Competitor-Mutualist Lotka-Volterra systems. We also construct an example of a system of this type with at least two limit cycles by using local Hopf bifurcation and analyse the scale of the parameters.

It also deserves to be noted that it is under the assumption $M_{12} = a_{11}a_{22} - a_{12}a_{21} < 0$ that Liang and Jiang [16] obtained the existence of the nontrivial limit cycle, generated by Hopf bifurcation, in Competitor-Competitor-Mutualist Lotka-Volterra systems [16, Theorem 5.5]. Hence in this case, in the competitive subcommunity of two species 1 and 2, at least one can resist invasion by the other. For the three-dimensional competitive Lotka-Volterra systems, van den Driessche and Zeeman [6] have shown that if none of the species can resist invasion by either of the others, then there is no periodic orbit and therefore limit cycles do not exist and global dynamics are known. For the system (2), it is obvious that none of the species can resist invasion by the other in the mutualistic subcommunity of two species 1 and 3, or 2 and 3. Therefore, it is a very interesting question whether there exist periodic orbits in Competitor-Competitor-Mutualist Lotka-Volterra systems if none of the species can resist invasion by the other in the competitive subcommunity of two species 1 and 2. In this paper, we shall answer this question by providing an example which has a stable limit cycle. Meanwhile, new amenable conditions are also given on the coefficients r_i, a_{ij}, under which system (1) has no periodic orbits if none of the species can resist invasion from either of the others. Thus all trajectories converge to equilibria. Based on this, we also present an example of global stability for a positive equilibrium, but the Volterra multipliers method [16, Theorem

5.6] cannot be applied.

The paper is organized as follows: In Section 2 we formulate the model and provide a review of some relevant related results. The main results are presented in Section 3 and full proofs are given in Section 4 and 5.

2 Background material

The Lotka-Volterra system (1) which models such a community takes the form

$$\begin{cases} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2 + a_{13}x_3) = x_1 f_1(x) \equiv F_1(x), \\ \dot{x}_2 = x_2(r_2 - a_{21}x_1 - a_{22}x_2 + a_{23}x_3) = x_2 F_2(x) \equiv F_2(x), \\ \dot{x_3} = x_3(r_3 + a_{31}x_1 + a_{32}x_2 - a_{33}x_3) = x_3 f_3(x) \equiv F_3(x), \end{cases} \tag{2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+ = \{x : x_i \geq 0 \text{ for } i = 1, 2, 3\}$, $r_i > 0$ and $a_{ij} > 0$ for $i, j = 1, 2, 3$. For obvious biological reasons, we restrict our attention to the closed positive orthant $\mathbb{R}^3_+$ and we denote the open positive orthant by $\mathrm{Int}\mathbb{R}^3_+$. Let $F = (F_1, F_2, F_3)$ in (2). It is easy to check that the Jacobian $DF(x)$ has the form

$$\begin{pmatrix} -D & B \\ C & -d \end{pmatrix},$$

in which D is a 2×2 matrix, B is a 2×1 matrix, C is a 1×2 matrix and d is a positive real number. Each off-diagonal element of D is nonnegative, and B and C are nonnegative matrices.

Note that the backward flow $\varphi_t(\cdot)$ of (2) is type-K monotone, that is, $x \leq_K y$ implies $\varphi_t x \leq_K \varphi_t y$ for all $t \leq 0$ and $x, y \in \mathbb{R}^3_+$ (where $x \leq_K y$ means $x_i \leq y_i$ for $i = 1, 2$ and $x_3 \geq y_3$) (see [22, 15]). Liang and Jiang [16] used the theory of monotone dynamical systems, together with the index theory of fixed points, to prove that there exists an invariant two-dimensional Lipschitz manifold V attracting all nontrivial orbits provided (2) is dissipative. Therefore the Poincaré-Bendixson theorem holds in Competitor-Competitor-Mutualist Lotka-Volterra systems . Based on this, Liang and Jiang [16] have also given a classification of all possible stable phase portraits of Competitor-Competitor-Mutualist Lotka-Volterra systems.

Motivated by the fundamental assumptions of the classification of three-dimensional competitive Lotka-Volterra systems in [26], we hereafter always assume that system (2) is dissipative and each equilibrium in $\mathbb{R}^3_+ \setminus \mathrm{Int}\mathbb{R}^3_+$ is hyperbolic (see [16]). The system (2) is called *dissipative* if there is a compact totally invariant set which uniformly attracts each compact set of initial values.

The restriction of (2) to the ith coordinate axis is the logistic equation $\dot{x}_i = x_i(r_i - a_{ii}x_i)$, which has an equilibrium at the carrying capacity $R_i = r_i/a_{ii}$. Furthermore, the assumption of dissipation implies that the following two subsystems of (2) are also dissipative:

$$\begin{cases} \dot{x}_1 = x_1(r_1 - a_{11}x_1 + a_{13}x_3), \\ \dot{x_3} = x_3(r_3 + a_{31}x_1 - a_{33}x_3), \end{cases} \tag{2_I}$$

$$\begin{cases} \dot{x}_2 = x_2(r_2 - a_{22}x_2 + a_{23}x_3), \\ \dot{x_3} = x_3(r_3 + a_{32}x_2 - a_{33}x_3), \end{cases} \tag{2_J}$$

where $I = \{1, 3\}$ and $J = \{2, 3\}$. Since systems (2_I) and (2_J) are cooperative, it follows from Smith [23] or Jiang [14] that (2_I) and (2_J) are dissipative if and only if

$$M_{13} = a_{11}a_{33} - a_{13}a_{31} > 0,$$

$$M_{23} = a_{22}a_{33} - a_{23}a_{32} > 0,$$

respectively. The systems(2_I) and (2_J) have the interior equilibria R_{13} and R_{23} which are globally asymptotically stable in $\mathrm{Int}H_I^+$ and $\mathrm{Int}H_J^+$, respectively. Here $\mathrm{Int}H_I^+ = \{x \in \mathbb{R}_+^3 : x_i > 0 \text{ for } i \in I \text{ and } x_i = 0 \text{ for } i \notin I\}$, similarly for $\mathrm{Int}H_J^+$. Since each subsystem of $\mathbb{R}_+^3$ is invariant under (2), we abuse notation somewhat and allow R_i to denote a point in $\mathbb{R}_+$, $\mathbb{R}_+^2$ or $\mathbb{R}_+^3$ and R_{13} and R_{23} to denote points in $\mathbb{R}_+^2$ or $\mathbb{R}_+^3$. By the assumption of hyperbolicity, the stability of R_{13} (R_{23}) can be determined by the sign of L_1 (L_2), where

$$\begin{aligned} L_1 &= r_1(a_{31}a_{23} - a_{21}a_{33}) + r_2(a_{11}a_{33} - a_{13}a_{31}) + r_3(a_{11}a_{23} - a_{13}a_{21}), \\ L_2 &= r_1(a_{22}a_{33} - a_{23}a_{32}) + r_2(a_{13}a_{32} - a_{12}a_{33}) + r_3(a_{13}a_{22} - a_{12}a_{23}). \end{aligned} \tag{3}$$

More precisely, R_{i3} is asymptotically stable (unstable) if and only if $L_i < 0$ ($L_i > 0$) for $i = 1, 2$. It follows from [14] that an equilibrium exists in $\mathrm{Int}\mathbb{R}_+^3$ if $L_1, L_2 > 0$.

The following theorem of Liang and Jiang [16] shows that there exists an invariant hypersurface V which is topologically and geometrically simple and for which all the orbits in $\mathrm{Int}\mathbb{R}_+^3$ are asymptotic to an orbit in V. In particular, all nontrivial periodic orbits lie on V.

First some notation. A vector v is called *positive* if $v \in \mathrm{Int}\mathbb{R}_+^3$. Let the set $K = \{x \in \mathbb{R}^3 : x_i \geq 0 \text{ for } i = 1, 2 \text{ and } x_3 \leq 0\}$ and $\mathrm{Int}K$ be the interior of K. Two points $u, v \in \mathbb{R}^3$ are *K-related* if either $u - v \in \mathrm{Int}K$ or $v - u \in \mathrm{Int}K$. A set S is called *K-balanced* if no two distinct points of S are K-related.

Theorem 2.1 (Liang and Jiang) *There exists an invariant K-balanced Lipschitz submanifold V, which is homeomorphic to $\mathbb{R}^2_+$, such that every orbit in $\mathrm{Int}\mathbb{R}^3_+$ of the dissipative system (2) is asymptotic to an orbit in V. In particular, all positive equilibria and nontrivial periodic orbits lie on V.*

Since the dynamics of a planar competitive or cooperative system are trivial (see [12]), the long term dynamics of (2) on $\mathbb{R}^3_+$ are completely determined by the dynamics on V. It is also important to note that Tereščák's result [24] implies that $V_0 = V \cap \mathrm{Int}\mathbb{R}^3_+$ is actually C^1. See also Benaim [2] the conditions under which V_0 is C^k ($k > 1$).

If there is no equilibrium in the interior of the invariant manifold V, then the dynamics of (2) is trivial. Therefore we are interested only in the case when (2) has a positive equilibrium in the interior of V and the equilibrium is found to be non-degenerative using the results in [16]. Without loss of generality, we can assume that $E = (1, 1, 1)$ is a positive equilibrium of (2) and has no zero eigenvalues. However, we should notice that the coefficients of F are strictly restricted in this case. Note that if E is a positive equilibrium of (2), then, by [16], (2) is dissipative provided that

$$\det M > 0, \qquad \text{where } M = \begin{pmatrix} a_{11} & 0 & -a_{13} \\ 0 & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}. \tag{4}$$

3 Statement of Main Results

We now state our main results. The proofs follow in Sections 4 and 5.

Theorem 3.1 *Assume that system* (2) *is dissipative with $a_{ij} > 0$ and $r_i > 0$, $i, j = 1, 2, 3$. Then the number of periodic orbits (and hence a fortiori number of limit cycles) is finite. Furthermore, there exist some $a_{ij} > 0$ and $r_i > 0$ such that* (2) *has at least two limit cycles.*

Remark 3.1. Theorem 3.1 shows that the number of periodic orbits of the dissipative system (2) is at most finite and that there is an example of (2) with at least two limit cycles. It is an interesting open problem to determine an upper bound (or better still, the maximum) of the number of limit cycles.

Remark 3.2. Note that the first statement of Theorem 3.1 is not a direct corollary of Zhu and Smith [27, Theorem 1.2], because the irreducible condition cannot hold on the boundary of $\mathbb{R}^3_+$, but the orbit of (2) could intersect with $\partial\mathbb{R}^3_+$.
Remark 3.3. According to Theorem 3.1, there does not exist a center of (2), and hence the dynamics on V is not Hamiltonian, which implies that the invariant hypersurface V cannot be flat (see [19]).
Remark 3.4. It is very interesting to ask whether the dynamics on V can be determined by its edges ([28]).

Liang and Jiang [16] proved that (2) has a periodic solution if $M_{12} = a_{11}a_{22} - a_{12}a_{21} < 0$. Note that if $M_{12} < 0$, then either λ_{12} or λ_{21} or both λ_{12} and λ_{21} are negative. The ecological interpretation of this is that at least one of the species 1 and 2 can resist invasion by the other. For three-dimensional competitive Lotka-Volterra systems, van den Driessche and Zeeman [6, 28] have shown that if none of the species can resist invasion by the others, then there is no periodic orbit, so global dynamics is known. Note that for (2) it is obvious that no species can resist invasion by the other species in the mutualistic subcommunity of species 1 and 3, or 2 and 3. The following theorem shows that the result of van den Driessche and Zeeman does not generalize to Competitor-Competitor-Mutualist Lotka-Volterra Systems. We also provide the sufficient conditions to guarantee that system (2) has no periodic orbits if none of the species can resist invasion by any of other species. Thus all orbits converge to equilibria.

Theorem 3.2 *It is possible that a nontrivial periodic orbit for the dissipative system* (2) *exists even if* $\lambda_{ij} = r_j - \dfrac{a_{ji}r_i}{a_{ii}} > 0$ *for* $1 \leq i \neq j \leq 2$. *However, if in addition,*

$$\max\{\frac{r_1a_{31}}{a_{11}}, \frac{r_2a_{32}}{a_{22}}\} < \min\{\frac{r_1a_{33}}{a_{13}}, \frac{r_2a_{33}}{a_{23}}\}, \tag{5}$$

then there are no periodic orbits for the dissipative system (2).

Remark 3.5. In [16, Theorem 5.6], Liang and Jiang also gave a sufficient condition which ensures the nonexistence of periodic orbits. However, their results do not be applied to the following example:
Example 1: Consider the matrix

$$A = \begin{pmatrix} 3 & 6 & -6 \\ 2 & 4.0001 & -4.00083 \\ -1 & -3 & 10 \end{pmatrix}$$

and $r = (3, 1.99927, 6)$ in the type-K competitive Lotka-Volterra system

$$\dot{x}_i = x_i(r_i - (Ax)_i), \qquad i = 1, 2, 3. \tag{6}$$

It is easy to see that there is a unique positive equilibrium E. Let

$$R = \begin{pmatrix} 2a_{11} & a_{12} + da_{21} \\ a_{12} + da_{21} & 2da_{22} \end{pmatrix}, \ S = \begin{pmatrix} 2A^*_{11} & A^*_{12} + dA^*_{21} \\ A^*_{12} + dA^*_{21} & 2dA^*_{22} \end{pmatrix},$$

where $A^* = (A^*_{ij})$ is the adjoint matrix of $-A$. A straightforward calculation yields that $\det R > 0$ if and only if $2.90662 < d < 3.09638$, while $\det S > 0$ if and only if $2.53289 < d < 2.72075$. Obviously, $\{d : \det R > 0\} \cap \{d : \det S > 0\} = \emptyset$, and hence the Volterra multipliers method in [16, Theorem 5.6] can not be applied in this case.

However, our Theorem 3.2 can be applied in this case. By evaluating (3) and (4), we obtain that system (6) is dissipative and uniformly persistent. Furthermore, system (6) also satisfies $\lambda_{12} > 0$ and $\lambda_{21} > 0$ and condition (5). Therefore, it follows from Theorem 3.2 and the Poincaré-Bendixson theorem that E is globally asymptotically stable in $\mathrm{Int}\mathbb{R}^3_+$.

4 Proof of Theorem 3.1

We first present an example of a Competitor-Competitor-Mutualist Lotka-Volterra System (2) with at least two limit cycles, followed by the proof of the finiteness of the number of nontrivial periodic orbits.

The idea for constructing such an example with two limit cycles is as follows: we consider a Competitor-Competitor-Mutualist Lotka-Volterra System which is uniformly persistent and for which the unique interior fixed point has a pair of purely imaginary eigenvalues, but is repelling on its center manifold. This implies the existence of an asymptotically stable (or a pair of semistable) limit cycle(s) by the Poincaré-Bendixson theorem. If we now change the parameters slightly, the fixed point will undergo a subcritical Hopf bifurcation. The interior equilibrium will become stable and will be surrounded by another, smaller, unstable limit cycle.

The same idea has been applied in [13, 25] to construct two limit cycles in three-dimensional competitive Lotka-Volterra systems. However, it is much more difficult to give an example of a Competitor-Competitor-Mutualist Lotka-Volterra System (2) with two limit cycles since many more essential conditions are restricted in our case, such as dissipation, uniform persistence and the conditions restricted to the coefficients if we assume that $E = (1, 1, 1)$ is the positive equilibrium, etc. Indeed, in our example the parameter range in which the two limit cycles coexist is rather small and so these two limit cycles would be very hard to find by numerical integration.

Consider the Competitor-Competitor-Mutualist Lotka-Volterra System

$$\dot{x}_i = x_i[A(E-x)]_i, \qquad i = 1,2,3, \tag{7}$$

where

$$A = (a_{ij}) = \begin{pmatrix} \frac{6}{5} & 1 & -\mu \\ \frac{2257}{850} & 1 & -\lambda \\ -\frac{1}{10} & -1 & \frac{6}{5} \end{pmatrix}$$

with two real parameters μ and λ. Note that μ and λ should be positive such that $r_i = (AE)_i > 0$ for all $i = 1,2,3$. According to linear algebra, the necessary conditions that $-A$ has a negative real eigenvalue and a pair of purely imaginary eigenvalues are

$$\det(A) = (M_{23} + M_{13} + M_{12}) \cdot \mathrm{tr} A,$$

where $\mathrm{tr}(A) = \sum_{i=1}^{3} a_{ii}$, $M_{23} = a_{22}a_{33} - a_{23}a_{32}$, $M_{13} = a_{11}a_{33} - a_{13}a_{31}$ and $M_{12} = a_{22}a_{11} - a_{12}a_{21}$. A simple calculation yields that $\lambda = \frac{1067}{425} - \frac{107}{85}\mu$. Let $y_i = x_i - 1$, $i = 1,2,3$, and set $z = Ty$. Then system (7) is transferred to a new one whose linear part is in the block diagonal form

$$linearpart = \begin{pmatrix} \frac{1187}{1070} & \frac{9409}{90950} - \frac{107\mu}{85} & 0 \\ \frac{197}{214} & -\frac{1187}{1070} & 0 \\ 0 & 0 & -\frac{17}{5} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

where $y = \mathrm{col}(y_1, y_2, y_3)$, $z = \mathrm{col}(z_1, z_2, z_3)$, and

$$T = \begin{pmatrix} -\frac{2257}{850} & \frac{12}{5} & \frac{1067}{425} - \frac{107\mu}{85} \\ \frac{1}{10} & 1 & \frac{11}{5} \\ \frac{1177}{425} + \frac{107\mu}{85} & \frac{11}{5} + \mu & -\frac{1067}{425} - \frac{97\mu}{85} \end{pmatrix}.$$

This can be reduced to the two-dimensional case by computing the center manifold

$$z_3 = G(z_1, z_2) = a_{11}z_1^2 + a_{12}z_1z_2 + a_{22}z_2^2 + h.o.t.,$$

where $h.o.t.$ denotes the terms with order greater than or equal to three. Solving for the $a_{ij}{}'$s and substituting leads to a rather complicated and lengthy expression of first focal value LV_1 as following

$$LV_1 = \frac{g_1(\mu)}{g_2(\mu)},$$

where

$$\begin{aligned} g_1(\mu) = \;& 856(57090363600000\mu^4 + 70552246385500\mu^3 \\ & -196136513479725\mu^2 - 123278835493330\mu + 194300080257591), \\ g_2(\mu) = \;& 14553375(1911343250\mu^3 + 36339777375\mu^2 \\ & +194642386740\mu + 201213446059). \end{aligned}$$

We computed LV_1 as a rational number using the computer algebraic system Maple. Now choose any $\mu \in (1.14422, 1.99439)$. We have $\mu > 0$, $\lambda = \frac{1067}{425} - \frac{107}{85}\mu > 0$, $r_i = (AE)_i > 0$ $(i = 1, 2, 3)$ and $LV_1 > 0$. A straightforward calculation also yields that $L_1 > 0, L_2 > 0$ in (3) and $\det M > 0$ in (4). Therefore, the Lotka-Volterra system (7) is dissipative and uniformly persistent. Moreover, E is repelling on its center manifold (which is on the hypersurface V). This implies the existence of an asymptotically stable (or a pair of semistable) limit cycle(s) on the hypersurface V. If we change the parameter μ slightly, E will undergo a *subcritical Hopf bifurcation*, which implies that E will become stable and will be surrounded by another smaller unstable limit cycle. Just as we mentioned above, the parameter range is rather small. We use the graphing capability of Maple to illustrate our findings (see Figure 1).

Now we focus on the finiteness of the number of nontrivial periodic orbits. According to the classification theorem ([16, Theorem 5.1]), the periodic orbits can only occur in the case of $L_1 > 0$ and $L_2 > 0$. Since system (2) is dissipative, we claim that φ_t is permanent if $L_1 > 0$ and $L_2 > 0$. Indeed, let M be the maximal compact invariant set for $\varphi|_{H^+_{\{1,2\}}}$. Then $f_3(x) > 0$ for every $x \in M$. So M is unsaturated (see [20, Definition 3.3]). Note that the maximal compact invariant set for $\varphi|_{\partial\mathbb{R}^3_+}$ is $K = M \cup \{0, R_3, R_{13}, R_{23}\}$. Since $L_1 > 0$ and $L_2 > 0$, it is easy to see that K admits an unsaturated Morse decomposition (see [20, Definition 4.1]). By [20, Theorem 4.3] , φ_t is permanent in $\mathbb{R}^3_+$, which means that there exists a bounded open subset D in $\mathbb{R}^3_+$ such that every orbit in $\mathbb{R}^3_+$ will enter into D. Now it is easy to check that $F = (x_1 f_1, x_2 f_2, x_3 f_3)$ satisfies the hypotheses $(H1) - (H4)$ in [27, p. 145] in D. Then, by [27, Theorem 1.2] with an alternative cone, there exist at most finite periodic orbits in D. Thus, we have completed the proof of Theorem 3.1.

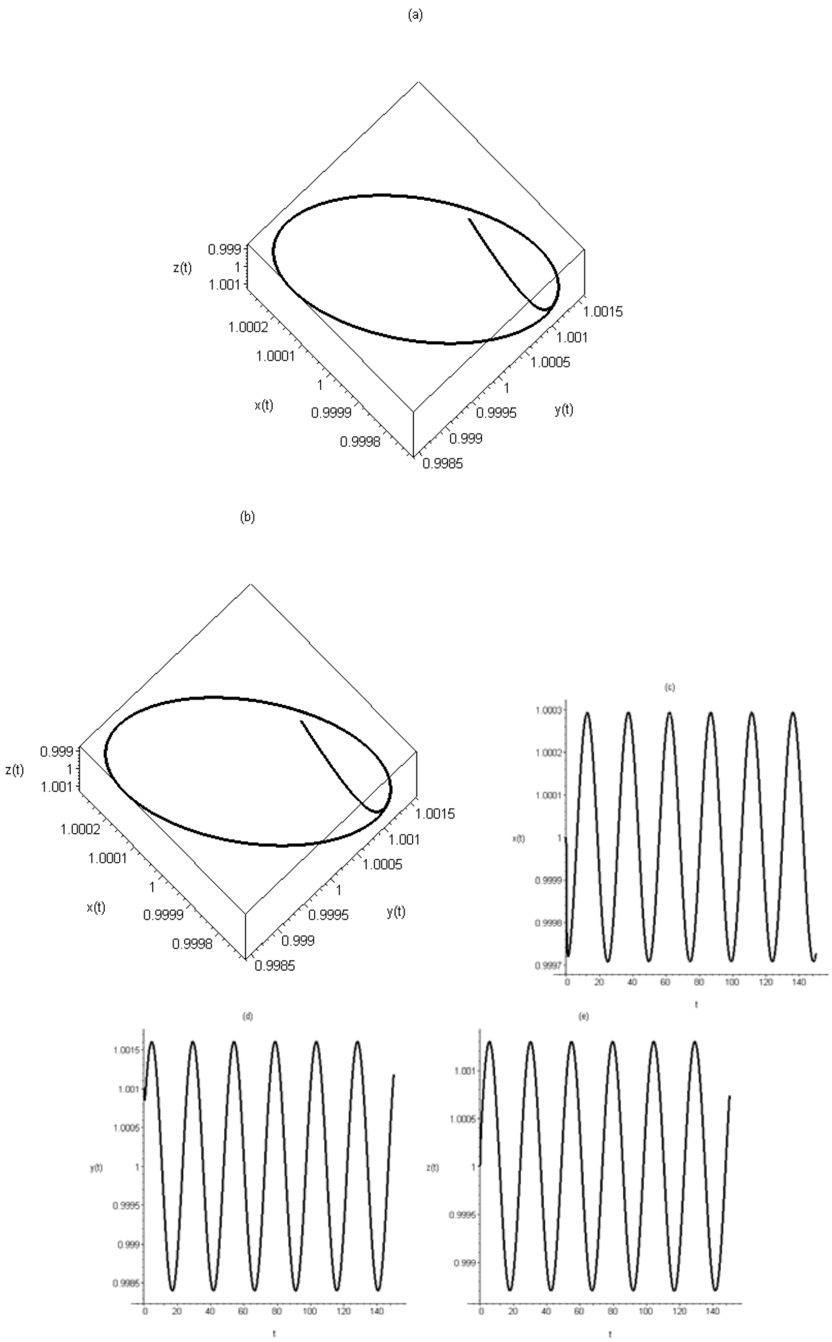

Figure 1: We take $\mu = 6/5, \lambda = 1$ and the initial value $(1, 1.001, 1)$. In (a), time t runs from 0 to 100. In (b), time t runs from 0 to 10000. (a) and (b) are the same, which implies that $E = (1, 1, 1)$ is repelling on its center manifold, and hence that there exists the limit cycle shown in this figure. We have to say that the limit cycle is very small. If we change the parameter $\mu = 6/5$ slightly, E will undergo a subcritical Hopf bifurcation, which implies that E will become stable and will be surrounded by another smaller unstable limit cycle, which is very hard to be displayed by numerical simulation. $(c), (d)$ and (e) show the variation of x, y, z as time t runs.

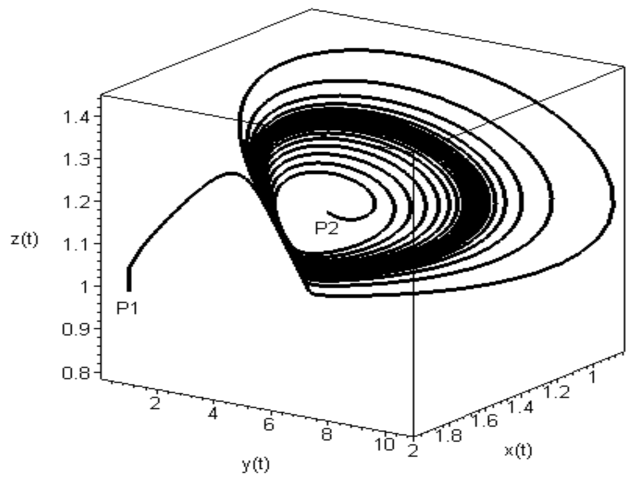

Figure 2: $P_1 = (2, 1, 1)$ and $P_2 = (0.9, 1, 1)$ are two initial values of Example 2.

5 Proof of Theorem 3.2

We first present an example which implies that there may exist a nontrivial periodic orbit in Competitor-Competitor-Mutualist Lotka-Volterra System (2) even if $\lambda_{12} > 0$ and $\lambda_{21} > 0$.

Example 2: Consider the Competitor-Competitor-Mutualist Lotka-Volterra System (6) with

$$A = \begin{pmatrix} 13 & 0.0125 & -11 \\ 280 & 3 & -6 \\ -7 & -1 & 9 \end{pmatrix}$$

and $r = (2.0125, 277, 1)$. Obviously, E is the unique positive equilibrium and $\lambda_{12} = 233.654$ and $\lambda_{21} = 0.858333$. It is also easy to compute that $L_1 > 0$ and $L_2 > 0$ in (2) and $\det M > 0$ in (2). Hence, system (6) is dissipative and uniformly persistent. Furthermore, E is hyperbolic and locally unstable on the hypersurface V, which implies that (6) has at least one stable nontrivial periodic orbit (see Figure 2).

Now we turn our attention to the second statement of Theorem 3.2. In order to prove Theorem 3.2, we first need the following lemma which is due to Busenberg and van den Driessche [4].

Lemma 5.1 *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a Lipschitz vector field and let $\gamma(t)$ be a closed piecewise C^1 curve bounding an orientable C^1 surface $S \subset \mathbb{R}^3$ with normal vector n. If there exists a vector field $G : \mathbb{R}^3 \to \mathbb{R}^3$, defined and C^1 in a neighborhood of S such that*

(i) $G \cdot F \leq 0$ (≥ 0) *on γ;*

(ii) $\operatorname{curl} G \cdot n \geq 0$ (≤ 0) *on S;*

(iii) $\operatorname{curl} G \cdot n \not\equiv 0$ *on S,*

then $\gamma(t)$ is not a periodic orbit of $\dot{x} = F(x)$ traversed in the positive direction with respect to n.

Suppose that system (2) has a nontrivial periodic orbit $\gamma(t)$. Then $\gamma(t)$ encloses a region $S \subset V_0 = \operatorname{Int}\mathbb{R}^3_+ \cap V$. Recall from section 2 that V_0 is C^1 and K-balanced, so for $x \in S$, the tangent plane T_xS to S at x exists and is K-balanced. Hence, we claim that the normal vector $n = (n_1, n_2, n_3)$ to S at x belongs to $\operatorname{Int}K \cup (-\operatorname{Int} K)$. Indeed, suppose, without loss of generality, that $n_1 > 0, n_2 \geq 0$ and $n_3 > 0$. Then $(n_1, n_2, n_3) \cdot (n_3, n_3, -(n_1 + n_2)) = 0$. So $(n_3, n_3.-(n_1+n_2)) \in T_xS$, contradicting the fact that T_xS is K-balanced. We have proved the claim. Now we may assume that

$$n = (n_1, n_2, n_3) \in \operatorname{Int}K \tag{8}$$

for every $x \in S$.

Let $F = (F_1, F_2, F_3)$ in (2). Define $G : \operatorname{Int}\mathbb{R}^3_+ \to \mathbb{R}^3$ by

$$G(x) = F(x) \times \frac{1}{x_1 x_2 x_3} \begin{pmatrix} r_1 x_1 \\ r_2 x_2 \\ -r^* x_3 \end{pmatrix},$$

where r^* satisfies the following inequality

$$\max\{\frac{r_1 a_{31}}{a_{11}}, \frac{r_2 a_{32}}{a_{22}}\} < r^* < \min\{\frac{r_1 a_{33}}{a_{13}}, \frac{r_2 a_{33}}{a_{23}}\}. \tag{9}$$

Obviously, $G \cdot F \equiv 0$ and a straightforward calculation yields

$$\operatorname{curl} G = \begin{pmatrix} \frac{1}{x_3}(r_1 a_{22} - r_2 a_{12}) + \frac{1}{x_2}(r_1 a_{33} - r^* a_{13}) \\ \frac{1}{x_1}(r_2 a_{33} - r^* a_{23}) + \frac{1}{x_3}(r_2 a_{11} - r_1 a_{21}) \\ \frac{1}{x_2}(-r^* a_{11} + r_1 a_{31}) + \frac{1}{x_1}(-r^* a_{22} + r_2 a_{32}) \end{pmatrix}.$$

Recall that $\lambda_{12} > 0$ and $\lambda_{21} > 0$, then it follows from (8) and (9) that $\operatorname{curl} G \cdot n > 0$ on S. Thus we may conclude from Lemma 5.1 that $\gamma(t)$ is not a nontrivial periodic orbit. This completes the proof.

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