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Varieties of Tree Languages

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Contents

Computer science is no more about computers than astronomy is about telescopes.

– Edsger Dijkstra

Chapter 1

Introduction and preliminaries

“Trees and terms are important structured objects that can be found almost everywhere in computer science, not only in connection with their mathematical foundations.” Jantzen [25]. Also, almost every working mathematician has heard of “trees” as this notion appears in many seemingly different areas of mathematics from graph theory to universal algebra to logic. In computer science trees are often regarded as a natural generalization of strings. Though it is not possible to present a complete history of the subject here, we quote the following from the survey paper [23]:

“The theory of tree automata and tree languages emerged in the middle of the 1960s quite naturally from the view of finite automata as unary algebras advocated by J. R. Büchi and J. B. Wright. From this perspective the generalization from strings to trees means simply that any finite algebra of finite type can be regarded as an automaton which as inputs accepts terms over the ranked alphabet formed by the operation symbols of the algebra, and these terms again can be seen as (formal representations of) labeled trees with a left-to-right ordering of the branches. Strings over a finite alphabet can then be regarded as terms over a unary ranked alphabet, and hence finite automata become special tree automata and string languages unary tree languages. The theory of tree automata and tree languages can thus be seen as an outgrowth of Büchi’s and Wright’s program which had as its goal a general theory that would encompass automata, universal algebra, equational logic, and formal languages. Some interesting vistas of this program and its development are opened by Büchi’s posthumous book [9] in which many of the ideas are traced back to people like Thue, Skolem, Post, and even Leibniz.”

It is true that the theory of tree automata and tree languages may have come into existence by generalizing string automata and languages, but “[o]f course, no branch of mathematics could stay alive very long as a mere generalization.” [55].

Apart from its intrinsic interest, the theory of tree automata and tree languages has found several applications and it offers new perspectives to various parts of mathematical linguistics. It has also been applied to some decision problems of logic, and it provides tools for syntactic pattern recognition (see [12] and [22]). "Actually using tree automata has proved to be a powerful approach to simplify and extend previously known results, and also to find new results. For instance recent works use tree automata for application in abstract interpretation using set constraints, rewriting, automated theorem proving and program verification, databases and XML schema languages." [12].

Mathematicians who have heard of trees may recall one or two definitions of them. Considering trees as terms over a ranked alphabet and a leaf alphabet has become a custom in some schools, especially here in Turku, Finland. An advantage of this approach is that the concepts and results of universal algebra become immediately usable.

It is worth noting that the impact of universal algebra on the theory of tree automata and tree languages has not been in one direction only; developments of tree automata and tree languages have suggested new problems and concepts of universal algebra. Also in this dissertation we have developed algebraic notions and proved theorems in universal algebra when the necessity has emerged. However, the recent book of Denecke and Wismath [15] is the first universal algebra text where tree automata and tree languages are explicitly studied (see Chapters 5 and 8 of [15]).

The main topic of this dissertation is the variety theory of tree languages. The history of variety theory begins with Eilenberg's celebrated variety theorem [17]. As Pin [39] puts it, "[t]he most important tool for classifying recognizable languages is Eilenberg's variety theorem [17], which gives a one-to-one correspondence between (pseudo-)varieties of finite semigroups and varieties of recognizable languages."

Eilenberg's theorem was motivated by characterizations of several families of string languages by syntactic monoids or semigroups (see [17, 38]), above all by Schützenberger's [50] theorem connecting star-free languages and aperiodic monoids. A fascinating feature of this variety theorem is the existence of its many instances. As a matter of fact, most of the interesting classes of algebraic structures are varieties, and similarly, most of the interesting families of tree or string languages studied in the literature turn out to be varieties of some kind. The aforementioned variety theorem connects these interesting families to each other.

Eilenberg's theorem has since then been extended in various directions. One of these extensions is Thérien's [57] notion of varieties of congruences on free monoids. Another extension is Pin's positive variety theorem [39] which establishes a bijective correspondence between positive varieties of string languages and varieties of ordered semigroups.

Concerning trees, which are studied in the field of universal algebra, Steinby's variety theorem [52] for varieties of recognizable subsets of free algebras and varieties of finite algebras was the first one of this kind. The correspondence with varieties of congruences, and some other generalizations, were added later by Almeida [1] and Steinby [53, 54]. Another variety theorem for trees is Ésik's [19] correspondence between families of tree languages and classes of theories (see also [20]).

As Ésik [19] notes, any variety theorem connects families of tree languages with classes of some structures via their "syntactic structures". One of these syntactic structures is the syntactic semigroup/monoid of a tree language introduced by Thomas [58] and further studied by Salomaa [48]. A different formalism, based on essentially the same concept, was considered by Nivat and Podelski [32, 42].

Several variety theorems for trees are proved in this dissertation:

The variety theorem for families of tree languages and varieties of finite algebras, provided by Steinby and Almeida, is generalized to many-sorted algebras in Chapter 2, which is joint work with Steinby [46].

Chapter 3, based on a joint paper with Petković [34], is inspired by Pin's theory of positive varieties of string languages and varieties of ordered monoids. We prove a variety theorem for positive varieties of tree languages and varieties of finite ordered algebras which correspond to each other via syntactic ordered algebras.

Tree languages definable by syntactic monoids are studied in Chapter 4. It was already known that any family of tree languages definable by syntactic monoids is a (generalized) variety of tree languages, though not every variety of tree languages is definable by syntactic monoids [54]. Characterizing the varieties of tree languages which are definable by syntactic monoids was a relatively long-standing open problem [54, 19]. Here we give an answer to this question by providing a variety theorem for families of tree languages and varieties of finite monoids which correspond to each other via syntactic monoids [45]. This characterization is generalized to a characterization of positive varieties of tree languages definable by syntactic ordered monoids in Chapter 5. This generalization was obtained together with Petković [34]. Also an instance of this positive variety theorem and the variety theorem in Chapter 4 is elaborated.

The rest of the dissertation is a study of Wilke's tree algebra formalism [60] for binary trees. A completeness theorem for the axiomatization of tree algebras and a variety theorem for families of binary tree languages and varieties of finite tree algebras is proved in Chapter 6. The first two sections of this chapter are based on a joint paper with Steinby [47]. Finally, a completeness property of Wilke's functions is presented without proofs in the last section. We have proved that term algebras over ranked alphabets with at least seven constant symbols are affine-complete, and that the free tree algebras over finite alphabets containing at least seven labels are affine-complete [43, 44].

The above mentioned results once again demonstrate the richness of the theory of tree automata and tree languages and they also suggest some new perspectives of the variety theory of string languages when words are viewed as unary trees. This, from a variety theory viewpoint, confirms our belief that not only trees are more than mere generalization of words, but also words are particular cases of trees.

We have made an effort to make the dissertation self-contained (except for the last chapter) for its expected readership. For basic notions of tree automata and tree languages, [23, 56] are more than enough, and [15, 25] provide the fundamental tools of universal algebra and term rewriting used throughout the dissertation.

Although we have tried to use a uniform notation throughout the thesis, some exceptions seemed inevitable. In particular, in Chapter 6 we have preserved most of Wilke's [60] notation, and thus many letters get meanings different from the ones they have in the previous chapters. For the reader's convenience an Index of Notation is provided.

Preliminaries

Strings over a finite alphabet X are often regarded as elements of the free monoid X^* generated by X. Similarly, any tree considered here may be viewed as an element of a term algebra. Also finite tree automata can be defined as finite algebras. Therefore, universal algebra provides a natural mathematical foundation for the theory of finite tree automata and recognizable tree languages. Here we first recall some basic notions of algebras and then we list formal definitions and concepts of trees as terms.

A *ranked alphabet* Σ is a finite set of function symbols each of which has a unique non-negative integer arity. For any $m \geq 0$, Σ_m denotes the elements of Σ with arity m. In particular, Σ_0 is the set of constant symbols of Σ. A Σ-*algebra* is a structure $\mathcal{A} = (A, \Sigma)$ where A is a non-empty set in which every symbol of Σ is realized, i.e., any $c \in \Sigma_0$ is realized by a constant $c^{\mathcal{A}} \in A$, and any $f \in \Sigma_m$ for $m > 0$ is realized by an m-ary function $f^{\mathcal{A}} : A^m \to A$. The algebra $\mathcal{A} = (A, \Sigma)$ is called finite if the set A is finite.

Recall that a binary *relation* on a set A is a subset $\theta \subseteq A \times A$. The fact that $(a, b) \in \theta$, for some $a, b \in A$, is often written as $a\ \theta\ b$. The *inverse* of the relation θ is $\theta^{-1} = \{(b, a) \mid (a, b) \in A\}$, and if θ' is another relation on the set A, the *composition* of θ and θ' is the relation

$$\theta \circ \theta' = \{(a, c) \mid (a, b) \in \theta \ \&\ (b, c) \in \theta' \text{ for some } b \in A\}.$$

The *diagonal* relation on A, $\{(a, a) \mid a \in A\}$, and the *universal* relation $A \times A$ are respectively denoted by Δ_A and ∇_A. The relation θ is called an *equivalence* on A if $\Delta_A, \theta^{-1}, \theta\circ\theta \subseteq \theta$. If θ is an equivalence on A, the *quotient set* A/θ is the set $\{a/\theta \mid a \in A\}$ where $a/\theta = \{b \in A \mid a \; \theta \; b\}$ is the θ*-class* of a. If A/θ is finite then θ is said to be of *finite index* or simply a *finite* relation. Note that Δ_A and ∇_A are the least and the greatest equivalence relations on A, respectively. For sets A and B, a mapping $\varphi : A \to B$ can be viewed also as a special relation, a subset of $A \times B$. The *image* $\varphi(a)$ of an $a \in A$ is often written as $a\varphi$. For subsets $C \subseteq A$ and $D \subseteq B$ the *image* of C and the *inverse image* of D are $C\varphi = \{a\varphi \in B \mid a \in C\}$ and $D\varphi^{-1} = \{a \in A \mid a\varphi \in D\}$, respectively.

Fix a ranked alphabet Σ and let $\mathcal{A} = (A, \Sigma)$, $\mathcal{B} = (B, \Sigma)$ be algebras.

The algebra $\mathcal{B}$ is said to be a *subalgebra* of $\mathcal{A}$, if $B \subseteq A$ and every function $f^{\mathcal{B}}$, for $f \in \Sigma$, is the restriction of $f^{\mathcal{A}}$ to B. In particular, $c^{\mathcal{B}} = c^{\mathcal{A}}$ for every constant symbol $c \in \Sigma_0$.

A mapping $\varphi : A \to B$ is called a *homomorphism*, if $c^{\mathcal{A}}\varphi = c^{\mathcal{B}}$ for any $c \in \Sigma_0$, and $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = f^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for every $f \in \Sigma_m$ $(m > 0)$ and every $a_1, \ldots, a_m \in A$. The fact that φ is a homomorphism is expressed by writing $\varphi : \mathcal{A} \to \mathcal{B}$. The *kernel* of φ is the relation $\ker\varphi = \{(a, c) \in A \times A \mid a\varphi = c\varphi\}$. A homomorphism is called a *monomorphism* if it is injective, and is an *epimorphism* if it is surjective. An *isomorphism* is a bijective homomorphism. Sometimes a homomorphism is called simply a *morphism*. We say that $\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$ and write $\mathcal{B} \leftarrow \mathcal{A}$ when there exists an epimorphism from $\mathcal{A}$ onto $\mathcal{B}$, and we write $\mathcal{A} \subseteq \mathcal{B}$ when there exists a monomorphism from $\mathcal{A}$ to $\mathcal{B}$. We say that $\mathcal{A}$ *divides* $\mathcal{B}$ and write $\mathcal{A} \preceq \mathcal{B}$ if for some algebra $\mathcal{C} = (C, \Sigma)$ there exist a monomorphism $\psi : \mathcal{C} \to \mathcal{B}$ and an epimorphism $\varphi : \mathcal{C} \to \mathcal{A}$. If there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, and we write $\mathcal{A} \cong \mathcal{B}$.

An equivalence relation θ on A is called a *congruence* on $\mathcal{A}$, if for all $f \in \Sigma_m$ $(m > 0)$ and all $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, if $a_1 \,\theta\, b_1, \ldots, a_m \,\theta\, b_m$ then $f^{\mathcal{A}}(a_1, \ldots, a_m) \,\theta\, f^{\mathcal{A}}(b_1, \ldots, b_m)$. It is easy to note that the kernel of any homomorphism is a congruence. If θ is a congruence on $\mathcal{A}$ then the *quotient algebra* is the structure $\mathcal{A}/\theta = (A/\theta, \Sigma)$ defined by $c^{\mathcal{A}/\theta} = c^{\mathcal{A}}/\theta$ for any $c \in \Sigma_0$, and $f^{\mathcal{A}/\theta}(a_1/\theta, \ldots, a_m/\theta) = f^{\mathcal{A}}(a_1, \ldots, a_m)/\theta$ for every $f \in \Sigma_m$ $(m > 0)$ and every $a_1, \ldots, a_m \in A$. Note that the condition of θ being a congruence on $\mathcal{A}$ ensures us that the above operations are well-defined on $\mathcal{A}/\theta$. The *natural mapping* $\theta^\natural : A \to A/\theta$, $a \mapsto a/\theta$, is seen to be an epimorphism. Note that $\ker\theta^\natural = \theta$. The Homomorphism Theorem in universal algebra states that if $\varphi : \mathcal{A} \to \mathcal{B}$ is an epimorphism, then $\mathcal{A}/\ker\varphi \cong \mathcal{B}$. This theorem can be generalized as follow. For any congruence θ' on $\mathcal{B}$ and any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, the relation $\varphi\circ\theta'\circ\varphi^{-1}$ is a congruence on $\mathcal{A}$. Now, if φ is an epimorphism, then $\mathcal{A}/\varphi\circ\theta'\circ\varphi^{-1} \cong \mathcal{B}/\theta'$. Note that by definition $\varphi \circ \varphi^{-1} = \ker\varphi$.

A mapping $p : A \to A$ is called an *elementary translation* of $\mathcal{A}$, if $p(\xi) = f^{\mathcal{A}}(a_1, \ldots, a_{i-1}, \xi, a_{i+1}, \ldots, a_m)$ for some $m > 0$, $f \in \Sigma_m$, $1 \leq i \leq m$ and

$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in A$, where ξ is a new variable ranging over A. The set $\mathrm{Tr}(\mathcal{A})$ of *translations* of $\mathcal{A}$ is the smallest set of unary operations on A that contains the *identity map* $1_A : A \to A$, $a \mapsto a$, and all the elementary translations of $\mathcal{A}$, and is closed under the composition. It is easy to verify that an equivalence θ on A is a congruence on $\mathcal{A}$ iff $p \circ \theta \circ p^{-1} \supseteq \theta$ for every translation $p \in \mathrm{Tr}(\mathcal{A})$.

The *direct product* $\mathcal{A} \times \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is the algebra $(A \times B, \Sigma)$ defined by $c^{\mathcal{A}\times\mathcal{B}} = (c^{\mathcal{A}}, c^{\mathcal{B}})$ for any $c \in \Sigma_0$, and

$$f^{\mathcal{A}\times\mathcal{B}}((a_1, b_1), \ldots, (a_m, b_m)) = (f^{\mathcal{A}}(a_1, \ldots, a_m), f^{\mathcal{B}}(b_1, \ldots, b_m))$$

for all $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_m \in B$. The following facts can be easily verified for any algebras $\mathcal{A}$ and $\mathcal{B}$, congruences θ and θ' on $\mathcal{A}$, and morphism $\varphi : \mathcal{B} \to \mathcal{A}$.

1. If $\theta \subseteq \theta'$, then $\mathcal{A}/\theta' \leftarrow \mathcal{A}/\theta$.
2. $\mathcal{A}/\theta \cap \theta' \subseteq \mathcal{A}/\theta \times \mathcal{A}/\theta'$.
3. $\mathcal{B}/\varphi \circ \theta \circ \varphi^{-1} \preceq \mathcal{A}/\theta$.

A class of finite Σ-algebras is called a *(pseudo-)variety*, if it is closed under subalgebras, homomorphic images and direct products. The classes we call *variety* here are sometimes called pseudo-variety to be distinguished from the term "variety" which is defined to be a class of (not necessarily finite) algebras closed under subalgebras, homomorphic images and (arbitrary, not necessarily finite) direct products. Birkhoff's theorem gives a logical characterization for those classes. For the pseudo-varieties, called simply *varieties* from now on, there exists an analogue characterization with ultimately definability by equations (see e.g. [2]), though we will not touch this subject in this dissertation. It is easy to see that the intersection of any class of varieties is a variety. So, for a collection of Σ-algebras $\mathbf{C}$, the intersection of all varieties containing $\mathbf{C}$ is a variety, called the variety generated by $\mathbf{C}$.

Now we review the theory of trees as terms. Roughly speaking, a tree is a structured object that is branched from a root which stands in the highest level and every node in the middle is either branched to other nodes or stands as a leaf. For a formal definition let Σ be a ranked alphabet and X be any finite set, called *leaf alphabet*. The set $\mathrm{T}(\Sigma, X)$ of *ΣX-trees* is defined to be the smallest set containing $\Sigma_0 \cup X$ such that $f(t_1, \ldots, t_m) \in \mathrm{T}(\Sigma, X)$ whenever $f \in \Sigma_m$ $(m > 0)$ and $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$. In this formalism the leaves of ΣX-trees are labelled by symbols from $\Sigma_0 \cup X$ and the inner nodes are labelled by the symbols in Σ with non-zero arities. Any subset of $\mathrm{T}(\Sigma, X)$ is called a *tree language*.

The algebra $\mathcal{T}(\Sigma, X) = (\mathrm{T}(\Sigma, X), \Sigma)$ is defined by $c^{\mathcal{T}(\Sigma,X)} = c$ for any $c \in \Sigma_0$ and $f^{\mathcal{T}(\Sigma,X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ for all $f \in \Sigma_m$ $(m > 0)$ and $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$. This is called the *$\Sigma X$-term algebra*, or simply a *term algebra*. We note that $\mathcal{T}(\Sigma, X)$ is the free Σ-algebra generated by

X, i.e., for any algebra $\mathcal{A} = (A, \Sigma)$, any mapping $\alpha : X \to A$ can uniquely be extended to a homomorphism $\alpha^{\mathcal{A}} : \mathcal{T}(\Sigma, X) \to \mathcal{A}$. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is said to be *recognized* by an algebra $\mathcal{A} = (A, \Sigma)$ when there exist a homomorphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and a subset $F \subseteq A$ such that $F\varphi^{-1} = T$. A tree language is called *recognizable* if a finite algebra recognizes it. A ΣX*-recognizer* is a triple $(\mathcal{A}, \alpha, F)$ where $\mathcal{A} = (A, \Sigma)$ is a finite algebra, $\alpha : X \to A$ is a map, and $F \subseteq A$ is a subset. The map α is called an *initial assignment*, and $\alpha^{\mathcal{A}} : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ is its extension to a homomorphism. The subset $F \subseteq A$ is called the *set of final states*, and the tree language recognized by $(\mathcal{A}, \alpha, F)$ is by definition $\{t \in \mathrm{T}(\Sigma, X) \mid t\alpha^{\mathcal{A}} \in F\}$.

Let ξ be a new symbol which does not appear in any ranked alphabet or leaf alphabet considered here. The set of ΣX*-contexts*, denoted by $\mathrm{C}(\Sigma, X)$, consists of the $\Sigma(X \cup \{\xi\})$-trees in which ξ appears exactly once. For contexts $P, Q \in \mathrm{C}(\Sigma, X)$ and tree $t \in \mathrm{T}(\Sigma, X)$, the context $Q \cdot P$, the composition of P and Q, results from P by replacing the special leaf ξ with Q, and the term $P(t)$, also denoted as $t \cdot P$, results from P by replacing ξ with t. For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ and context $P \in \mathrm{C}(\Sigma, X)$, the *inverse translation* of T under P is $P^{-1}(T) = \{t \in \mathrm{T}(\Sigma, X) \mid t \cdot P \in T\}$.

We shall now outline the basic theory of varieties of recognizable tree languages that is the general starting point of this work. For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the *syntactic congruence* $\approx^T$ of T [53] is defined by

$$t \approx^T s \iff \forall P \in \mathrm{C}(\Sigma, X)\big(t \cdot P \in T \leftrightarrow s \cdot P \in T\big) \quad (t, s \in \mathrm{T}(\Sigma, X)).$$

It can be easily seen that the relation $\approx^T$ is really a congruence on $\mathcal{T}(\Sigma, X)$. A tree language is recognizable iff its syntactic congruence is of finte index. The *syntactic algebra* $\mathrm{SA}(T)$ of T is the quotient Σ-algebra $\mathcal{T}(\Sigma, X)/\approx^T$. It can be shown that a finite algebra $\mathcal{A}$ recognizes a tree language T iff $\mathrm{SA}(T) \preceq \mathcal{A}$. Thus, $\mathrm{SA}(T)$ is the smallest agebra recognizing T. The following relations hold for any leaf alphabets X, Y, tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$, homomorphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, and context $P \in \mathrm{C}(\Sigma, X)$ (see Propositions 3.3 and 3.4 in [53]).

1. $\approx^{\mathrm{T}(\Sigma,X)\setminus T} = \approx^T$ and $\mathrm{SA}(\mathrm{T}(\Sigma, X) \setminus T) \cong \mathrm{SA}(T)$.
2. $\approx^{T \cap T'} \supseteq \approx^T \cap \approx^{T'}$ and $\mathrm{SA}(T \cap T') \subseteq \mathrm{SA}(T) \times \mathrm{SA}(T')$.
3. $\approx^{P^{-1}(T)} \supseteq \approx^T$ and $\mathrm{SA}(P^{-1}(T)) \leftarrow \mathrm{SA}(T)$.
4. $\approx^{T\varphi^{-1}} \supseteq \varphi \circ \approx^T \circ \varphi^{-1}$ and $\mathrm{SA}(T\varphi^{-1}) \preceq \mathrm{SA}(T)$.
5. Moreover, if φ is an epimorphism, then $\approx^{T\varphi^{-1}} = \varphi \circ \approx^T \circ \varphi^{-1}$ and $\mathrm{SA}(T\varphi^{-1}) \cong \mathrm{SA}(T)$.

A *family* $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ of recognizable tree languages is a mapping which assigns to every leaf alphabet X a collection $\mathscr{V}(\Sigma, X)$ of recognizable ΣX-tree languages. Similarly, a *family* $\Gamma = \{\Gamma(X)\}$ of finite congruences is a mapping which assigns to every leaf alphabet X a collection $\Gamma(X)$ of finite congruences on $\mathcal{T}(\Sigma, X)$. A *variety of tree languages* is a family of recognizable tree languages closed under Boolean operations (complements and intersections), inverse translations and inverse morphisms. That is to say, a family $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ is a variety of tree languages, if for any leaf alphabets X, Y, tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$, homomorphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, and context $P \in \mathrm{C}(\Sigma, X)$, if $T, T' \in \mathscr{V}(\Sigma, X)$, then $\mathrm{T}(\Sigma, X) \setminus T, T \cap T', P^{-1}(T) \in \mathscr{V}(\Sigma, X)$ and $T\varphi^{-1} \in \mathscr{V}(\Sigma, Y)$. A *variety of congruences* is a family of finite congruences $\Gamma = \{\Gamma(X)\}$ such that for any leaf alphabets X, Y and homomorphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, whenever $\theta, \theta' \in \Gamma(X)$ then $\theta \cap \theta' \in \Gamma(X)$ and $\varphi \circ \theta \circ \varphi^{-1} \in \Gamma(Y)$, and moreover, if ϑ is another congruence on $\mathcal{T}(\Sigma, X)$ such that $\theta \subseteq \vartheta$, then $\vartheta \in \Gamma(X)$.

Let $\mathbf{VFA}(\Sigma)$, $\mathbf{VTL}(\Sigma)$, and $\mathbf{VFC}(\Sigma)$ denote the class of all varieties of finite Σ-algebras, the class of all varieties of tree languages, and the class of all varieties of congruences, respectively. These three classes are easily seen to be complete lattices with respect to the inclusion ($\subseteq$) relation.

For a class $\mathbf{K}$ of finite Σ-algebras define the family $\mathbf{K}^t = \{\mathbf{K}^t(\Sigma, X)\}$ of tree languages and the family $\mathbf{K}^c = \{\mathbf{K}^c(\Sigma, X)\}$ of congruences by, respectively, $\mathbf{K}^t(\Sigma, X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SA}(T) \in \mathbf{K}\}$ and $\mathbf{K}^c(\Sigma, X) = \{\theta \mid \mathcal{T}(\Sigma, X)/\theta \in \mathbf{K}\}$ for any leaf alphabet X. For a family $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ of tree languages, let $\mathscr{V}^a$ be the variety of finite Σ-algebras generated by the collection $\{\mathrm{SA}(T) \mid T \in \mathscr{V}(\Sigma, X)\}$, and let $\mathscr{V}^c = \{\mathscr{V}^c(\Sigma, X)\}$ be the family of finite congruences defined by setting for each leaf alphabet X, $\mathscr{V}^c(\Sigma, X) = \{\theta \mid \theta \supseteq \approx^{T_1} \cap \cdots \cap \approx^{T_m} \text{ for some } T_1, \ldots, T_m \in \mathscr{V}(\Sigma, X)\}$. Finally, for a class $\Gamma = \{\Gamma(X)\}$ of finite congruences, let Γ^a be the variety of finite Σ-algebras generated by the collection $\{\mathcal{T}(\Sigma, X)/\theta \mid \theta \in \Gamma(\Sigma, X)\}$, and let $\Gamma^t = \{\Gamma^t(\Sigma, X)\}$ be the family of recognizable tree languages defined by $\Gamma^t(\Sigma, X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \approx^T \in \Gamma(\Sigma, X)\}$ for any leaf alphabet X.

It can be proved that these operations map varieties to varieties, and are isotone, i.e., for $\mathbf{K}, \mathbf{L} \in \mathbf{VFA}(\Sigma)$, $\mathscr{V}, \mathscr{W} \in \mathbf{VTL}(\Sigma)$, and $\Gamma, \Psi \in \mathbf{VFC}(\Sigma)$, we have $\mathbf{K}^t, \Gamma^t \in \mathbf{VTL}(\Sigma)$, $\mathbf{K}^c, \mathscr{V}^c \in \mathbf{VFC}(\Sigma)$ and $\mathscr{V}^a, \Gamma^a \in \mathbf{VFA}(\Sigma)$, and if $\mathbf{K} \subseteq \mathbf{L}$, $\mathscr{V} \subseteq \mathscr{W}$ and $\Gamma \subseteq \Psi$, then $\mathbf{K}^t \subseteq \mathbf{L}^t$, $\mathbf{K}^c \subseteq \mathbf{L}^c$, $\mathscr{V}^a \subseteq \mathscr{W}^a$, $\mathscr{V}^c \subseteq \mathscr{W}^c$, $\Gamma^a \subseteq \Psi^a$, and $\Gamma^t \subseteq \Psi^t$. The variety theorem (Proposition 3.7 of [53]) states that the lattices $\mathbf{VFA}(\Sigma)$, $\mathbf{VTL}(\Sigma)$ and $\mathbf{VFC}(\Sigma)$ are isomorphic with the above mappings, i.e., $\mathbf{K}^{ta} = \mathbf{K}$, $\mathbf{K}^{ca} = \mathbf{K}$, $\mathscr{V}^{at} = \mathscr{V}$, $\mathscr{V}^{ct} = \mathscr{V}$, $\Gamma^{ac} = \Gamma$, and $\Gamma^{tc} = \Gamma$, and moreover these operations are compatible with each other, i.e., $\mathbf{K}^{ct} = \mathbf{K}^t$, $\mathbf{K}^{tc} = \mathbf{K}^c$, $\mathscr{V}^{ca} = \mathscr{V}^a$, $\mathscr{V}^{ac} = \mathscr{V}^c$, $\Gamma^{ta} = \Gamma^a$, and $\Gamma^{at} = \Gamma^t$ for any $\mathbf{K} \in \mathbf{VFA}(\Sigma)$, $\mathscr{V} \in \mathbf{VTL}(\Sigma)$, and $\Gamma \in \mathbf{VFC}(\Sigma)$.

Chapter 2

Many-sorted variety theorem

Many-sorted algebras have found their way into computer science through abstract data type specifications. Many-sorted algebras and their specifications in terms of equations or other axioms are the mathematical fundament of rigorous approaches to abstract data types in programming and specification languages. It is widely believed that many-sorted algebras are the right mathematical tools to explain what abstract data types are (see [16]).

It appears that Maibaum [29] was the first one to consider many-sorted tree languages, while the idea of recognizable subsets of arbitrary algebras goes back to Mezei and Wright [31]. Many-sorted trees are used also by Engelfriet and Schmidt [18] in their study of the equational semantics of context-free tree languages. Recognizable subsets of general many-sorted algebras were studied by Courcelle [13, 14].

In this chapter, we consider varieties of recognizable subsets of many-sorted finitely generated free algebras over a given variety, varieties of congruences of such algebras, and varieties of finite many-sorted algebras. A variety theorem that establishes bijections between the classes of these three types of varieties is proved. For this, appropriate notions of many-sorted syntactic congruences and algebras are needed. Indeed, by developing a theory of varieties of recognizable subsets of free many-sorted algebras we generalize the theories of [52, 53] and [1] to the many-sorted case.

In Section 2.1 we present some basic definitions and our notation for many-sorted algebras. Also some more specialized notions relevant to our work are introduced. The references [16, 28, 30] may be consulted for general treatments of the theory of many-sorted algebras.

In Section 2.2 recognizable subsets of many-sorted algebras are considered. There are actually two types of these, recognizable sorted subsets and the "pure" recognizable subsets considered in [14, 18, 29] in which all elements are of some given sort. We mainly consider the former type in this thesis. Also, syntactic congruences and syntactic algebras of subsets of many-sorted algebras are introduced, and it is shown that they enjoy all the same general properties as their counterparts for monoids [17, 38] or term algebras, or one-sorted algebras in general [1, 52, 53].

In Section 2.3 we define our varieties of recognizable sets and varieties of congruences. For this a finite set of sorts S and variety $\mathbf{V}$ of some finite S-sorted type Ω are fixed. A variety of recognizable $\mathbf{V}$-sets consists then of recognizable subsets of finitely generated free algebras over $\mathbf{V}$. Similarly, a variety of $\mathbf{V}$-congruences consists of congruences of finite index on these algebras. Finally, a $\mathbf{V}$-variety of finite algebras is defined as a variety of finite algebras contained in $\mathbf{V}$. We define six mappings that transform varieties of recognizable $\mathbf{V}$-sets, varieties of $\mathbf{V}$-congruences and $\mathbf{V}$-varieties of finite algebras to each other. Then we prove our Variety Theorem that essentially says that these six mappings form three pairs of mutually inverse isomorphisms between the complete lattices of these three kinds of varieties.

2.1 Many-sorted algebras

In what follows, S is always a non-empty set of *sorts*. We consider various families of objects indexed by S. Such families are said to be *S-sorted*, or just *sorted*. The sort of an object is usually shown as a subscript or in parentheses (to avoid multiple subscripts). An *S-sorted set* $A = \langle A_s \rangle_{s \in S}$ is an S-indexed family of sets; for each $s \in S$, A_s is the set of *elements of sort s in A*, and we write it also as $A(s)$. The basic set-theoretic notions are defined for S-sorted sets in the natural sortwise manner. In particular, for any S-sorted sets $A = \langle A_s \rangle_{s \in S}$ and $B = \langle B_s \rangle_{s \in S}$, $A \subseteq B$ means that $A_s \subseteq B_s$ for every $s \in S$, $A \cup B = \langle A_s \cup B_s \rangle_{s \in S}$ and $A \cap B = \langle A_s \cap B_s \rangle_{s \in S}$, and general sorted unions and intersections are defined similarly. The notation $\emptyset$ is used also for the S-sorted empty set $\langle \emptyset \rangle_{s \in S}$.

We shall also consider subsets of one given sort of sorted sets. With any subset $T \subseteq A_u$ of some sort $u \in S$ of an S-sorted set $A = \langle A_s \rangle_{s \in S}$ we associate the sorted subset $\langle T \rangle \subseteq A$ such that $\langle T \rangle_u = T$ and $\langle T \rangle_s = \emptyset$ for every $s \in S$, $s \neq u$.

A *sorted relation* $\theta = \langle \theta_s \rangle_{s \in S}$ on an S-sorted set $A = \langle A_s \rangle_{s \in S}$ is an S-sorted family of relations such that for each $s \in S$, θ_s is a relation on A_s. A *sorted equivalence* on $A = \langle A_s \rangle_{s \in S}$ is a sorted relation $\theta = \langle \theta_s \rangle_{s \in S}$ where θ_s is an equivalence relation on A_s for each $s \in S$. If $\theta = \langle \theta_s \rangle_{s \in S}$ is an equivalence on A, then the corresponding *quotient set* is the S-sorted set $A/\theta = \langle A_s/\theta_s \rangle_{s \in S}$, where $A_s/\theta_s = \{a/\theta_s \mid a \in A_s\}$ $(s \in S)$.

The *sorted diagonal relation* and the *sorted universal relation* on $A = \langle A_s \rangle_{s \in S}$ are $\Delta_A = \langle \Delta_{A(s)} \rangle_{s \in S}$ and $\nabla_A = \langle \nabla_{A(s)} \rangle_{s \in S}$, respectively, where $\Delta_{A(s)} = \{(a, a) \mid a \in A(s)\}$ and $\nabla_{A(s)} = A(s) \times A(s)$ for each $s \in S$.

A *sorted mapping* $\varphi \colon A \to B$ from an S-sorted set $A = \langle A_s \rangle_{s \in S}$ to an

S-sorted set $B = \langle B_s \rangle_{s\in S}$ is an S-sorted family $\varphi = \langle \varphi_s \rangle_{s\in S}$ of mappings $\varphi_s \colon A_s \to B_s$ $(s \in S)$. The *kernel* of φ is the sorted equivalence $\ker\varphi = \langle \ker\varphi_s \rangle_{s\in S}$ on A. For any sorted subset $H = \langle H_s \rangle_{s\in S}$ of A, $H\varphi$ denotes the sorted subset $\langle H_s\varphi_s \rangle_{s\in S}$ of B. Similarly, if $H = \langle H_s \rangle_{s\in S}$ is a sorted subset of B, then $H\varphi^{-1}$ denotes the sorted subset $\langle H_s\varphi_s^{-1} \rangle_{s\in S}$ of A. The *composition* of two S-sorted mappings $\varphi \colon A \to B$ and $\psi \colon B \to C$, where $C = \langle C_s \rangle_{s\in S}$ is also an S-sorted set, is defined as the sorted mapping $\varphi\psi \colon A \to C$ such that $(\varphi\psi)_s = \varphi_s\psi_s$ for each $s \in S$. Here the mappings were composed from left to right, as we shall do especially with homomorphisms. Hence, $\varphi_s\psi_s \colon a \mapsto (a\varphi_s)\psi_s$ for all $s \in S$ and $a \in A_s$.

Treating S as an alphabet, S^* denotes the set of finite strings over S, including the empty string e, and S^+ is the set of non-empty strings over S. An *S-sorted signature* Ω is a set of operation symbols each of which has been assigned a *type* that is an element of $S^* \times S$. For any $(w, s) \in S^* \times S$, let $\Omega_{w,s}$ be the set of symbols of type (w, s), and Ω may be given by specifying the non-empty sets $\Omega_{w,s}$. If $f \in \Omega_{w,s}$, then w is the *domain type* of f, and s is its *sort*. In particular, every element of $\Omega_{e,s}$, for the empty string e, is a *constant symbol* of sort s. The fact that $f \in \Omega_{w,s}$ is expressed also by writing $f \colon w \to s$. For a finite S, a finite S-sorted signature is called an *S-sorted ranked alphabet*. Later S is assumed to be finite and Ω is always an S-sorted ranked alphabet. However, the following basic definitions and facts do not depend on this assumption.

An *Ω-algebra* $\mathcal{A} = (A, \Omega)$ consists of an S-sorted set $A = \langle A_s \rangle_{s\in S}$, where $A_s \neq \emptyset$ for every $s \in S$, equipped with constants and operations as follows:

(1) for each constant symbol $c \in \Omega_{e,s}$ of sort $s \in S$, an element $c^{\mathcal{A}} \in A_s$ of sort s is specified;

(2) for any function symbol $f \in \Omega_{w,s}$ with $w \in S^+$ and $s \in S$, there is an operation $f^{\mathcal{A}} \colon A^w \to A_s$ of *type* (w, s), *domain type* w and *sort* s. Here $A^w = A_{s(1)} \times \cdots \times A_{s(m)}$ for $w = s(1) \cdots s(m)$.

Such an algebra $\mathcal{A}$ is said to be *S-sorted*. For each $s \in S$, A_s is the set of elements of $\mathcal{A}$ of *sort* s. The algebra $\mathcal{A}$ is *trivial* if every A_s $(s \in S)$ is a singleton set. We may write $\mathcal{A} = (\langle A_s \rangle_{s\in S}, \Omega)$ to emphasize the fact that $\mathcal{A}$ is S-sorted. However, when we speak about the Ω-algebras $\mathcal{A} = (A, \Omega)$, $\mathcal{B} = (B, \Omega)$ and $\mathcal{C} = (C, \Omega)$, it will usually be assumed that $A = \langle A_s \rangle_{s\in S}$, $B = \langle B_s \rangle_{s\in S}$ and $C = \langle C_s \rangle_{s\in S}$.

An Ω-algebra $\mathcal{B} = (B, \Omega)$ such that $B \subseteq A$ is a *subalgebra* of $\mathcal{A} = (A, \Omega)$, and this we may express by writing $\mathcal{B} \subseteq \mathcal{A}$, if

(1) $c^{\mathcal{B}} = c^{\mathcal{A}}$ whenever $c \in \Omega_{e,s}$ for some $s \in S$, and
(2) $f^{\mathcal{B}} = f^{\mathcal{A}}|_{B^w}$ for any $f \in \Omega_{w,s}$ with $w \in S^+$ and $s \in S$.

A sorted equivalence $\theta = \langle \theta_s \rangle_{s\in S}$ on A is a *congruence* on $\mathcal{A} = (A, \Omega)$ if

$$a_1\,\theta_{s(1)}\,b_1, \ldots, a_m\,\theta_{s(m)}\,b_m \;\Rightarrow\; f^{\mathcal{A}}(a_1, \ldots, a_m)\,\theta_s\,f^{\mathcal{A}}(b_1, \ldots, b_m),$$

whenever $f\colon s(1)\cdots s(m)\to s$ and $a_1,b_1\in A_{s(1)},\ldots,a_m,b_m\in A_{s(m)}$. The corresponding *quotient algebra* $\mathcal{A}/\theta=(A/\theta,\Omega)$ is defined by setting

(1) $c^{\mathcal{A}/\theta}=c^{\mathcal{A}}/\theta_s$ for any $c\in\Omega_{e,s}$, and
(2) $f^{\mathcal{A}/\theta}(a_1/\theta_{s(1)},\ldots,a_m/\theta_{s(m)})=f^{\mathcal{A}}(a_1,\ldots,a_m)/\theta_s$ for any sorted function symbol $f\colon s(1)\ldots s(m)\to s$ and any $a_1\in A_{s(1)},\ldots,a_m\in A_{s(m)}$.

Since θ is a congruence, the operations $f^{\mathcal{A}/\theta}$ are well-defined.

A sorted mapping $\varphi\colon A\to B$ is a *homomorphism* from $\mathcal{A}=(A,\Omega)$ to $\mathcal{B}=(B,\Omega)$, and we express this by writing $\varphi\colon\mathcal{A}\to\mathcal{B}$, if

(1) $c^{\mathcal{A}}\varphi_s=c^{\mathcal{B}}$ whenever $c\in\Omega_{e,s}$ for some $s\in S$, and
(2) $f^{\mathcal{A}}(a_1,\ldots,a_m)\varphi_s=f^{\mathcal{B}}(a_1\varphi_{s(1)},\ldots,a_m\varphi_{s(m)})$ for any function symbol $f\colon s(1)\cdots s(m)\to s$ and any $a_1\in A_{s(1)},\ldots,a_m\in A_{s(m)}$.

A homomorphism φ is a *monomorphism*, an *epimorphism* or an *isomorphism*, if every φ_s $(s\in S)$ is injective, surjective or bijective, respectively. If there exists an epimorphism $\mathcal{A}\to\mathcal{B}$, then $\mathcal{B}$ is an *image* of $\mathcal{A}$, and we write $\mathcal{B}\leftarrow\mathcal{A}$. If there is an isomorphism $\varphi\colon\mathcal{A}\to\mathcal{B}$, the algebras are *isomorphic*, $\mathcal{A}\cong\mathcal{B}$ in symbols. An Ω-algebra $\mathcal{A}$ *divides* an Ω-algebra $\mathcal{B}$, and we write $\mathcal{A}\preceq\mathcal{B}$, if $\mathcal{A}$ is an image of a subalgebra of $\mathcal{B}$. We shall write $\mathcal{A}\subseteq\mathcal{B}$ also when $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{B}$. We observe that $\mathcal{A}\preceq\mathcal{B}$ when there exist an Ω-algebra $\mathcal{C}$, a monomorphism $\varphi\colon\mathcal{C}\to\mathcal{B}$ and an epimorphism $\psi\colon\mathcal{C}\to\mathcal{A}$.

The *natural map* corresponding to a sorted equivalence $\theta=\langle\theta_s\rangle_{s\in S}$ on a sorted set A is the sorted map $\theta^\natural\colon A\to A/\theta$, where $\theta^\natural_s\colon A_s\to A_s/\theta_s$, $a\mapsto a/\theta_s$, for each $s\in S$. It is easy to verify that if θ is a congruence on an Ω-algebra $\mathcal{A}$, then $\theta^\natural$ is an epimorphism from $\mathcal{A}$ onto $\mathcal{A}/\theta$. Moreover, the Homomorphism Theorem extends in a straightforward manner to many-sorted algebras as follows (cf. [30], for example).

Proposition 2.1.1 *If $\varphi\colon\mathcal{A}\to\mathcal{B}$ is a homomorphism of Ω-algebras, then* $\ker\varphi$ *is a congruence on $\mathcal{A}$ and $\psi\colon\mathcal{A}/\ker\varphi\to\mathcal{B}$, $a/\ker\varphi_s\mapsto a\varphi_s$, is a monomorphism such that* $(\ker\varphi)^\natural\psi=\varphi$. *If φ is an epimorphism, then ψ is an isomorphism.* □

Next we introduce the many-sorted version of a notion that has proved very useful for dealing with congruences.

Let $\mathcal{A}=(A,\Omega)$ be an Ω-algebra. For any pair $s,s'\in S$ of sorts, an *elementary s,s'-translation* is any mapping $A_s\to A_{s'}$ of the form

$$\alpha(\xi_s)=f^{\mathcal{A}}(a_1,\ldots a_{j-1},\xi_s,a_{j+1}\ldots,a_m),$$

where $m\ge 1$, $f\colon s(1)\ldots s(m)\to s'$, $1\le j\le m$, $s(j)=s$, and $a_i\in A_{s(i)}$ for every $i\neq j$. Here and later, ξ_s is a variable of sort s that does not appear in the other alphabets considered.

Let $\mathrm{ETr}(\mathcal{A},s,s')$ denote the set of all elementary s,s'-translations of $\mathcal{A}$. The $S\times S$-sorted set $\mathrm{Tr}(\mathcal{A})=\langle\mathrm{Tr}(\mathcal{A},s,s')\rangle_{s,s'\in S}$ of all *translations* of $\mathcal{A}$ is now defined inductively by the following clauses:

(1) $\mathrm{ETr}(\mathcal{A}, s, s') \subseteq \mathrm{Tr}(\mathcal{A}, s, s')$ for all $s, s' \in S$;
(2) for each $s \in S$, the identity map $1_{A(s)} \colon A(s) \to A(s)$ is in $\mathrm{Tr}(\mathcal{A}, s, s)$;
(3) if $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ and $\beta(\xi_{s'}) \in \mathrm{Tr}(\mathcal{A}, s', s'')$, for some $s, s', s'' \in S$, then $\beta(\alpha(\xi_s)) \in \mathrm{Tr}(\mathcal{A}, s, s'')$.

For any $s, s' \in S$, the elements of $\mathrm{Tr}(\mathcal{A}, s, s')$ are the s, s'-*translations* of $\mathcal{A}$.

The following lemma is an immediate generalization of the corresponding fact about one-sorted algebras (see e.g. [10, 11, 15]).

Lemma 2.1.2 *Let $\mathcal{A} = (A, \Omega)$ be an Ω-algebra. Every congruence $\theta = \langle\theta_s\rangle_{s\in S}$ on $\mathcal{A}$ is invariant with respect to all translations of $\mathcal{A}$, that is to say, $a\,\theta_s\,b$ implies $\alpha(a)\,\theta_{s'}\,\alpha(b)$ for all $s, s' \in S$, $a, b \in A_s$ and $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$. On the other hand, a sorted equivalence θ on A is a congruence if it is invariant with respect to every elementary translation of $\mathcal{A}$.* □

The following generalization of a lemma from the one-sorted case [52, 53] is frequently needed.

Lemma 2.1.3 *Let $\varphi \colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of Ω-algebras from $\mathcal{A} = (A, \Omega)$ to $\mathcal{B} = (B, \Omega)$. For any $s, s' \in S$ and every $\alpha(\xi_s)$ in $\mathrm{Tr}(\mathcal{A}, s, s')$, there exists a translation $\alpha_\varphi(\xi_s) \in \mathrm{Tr}(\mathcal{B}, s, s')$ such that $\alpha(a)\varphi_{s'} = \alpha_\varphi(a\varphi_s)$ for every $a \in A_s$. If φ is an epimorphism, then for all $s, s' \in S$ and every $\beta(\xi_s)$ in $\mathrm{Tr}(\mathcal{B}, s, s')$ there exists an $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ such that $\beta = \alpha_\varphi$.*

Proof. Because the claim clearly holds for the identity translations and all other non-elementary translations are products of elementary translations, it suffices to note that for any elementary s, s'-translation

$$\alpha(\xi_s) = f^{\mathcal{A}}(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m)$$

with $f \colon s(1) \ldots s(m) \to s'$ and $s(j) = s$, we may choose

$$\alpha_\varphi(\xi_s) = f^{\mathcal{B}}(a_1\varphi_{s(1)}, \ldots a_{j-1}\varphi_{s(j-1)}, \xi_s, a_{j+1}\varphi_{s(j+1)} \ldots, a_m\varphi_{s(m)}).$$

If φ is surjective, then every elementary translation of $\mathcal{B}$ can be obtained this way, which also then holds for all translations of $\mathcal{B}$. □

Translations of an Ω-algebra $\mathcal{A} = (A, \Omega)$ and their inverses are applied to subsets of a given sort and to sorted subsets as follows. Let $\alpha(\xi_s)$ be a translation in $\mathrm{Tr}(\mathcal{A}, s, s')$ for some $s, s' \in S$. For any $u \in S$ and $T \subseteq A_u$, let

- $\alpha(T) = \{\alpha(a) \mid a \in T\}$ $(\subseteq A_{s'})$ if $u = s$, and $\alpha(T) = \emptyset$ if $u \neq s$;
- $\alpha^{-1}(T) = \{a \in A_s \mid \alpha(a) \in T\}$ if $u = s'$, and $\alpha^{-1}(T) = \emptyset$ if $u \neq s'$.

Furthermore, for any sorted subset $L = \langle L_s\rangle_{s\in S}$ of A, we set

- $\alpha(L) = \langle K_u\rangle_{u\in S}$, where $K_{s'} = \alpha(L_s)$, and $K_u = \emptyset$ for each $u \neq s'$;
- $\alpha^{-1}(L) = \langle K_u\rangle_{u\in S}$, where $K_s = \alpha^{-1}(L_{s'})$, and $K_u = \emptyset$ for each $u \neq s$.

The *direct product* of two Ω-algebras $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ is the Ω-algebra $\mathcal{A} \times \mathcal{B} = (A \times B, \Omega)$, where

(1) $A \times B = \langle A_s \times B_s \rangle_{s \in S}$,
(2) $c^{\mathcal{A}\times\mathcal{B}} = (c^{\mathcal{A}}, c^{\mathcal{B}})$ for any $s \in S$ and $c \in \Omega_{e,s}$, and
(3) $f^{\mathcal{A}\times\mathcal{B}}((a_1,b_1),\ldots,(a_m,b_m)) = (f^{\mathcal{A}}(a_1,\ldots,a_m), f^{\mathcal{B}}(b_1,\ldots,b_m))$ for any $a_1 \in A_{s(1)}, b_1 \in B_{s(1)}, \ldots, a_m \in A_{s(m)}, b_m \in B_{s(m)}$ and any function symbol $f\colon s(1)\ldots s(m) \to s$.

The direct product $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ of any finite family $\mathcal{A}_1, \ldots, \mathcal{A}_n$, or the direct product $\prod_{i\in I} \mathcal{A}_i$ of a general family $\mathcal{A}_i$ $(i \in I)$ of Ω-algebras, are defined correspondingly.

If $\varphi\colon A \to B$ is a sorted mapping from an S-sorted set $A = \langle A_s \rangle_{s\in S}$ to an S-sorted set $B = \langle B_s \rangle_{s\in S}$ and $\theta = \langle \theta_s \rangle_{s\in S}$ is a sorted equivalence on B, then $\varphi \circ \theta \circ \varphi^{-1}$ is the sorted equivalence on A defined by the condition

$$a_1\,(\varphi \circ \theta \circ \varphi^{-1})_s\, a_2 \;\Leftrightarrow\; a_1\varphi_s\,\theta_s\,a_2\varphi_s \qquad (s \in S,\ a_1, a_2 \in A_s).$$

In the following lemma we note a few basic facts about quotient algebras.

Lemma 2.1.4 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be Ω-algebras, θ, θ' be congruences on $\mathcal{A}$, ρ be a congruence on $\mathcal{B}$, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then the following hold.*

(1) *If $\theta \subseteq \theta'$, then $\mathcal{A}/\theta' \leftarrow \mathcal{A}/\theta$.*
(2) *$\mathcal{A}/\theta \cap \theta' \subseteq \mathcal{A}/\theta \times \mathcal{A}/\theta'$.*
(3) *The relation $\varphi\circ\rho\circ\varphi^{-1}$ is a congruence on $\mathcal{A}$, and $\mathcal{A}/\varphi\circ\rho\circ\varphi^{-1} \preceq \mathcal{B}/\rho$. If φ is an epimorphism, then $\mathcal{A}/\varphi \circ \rho \circ \varphi^{-1} \cong \mathcal{B}/\rho$*

Proof. Statements (1) and (2) are direct generalizations of well-known facts. In the many-sorted case they follow, for example, from Theorem 3.4.20 and Lemma 4.1.5 of [30].

Let us prove (3). If $a\,(\varphi \circ \rho \circ \varphi^{-1})_s\, b$, for some $s \in S$ and $a, b \in A_s$, then $a\varphi_s\,\rho_s\,b\varphi_s$. By Lemma 2.1.3, for any $s' \in S$ and every $\alpha \in \mathrm{Tr}(\mathcal{A}, s, s')$, there is an $\alpha_\varphi \in \mathrm{Tr}(\mathcal{B}, s, s')$ such that $\alpha_\varphi(d\varphi_s) = \alpha(d)\varphi_{s'}$ for every $d \in A_s$. Since $\alpha_\varphi(a\varphi_s)\,\rho_{s'}\,\alpha_\varphi(b\varphi_s)$ by Lemma 2.1.2, we also have $\alpha(a)\varphi_{s'}\,\rho_{s'}\,\alpha(b)\varphi_{s'}$, that is to say, $\alpha(a)\,(\varphi\circ\rho\circ\varphi^{-1})_{s'}\,\alpha(b)$. Hence, $\varphi\circ\rho\circ\varphi^{-1}$ is a congruence relation on $\mathcal{A}$ by Lemma 2.1.2. It is now easy to see that $\psi\colon \mathcal{A}/(\varphi \circ \rho \circ \varphi^{-1}) \to \mathcal{B}/\rho$ is a monomorphism if we define

$$\psi_s\colon A_s/(\varphi \circ \rho \circ \varphi^{-1})_s \to B_s/\rho_s, \quad a/(\varphi \circ \rho \circ \varphi^{-1})_s \mapsto a\varphi_s/\rho_s,$$

for each $s \in S$. Finally, we note that if φ is surjective, then so is ψ. □

The class operators S, H, P and $\mathrm{P_f}$ are defined exactly as in the one-sorted case: for any class $\mathbf{K}$ of Ω-algebras and any Ω-algebra $\mathcal{A}$,

(1) $\mathcal{A} \in \mathrm{S}(\mathbf{K})$ iff $\mathcal{A}$ is isomorphic to a subalgebra of a member of $\mathbf{K}$,
(2) $\mathcal{A} \in \mathrm{H}(\mathbf{K})$ iff $\mathcal{A}$ is an image of some member of $\mathbf{K}$,
(3) $\mathcal{A} \in \mathrm{P}(\mathbf{K})$ iff $\mathcal{A}$ is isomorphic to the direct product of a family of algebras in $\mathbf{K}$, and

(4) $\mathcal{A} \in \mathrm{P_f}(\mathbf{K})$ iff $\mathcal{A}$ is isomorphic to the direct product of a finite family of algebras in $\mathbf{K}$.

A class $\mathbf{K}$ of Ω-algebras is a *variety* if $\mathrm{S}(\mathbf{K}), \mathrm{H}(\mathbf{K}), \mathrm{P}(\mathbf{K}) \subseteq \mathbf{K}$. Birkhoff's well-known theorem [6] by which a class of algebras is definable by equations iff it is a variety, holds also for many-sorted algebras (cf. Section 5 of [30]).

A class $\mathbf{K}$ of finite Ω-algebras is called a *variety of finite Ω-algebras*, an Ω-VFA for short, if it is closed under subalgebras, homomorphic images, and finite direct products, i.e., if $\mathrm{S}(\mathbf{K}), \mathrm{H}(\mathbf{K}), \mathrm{P_f}(\mathbf{K}) \subseteq \mathbf{K}$. It is easy to show that a class $\mathbf{K}$ of finite Ω-algebras is an Ω-VFA iff $\mathcal{A} \in \mathbf{K}$ whenever $\mathcal{A} \preceq \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ for some $n \geq 0$ and $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathbf{K}$. When we deal with varieties of finite Ω-algebras, both S and Ω are assumed to be finite.

Let $X = \langle X_s \rangle_{s \in S}$ be an S-sorted alphabet disjoint from Ω. The S-sorted set $T_\Omega(X) = \langle T_\Omega(X,s) \rangle_{s \in S}$ of *Ω-terms with variables in X* is defined inductively by:

(1) $\Omega_{e,s} \cup X_s \subseteq T_\Omega(X,s)$ for every $s \in S$, and
(2) $f(t_1, \ldots, t_m) \in T_\Omega(X,s)$ for any function symbol $f\colon s_1 \ldots s_m \to s$ and terms $t_1 \in T_\Omega(X,s_1)$, $\ldots$, $t_m \in T_\Omega(X,s_m)$.

The alphabet X is said to be *full* for Ω if $T_\Omega(X,s) \neq \emptyset$ for every sort $s \in S$. Note that a given $T_\Omega(X,s)$ may be non-empty even when $X_s = \Omega_{e,s} = \emptyset$. If $X = \langle X_s \rangle_{s \in S}$ is full for Ω, then the *ΩX-term algebra* $\mathcal{T}_\Omega(X) = (T_\Omega(X), \Omega)$ is defined in the natural way:

(1) $c^{\mathcal{T}_\Omega(X)} = c$ for any $s \in S$ and $c \in \Omega_{e,s}$, and
(2) $f^{\mathcal{T}_\Omega(X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ whenever $m > 0$, $f\colon s_1 \ldots s_m \to s$ and $t_1 \in T_\Omega(X,s_1)$, $\ldots$, $t_m \in T_\Omega(X,s_m)$.

Of course, $\mathcal{T}_\Omega(X)$ is *freely generated* by X over the class of all Ω-algebras, that is to say, for any Ω-algebra $\mathcal{A} = (A, \Omega)$, any sorted mapping $\alpha : X \to A$ has a unique extension to a homomorphism $\alpha^{\mathcal{A}}\colon \mathcal{T}_\Omega(X) \to \mathcal{A}$.

In a more general setting, if $\mathbf{V}$ is a class of Ω-algebras, an Ω-algebra $\mathcal{F} = (\langle F_s \rangle_{s \in S}, \Omega)$ is *freely generated* over $\mathbf{V}$ by a sorted subset $G \subseteq F$, if $\mathcal{F} \in \mathbf{V}$, $\mathcal{F}$ is generated by G, and for any $\mathcal{A} = (A, \Omega)$ in $\mathbf{V}$, any sorted mapping $\varphi_0\colon G \to A$ can be extended to a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{A}$. If such an $\mathcal{F}$ exists, it is determined uniquely up to isomorphism by $\mathbf{V}$ and G, and we denote it $\mathcal{F}_{\mathbf{V}}(G) = (\mathrm{F}_{\mathbf{V}}(G), \Omega)$ with $\mathrm{F}_{\mathbf{V}}(G) = \langle \mathrm{F}_{\mathbf{V}}(G,s) \rangle_{s \in S}$.

Let Ω be an S-sorted ranked alphabet and let X be an S-sorted alphabet disjoint from Ω. For each $s \in S$, let ξ_s be again a special symbol of sort s. The $S \times S$-sorted set $C_\Omega(X) = \langle C_\Omega(X,s,s') \rangle_{s,s' \in S}$ of *ΩX-contexts* is defined inductively by the conditions

(1) $\xi_s \in C_\Omega(X,s,s)$ for each $s \in S$, and
(2) $f(t_1, \ldots, t_{j-1}, p, t_{j+1} \ldots, t_m) \in C_\Omega(X,s,s')$ for $s, s', s_1, \ldots, s_m \in S$, $m \geq 1$, $f\colon s_1 \ldots s_m \to s'$, $1 \leq j \leq m$, $p \in C_\Omega(X,s,s_j)$, and terms $t_i \in T_\Omega(X,s_i)$ where $i \neq j$.

The *composition* $p \cdot q = q(p)$ of two ΩX-contexts $p \in C_\Omega(X, s, s')$ and $q \in C_\Omega(X, s', s'')$ (for some $s, s', s'' \in S$) is the ΩX-context in $C_\Omega(X, s, s'')$ obtained from q when $\xi_{s'}$ is replaced with p.

Let $\mathcal{A} = (A, \Omega)$ be any Ω-algebra. Every translation of $\mathcal{A}$ is *represented* in a natural way by an ΩA-context of a matching type:

(1) an elementary translation $\alpha(\xi_s) = f^{\mathcal{A}}(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m)$ is represented by the ΩA-context $f(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m)$,
(2) the identity map $1_{A(s)} \colon A(s) \to A(s)$ is represented by ξ_s, and
(3) if $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ and $\beta(\xi_{s'}) \in \mathrm{Tr}(\mathcal{A}, s', s'')$ are represented by the ΩA-contexts $p(\xi_s) \in C_\Omega(A, s, s')$ and $q(\xi_{s'}) \in C_\Omega(A, s', s'')$, respectively, then $\beta(\alpha(\xi_s))$ is represented by $q(p(\xi_s)) \in C_\Omega(A, s, s'')$.

That a translation $\alpha(\xi_s)$ is represented by a context $p(\xi_s)$ means that α is the polynomial function (cf. [10], for example) defined by p in $\mathcal{A}$, when p is interpreted as a polynomial symbol with ξ_s as the only variable.

2.2 Syntactic congruences and algebras

An equivalence θ on a set A *saturates* a subset L of A if L is the union of some θ-classes, and θ is said to be of *finite index* if it has a finite number of equivalence classes. Mezei and Wright [31] call a subset L of an algebra $\mathcal{A}$ *recognizable* if it is saturated by a congruence of finite index on $\mathcal{A}$. Clearly, L is recognizable if and only if there exist a finite algebra $\mathcal{B}$, a homomorphism $\varphi \colon \mathcal{A} \to \mathcal{B}$ and a subset H of $\mathcal{B}$ such that $L = H\varphi^{-1}$. We use this condition, where $\mathcal{B}$ may be viewed as a "recognizer" of L, for defining recognizability in many-sorted algebras. There are two natural types of recognizable subsets of a sorted algebra: the *recognizable sorted subsets* and the *recognizable subsets of a given sort.*

In what follows, S is always a finite set of sorts and Ω is an S-sorted ranked alphabet. An S-sorted set $A = \langle A_s \rangle_{s \in S}$ is said to be *finite* if every A_s $(s \in S)$ is finite, and similarly, an Ω-algebra $\mathcal{A} = (A, \Omega)$ is *finite* if $A = \langle A_s \rangle_{s \in S}$ is finite.

Definition 2.2.1 A sorted subset $L \subseteq A$ of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is *recognizable* if there exist a finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi \colon \mathcal{A} \to \mathcal{B}$ and a sorted subset H of B such that $L = H\varphi^{-1}$, and we say that $\mathcal{B}$ *recognizes* L. Let $\mathrm{Rec}(\mathcal{A})$ be the set of all recognizable subsets of $\mathcal{A}$.

For any $s \in S$, a subset T of A_s is said to be *recognizable* in $\mathcal{A}$ if there exist a finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi \colon \mathcal{A} \to \mathcal{B}$ and a subset H of B_s such that $T = H\varphi_s^{-1}$. Let $\mathrm{Rec}(\mathcal{A}, s)$ denote the set of all such subsets of A_s. We call such sets *pure recognizable sets.*

The recognizable tree languages of sort $s \in S$ considered by Maibaum [29] are the pure recognizable subsets of the term algebra $\mathcal{T}_\Omega(\emptyset)$ of sort s, i.e., the elements of $\mathrm{Rec}(\mathcal{T}_\Omega(\emptyset), s)$. Courcelle [13, 14] extends this notion to any S-sorted algebra $\mathcal{A} = (\langle A_s \rangle_{s\in S}, \Omega)$, without assuming the finiteness of S or Ω, by calling a subset $T \subseteq A_s$ recognizable if there exist a locally finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a subset H of B_s such that $T = H\varphi_s^{-1}$; an algebra $\mathcal{B} = (\langle B_s \rangle_{s\in S}, \Omega)$ is *locally finite* if every B_s is finite ($s \in S$). Since we assume that S is finite, this "locally finite" means here just "finite".

A sorted equivalence $\theta = \langle \theta_s \rangle_{s\in S}$ on an S-sorted set $A = \langle A_s \rangle_{s\in S}$ is said to *saturate* a sorted subset $L = \langle L_s \rangle_{s\in S}$ of A if every L_s is the union of some θ_s-classes ($s \in S$), and θ is of *finite index* if every θ_s ($s \in S$) is of finite index. The following lemma is an obvious generalization of the above fact.

Lemma 2.2.2 *A sorted subset of an Ω-algebra $\mathcal{A}$ is recognizable iff it is saturated by a congruence on $\mathcal{A}$ of finite index. Similarly, a subset $T \subseteq A_u$ of some sort $u \in S$ is recognizable iff it is saturated by θ_u for some congruence $\theta = \langle \theta_s \rangle_{s\in S}$ on $\mathcal{A}$ of finite index.* □

Next we present a few closure properties that are well-known for recognizable subsets of one-sorted algebras.

Proposition 2.2.3 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be any Ω-algebras.*

(1) $\emptyset, A \in \mathrm{Rec}(\mathcal{A})$.
(2) *If $K, L \in \mathrm{Rec}(\mathcal{A})$, then $K \cup L, K \cap L, K - L \in \mathrm{Rec}(\mathcal{A})$.*
(3) *If $L \in \mathrm{Rec}(\mathcal{A})$ and $\alpha \in \mathrm{Tr}(\mathcal{A}, s, s')$ for some $s, s' \in S$, then $\alpha^{-1}(L)$ belongs to* $\mathrm{Rec}(\mathcal{A})$.
(4) *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism and $L \in \mathrm{Rec}(\mathcal{B})$, then $L\varphi^{-1}$ belongs to* $\mathrm{Rec}(\mathcal{A})$.

Proof. Assertion (1) is trivial, and (2) can be proved as usual by defining the direct product of any two finite algebras recognizing K and L.

For (3), we recall first that $\alpha^{-1}(L)_s = \alpha^{-1}(L_{s'})$ and $\alpha^{-1}(L)_{s''} = \emptyset$ for every $s'' \neq s$. Assume now that $L = H\varphi^{-1}$, where $\varphi\colon \mathcal{A} \to \mathcal{C}$ is a homomorphism to a finite algebra Ω-algebra $\mathcal{C} = (C, \Omega)$, and $H \subseteq C$. By Lemma 2.1.3 there is a translation $\alpha_\varphi \in \mathrm{Tr}(\mathcal{C}, s, s')$ such that $\alpha(a)\varphi_{s'} = \alpha_\varphi(a\varphi_s)$ for every $a \in L_s$. Now it is easy to see that $\alpha^{-1}(L) = G\varphi^{-1}$ for the sorted subset G of C defined in such a way that $G_s = \alpha_\varphi^{-1}(H_{s'})$ and $G_{s''} = \emptyset$ for every $s'' \neq s$.

To prove (4), assume that $L = H\psi^{-1}$, where $\psi\colon \mathcal{B} \to \mathcal{C}$ is a homomorphism to a finite algebra Ω-algebra $\mathcal{C} = (C, \Omega)$ and $H \subseteq C$. Then $L\varphi^{-1} = H(\varphi\psi)^{-1} \in \mathrm{Rec}(\mathcal{A})$ as claimed. □

Let us clarify here the relationship between the two notions of recognizable subsets, recognizable sorted subsets and pure recognizable subsets.

The following fact can be derived directly from Definition 2.2.1.

Lemma 2.2.4 *Let $\mathcal{A} = (A, \Omega)$ be an S-sorted algebra. For any $s \in S$ and $T \subseteq A_s$, $T \in \mathrm{Rec}(\mathcal{A}, s)$ iff $\langle T \rangle \in \mathrm{Rec}(\mathcal{A})$.* □

The forward direction of the following proposition is again a direct consequence of Definition 2.2.1, and the converse part follows from Lemma 2.2.4 and Proposition 2.2.3(2).

Proposition 2.2.5 *A sorted subset $L = \langle L_s \rangle_{s \in S}$ of an S-sorted algebra $\mathcal{A} = (A, \Omega)$ is recognizable iff $L_s \in \mathrm{Rec}(\mathcal{A}, s)$ for every $s \in S$.* □

We shall now present a theory of syntactic congruences and syntactic algebras for S-sorted algebras similar to those known for semigroups, monoids (cf. [17, 38, 40]) or general one-sorted algebras (cf. [1, 52, 53]).

Definition 2.2.6 The *syntactic congruence* $\approx^L = \langle \approx^L_s \rangle_{s \in S}$ of a sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is defined by

$$a \approx^L_s b \Leftrightarrow (\forall s' \in S)(\forall \alpha \in \mathrm{Tr}(\mathcal{A}, s, s'))(\alpha(a) \in L_{s'} \leftrightarrow \alpha(b) \in L_{s'})$$

for every $s \in S$ and $a, b \in A_s$.

The following basic property of syntactic congruences can be verified exactly as in the one-sorted case.

Lemma 2.2.7 *The syntactic congruence $\approx^L$ of any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is the greatest congruence on $\mathcal{A}$ that saturates L.* □

Of course, we have also the following Nerode-Myhill type theorem.

Proposition 2.2.8 *For any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$, the following are equivalent:*

(1) *$L \in \mathrm{Rec}(\mathcal{A})$;*
(2) *L is saturated by a congruence on $\mathcal{A}$ of finite index;*
(3) *$\approx^L$ is of finite index.*

Proof. If there exist a finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a sorted subset H of B such that $L = H\varphi^{-1}$, then $\ker \varphi$ is a congruence on $\mathcal{A}$ of finite index saturating L. On the other hand, if L is saturated by a congruence θ on $\mathcal{A}$ of finite index, then L is recognized by the finite Ω-algebra $\mathcal{A}/\theta$. Hence, (1) and (2) are equivalent. Conditions (2) and (3) are equivalent by Lemma 2.2.7. □

Also the following facts can be proved similarly as their counterparts in the one-sorted theory ([53]). Note that K and L are always sorted subsets.

Proposition 2.2.9 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be Ω-algebras.*
(1) $\approx^{A-L} = \approx^L$, *for every $L \subseteq A$.*
(2) $\approx^K \cap \approx^L \subseteq \approx^{K\cap L}$, *for every $K, L \subseteq A$.*
(3) $\approx^L \subseteq \approx^{\alpha^{-1}(L)}$, *for every $L \subseteq A$ and translation $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$.*
(4) *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $\varphi\circ\approx^L\circ\varphi^{-1} \subseteq \approx^{L\varphi^{-1}}$ for every $L \subseteq B$. If φ is an epimorphism, then $\varphi\circ\approx^L\circ\varphi^{-1} = \approx^{L\varphi^{-1}}$.* □

For any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$, let $A/L = \langle A_s/L\rangle_{s\in S}$, where $A_s/L = A_s/\approx^L_s$ for each sort $s \in S$. Moreover, for any $s \in S$ and any $a \in A_s$, let a/L be a shorthand for $a/\approx^L_s$.

Definition 2.2.10 The *syntactic algebra* $\mathcal{A}/L = (A/L, \Omega)$ of a subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is the quotient algebra $\mathcal{A}/\approx^L$, and the corresponding canonical homomorphism $\varphi^L = \langle \varphi^L_s\rangle_{s\in S}$, where for each $s \in S$,

$$\varphi^L_s\colon A_s \to A_s/L,\ a \mapsto a/L, \quad (a \in A_s),$$

is called the *syntactic homomorphism* of L.

It is clear that any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is recognized by its syntactic algebra. Indeed, $L = L\varphi^L(\varphi^L)^{-1}$ for the syntactic homomorphism $\varphi^L\colon \mathcal{A} \to \mathcal{A}/L$. It follows from Lemma 2.2.7 that $\mathcal{A}/L$ is in the following sense the least algebra recognizing L.

Lemma 2.2.11 *A sorted subset L of an Ω-algebra $\mathcal{A}$ is recognizable iff the syntactic algebra $\mathcal{A}/L$ is finite. An Ω-algebra $\mathcal{B}$ recognizes L iff $\mathcal{A}/L \preceq \mathcal{B}$.*□

Proposition 2.2.12 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be any Ω-algebras.*
(1) $\mathcal{A}/(A - L) = \mathcal{A}/L$, *for any $L \subseteq A$.*
(2) $\mathcal{A}/K \cap L \preceq \mathcal{A}/K \times \mathcal{A}/L$, *for any $K, L \subseteq A$.*
(3) $\mathcal{A}/\alpha^{-1}(L) \preceq \mathcal{A}/L$, *for any $L \subseteq A$, $s, s' \in S$ and $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$.*
(4) $\mathcal{A}/L\varphi^{-1} \preceq \mathcal{B}/L$, *for any homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and any $L \subseteq B$. Moreover, if φ is an epimorphism, then $\mathcal{A}/L\varphi^{-1} \cong \mathcal{B}/L$.*

Proof. Assertions (1) and (3) follow immediately by the corresponding parts of Proposition 2.2.9 and Lemma 2.1.4. For (2) it suffices to note that

$$\mathcal{A}/K \cap L \leftarrow \mathcal{A}/(\approx^K \cap \approx^L) \subseteq \mathcal{A}/K \times \mathcal{A}/L$$

by Proposition 2.2.9(2) and Lemma 2.1.4.
To prove (4), let us first assume that φ is an epimorphism and show that

$$\psi_s\colon A_s/L\varphi^{-1} \to B_s/L,\ a/L\varphi^{-1} \mapsto a\varphi_s/L, \quad (s \in S,\ a \in A_s)$$

defines an isomorphism $\psi = \langle\psi_s\rangle_{s\in S}$ between $\mathcal{A}/L\varphi^{-1}$ and $\mathcal{B}/L$. First we verify that ψ is well-defined and injective: for each $s \in S$ and any $a, a' \in A_s$,

$$\begin{aligned}
(a/L)\psi_s = (a'/L)\psi_s \quad &\Leftrightarrow \quad a\varphi_s \approx^L_s a'\varphi_s \\
&\Leftrightarrow \quad (\forall s')(\forall\beta)[\beta(a\varphi_s) \in L_{s'} \leftrightarrow \beta(a'\varphi_s) \in L_{s'}] \\
&\Leftrightarrow \quad (\forall s')(\forall\alpha)[\alpha_\varphi(a\varphi_s) \in L_{s'} \leftrightarrow \alpha_\varphi(a'\varphi_s) \in L_{s'}] \\
&\Leftrightarrow \quad (\forall s')(\forall\alpha)[\alpha(a)\varphi_{s'} \in L_{s'} \leftrightarrow \alpha(a')\varphi_{s'} \in L_{s'}] \\
&\Leftrightarrow \quad (\forall s')(\forall\alpha)[\alpha(a) \in L_{s'}\varphi^{-1}_{s'} \leftrightarrow \alpha(a') \in L_{s'}\varphi^{-1}_{s'}] \\
&\Leftrightarrow \quad a/L\varphi^{-1} = a'/L\varphi^{-1},
\end{aligned}$$

where s' ranges over S, α over $\mathrm{Tr}(\mathcal{A}, s, s')$ and β over $\mathrm{Tr}(\mathcal{B}, s, s')$.
Consider now a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ that is not necessarily onto, and let $\mathcal{C} = (\langle A_s\varphi_s\varphi_s^L\rangle_{s\in S}, \Omega)$ be the subalgebra of $\mathcal{B}/L$ obtained as the image of $\mathcal{B}$ under the homomorphism $\varphi\varphi^L : \mathcal{A} \to \mathcal{B}/L$. Then $\eta\colon \mathcal{A} \to \mathcal{C}, a \mapsto a\varphi\varphi^L$, is an epimorphism, and hence $\mathcal{A}/L\varphi^{-1}\eta\eta^{-1} \cong \mathcal{C}/L\varphi^{-1}\eta$. However, this implies $\mathcal{A}/L\varphi^{-1} \preceq \mathcal{B}/L$ since $L\varphi^{-1}\eta\eta^{-1} = L\varphi^{-1}$ and $\mathcal{C} \subseteq \mathcal{B}/L$. □

Lemma 2.2.13 *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism of Ω-algebras and $L \subseteq B$, then for every $s \in S$,*

$$\varphi_s\circ\approx_s^L\circ\varphi_s^{-1} \;\subseteq\; \bigcap\{\approx_s^{\beta^{-1}(L)\varphi^{-1}} \mid \beta \in \mathrm{Tr}(\mathcal{B}, s, s'),\ s' \in S\},$$

and if φ is an epimorphism, equality holds.

Proof. Let $\rho = \bigcap\{\approx_s^{\beta^{-1}(L)\varphi^{-1}} \mid \beta \in \mathrm{Tr}(\mathcal{B}, s, s'),\ s' \in S\}$. Parts (3) and (4) of 2.2.9 yield for every $\beta \in \mathrm{Tr}(\mathcal{B}, s, s')$,

$$\varphi_s\circ \approx_s^L \circ\varphi_s^{-1} \;\subseteq\; \varphi_s\circ \approx_s^{\beta^{-1}(L)} \circ\varphi_s^{-1} \;\subseteq \approx_s^{\beta^{-1}(L)\varphi^{-1}}.$$

Hence $\varphi_s\circ \approx_s^L \circ\varphi_s^{-1} \subseteq \rho$. Assume now that φ is surjective. The converse inclusion is then obtained by the following chain of implications, where $a, a' \in A_s$, s' and s'' range over S, β and γ are translations of $\mathcal{B}$, and $(\forall\beta)_{s,s'}$ is a shorthand for $(\forall\beta \in \mathrm{Tr}(\mathcal{B}, s, s'))$ etc.:

$$\begin{array}{rll}
a\,\rho\,a' & \Rightarrow & (\forall s')(\forall\beta)_{s,s'}[a \approx_s^{\beta^{-1}(L)\varphi^{-1}} a'] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}[a\varphi_s \approx_s^{\beta^{-1}(L)} a'\varphi_s] \\
& \Rightarrow & (\forall s', s'')(\forall\beta)_{s,s'}(\forall\gamma)_{s,s''}[\gamma(a\varphi_s) \in \beta^{-1}(L)_{s''} \leftrightarrow \gamma(a'\varphi_s) \in \beta^{-1}(L)_{s''}] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}(\forall\gamma)_{s,s}[\gamma(a\varphi_s) \in \beta^{-1}(L_{s'}) \leftrightarrow \gamma(a'\varphi_s) \in \beta^{-1}(L_{s'})] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}(\forall\gamma)_{s,s}[\beta(\gamma(a\varphi_s)) \in L_{s'} \leftrightarrow \beta(\gamma(a'\varphi_s)) \in L_{s'}] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}[\beta(a\varphi_s) \in L_{s'} \leftrightarrow \beta(a'\varphi_s) \in L_{s'}] \\
& \Rightarrow & a\varphi_s \approx_s^L a'\varphi_s \\
& \Rightarrow & a\,\varphi_s\circ \approx_s^L \circ\varphi_s^{-1}\, a'.
\end{array}$$

Here we used also the fact that $\beta^{-1}(L)_{s''} = \emptyset$ for every $s'' \neq s$. □

Let us now present the natural generalizations of some basic facts known for monoids [17, 38] and algebras in general in the one-sorted case [52, 53].

Lemma 2.2.14 *Let $L = \langle L_s\rangle_{s\in S}$ be a sorted subset of an Ω-algebra $\mathcal{A} = (A, \Omega)$. For any $s \in S$ and $a \in A_s$,*

$$a/L \;=\; \bigcap\{\alpha^{-1}(L_{s'}) \mid \alpha(a_s) \in L_{s'}\} \setminus \bigcup\{\alpha^{-1}(L_{s'}) \mid \alpha(a_s) \notin L_{s'}\},$$

where s' ranges over S and α over $\mathrm{Tr}(\mathcal{A}, s, s')$. □

Lemma 2.2.15 *Any congruence θ on an algebra $\mathcal{A} = (A, \Omega)$ is the intersection of some syntactic congruences. In particular,*

$$\theta \;=\; \bigcap\{\approx^{\langle a/\theta\rangle} \mid s \in S, a \in A_s\}.$$

□

Let us call an Ω-algebra $\mathcal{A}$ *syntactic*, if $\mathcal{A} \cong \mathcal{B}/L$ for some Ω-algebra $\mathcal{B}$ and some sorted subset L of $\mathcal{B}$. A sorted subset D of an Ω-algebra $\mathcal{A}$ is *disjunctive* if $\approx^D = \Delta_A$.

Proposition 2.2.16 *An Ω-algebra $\mathcal{A}$ is syntactic if and only if it has a disjunctive subset.* □

Subdirect products of Ω-algebras are defined (cf. [30], Section 4.1, or [28], p. 159) exactly as for one-sorted algebras, and by generalizing in an obvious way a well-known theorem of Birkhoff (cf. [10], for example), we may say that an Ω-algebra $\mathcal{A} = (A, \Omega)$ is *subdirectly irreducible* if the intersection of all non-trivial congruences on $\mathcal{A}$ is the diagonal relation Δ_A. By applying Lemma 2.2.15 to the diagonal relation we get the following result.

Corollary 2.2.17 *Every subdirectly irreducible Ω-algebra is syntactic.* □

Since it is clear that also varieties of many-sorted algebras are generated by their subdirectly irreducible members, Corollary 2.2.17 implies the following important lemma which follows also directly from Lemma 2.2.15: $\mathcal{A} \subseteq \prod\{\mathcal{A}/\approx^{\{a\}} \mid a \in A\}$ for any finite $\mathcal{A} = (A, \Omega)$ because we already have $\Delta_A = \bigcap\{\approx^{\{a\}} \mid a \in A\}$.

Lemma 2.2.18 *Every Ω-VFA is generated by syntactic algebras. Hence, if $\mathbf{K}$ is an Ω-VFA and $\mathcal{A}$ any finite Ω-algebra, then $\mathcal{A} \in \mathbf{K}$ iff $\mathcal{A}$ divides the product $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ for some $n \geq 0$ and some syntactic algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathbf{K}$.* □

2.3 The Variety Theorem

Let S and Ω be again a finite set of sorts and an S-sorted ranked alphabet, respectively. We shall consider varieties of recognizable subsets of finitely generated free algebras over a given variety $\mathbf{V}$ of Ω-algebras. If $\mathbf{V}$ is the class of all Ω-algebras, we are actually dealing with varieties of many-sorted tree languages. In what follows, we call finite S-sorted alphabets full for Ω simply *full alphabets*, and $X = \langle X_s \rangle_{s \in S}$ and $Y = \langle Y_s \rangle_{s \in S}$ are always such full alphabets.

The free algebra $\mathcal{F}_{\mathbf{V}}(X) = (\mathrm{F}_{\mathbf{V}}(X), \Omega)$ exists for every full alphabet X, and we call the recognizable subsets of $\mathcal{F}_{\mathbf{V}}(X)$ *recognizable* $\mathbf{V}$*-sets*. The syntactic algebra $\mathcal{F}_{\mathbf{V}}(X)/L$ of a sorted subset L of $\mathcal{F}_{\mathbf{V}}(X)$ is denoted simply $\mathrm{SA}(L)$. It is clear that $\mathrm{SA}(L) \in \mathbf{V}$.

We shall also need the following fact that can be proved similarly as its one-sorted counterpart [17, 52, 53].

Lemma 2.3.1 *Let $\mathcal{A}$ be a finite algebra in* $\mathbf{V}$ *and let X be a full alphabet such that for some generating set $G = \langle G_s \rangle_{s \in S}$ of $\mathcal{A}$, $|G_s| \leq |X_s|$ for every $s \in S$. Then $\mathcal{A}$ is syntactic iff $\mathcal{A} \cong \mathrm{SA}(L)$ for some $L \in \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X))$.* □

A *family of recognizable* $\mathbf{V}$*-sets* is a mapping $\mathcal{R}$ that assigns to each full alphabet X a set $\mathcal{R}(X) \subseteq \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X))$ of recognizable $\mathbf{V}$-sets. We write then $\mathcal{R} = \{\mathcal{R}(X)\}_X$ with the understanding that X ranges over all full alphabets. The inclusion relation and the basic set-operations are defined for families of recognizable $\mathbf{V}$-sets by the natural componentwise conditions. For example, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are any families of recognizable $\mathbf{V}$-sets, then $\mathcal{R}_1 \subseteq \mathcal{R}_2$ iff $\mathcal{R}_1(X) \subseteq \mathcal{R}_2(X)$ for every X.

For any X and $L \subseteq \mathrm{F}_{\mathbf{V}}(X)$, the complement of $\overline{L}$ is $\mathrm{F}_{\mathbf{V}}(X) \setminus L$.

Definition 2.3.2 A family of recognizable $\mathbf{V}$-sets $\mathcal{R} = \{\mathcal{R}(X)\}_X$ is a *variety of recognizable* $\mathbf{V}$*-sets*, a $\mathbf{V}$-VRS for short, if for all full alphabets X and Y, the following hold.

(1) $\mathcal{R}(X) \neq \emptyset$,
(2) $K, L \in \mathcal{R}(X)$ implies $K \cap L, \mathrm{F}_{\mathbf{V}}(X) \setminus L \in \mathcal{R}(X)$,
(3) if $L \in \mathcal{R}(X)$, then $\alpha^{-1}(L) \in \mathcal{R}(X)$ for every $\alpha \in \mathrm{Tr}(\mathcal{F}_{\mathbf{V}}(X))$, and
(4) if $L \in \mathcal{R}(Y)$, then $L\varphi^{-1} \in \mathcal{R}(X)$ for every $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$.

Let $\mathrm{VRS}(\mathbf{V})$ denote the class of all varieties of recognizable $\mathbf{V}$-sets.

It is clear that the intersection of any family of varieties of recognizable $\mathbf{V}$-sets is again a $\mathbf{V}$-VRS, and hence $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ is a complete (in fact, algebraic) lattice.

If $L = \langle L_s \rangle_{s \in S}$ is a sorted subset of any algebra $\mathcal{A} = (A, \Omega)$ and $s \in S$ is any sort, then $\langle L_s \rangle = 1_{A(s)}^{-1}(L)$ where $1_{A(s)}$ is the identity map on $A(s)$. Applied to the algebras $\mathcal{F}_{\mathbf{V}}(X)$, this observation yields the following fact.

Lemma 2.3.3 *Let $\mathcal{R} = \{\mathcal{R}(X)\}_X$ be a* $\mathbf{V}$*-VRS. If $L = \langle L_s \rangle_{s \in S} \in \mathcal{R}(X)$ for some X, then $\langle L_s \rangle \in \mathcal{R}(X)$ for every $s \in S$.* □

From Lemma 2.3.3 and Lemma 2.2.14 we get directly the following fact.

Lemma 2.3.4 *If $\mathcal{R} = \{\mathcal{R}(X)\}_X$ is a* $\mathbf{V}$*-VRS and $L \in \mathcal{R}(X)$ for some X, then $\langle a/L \rangle \in \mathcal{R}(X)$ for any $s \in S$ and any $a \in F_V(X, s)$.* □

For any full alphabet X, let $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ denote the set of all congruences on $\mathcal{F}_{\mathbf{V}}(X)$ of finite index. Such congruences are called $\mathbf{V}$*-congruences*. A *family of* $\mathbf{V}$*-congruences* is a map Γ that assigns to each X a set $\Gamma(X) \subseteq \mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. We represent such a family in the form $\Gamma = \{\Gamma(X)\}_X$.

Definition 2.3.5 A family of $\mathbf{V}$-congruences $\Gamma = \{\Gamma(X)\}_X$ is a *variety of* $\mathbf{V}$*-congruences*, a $\mathbf{V}$-VFC for short, if for all X and Y,

(1) $\Gamma(X) \neq \emptyset$,
(2) if $\theta, \theta' \in \Gamma(X)$, then $\theta \cap \theta' \in \Gamma(X)$,
(3) if $\theta \in \Gamma(X)$ and $\theta \subseteq \theta'$, then $\theta' \in \Gamma(X)$, and
(4) if $\theta \in \Gamma(Y)$, then $\varphi \circ \theta \circ \varphi^{-1} \in \Gamma(X)$ for any $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$.

Let VFC($\mathbf{V}$) denote the class of all varieties of $\mathbf{V}$-congruences.

In other words, a variety of $\mathbf{V}$-congruences is a family of filters of the congruences (closed under intersections and supersets) which is closed under inverse homomorphisms. It is again easy to see that (VFC($\mathbf{V}$), $\subseteq$) is an algebraic lattice.

Let S, Ω and $\mathbf{V}$ be as in the previous section. By definition, a *variety of finite* $\mathbf{V}$*-algebras*, a $\mathbf{V}$-VFA for short, is a variety of finite Ω-algebras contained in $\mathbf{V}$. Let VFA($\mathbf{V}$) be the class of all $\mathbf{V}$-VFAs. We shall prove a Variety Theorem that establishes a triple of bijective correspondences between all varieties of recognizable $\mathbf{V}$-sets, all varieties of finite $\mathbf{V}$-algebras, and all varieties of $\mathbf{V}$-congruences. The proof is similar to those of various other Variety Theorems, and in particular to the one of [53]. We however present a rather detailed proof.

Let us now introduce the six mappings that will yield the Variety Theorem in the form of three pairs of mutually inverse isomorphisms between the three complete lattices (VFA($\mathbf{V}$), $\subseteq$), (VRS($\mathbf{V}$), $\subseteq$) and (VFC($\mathbf{V}$), $\subseteq$).

Definition 2.3.6 For any $\mathbf{V}$-VFA $\mathbf{K}$, $\mathbf{V}$-VRS $\mathcal{R}$, and $\mathbf{V}$-VFC Γ, let

(1) $\mathbf{K}^r$ be the family of recognizable $\mathbf{V}$-sets such that for each X,
$$\mathbf{K}^r(X) = \{L \subseteq \mathrm{F}_{\mathbf{V}}(X) \mid \mathrm{SA}(L) \in \mathbf{K}\},$$
(2) $\mathbf{K}^c$ be the family of $\mathbf{V}$-congruences such that for each X,
$$\mathbf{K}^c(X) = \{\theta \in \mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X)) \mid \mathcal{F}_{\mathbf{V}}(X)/\theta \in \mathbf{K}\},$$
(3) $\mathcal{R}^a$ be the $\mathbf{V}$-VFA generated by the syntactic algebras SA(L) where $L \in \mathcal{R}(X)$ for some X,
(4) $\mathcal{R}^c$ be the family of $\mathbf{V}$-congruences such that for each X, $\mathcal{R}^c(X)$ is the set of all congruences θ in FCon($\mathcal{F}_{\mathbf{V}}(X)$) such that $\theta \supseteq \approx^{L_1} \cap \cdots \cap \approx^{L_n}$ holds for some $L_1, \ldots, L_n \in \mathcal{R}(X)$,
(5) Γ^a be the $\mathbf{V}$-VFA generated by all algebras $\mathcal{F}_{\mathbf{V}}(X)/\theta$ such that θ belongs to $\Gamma(X)$ for some X, and let
(6) Γ^r be the family of recognizable $\mathbf{V}$-sets such that for each X,
$$\Gamma^r(X) = \{L \subseteq \mathrm{F}_{\mathbf{V}}(X) \mid \approx^L \in \Gamma(X)\}.$$

We note that these operations map varieties to varieties and are isotone.

Lemma 2.3.7 *For any* $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$*,* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$ *and* $\Gamma \in \mathrm{VFC}(\mathbf{V})$*,*

(1) $\mathcal{R}^a, \Gamma^a \in \mathrm{VFA}(\mathbf{V})$*,*
(2) $\mathbf{K}^r, \Gamma^r \in \mathrm{VRS}(\mathbf{V})$*, and*

(3) $\mathbf{K}^c, \mathcal{R}^c \in \mathrm{VFC}(\mathbf{V})$.

Moreover, each of the mappings $\mathbf{K} \mapsto \mathbf{K}^r$, $\mathbf{K} \mapsto \mathbf{K}^c$, $\mathcal{R} \mapsto \mathcal{R}^a$, $\mathcal{R} \mapsto \mathcal{R}^c$, $\Gamma \mapsto \Gamma^a$ *and* $\Gamma \mapsto \Gamma^r$ *is inclusion-preserving.*

Proof. By definition, $\mathcal{R}^a$ and Γ^a are $\mathbf{V}$-VFA's. The families $\mathbf{K}^r$ and Γ^r are $\mathbf{V}$-VRS's by Propositions 2.2.12 and 2.2.9. Finally, Lemmas 2.1.4 and 2.2.13, and Proposition 2.2.9 imply that $\mathbf{K}^c$ and $\mathcal{R}^c$ are $\mathbf{V}$-VFC's. □

We shall show that the six mappings introduced above form three pairs of mutually inverse isomorphisms between the complete lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$, $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$. Since we already know that all the maps are isotone, it suffices to show that they are pairwise inverses of each other.

Proposition 2.3.8 *The lattices* $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ *are isomorphic as*

(1) $\mathbf{K}^{ra} = \mathbf{K}$ *for every* $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$, *and*
(2) $\mathcal{R}^{ar} = \mathcal{R}$ *for every* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$.

Proof. It suffices to prove (1) and (2).

Since $\mathbf{K}^{ra}$ is generated by syntactic algebras belonging to $\mathbf{K}$, the inclusion $\mathbf{K}^{ra} \subseteq \mathbf{K}$ is obvious. For the converse inclusion, let us consider any syntactic algebra $\mathcal{A} \in \mathbf{K}$. By Lemma 2.3.1 there exists an X such that $\mathcal{A} \cong \mathrm{SA}(L)$ for some $L \in \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X))$. Then $L \in \mathbf{K}^r(X)$ and hence $\mathcal{A} \in \mathbf{K}^{ra}$. This implies $\mathbf{K} \subseteq \mathbf{K}^{ra}$ because, by Lemma 2.2.18, $\mathbf{K}$ is generated by syntactic algebras.

The inclusion $\mathcal{R} \subseteq \mathcal{R}^{ar}$ is obvious: if $L \in \mathcal{R}(X)$ for some X, then $\mathrm{SA}(L) \in \mathcal{R}^a$ and hence $L \in \mathcal{R}^{ar}(X)$. Assume then that $L \in \mathcal{R}^{ar}(X)$ for some X. Then $\mathrm{SA}(L) \in \mathcal{R}^a$ implies that $\mathrm{SA}(L) \preceq \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ for some $k \geq 1$, some full alphabets $X_i = \langle X_i(s) \rangle_{s \in S}$ and sets $L_i \in \mathcal{R}(X_i)$ (where $i = 1, \ldots, k$). For each $i = 1, \ldots, k$, let φ_i denote the syntactic homomorphisms $\varphi^{L_i} \colon \mathcal{F}_{\mathbf{V}}(X_i) \to \mathrm{SA}(L_i)$. Then there is a homomorphism

$$\eta \colon \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$$

such that for every $i = 1, \ldots, k$, $\eta\pi_i = \tau_i\varphi_i$, where

$$\pi_i \colon \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k) \longrightarrow \mathrm{SA}(L_i),$$

and

$$\tau_i \colon F_{\mathbf{V}}(X_1) \times \cdots \times F_{\mathbf{V}}(X_k) \longrightarrow F_{\mathbf{V}}(X_i)$$

are the respective projection functions. By Lemma 2.2.11 there exist a homomorphism $\varphi \colon \mathcal{F}_{\mathbf{V}}(X) \to \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ and a subset H of $\mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ such that $L = H\varphi^{-1}$. Since η is an epimorphism, there is a homomorphism $\psi \colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ such that $\psi\eta = \varphi$. Because H is finite, $L = \bigcup_{u \in H} u\varphi^{-1}$ is the union of finitely many sets $u\varphi^{-1}$ with $u = (u_1, \ldots, u_k) \in \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$. For each such $u \in H$, $u\varphi^{-1} = \bigcap\{u_i(\varphi\pi_i)^{-1} \mid 1 \leq i \leq k\} = \bigcap\{u_i\varphi_i^{-1}(\psi\tau_i)^{-1} \mid 1 \leq i \leq k\}$.

By Lemma 2.3.4, $u_i\varphi_i^{-1} \in \mathcal{R}(X_i)$ for each $i \leq k$, and thus $L \in \mathcal{R}(X)$. □

Lemma 2.3.9 *For any* $\mathbf{V}$-VFC Γ *and any finite algebra* $\mathcal{A} \in \mathbf{V}$, $\mathcal{A} \in \Gamma^{\mathsf{a}}$ *iff there exist a finite set* X *and an epimorphism* $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{A}$ *such that* $\ker\varphi \in \Gamma(X)$.

Proof. If $\mathcal{A} \in \Gamma^a$, then $\mathcal{A} \preceq \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$ for some full alphabets $X_1, \ldots, X_k$ and congruences $\theta_1 \in \Gamma(X_1), \ldots, \theta_k \in \Gamma(X_k)$ where $k \geq 1$. This means that for some algebra $\mathcal{B}$ there exist an epimorphism $\eta\colon \mathcal{B} \to \mathcal{A}$ and a monomorphism $\varphi\colon \mathcal{B} \to \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times\cdots\times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$. The algebras $\mathcal{F}_{\mathbf{V}}(X_i)/\theta_i$ are finite members of $\mathbf{V}$ and hence there is for some X an epimorphism $\psi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{B}$. By setting $(a_1, \ldots, a_k)\chi = (a_1/\theta_1, \ldots, a_k/\theta_k)$ we define an epimorphism

$$\chi\colon \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k.$$

For each $i = 1, \ldots, k$, let $\pi_i\colon F_{\mathbf{V}}(X_1) \times \cdots \times F_{\mathbf{V}}(X_k) \to F_{\mathbf{V}}(X_i)$ be the i-th projection, and let $\omega\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ be the homomorphism such that $\omega\chi = \psi\varphi$. Then $\psi\eta\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{A}$ is an epimorphism, and $\ker\psi\eta \supseteq \ker\psi\varphi = \ker\omega\chi = \bigcap\{\omega\pi_i \circ \theta_i \circ (\omega\pi_i)^{-1} \mid 1 \leq i \leq k\}$ shows that $\ker\psi\eta \in \Gamma(X)$.
The converse implication is immediately clear by the definition of Γ^a. □

Proposition 2.3.10 *The lattices* $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ *are isomorphic as*

(1) $\mathbf{K}^{ca} = \mathbf{K}$ *for every* $\mathbf{V}$-VFA $\mathbf{K}$, *and*
(2) $\Gamma^{ac} = \Gamma$ *for every* $\mathbf{V}$-VFC Γ.

Proof. (1) The inclusion $\mathbf{K} \subseteq \mathbf{K}^c$ is straightforward. For the opposite inclusion, suppose $\mathcal{A} \in \mathbf{K}^{ca}$. Then by Lemma 2.3.9 there is an X and an epimorphism $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{A}$ such that $\ker\varphi \in \mathbf{K}^c$, and thus $\mathcal{F}_{\mathbf{V}}(X)/\ker\varphi \cong \mathcal{A}$ by Proposition 2.1.1, which is the case exactly when $\mathcal{A} \in \mathbf{K}$.

(2) Consider any X and $\theta \in \mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. If $\theta \in \Gamma^{ac}(X)$, then by Lemma 2.3.9, there exist a Y and an epimorphism $\psi\colon \mathcal{F}_{\mathbf{V}}(Y) \to \mathcal{F}_{\mathbf{V}}(X)/\theta$ such that $\ker\psi \in \Gamma(Y)$. Since ψ is surjective, there is for any $s \in S$ and every $x \in X_s$ an element $t_s^x \in \mathrm{F}_{\mathbf{V}}(Y)_s$ such that $t_s^x\psi_s = x/\theta_s$. If $\varphi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$ is the homomorphism such that $x\varphi = t_s^x$ for all $s \in S$ and $x \in X_s$, then $\varphi\psi = \theta^\natural$, where $\theta^\natural\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X)/\theta$ is the canonical epimorphism. Hence $\theta = \ker\varphi\psi = \varphi \circ (\ker\psi) \circ \varphi^{-1} \in \Gamma(X)$. The converse inclusion is obvious: if $\theta \in \Gamma(X)$, then $\mathcal{F}_{\mathbf{V}}(X)/\theta \in \Gamma^a$ implies $\theta \in \Gamma^{ac}$. □

Propositions 2.3.8 and 2.3.10 already show that the lattices $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ are isomorphic, but the following composition laws imply also that the mappings $\mathcal{R} \mapsto \mathcal{R}^c$ and $\Gamma \mapsto \Gamma^r$ form a pair of mutually inverse isomorphisms between them.

Proposition 2.3.11 *For any* $\mathbf{V}$-VFA $\mathbf{K}$*,* $\mathbf{V}$-VRS $\mathcal{R}$*, and* $\mathbf{V}$-VFC Γ*,*

(1) $\mathbf{K}^{cr} = \mathbf{K}^r$*,*
(2) $\mathcal{R}^{ac} = \mathcal{R}^c$*, and*
(3) $\Gamma^{ra} = \Gamma^a$*.*

Proof. For (1) it suffices to note that

$$L \in \mathbf{K}^r(X) \Leftrightarrow \mathrm{SA}(L) \in \mathbf{K} \Leftrightarrow \approx^L \in \mathbf{K}^c(X) \Leftrightarrow L \in \mathbf{K}^{cr}(X),$$

for any X and $L \subseteq \mathrm{F}_{\mathbf{V}}(X)$.

To prove (2), let us consider any X and $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. If $\theta \in \mathcal{R}^c(X)$, then $\approx^{L_1} \cap \cdots \cap \approx^{L_k} \subseteq \theta$ for some $k \geq 1$ and $L_1, \cdots, L_k \in \mathcal{R}(X)$. This implies that $\mathcal{F}_{\mathbf{V}}(X)/\theta \in \mathcal{R}^a$ since $\mathcal{F}_{\mathbf{V}}(X)/\theta \preceq \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$, and therefore $\theta \in \mathcal{R}^{ac}$.

If $\theta \in \mathcal{R}^{ac}(X)$, then $\mathcal{F}_{\mathbf{V}}(X)/\theta \preceq \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ for some full alphabets $X_1, \cdots, X_k$ and sorted sets $L_1 \in \mathcal{R}(X_1), \cdots, L_k \in \mathcal{R}(X_k)$, where $k > 0$. Hence, there is an Ω-algebra $\mathcal{B}$ such that there exist an epimorphism $\psi\colon \mathcal{B} \to \mathcal{F}_{\mathbf{V}}(X)/\theta$ and a monomorphism $\eta\colon \mathcal{B} \to \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$. We may also assume that there is an epimorphism $\varphi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{B}$ such that $\varphi\psi = \theta^\natural$ (if not, we replace $\mathcal{B}$ with a suitable subalgebra). For each $i = 1, \ldots, k$, let π_i be the i-th projection from $\mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ onto $\mathcal{F}_{\mathbf{V}}(X_i)$, and let

$$\pi\colon \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$$

be defined by $(t_1, \ldots, t_k)\pi_s = (t_1/L_1, \ldots, t_k/L_k)$ for all sorts $s \in S$ and all terms $t_1 \in \mathcal{F}_{\mathbf{V}}(X_1, s), \ldots, t_k \in \mathcal{F}_{\mathbf{V}}(X_k, s)$. Since π clearly is surjective, we may define a homomorphism $\gamma\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ for which $\gamma\pi = \varphi\eta$. Then

$$\theta = \ker \varphi\psi \supseteq \ker \varphi\eta = \ker \gamma\pi = \bigcap\{\gamma\pi_i \circ \approx^{L_i} \circ (\gamma\pi_i)^{-1} \mid 1 \leq i \leq k\},$$

and hence $\theta \in \mathcal{R}^c(X)$.

To prove (3), consider any finite algebra $\mathcal{A} = (A, \Omega)$. Now, $\mathcal{A}$ belongs to Γ^a iff $\mathcal{A} \preceq \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$, for some full alphabets $X_1, \ldots, X_k$ and some $\theta_1 \in \Gamma(X_1), \ldots, \theta_k \in \Gamma(X_k)$ ($k \geq 1$). Since any $\Gamma(X)$ is generated by syntactic congruences by Lemma 2.2.15, we can assume that each θ_i is the syntactic congruence of some $L_i \subseteq \mathrm{F}_{\mathbf{V}}(X_i)$, and then L_i belongs to $\Gamma^r(X_i)$, and so $\mathcal{A} \in \Gamma^a$ iff $\mathcal{A} \in \Gamma^{ra}$. □

So far, we have shown that the lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ and $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ are isomorphic, also the lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ are isomorphic. So we get the following result.

Proposition 2.3.12 *The lattices* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ *are isomorphic as*

(1) $\mathcal{R}^{cr} = \mathcal{R}$ *for every* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$*, and*
(2) $\Gamma^{rc} = \Gamma$ *for every* $\Gamma \in \mathrm{VFC}(\mathbf{V})$*.*

Proof. By using the previous three propositions we can see that $\mathcal{R}^{cr} = \mathcal{R}^{acr} = \mathcal{R}^{ar} = \mathcal{R}$ for every $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$. Similarly, $\Gamma^{rc} = \Gamma^{rac} = \Gamma^{ac} = \Gamma$ for every $\Gamma \in \mathrm{VFC}(\mathbf{V})$. □

Let us note that Proposition 2.3.12 could be obtained also directly in a similar way as the analogous facts are proved in [1] (Lemma 2.15 and Proposition 3.8).

Lemma 2.3.13 *For any* $\mathbf{V}$*-VRS* $\mathcal{R}$ *and any congruence* θ *of finite index on* $\mathcal{F}_{\mathbf{V}}(X)$*,* θ *is in* $\mathcal{R}^{\mathrm{c}}(X)$ *iff all* θ*-classes are in* $\mathcal{R}(X)$*.*

Proof. By Lemma 2.2.15, if all θ-classes are in $\mathcal{R}(X)$, then $\theta \in \mathcal{R}^{\mathrm{c}}(X)$. For the converse, suppose $\theta \in \mathcal{R}^{\mathrm{c}}(X)$. So there are $T^{(1)}, \ldots, T^{(k)} \in \mathcal{R}(X)$ $(k \geq 1)$ such that $\theta \supseteq\, \approx^{T^{(1)}} \cap \cdots \cap \approx^{T^{(k)}}$. Then it can be easily shown that every θ-class is a Boolean combination of some $T^{(n)}$-classes $(n \leq k)$:

$$a/\theta = \bigcup_{b \in a/\theta} (b/T^{(1)} \cap \cdots \cap b/T^{(k)}).$$

By Lemma 2.2.14, $b/T^{(n)} \in \mathcal{R}(X)$ for all $n \leq k$, hence $a/\theta \in \mathcal{R}(X)$. □

Now, an alternative proof of Proposition 2.3.12 is as follows.

Proof. The inclusion $\mathcal{R} \subseteq \mathcal{R}^{cr}$ of (1) is obvious. For the opposite inclusion, suppose $T \in \mathcal{R}^{cr}(X)$ for some X. Then $\approx^T \in \mathcal{R}^{c}(X)$, and so by Lemma 2.3.13, every $\approx^T$-class is in $\mathcal{R}(X)$. Since T is the union of some $\approx^T$-classes, then $T \in \mathcal{R}(X)$. Again the inclusion $\Gamma \subseteq \Gamma^{rc}$ of (2) is clear. For the opposite, take a $\theta \in \Gamma^{rc}(X)$ for some X. By Lemma 2.3.13, every θ-class a/θ belongs to $\Gamma^{r}(X)$. Then the syntactic congruences $\approx^{a/\theta}$ of those θ-classes are in $\Gamma(X)$. By Lemma 2.2.15, θ is the intersection of these syntactic congruences of θ-classes. Thus $\theta \in \Gamma(X)$. □

We may sum up the results of this section as follows.

Proposition 2.3.14 (Variety Theorem) *The mappings*

$$\begin{array}{ll} \mathrm{VFA}(\mathbf{V}) \to \mathrm{VRS}(\mathbf{V}),\ \mathbf{K} \mapsto \mathbf{K}^r, & \mathrm{VRS}(\mathbf{V}) \to \mathrm{VFA}(\mathbf{V}),\ \mathcal{R} \mapsto \mathcal{R}^a, \\ \mathrm{VFA}(\mathbf{V}) \to \mathrm{VFC}(\mathbf{V}),\ \mathbf{K} \mapsto \mathbf{K}^c, & \mathrm{VFC}(\mathbf{V}) \to \mathrm{VFA}(\mathbf{V}),\ \Gamma \mapsto \Gamma^a, \textit{ and} \\ \mathrm{VRS}(\mathbf{V}) \to \mathrm{VFC}(\mathbf{V}),\ \mathcal{R} \mapsto \mathcal{R}^c, & \mathrm{VFC}(\mathbf{V}) \to \mathrm{VRS}(\mathbf{V}),\ \Gamma \mapsto \Gamma^r, \end{array}$$

form three pairs of isomorphisms that are inverses of each other between the lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$*,* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$*, and* $(\mathrm{VFC}(\mathbf{V}), \subseteq)$*. Moreover,* $\mathbf{K}^{cr} = \mathbf{K}^r$*,* $\mathbf{K}^{rc} = \mathbf{K}^c$*,* $\mathcal{R}^{ca} = \mathcal{R}^a$*,* $\mathcal{R}^{ac} = \mathcal{R}^c$*,* $\Gamma^{ra} = \Gamma^a$*, and* $\Gamma^{ar} = \Gamma^r$*, for any* $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$*,* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$*, and* $\Gamma \in \mathrm{VFC}(\mathbf{V})$*.* □

Chapter 3

Positive varieties of tree languages

Roughly speaking, a variety of tree languages is a family of tree languages closed under finite Boolean operations (complements, finite intersections and finite unions), inverse translations and inverse morphisms. However, there are also some interesting families of tree languages that do not possess all of these closure properties. Some of those families are so-called positive varieties of tree languages which are families of tree languages closed under finite positive Boolean operations (intersections and unions), inverse translations and inverse morphisms. One example is the family of finite tree languages. These families can not be characterized by algebras, but there is a characterization for them by richer structures, namely by ordered algebras. The theory of ordered algebras is a useful and interesting area in itself. Actually, ordered algebras play an important role in theoretical computer science, as Bloom and Wright [8] put it "[e]ver since Scott popularized their use in [51], ordered algebras have been used in many places in theoretical computer science". Here we prove a Variety Theorem for positive varieties of tree languages and varieties of ordered algebras. This result is inspired by Pin's positive variety theorem [39] which established a bijective correspondence between positive varieties of string languages and varieties of ordered semigroups; see also [24, 41].

In Section 3.1 we review some basic notions of ordered algebras, ideals and quotient ordered algebras.

In Section 3.2 we introduce positive varieties of tree languages and prove a variety theorem for these varieties and varieties of finite ordered algebras.

In Section 3.3 we extend the positive variety theorem to generalized varieties. A generalized family of tree languages is a mapping that assigns a set of recognizable ΣX-tree languages to any ranked alphabet Σ and any leaf alphabet X. A generalized positive variety, is a generalized family of tree languages closed under the positive Boolean operations, inverse translations, and inverse generalized morphisms. A generalized variety of finite ordered algebras is a mapping that assigns a class of finite ordered algebras over Σ for any ranked alphabet Σ, and is closed under generalized subalgebras, generalized homomorphic images and generalized finite direct products. Our generalized positive variety theorem establishes a bijective correspondence between generalized positive varieties of tree languages and generalized varieties of finite ordered algebras.

The theory will also be illustrated by some examples.

3.1 Ordered algebras

In this section, after reviewing the terminology of ordered sets and ordered algebras, we define the notions of ideals, quotient ordered algebras, and syntactic ordered algebras, cf. [7].

A binary relation on a set A is a *quasi-order* if it is reflexive and transitive, an *order* if it is reflexive, anti-symmetric and transitive, and an *equivalence* if it is reflexive, symmetric and transitive; cf. Preliminaries in Chapter 1. We note that quasi-orders have also been called "pre-orders", and reflexive, anti-symmetric, and transitive relations are often called partial orders. If the union of an order with its inverse is the universal relation, then we call it *linear order*; this is what some authors call "total order".

For a quasi-order $\preccurlyeq$ on a set A, the relation $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ can be shown to be an equivalence relation on A, called the *equivalence relation of* $\preccurlyeq$, and the relation $\leqslant$ defined on the quotient set A/θ by $a/\theta \leqslant b/\theta \iff a \preccurlyeq b$, is a well-defined order on A/θ. This order $\leqslant$ on A/θ is called the *order induced by the quasi-order* $\preccurlyeq$.

Let Σ be a ranked alphabet. An *ordered* Σ*-algebra* is a structure $\mathcal{A} = (A, \Sigma, \leqslant)$ where (A, Σ) is an algebra and $\leqslant$ is an order on A compatible with the operations of $\mathcal{A}$, that is to say, for any $f \in \Sigma_m$ $(m > 0)$ and any elements $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, whenever $a_1 \leqslant b_1, \ldots, a_m \leqslant b_m$ then $f^{\mathcal{A}}(a_1, \ldots, a_m) \leqslant f^{\mathcal{A}}(b_1, \ldots, b_m)$. We note that any algebra (A, Σ) in the classical sense is an ordered algebra (A, Σ, Δ_A) in which the order relation is equality.

Let us agree that $\mathcal{A}$ and $\mathcal{B}$ denote the ordered algebras $(A, \Sigma, \leqslant)$ and $(B, \Sigma, \leqslant')$, respectively.

The ordered algebra $\mathcal{B}$ is an *order subalgebra* of $\mathcal{A}$, if (B, Σ) is a subalgebra of (A, Σ) and $\leqslant'$ is the restriction of $\leqslant$ to B.

A mapping $\varphi : A \to B$ is an *order morphism* if $c^{\mathcal{A}}\varphi = c^{\mathcal{B}}$ for any $c \in \Sigma_0$, $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = f^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$, and moreover it is order preserving: for any $a, b \in A$ if $a \leqslant b$ then $a\varphi \leqslant' b\varphi$. In that case we write $\varphi : \mathcal{A} \to \mathcal{B}$. An order morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is an *order epimorphism* if it is surjective, and then $\mathcal{B}$ is an *order epimorphic image* of $\mathcal{A}$, in notation $\mathcal{B} \leftarrow \mathcal{A}$. If $\mathcal{B}$ is an order epimorphic image of an order subalgebra of $\mathcal{A}$, then $\mathcal{B}$ is said to *divide* $\mathcal{A}$, in notation $\mathcal{B} \preceq \mathcal{A}$. If φ is injective then it is an *order monomorphism*. When φ is bijective and its inverse is also an order morphism, then it is an *order isomorphism*. We write $\mathcal{A} \cong \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic, and write $\mathcal{A} \subseteq \mathcal{B}$ when $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{B}$.

The direct product of $\mathcal{A}$ and $\mathcal{B}$ is the structure $(A\times B, \Sigma, \leqslant \times \leqslant')$ where $(A \times B, \Sigma)$ is the usual direct product of the algebras (A, Σ) and (B, Σ), and the relation $\leqslant \times \leqslant'$ is defined on $A \times B$ by $(a, b) \leqslant \times \leqslant' (c, d) \iff a \leqslant c \;\&\; b \leqslant' d$ for $(a, b),(c, d) \in A \times B$. It is easy to see that the structure $(A \times B, \Sigma, \leqslant \times \leqslant')$ is an ordered algebra and it is denoted by $\mathcal{A} \times \mathcal{B}$.

A *variety of finite ordered algebras* is a class of finite ordered algebras closed under order subalgebras, order epimorphic images, and direct products. The abbreviation VFOA stands for "variety of finite ordered algebras".

Definition 3.1.1 A *quasi-order* on the ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is a quasi-order $\preccurlyeq$ on the set A such that $\preccurlyeq \supseteq \leqslant$, and for any $f \in \Sigma_m$ $(m > 0)$ and any $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, $f^{\mathcal{A}}(a_1, \ldots, a_m) \preccurlyeq f^{\mathcal{A}}(b_1, \ldots, b_m)$ holds whenever $a_1 \preccurlyeq b_1, \ldots, a_m \preccurlyeq b_m$.

Let $\preccurlyeq$ be a quasi-order on $\mathcal{A}$. The relation $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ is a congruence on (A, Σ) and so, the quotient structure $(A/\theta, \Sigma)$ is a Σ-algebra. Moreover, the relation $\leq$ defined on A/θ by $a/\theta \leq b/\theta$ iff $a \preccurlyeq b$ for $a, b \in A$ is a well-defined order. Moreover, the structure $(A/\theta, \Sigma, \leq)$ can be shown to be an ordered algebra. It can be noticed that quasi-orders on ordered algebras play the same role as congruences for ordinary algebras.

Definition 3.1.2 For a quasi-order $\preccurlyeq$ on $\mathcal{A}$, the *quotient* of $\mathcal{A}$ under $\preccurlyeq$ is the structure $\mathcal{A}/{\preccurlyeq} = (A/\theta, \Sigma, \leq)$ where $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ is the Σ-congruence induced by $\preccurlyeq$ and $\leq$ is the order induced by $\preccurlyeq$.

Lemma 3.1.3 *Let $\varphi : \mathcal{B} \to \mathcal{A}$ be an order morphism. If $\preccurlyeq$ is a quasi-order on $\mathcal{A}$, then the relation $\varphi\circ \preccurlyeq \circ\varphi^{-1}$ is a quasi-order on $\mathcal{B}$ such that for all $b, d \in B$, $b \;\varphi\circ \preccurlyeq \circ\varphi^{-1}\; d \iff b\varphi \preccurlyeq d\varphi$. Moreover, if θ is the congruence on $\mathcal{A}$ induced by $\preccurlyeq$, then the congruence on $\mathcal{B}$ induced by $\varphi\circ \preccurlyeq \circ\varphi^{-1}$ is $\varphi \circ \theta \circ \varphi^{-1}$.* □

Proposition 3.1.4 *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras, $\preccurlyeq$ be a quasi-order on $\mathcal{B}$, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism.*

(1) *The image of $\mathcal{A}$, $\mathcal{A}\varphi = (A\varphi, \Sigma, \leqslant'')$ where $\leqslant''$ is the restriction of $\leqslant'$ to $A\varphi$, is an order subalgebra of $\mathcal{B}$.*

(2) *$\mathcal{A}/\varphi\circ \preccurlyeq \circ\varphi^{-1} \cong \mathcal{A}\varphi/{\preccurlyeq'}$ where $\preccurlyeq'$ is the restriction of $\preccurlyeq$ to $A\varphi$.*

(3) *If* φ *is an order epimorphism then* $\mathcal{A}/\varphi\circ \preccurlyeq \circ\varphi^{-1} \cong \mathcal{B}/\preccurlyeq$.

Proof. The statement (1) is straightforward and (3) follows from (2). For (2) we let $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ and note that $\psi : A/\varphi\circ \preccurlyeq \circ\varphi^{-1} \to A\varphi/\preccurlyeq'$ defined by $(a/\varphi \circ \theta \circ \varphi^{-1})\psi = a\varphi/\theta$ for $a \in A$, is an order isomorphism. □

The particular case of the Proposition 3.1.4 when $\preccurlyeq = \leqslant'$ is of interest: then we have $\theta = \Delta_B$ and $\varphi \circ \theta \circ \varphi^{-1} = \ker \varphi$, hence we get the homomorphism theorem for ordered algebras, namely $\mathcal{A}/\ker \varphi \cong \mathcal{A}\varphi$. Similar results for semigroups can be found in [27].

Proposition 3.1.5 *Let* $\mathcal{A} = (A, \Sigma, \leqslant)$ *be an ordered algebra, and* $\preccurlyeq, \preccurlyeq'$ *be two quasi-orders on* $\mathcal{A}$.

(1) *If* $\preccurlyeq \subseteq \preccurlyeq'$ *then* $\mathcal{A}/\preccurlyeq' \leftarrow \mathcal{A}/\preccurlyeq$.

(2) *The relation* $\preccurlyeq \cap \preccurlyeq'$ *is a quasi-order on* $\mathcal{A}$ *and* $\mathcal{A}/\preccurlyeq \cap \preccurlyeq'$ *is an order subalgebra of* $\mathcal{A}/\preccurlyeq \times \mathcal{A}/\preccurlyeq'$. □

Recall the notion of $\mathrm{Tr}(\mathcal{A})$ the set of translations of an algebra $\mathcal{A} = (A, \Sigma)$ from the Preliminaries in Chapter 1. The composition of translations p and q is denoted by $q \cdot p$, that is $(q \cdot p)(a) = p(q(a))$ for all $a \in A$. We note that the set $\mathrm{Tr}(\mathcal{A})$ equipped with the composition operation is a monoid, called the *translation monoid* of $\mathcal{A}$. For an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ a subset $I \subseteq \mathcal{A}$ is an *ideal* of $\mathcal{A}$, in notation $I \trianglelefteq \mathcal{A}$, if $a \leqslant b \in I$ implies $a \in I$ for every $a, b \in A$. We note that ideals can be empty, i.e., $\emptyset \trianglelefteq \mathcal{A}$. For any $a \in A$, $(a] = \{b \in A \mid b \leqslant a\}$ is the ideal of $\mathcal{A}$ *generated* by a.

Definition 3.1.6 The *syntactic quasi-order* $\preccurlyeq_I$ of an ideal I of an ordered algebra $\mathcal{A}$ is defined by

$$a \preccurlyeq_I b \Longleftrightarrow \big(\forall p \in \mathrm{Tr}(\mathcal{A})\big)\big(p(b) \in I \Rightarrow p(a) \in I\big) \quad (a, b \in A).$$

The *syntactic ordered algebra* of I is the quotient ordered algebra $\mathrm{SOA}(I) = \mathcal{A}/\preccurlyeq_I$, also denoted by $\mathcal{A}/I$ (cf. [39]).

We note that for any ideal I, $\preccurlyeq_I$ is a quasi-order on $\mathcal{A}$ and the equivalence relation $\approx^I$ of $\preccurlyeq_I$ is the syntactic congruence of I in the classical sense (see e.g. [52, 53]): $a \approx^I b \iff (\forall p \in \mathrm{Tr}(\mathcal{A}))(p(a) \in I \Leftrightarrow p(b) \in I)$.

It is known that the syntactic congruence of I is the greatest congruence that saturates I ([52, 53]). Correspondingly, the syntactic quasi-order of I is the greatest quasi-order on $\mathcal{A}$ that satisfies $a \preccurlyeq b \in I \Rightarrow a \in I$ for all $a, b \in A$.

Trivially, any subset $I \subseteq A$ of the ordered algebra $\mathcal{A} = (A, \Sigma, \Delta_A)$ is an ideal of $\mathcal{A}$. The following is essentially Lemma 3.2 of [53]; cf. Lemma 2.1.3 in Chapter 2.

Proposition 3.1.7 *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism. The mapping φ induces a monoid morphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_\varphi$, such that $p(a)\varphi = p_\varphi(a\varphi)$ for all $a \in A$. Moreover, if φ is an order epimorphism then the induced map is a monoid epimorphism.*

Proof. For any elementary translation $p = f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ of $\mathcal{A}$ where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$, the unary function p_φ on B defined by $b \mapsto f^{\mathcal{B}}(a_1\varphi, \ldots, b, \ldots, a_m\varphi)$ is an elementary translation of $\mathcal{B}$, and if φ is surjective then every elementary translation of $\mathcal{B}$ is of this form. The mapping $p \mapsto p_\varphi$ can be inductively extended to all translations by setting $(1_A)_\varphi = 1_B$ and $(p \cdot q)_\varphi = p_\varphi \cdot q_\varphi$. This extension is well-defined and the identity $p_\varphi(a\varphi) = p(a)\varphi$ obviously holds for any $a \in A$ and $p \in \mathrm{Tr}(\mathcal{A})$. □

For a subset $D \subseteq A$ and a translation $p \in \mathrm{Tr}(\mathcal{A})$, the inverse translation of D under p is $p^{-1}(D) = \{a \in A \mid p(a) \in D\}$, and for an order morphism $\varphi : \mathcal{B} \to \mathcal{A}$, the inverse image of D under φ is $D\varphi^{-1} = \{b \in B \mid b\varphi \in D\}$.

Positive Boolean operations are intersection and union of sets, while Boolean operations also include the complement operation. It can be easily proved that for ordered algebras $\mathcal{A}$ and $\mathcal{B}$, ideals $I, J \trianglelefteq \mathcal{A}$, $K \trianglelefteq \mathcal{B}$, translation $p \in \mathrm{Tr}(\mathcal{A})$, and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$, the sets $I \cap J, I \cup J, p^{-1}(I)$ and $K\varphi^{-1}$ are ideals of $\mathcal{A}$. This is formulated in the following lemma whose proof is straightforward (cf. [39]). Note that the complement of an ideal is not necessarily an ideal.

Lemma 3.1.8 *The collection of all ideals of a fixed ordered algebra is closed under positive Boolean operations, inverse translations and inverse order morphisms.* □

Proposition 3.1.9 *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be ordered algebras, $I, J \trianglelefteq \mathcal{A}$, $K \trianglelefteq \mathcal{B}$ be ideals, $p \in \mathrm{Tr}(\mathcal{A})$ be a translation, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism. Then the following inclusions hold:*

(1) $\preccurlyeq_{I \cap J}, \preccurlyeq_{I \cup J} \supseteq \preccurlyeq_I \cap \preccurlyeq_J$;

(2) $\preccurlyeq_{p^{-1}(I)} \supseteq \preccurlyeq_I$;

(3) $\preccurlyeq_{K\varphi^{-1}} \supseteq \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$, *and if φ is an order epimorphism then the equality holds:* $\preccurlyeq_{K\varphi^{-1}} = \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$.

Proof. The statements (1) and (2) are obvious. For (3) assume $(a, b) \in \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$ for some $a, b \in A$. Then $a\varphi \preccurlyeq_K b\varphi$. Hence, for any $p \in \mathrm{Tr}(\mathcal{A})$, if $p(b) \in K\varphi^{-1}$ then $p(b)\varphi \in K$ what means $p_\varphi(b\varphi) \in K$. This implies now $p_\varphi(a\varphi) \in K$, i.e., $p(a)\varphi \in K$, and so $p(a) \in K\varphi^{-1}$.

Therefore $a \preccurlyeq_{K\varphi^{-1}} b$, and hence $\varphi \circ \preccurlyeq_K \circ \varphi^{-1} \subseteq \preccurlyeq_{K\varphi^{-1}}$. When φ is surjective, we note that by Proposition 3.1.7 every translation $q \in \mathrm{Tr}(\mathcal{B})$

is of the form p_φ for some $p \in \mathrm{Tr}(\mathcal{A})$. Thus in this case the inclusion $\preccurlyeq_{K\varphi^{-1}} \subseteq \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$ holds, whence $\preccurlyeq_{K\varphi^{-1}} = \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$. □

Combining Propositions 3.1.9, 3.1.5 and 3.1.4 we get the following corollary.

Corollary 3.1.10 *For any ordered algebras $\mathcal{A}$ and $\mathcal{B}$, ideals $I, J \trianglelefteq \mathcal{A}$ and $K \trianglelefteq \mathcal{B}$, translation $p \in \mathrm{Tr}(\mathcal{A})$, and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$,*

(1) $\mathrm{SOA}(I \cap J), \mathrm{SOA}(I \cup J) \preceq \mathrm{SOA}(I) \times \mathrm{SOA}(J)$;

(2) $\mathrm{SOA}(p^{-1}(I)) \leftarrow \mathrm{SOA}(I)$;

(3) $\mathrm{SOA}(K\varphi^{-1}) \preceq \mathrm{SOA}(K)$ *and, moreover, if φ is an order epimorphism, then* $\mathrm{SOA}(K\varphi^{-1}) \cong \mathrm{SOA}(K)$. □

Here we consider some examples of ordered algebras and prove some of their elementary properties which will be used later.

For an algebra $\mathcal{A} = (A, \Sigma)$, the *translation semigroup* $\mathrm{TrS}(\mathcal{A})$ of $\mathcal{A}$ consists of the elementary translations $f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$, where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$, and their compositions. We note that $\mathrm{TrS}(\mathcal{A})$ does not automatically include the identity translation $1_A : A \to A$.

Definition 3.1.11 An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *n-nilpotent* $(n \geq 0)$ if $p_1 \cdots p_n(a) \leqslant b$ holds for all $a, b \in A$ and all $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$.

An ordered algebra $\mathcal{A}$ is called *n-conilpotent* $(n \geq 0)$ if $b \leqslant p_1 \cdots p_n(a)$ holds for all $a, b \in A$ and all $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$.

An ordered algebra is called (co)nilpotent if it is n-(co)nilpotent for some $n \geq 0$.

The classes of all nilpotent Σ-algebras and conilpotent Σ-algebras are denoted by $\mathbf{Nil}(\Sigma)$ and $\mathbf{coNil}(\Sigma)$, respectively.

An element $a_0 \in A$ is a *trap* of $\mathcal{A}$, if $p(a_0) = a_0$ holds for any $p \in \mathrm{Tr}(\mathcal{A})$.

Lemma 3.1.12 *Every n-(co)nilpotent ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ has a unique trap which is the least (greatest) element of the algebra.*

Proof. We prove only the nilpotent case. For every $p_1, \ldots, p_n, q_1, \ldots, q_n$ in $\mathrm{TrS}(\mathcal{A})$ and $a, b \in A$ we have

$$p_1 \cdots p_n(a) \leqslant q_1 \cdots q_n(b) \leqslant p_1 \cdots p_n(a).$$

Thus $p_1 \cdots p_n(a) = q_1 \cdots q_n(b)$, and let a_0 be this element. Clearly we have $p(a_0) = a_0$ for every $p \in \mathrm{TrS}(\mathcal{A})$ and $a_0 \leqslant a$ for every $a \in A$. So, a_0 is the unique trap of $\mathcal{A}$ which is the least element. □

Proposition 3.1.13 *Class $\mathbf{Nil}(\Sigma)$ of all nilpotent ordered Σ-algebras and class $\mathbf{coNil}(\Sigma)$ of all conilpotent ordered Σ-algebras are VFOA.*

Proof. It can be easily seen that the class of n-nilpotent ordered algebras is closed under order subalgebras and direct products. To see that it is closed under order epimorphic images, let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras such that $\mathcal{A}$ is n-nilpotent and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an order epimorphism. Let $b, d \in B$ be two elements and $q_1, \ldots, q_n \in \mathrm{TrS}(\mathcal{B})$. There are $a, c \in A$ such that $b = a\varphi$ and $d = c\varphi$, and by Proposition 3.1.7, there are $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$ such that $(p_j)_\varphi = q_j$ for all $j = 1, \ldots, n$. It follows from $p_1 \cdots p_n(a) \leqslant c$ that $p_1 \cdots p_n(a)\varphi \leqslant' c\varphi$, so $(p_1)_\varphi \cdots (p_n)_\varphi(a\varphi) \leqslant' c\varphi$, and thus $q_1 \cdots q_n(b) \leqslant' d$ holds. Hence, $\mathcal{B}$ is n-nilpotent.

Finally, the claim that $\mathbf{Nil}(\Sigma)$ is a VFOA follows from the fact that an n-nilpotent ordered algebra is an $(n+1)$-nilpotent ordered algebra as well.

By a dual argument one can show that $\mathbf{coNil}(\Sigma)$ is a VFOA. □

A semigroup is called *n-nilpotent* if it contains a zero element and the product of every n elements is zero. It is called *nilpotent* if it is n-nilpotent for a natural n.

Lemma 3.1.14 *If $\mathcal{A} = (A, \Sigma, \leqslant)$ is an order n-nilpotent or n-conilpotent algebra, then the translation semigroup* $\mathrm{TrS}(\mathcal{A})$ *is a nilpotent semigroup.*

Proof. For every $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$ we have
$$p_1 \cdots p_n(a) \leqslant q_1 \cdots q_n(a) \leqslant p_1 \cdots p_n(a).$$
Thus $p_1 \cdots p_n = q_1 \cdots q_n$, so $p_1 \cdots p_n \in \mathrm{TrS}(\mathcal{A})$ is the zero element of $\mathrm{TrS}(\mathcal{A})$ and the product of every n elements of this semigroup is zero. □

3.2 Positive variety theorem

Recall the notions of ΣX-terms and ΣX-contexts from the Preliminaries in Chapter 1. Note that $\mathrm{C}(\Sigma, X)$ forms a monoid with respect to the composition operation, and that $t \cdot (Q \cdot P) = (t \cdot Q) \cdot P$ holds for all $P, Q \in \mathrm{C}(\Sigma, X)$, $t \in \mathrm{T}(\Sigma, X)$. There is a bijective correspondence between $\mathrm{C}(\Sigma, X)$ and the translations of the term algebra $\mathcal{T}(\Sigma, X)$ in a natural way: an elementary context $P = f(t_1, \ldots, \xi, \ldots, t_m)$ corresponds to $P^{\mathcal{T}(\Sigma,X)} = f^{\mathcal{T}(\Sigma,X)}(t_1, \ldots, \xi, \ldots, t_m)$, and the composition $P \cdot Q$ of the contexts P and Q corresponds to the composition $P^{\mathcal{T}(\Sigma,X)} \cdot Q^{\mathcal{T}(\Sigma,X)}$ of translations.

Every tree language $T \subseteq \mathrm{T}(\Sigma, X)$ can be regarded as an ideal of the ordered algebra $\mathcal{T}(\Sigma, X) = (\mathrm{T}(\Sigma, X), \Sigma, =)$. Thus one can define the *syntactic quasi-order* $\preccurlyeq_T$ of T as the syntactic quasi-order of an ideal by Definition 3.1.6. We note that it satisfies the following for any $t, s \in \mathrm{T}(\Sigma, X)$
$$t \preccurlyeq_T s \iff (\forall P \in \mathrm{C}(\Sigma, X))\big(s \cdot P \in T \Rightarrow t \cdot P \in T\big).$$
The corresponding equivalence relation $\approx^T = \preccurlyeq_T \cap \preccurlyeq_T^{-1}$ of $\preccurlyeq_T$ is the *syntactic congruence* of T:

$$t \approx^T s \iff (\forall P \in \mathrm{C}(\Sigma, X))\big(t \cdot P \in T \Leftrightarrow s \cdot P \in T\big).$$

The *syntactic ordered algebra* of T is $\mathrm{SOA}(T) = (\mathrm{T}(\Sigma, X)/{\approx^T}, \Sigma, \leqslant_T)$, where $\leqslant_T$ the order induced by $\preccurlyeq_T$ satisfies the following for $t, s \in \mathrm{T}(\Sigma, X)$,

$$t/{\approx^T} \leqslant_T s/{\approx^T} \iff t \preccurlyeq_T s.$$

The *syntactic morphism* of T is the mapping $\varphi^T : \mathcal{T}(\Sigma, X) \to \mathrm{SOA}(T)$ defined by $t\varphi^T = t/{\approx^T}$ for $t \in \mathrm{T}(\Sigma, X)$.

It can be easily seen that not every ordered algebra is the syntactic ordered algebra of a tree language. However, these syntactic ordered algebras can be characterized as follows (cf. [53] Proposition 3.6).

Proposition 3.2.1 *A finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is order isomorphic to the syntactic ordered algebra of a tree language if and only if there exists an ideal $I \trianglelefteq \mathcal{A}$ such that $\preccurlyeq_I = \leqslant$.*

Proof. First, suppose $\mathcal{A} \cong \mathrm{SOA}(T)$ for some tree language T. Then the subset $I = \{t/{\approx^T} \mid t \in T\}$ is an ideal of $\mathrm{SOA}(T)$ and $\preccurlyeq_I = \leqslant_T$.

Conversely, suppose $\preccurlyeq_I = \leqslant$ holds for some ideal $I \trianglelefteq \mathcal{A}$. Let the morphism $\varphi : \mathcal{T}(\Sigma, A) \to \mathcal{A}$ be the unique extension of the identity map $1_A : A \to A$. Since φ is an epimorphism $\preccurlyeq_{I\varphi^{-1}} = \varphi \circ \preccurlyeq_I \circ \varphi^{-1}$ by Proposition 3.1.9(3). So, Proposition 3.1.4 implies $\mathcal{T}(\Sigma, A)/{\preccurlyeq_{I\varphi^{-1}}} \cong \mathcal{A}/{\preccurlyeq_I}$, and since $\preccurlyeq_I = \leqslant$ by assumption, then $\mathrm{SOA}(I\varphi^{-1}) \cong \mathcal{A}$. □

Let Σ be a ranked alphabet, X be a leaf alphabet, and $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *recognized* by $\mathcal{A}$ if there exists an ideal $I \trianglelefteq \mathcal{A}$ and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $T = I\varphi^{-1}$.

Proposition 3.2.2 *For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ and an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$, $\mathrm{SOA}(T) \preceq \mathcal{A}$ if and only if T is recognized by $\mathcal{A}$.*

Proof. Suppose $T = I\varphi^{-1}$ for a morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and an ideal $I \trianglelefteq \mathcal{A}$. Let the ordered Σ-algebra $\mathcal{B}$ be the φ-image of $\mathcal{T}(\Sigma, X)$, and define the mapping $\psi : \mathcal{B} \to \mathrm{SOA}(T)$ by $(t\varphi)\psi = t/{\approx^T}$ for $t \in \mathrm{T}(\Sigma, X)$. We show that $t\varphi \leqslant s\varphi$ implies $t \preccurlyeq_T s$ for all $t, s \in \mathrm{T}(\Sigma, X)$. This also proves that ψ is well-defined. Suppose $t\varphi \leqslant s\varphi$, then $t\varphi \preccurlyeq_I s\varphi$ since $\leqslant \subseteq \preccurlyeq_I$. Now, for any translation $p \in \mathrm{Tr}(\mathcal{A})$,

$$p(s) \in T \Rightarrow p(s)\varphi \in I \Rightarrow p_\varphi(s\varphi) \in I \Rightarrow p_\varphi(t\varphi) \in I \Rightarrow p(t)\varphi \in I \Rightarrow p(t) \in T,$$

so $t \preccurlyeq_T s$. It can also be seen that ψ is a Σ-morphism. Thus ψ is an order epimorphism, and hence $\mathrm{SOA}(T) \leftarrow \mathcal{B} \subseteq \mathcal{A}$.

Now suppose that $\mathrm{SOA}(T) \leftarrow \mathcal{B} \subseteq \mathcal{A}$ holds for an ordered algebra $\mathcal{B}$ and let $\psi : \mathcal{B} \to \mathrm{SOA}(T)$ be an order epimorphism. There exists a Σ-morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $(x\varphi)\psi = x/{\approx^T}$ for every $x \in X \cup \Sigma_0$. By induction on t it can be shown that $t\varphi\psi = t/{\approx^T}$ holds for every $t \in \mathrm{T}(\Sigma, X)$.

The set $\{t/{\approx^T} \in \mathrm{SOA}(T) \mid t \in T\}\psi^{-1}$ is an ideal of $\mathcal{B}$. If I is the ideal of $\mathcal{A}$ generated by this set, then $T = I\varphi^{-1}$. □

From Proposition 3.2.2 it follows that the syntactic ordered algebra of a tree language is the least ordered algebra which recognizes the tree language.

Let us recall that for a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, a context P in $\mathrm{C}(\Sigma, X)$, and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, the inverse translation of T under P is $P^{-1}(T) = \{t \in \mathrm{T}(\Sigma, X) \mid t \cdot P \in T\}$, and the inverse morphism of T under φ is $T\varphi^{-1} = \{t \in \mathrm{T}(\Sigma, Y) \mid t\varphi \in T\}$.

The following is an immediate consequence of Corollary 3.1.10.

Corollary 3.2.3 *For tree languages* $T, T' \subseteq \mathrm{T}(\Sigma, X)$, *context* $P \in \mathrm{C}(\Sigma, X)$, *and* Σ*-morphism* $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$,

(1) $\mathrm{SOA}(T \cap T'), \mathrm{SOA}(T \cup T') \preceq \mathrm{SOA}(T) \times \mathrm{SOA}(T')$;

(2) $\mathrm{SOA}(P^{-1}(T)) \leftarrow \mathrm{SOA}(T)$;

(3) $\mathrm{SOA}(T\varphi^{-1}) \preceq \mathrm{SOA}(T)$ *and moreover, when* φ *is surjective then* $\mathrm{SOA}(T\varphi^{-1}) \cong \mathrm{SOA}(T)$. □

Let Σ be a fixed ranked alphabet. Let us recall that a class of finite ordered Σ-algebras is a variety (of finite ordered algebras) if it is closed under order subalgebras, order epimorphic images, and finite direct products. In what follows we consider families of recognizable tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, where for any leaf alphabet X, $\mathscr{V}(\Sigma, X)$ is a set of recognizable ΣX-tree languages.

Definition 3.2.4 A *positive variety of tree languages* is a family of recognizable tree languages closed under finite positive Boolean operations (finite intersections and finite unions), inverse translations and inverse morphisms. That is to say, a family $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ of recognizable tree languages, is a positive variety of tree languages, a PVTL for short, if for any leaf alphabets X, Y, tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$, homomorphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, and context $P \in \mathrm{C}(\Sigma, X)$, if $T, T' \in \mathscr{V}(\Sigma, X)$, then $T \cup T', T \cap T', P^{-1}(T) \in \mathscr{V}(\Sigma, X)$ and also $T\varphi^{-1} \in \mathscr{V}(\Sigma, Y)$.

For any PVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ always $\emptyset, \mathrm{T}(\Sigma, X) \in \mathscr{V}(\Sigma, X)$ holds, since $\emptyset$ is the empty union of tree languages, and $\mathrm{T}(\Sigma, X)$ is the empty intersection of tree languages.

Definition 3.2.5 For a variety of finite ordered algebras $\mathbf{K}$, let the indexed family $\mathbf{K}^t = \{\mathbf{K}^t(X)\}$ be the family of tree languages whose syntactic ordered algebras are in $\mathbf{K}$, that is

$$\mathbf{K}^t(X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOA}(T) \in \mathbf{K}\}.$$

For a positive variety of tree languages $\mathscr{V}$, let $\mathscr{V}^a$ be the variety of finite ordered algebras generated by syntactic ordered algebras of tree languages in $\mathscr{V}$, that is to say $\mathscr{V}^a$ is the VFOA generated by the class

$$\{\mathrm{SOA}(T) \mid T \in \mathscr{V}(X) \text{ for a leaf alphabet } X\}.$$

By Corollary 3.2.3, for a variety of finite ordered algebras $\mathbf{K}$, the family $\mathbf{K}^t$ is a positive variety of tree languages.

Lemma 3.2.6 (1) *The operations* $\mathbf{K} \mapsto \mathbf{K}^t$ *and* $\mathscr{V} \mapsto \mathscr{V}^a$ *are monotone, i.e., if* $\mathbf{K} \subseteq \mathbf{L}$ *and* $\mathscr{V} \subseteq \mathscr{W}$*, then* $\mathbf{K}^t \subseteq \mathbf{L}^t$ *and* $\mathscr{V}^a \subseteq \mathscr{W}^a$.

(2) $\mathscr{V} \subseteq \mathscr{V}^{at}$*, and* $\mathbf{K}^{ta} \subseteq \mathbf{K}$.

Proof. The statement (1) and the inclusion $\mathscr{V} \subseteq \mathscr{V}^{at}$ are obvious. In order to prove $\mathbf{K}^{ta} \subseteq \mathbf{K}$, we note that if $\mathcal{A} \in \mathbf{K}^{ta}$ then for some $T_1, \ldots, T_n$ in $\mathbf{K}^t$, $\mathcal{A} \preceq \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_n)$ holds, what by definition means that $\mathrm{SOA}(T_j) \in \mathbf{K}$ for every j, and hence $\mathcal{A} \in \mathbf{K}$. □

The following was proved for classical algebras in [48].

Lemma 3.2.7 *For any finite ordered algebra* $\mathcal{A} = (A, \Sigma, \leqslant)$*, there are tree languages* $T_1, \ldots, T_m$ *recognizable by* $\mathcal{A}$ *such that*

$$\mathcal{A} \subseteq \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_m).$$

Proof. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra, and suppose the epimorphism $\psi : \mathcal{T}(\Sigma, A) \to \mathcal{A}$ is obtained by extending the identity mapping $1_A : A \to A$. Recall that for any $a \in A$, $(a] = \{b \in A \mid b \leqslant a\}$ is the ideal of $\mathcal{A}$ generated by a. By Corollary 3.1.10(3), $\mathrm{SOA}((a]\psi^{-1}) \cong \mathcal{A}/(a]$ for every $a \in A$. We show $\mathcal{A} \subseteq \prod_{a\in A} \mathcal{A}/(a]$. This will finish the proof since $(a]\psi^{-1}$ is recognizable by $\mathcal{A}$. Define $\phi : \mathcal{A} \to \prod_{a\in A} \mathcal{A}/(a]$ by $u\phi = \big(u/{\approx^{(a]}}\big)_{a\in A}$ for $u \in A$. Clearly ϕ is an order morphism. It suffices to show that ϕ is injective. Suppose $u\phi = v\phi$ for $u, v \in A$. Then $u/{\approx^{(a]}} = v/{\approx^{(a]}}$ for every $a \in A$. In particular, $u/{\approx^{(u]}} = v/{\approx^{(u]}}$ and $u/{\approx^{(v]}} = v/{\approx^{(v]}}$, which imply $v \in (u]$ and $u \in (v]$, respectively. So, $u \leqslant v$ and $v \leqslant u$, whence $u = v$. □

Corollary 3.2.8 (1) *Every VFOA is generated by syntactic ordered algebras of some tree languages.*

(2) *For any PVTL* $\mathscr{V}$ *and any finite ordered algebra* $\mathcal{A}$*, if every tree language recognizable by* $\mathcal{A}$ *belongs to* $\mathscr{V}$*, then* $\mathcal{A} \in \mathscr{V}^a$. □

Lemma 3.2.9 *For every variety of finite ordered algebras* $\mathbf{K}$, $\mathbf{K} \subseteq \mathbf{K}^{ta}$.

Proof. By Corollary 3.2.8(1), it is enough to show that the syntactic ordered algebras of tree languages that belong to $\mathbf{K}$ are in $\mathbf{K}^{ta}$. Suppose that for a tree language T, $\mathrm{SOA}(T) \in \mathbf{K}$. Then T is in $\mathbf{K}^t$ by definition, so $\mathrm{SOA}(T)$ belongs to $\mathbf{K}^{ta}$ which finishes the proof. □

The essential part of the positive variety theorem is the following.

Lemma 3.2.10 *For every positive variety of tree languages $\mathscr{V}$, $\mathscr{V}^{at} \subseteq \mathscr{V}$.*

Proof. If $T \in \mathscr{V}^{at}(X)$, then there are leaf alphabets $X_1, \ldots, X_n$ and tree languages $T_1 \in \mathscr{V}(X_1), \ldots, T_n \in \mathscr{V}(X_n)$ such that $\mathrm{SOA}(T)$ divides the product $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ where $\mathcal{A}_j = (A_j, \Sigma, \leqslant_j) = \mathrm{SOA}(T_j)$ for $j = 1, \ldots, n$. Thus, by Proposition 3.2.2, T is recognized by $\mathcal{A}$, and so there is an order morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and an ideal $I \trianglelefteq \mathcal{A}$ such that $T = I\varphi^{-1}$. For any n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \prod_i A_i$ we have $(\mathbf{a}] = (a_1] \times \cdots \times (a_n]$. Let $\varphi_j : \mathcal{T}(\Sigma, X) \to \mathcal{A}_j$ be the composition of φ with the j-th projection mapping $\prod_i A_i \to A_j$. Then $T = I\varphi^{-1} = \bigcup_{\mathbf{a} \in I} (\mathbf{a}]\varphi^{-1} = \bigcup_{(a_1, \ldots, a_n) \in I} \bigcap_{j \leq n} (a_j]\varphi_j^{-1}$. It is now enough to show $(a_j]\varphi_j^{-1} \in \mathscr{V}(X)$ for every $1 \leq j \leq n$. Fix a $j \leq n$. Let $\psi_j = \varphi^{T_j} : \mathcal{T}(\Sigma, X_j) \to \mathcal{A}_j$ be the syntactic morphism of T_j. One can construct a Σ-morphism $\chi_j : \mathcal{T}(\Sigma, X) \to \mathcal{T}(\Sigma, X_j)$ such that $\chi_j \psi_j = \varphi_j$. Then $(a_j]\varphi_j^{-1} = (a_j]\psi_j^{-1}\chi_j^{-1}$ and, since $\mathscr{V}$ is closed under inverse morphisms, for showing $(a_j]\varphi_j^{-1} \in \mathscr{V}(X)$ it suffices to show that $(a_j]\psi_j^{-1} \in \mathscr{V}(X_j)$. Choose a $t \in \mathrm{T}(\Sigma, X_j)$ such that $a_j = t\psi_j$. We show $(a_j]\psi_j^{-1} = \bigcap \{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma, X_j), P(t) \in T_j\}$.

The intersection on the righthand side is finite since T_j is recognizable. For any $s \in \mathrm{T}(\Sigma, X_j)$, we have that $s \in (a_j]\psi_j^{-1}$ iff $s\psi_j \leqslant_j a_j = t\psi_j$, i.e., $s \preccurlyeq_{T_j} t$, what by definition means that $P(t) \in T_j$ implies $P(s) \in T_j$ for any $P \in \mathrm{C}(\Sigma, X_j)$. This is further equivalent to $s \in P^{-1}(T_j)$ whenever $P(t) \in T_j$ for any $P \in \mathrm{C}(\Sigma, X_j)$, what finally means

$$s \in \bigcap \{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma, X_j), P(t) \in T_j\}.$$

From $T_j \in \mathscr{V}(X_j)$ and the fact that $\mathscr{V}$ is closed under inverse translations and positive Boolean operations, it follows that $(a_j]\psi_j^{-1} \in \mathscr{V}(X_j)$. Therefore, $(a_j]\varphi_j^{-1}$ belongs to $\mathscr{V}(X)$ for all j, and thus $T \in \mathscr{V}(X)$. □

Summing up, we have shown the following.

Proposition 3.2.11 (Positive Variety Theorem) *The variety operations $\mathbf{K} \mapsto \mathbf{K}^t$ and $\mathscr{V} \mapsto \mathscr{V}^a$ are mutually inverse lattice isomorphisms between the class of all varieties of finite ordered algebras and the class of all positive varieties of recognizable tree languages, i.e., $\mathscr{V}^{at} = \mathscr{V}$ and $\mathbf{K}^{ta} = \mathbf{K}$.* □

Let us consider some families of tree languages and provide some instances for Positive Variety Theorem (Proposition 3.2.11).

Definition 3.2.12 A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *cofinite* if either $T = \emptyset$ or its complement $\mathrm{T}(\Sigma, X) \setminus T$ is finite. The family of cofinite ΣX-tree languages is denoted by $\mathrm{Cof}(\Sigma, X)$, and $\mathrm{Cof}_\Sigma = \{\mathrm{Cof}(\Sigma, X)\}$ is the family of cofinite tree languages for all leaf alphabets X. Similarly, $\mathrm{Fin}(\Sigma, X)$ is the family of finite ΣX-tree languages, and $\mathrm{Fin}_\Sigma = \{\mathrm{Fin}(\Sigma, X)\}$ is the family of finite tree languages for all leaf alphabets X.

Proposition 3.2.13 *A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is cofinite if and only if it can be recognized by a finite nilpotent ordered algebra. Similarly, a tree language is finite if and only if it can be recognized by a finite conilpotent ordered algebra.*

Proof. We show the cofinite case, a dual proof works for the finite case.

Suppose $T \subseteq \mathrm{T}(\Sigma, X)$ is cofinite. If $T = \emptyset$ then there is nothing to prove. Otherwise there exists an $n \in \mathbb{N}$ such that $P_1 \cdots P_n(t) \in T$ holds for all $P_1, \ldots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t \in \mathrm{T}(\Sigma, X)$. Therefore, $P_1 \cdots P_n(t) \preccurlyeq_T s$ holds for all $P_1, \ldots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and all $t, s \in \mathrm{T}(\Sigma, X)$. This immediately implies that the syntactic algebra $\mathrm{SOA}(T)$ of T satisfies $p_1 \cdots p_n(a) \leqslant_T b$ for all $p_1, \ldots, p_n \in \mathrm{TrS}(\mathrm{SOA}(T))$ and all $a, b \in \mathrm{SOA}(T)$. Thus, $\mathrm{SOA}(T)$ is an n-nilpotent ordered algebra.

Conversely, suppose that a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is recognized by an n-nilpotent ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$. Let $\varphi : \mathrm{T}(\Sigma, X) \to \mathcal{A}$ be an order morphism and $I \trianglelefteq A$ be an ideal such that $T = I\varphi^{-1}$. The mapping $\varphi_* : \mathrm{C}(\Sigma, X) \setminus \{\xi\} \to \mathrm{TrS}(\mathcal{A})$ obtained by setting

$$f(t_1, \ldots, \xi, \ldots, t_m)\varphi_* = f^{\mathcal{A}}(t_1\varphi, \ldots, \xi, \ldots, t_m\varphi)$$

for all function symbols $f \in \Sigma_m$ $(m > 0)$ and trees $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$, and $(P \cdot Q)\varphi_* = P\varphi_* \cdot Q\varphi_*$, is a semigroup morphism which satisfies $P\varphi_*(t\varphi) = P(t)\varphi$ for all $t \in \mathrm{T}(\Sigma, X)$, $P \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$. Since $\mathcal{A}$ is an n-nilpotent ordered algebra, then $p_1 \cdots p_n(a) \in I$ holds for all $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$. In particular, $P_1\varphi_* \cdots P_n\varphi_*(t\varphi) \in I$ holds for all $P_1, \ldots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t \in \mathrm{T}(\Sigma, X)$. The statement $P_1\varphi_* \cdots P_n\varphi_*(t\varphi) \in I$ is equivalent to $P_1 \cdots P_n(t)\varphi \in I$ and $P_1 \cdots P_n(t) \in I\varphi^{-1} = T$. It follows that T is cofinite. □

Corollary 3.2.14 *Families* Cof_Σ *and* Fin_Σ *are PVTL and moreover the identities* $\mathrm{Cof}_\Sigma = \mathbf{Nil}(\Sigma)^t$ *and* $\mathrm{Fin}_\Sigma = \mathbf{coNil}(\Sigma)^t$ *hold.*

Proof. This follows immediately from Propositions 3.2.13, 3.1.13 and 3.2.11. However, it can be verified directly that the families of finite and cofinite tree languages are closed under finite unions and intersections, inverse translations and inverse morphisms. □

3.3 Generalized positive variety theorem

Generalized varieties of tree languages and generalized varieties of finite algebras were introduced by Steinby [54] who proved a generalized variety theorem for these classes. A variety of finite algebras is a class of finite algebras over a fixed ranked alphabet, and subalgebras, homomorphic images and direct products are defined for algebras over the same ranked alphabet. These notions can be generalized for algebras over different ranked alphabets. A generalized variety of finite algebras is a class of finite algebras (over any ranked alphabet) that satisfies certain closure properties. Similarly, a generalized variety of tree languages is defined. In this section we generalize our Positive Variety Theorem (Proposition 3.2.11) to generalized positive varieties of tree languages and generalized varieties of finite ordered algebras. This will be used for another variety theorem in Chapter 5 which gives a characterization for families of tree languages definable by syntactic ordered monoids. The following definition is the ordered version of Definitions 3.1, 3.2, 3.3, 3.14 from [54].

Definition 3.3.1 Suppose that $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ are ordered algebras.

(a) The ordered algebra $\mathcal{B}$ is said to be an *order g-subalgebra* of $\mathcal{A}$, in notation $\mathcal{B} \subseteq_g \mathcal{A}$, if $B \subseteq A$, $\Omega_m \subseteq \Sigma_m$ for all $m \geq 0$, $f^{\mathcal{B}}$ is the restriction of $f^{\mathcal{A}}$ to B for every $f \in \Omega$, and $\leqslant'$ is the restriction of $\leqslant$ to B.

(b) An *assignment* is a mapping $\kappa : \Sigma \to \Omega$ such that $\kappa(\Sigma_m) \subseteq \Omega_m$ for all $m \geq 0$. An *order g-morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair (κ, φ) where the mapping $\kappa : \Sigma \to \Omega$ is an assignment and $\varphi : A \to B$ is an order preserving mapping satisfying $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = (f\kappa)^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for any $m \geq 0$, $f \in \Sigma_m$, and $a_1, \ldots, a_m \in A$. Note that by "$\varphi : A \to B$ is order preserving" we mean that $a \leqslant b$ implies $a\varphi \leqslant' b\varphi$ for all $a, b \in A$. If both κ and φ are surjective, then (κ, φ) is an *order g-epimorphism*, and in that case we write $\mathcal{B} \leftarrow_g \mathcal{A}$ meaning that $\mathcal{B}$ is an *order g-epimorphic image* of $\mathcal{A}$. When $\mathcal{B}$ is an order g-epimorphic image of an order g-subalgebra of $\mathcal{A}$, we write $\mathcal{B} \preceq_g \mathcal{A}$. When both κ and φ are bijective and $(\kappa^{-1}, \varphi^{-1})$ is an order g-morphism, (κ, φ) is an *order g-isomorphism*, and $\mathcal{B} \cong_g \mathcal{A}$ means that $\mathcal{B}$ and $\mathcal{A}$ are order g-isomorphic.

(c) Let $\Sigma^1, \ldots, \Sigma^n$ and Γ be ranked alphabets. The product $\Sigma^1 \times \cdots \times \Sigma^n$ is a ranked alphabet such that $(\Sigma^1 \times \cdots \times \Sigma^n)_m = \Sigma^1_m \times \cdots \times \Sigma^n_m$ for every $m \geq 0$. For any assignment $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ and any finite number of ordered algebras $\mathcal{A}_1 = (A_1, \Sigma^1, \leqslant_1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n, \leqslant_n)$, the *$\kappa$-product* of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is the ordered Γ-algebra

$$\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (A_1 \times \cdots \times A_n, \Gamma, \leqslant_1 \times \cdots \times \leqslant_n)$$

defined by the following identities for any $c \in \Gamma_0$, $f \in \Gamma_m$ $(m > 0)$ and $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in A_1 \times \cdots \times A_n$ $(i \leq n)$,

(1) $c^{\kappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)} = (c_1^{\mathcal{A}_1}, \ldots, c_n^{\mathcal{A}_n})$ where $c\kappa = (c_1, \ldots, c_n)$,

(2) $f^{\kappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = (f_1^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, f_n^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn}))$ where $f\kappa = (f_1, \ldots, f_n)$, and

(3) $\mathbf{a}_1 \leqslant_1 \times \cdots \times \leqslant_n \mathbf{a}_2 \iff a_{11} \leqslant_1 a_{21} \,\&\, \ldots \,\&\, a_{1n} \leqslant_n a_{2n}$.

Without specifying the assignment κ, such algebras are called *g-products*.

A *generalized variety of finite ordered algebras*, a gVFOA for short, is a mapping $\mathbf{K} = \{\mathbf{K}(\Sigma)\}$ which assigns a class of finite ordered Σ-algebras $\mathbf{K}(\Sigma)$ to any ranked alphabet Σ, and is closed under order g-subalgebras, order g-epimorphic images, and g-products.

Proposition 3.3.2 *Suppose that $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ are ordered algebras, $\preccurlyeq$ is a quasi-order on $\mathcal{B}$ and $(\kappa, \varphi) : \mathcal{A} \to \mathcal{B}$ is an order g-morphism. Then*

(1) *the image of $\mathcal{A}$, $\mathcal{A}(\kappa, \varphi) = (A\varphi, \Sigma\kappa, \leqslant'')$ where $\leqslant''$ is the restriction of $\leqslant'$ to $A\varphi$, is an order g-subalgebra of $\mathcal{B}$,*
(2) *$\varphi \circ \preccurlyeq \circ \varphi^{-1}$ is a quasi-order on $\mathcal{A}$ and $\mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \cong_g \mathcal{A}\varphi/\preccurlyeq'$, where $\preccurlyeq'$ is the restriction of $\preccurlyeq$ to $A\varphi$, and*
(3) *if φ is an order g-epimorphism, then $\mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \cong_g \mathcal{B}/\preccurlyeq$.* □

The proof is a direct generalization of that of Proposition 3.1.4.

Also, many of the already presented results have their "generalized" counterparts with slightly different proofs. For example, a result analogous to Proposition 3.1.9 can be proved. As a corollary, it follows that for any g-morphism $(\kappa, \varphi) : \mathcal{T}(\Omega, Y) \to \mathcal{T}(\Sigma, X)$ and tree language $T \subseteq \mathrm{T}(\Sigma, X)$, $\mathrm{SOA}(T\varphi^{-1}) \preceq_g \mathrm{SOA}(T)$ holds, and if (κ, φ) is a g-epimorphism then $\mathrm{SOA}(T\varphi^{-1}) \cong_g \mathrm{SOA}(T)$.

Let Σ and Ω be ranked alphabets, X be a leaf alphabet, and $\mathcal{A} = (A, \Omega, \leqslant)$ be an ordered algebra. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is said to be *g-recognized* by $\mathcal{A}$ if there exist an ideal $I \trianglelefteq \mathcal{A}$ and an order g-morphism $(\kappa, \varphi) : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $T = I\varphi^{-1}$. Similarly to Proposition 3.2.2, it can be proved that a tree language T is g-recognized by $\mathcal{A}$ if $\mathrm{SOA}(T) \preceq_g \mathcal{A}$. Contrary to Proposition 3.2.2, the converse of this statement is not true in general, for more details see the definition of reduced syntactic algebra in Section 6 of [54].

Definition 3.3.3 A *family* of recognizable tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, where $\mathscr{V}(\Sigma, X)$ consists of recognizable ΣX-tree languages for any ranked alphabet Σ and leaf alphabet X, is said to be a *generalized positive variety of tree languages*, abbreviated by gPVTL, if it is closed under finite positive Boolean operations (finite intersections and unions), inverse translations, and inverse g-morphisms.

Definition 3.3.4 Let $\mathbf{K} = \{\mathbf{K}(\Sigma)\}$ be a gVFOA. Define the family $\mathbf{K}^t = \{\mathbf{K}^t(\Sigma, X)\}$ to be the family of tree languages whose syntactic ordered algebras are in $\mathbf{K}$, that is $\mathbf{K}^t(\Sigma, X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOA}(T) \in \mathbf{K}(\Sigma)\}$.

For a gPVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, let $\mathscr{V}^a = \{\mathscr{V}^a(\Sigma)\}$ be the gVFOA generated by the class $\{\mathrm{SOA}(T) \mid T \in \mathscr{V}(\Sigma, X) \text{ for some } \Sigma, X\}$.

It can be proved similarly to Lemmas 3.2.6, 3.2.9 and Corollary 3.2.8 (or Proposition 6.13 of [53]) that every gVFOA is generated by syntactic ordered algebras of some tree languages and that if every tree language recognizable by a finite ordered algebra $\mathcal{A}$ belongs to a gPVTL $\mathscr{V}$, then $\mathcal{A} \in \mathscr{V}^a$.

Proposition 3.3.5 (Generalized Positive Variety Theorem) *The operations $\mathbf{K} \mapsto \mathbf{K}^t$ and $\mathscr{V} \mapsto \mathscr{V}^a$ are mutually inverse lattice isomorphisms between the class of all gVFOA's and the class of gPVTL's, i.e., $\mathscr{V}^{at} = \mathscr{V}$ and $\mathbf{K}^{ta} = \mathbf{K}$.*

Proof. The facts that for a gVFOA $\mathbf{K}$ the family $\mathbf{K}^t$ is a gPVTL and that the mappings $\mathbf{K} \mapsto \mathbf{K}^t$ and $\mathscr{V} \mapsto \mathscr{V}^a$ are monotone, as well as the relations $\mathscr{V} \subseteq \mathscr{V}^{at}$ and $\mathbf{K}^{ta} = \mathbf{K}$, can be proved in a way similar to the proofs of the corresponding claims in Section 3.2. We show only the inclusion $\mathscr{V}^{at} \subseteq \mathscr{V}$.

Suppose $T \in \mathscr{V}^{at}(\Sigma, X)$. There are ranked alphabets $\Sigma^1, \ldots, \Sigma^n$, leaf alphabets $X_1, \ldots, X_n$ and languages $T_1 \in \mathscr{V}(\Sigma^1, X_1), \ldots, T_n \in \mathscr{V}(\Sigma^n, X_n)$, for some $n > 0$, such that $\mathrm{SOA}(T) \preceq_g \kappa(\mathrm{SOA}(T_1), \ldots, \mathrm{SOA}(T_n))$ for a ranked alphabet Γ and an assignment $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$. Let $\mathcal{A}_j = \mathrm{SOA}(T_j)$ for $j = 1, \ldots, n$. Then T is g-recognized by $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, and which means that there are an order g-morphism $(\lambda, \varphi) : \mathcal{T}(\Sigma, X) \to \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and an ideal $I \trianglelefteq \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ such that $T = I\varphi^{-1}$. Let $\varphi_j : \mathrm{T}(\Sigma, X) \to A_j$ be the composition of φ with the j-th projection function $\prod_i A_i \to A_j$, and let $\lambda_j : \Sigma \to \Sigma^j$ be the composition of $\lambda\kappa : \Sigma \to \Sigma^1 \times \cdots \times \Sigma^n$ with the j-th projection function $\Sigma^1 \times \cdots \times \Sigma^n \to \Sigma^j$. Then $(\lambda_j, \varphi_j) : \mathcal{T}(\Sigma, X) \to \mathcal{A}_j$ is an order g-morphism, and similarly to the proof of Lemma 3.2.10,

$$T = I\varphi^{-1} = \bigcup_{\mathbf{a} \in I} (\mathbf{a}]\varphi^{-1} = \bigcup_{(a_1, \ldots, a_n) \in I} \bigcap_{j \leq n} (a_j]\varphi_j^{-1}.$$

For showing $T \in \mathscr{V}(\Sigma, X)$ it suffices to show that $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ for every $j \leq n$. Fix a $j \leq n$. Let $\psi_j = \varphi^{T_j} : \mathcal{T}(\Sigma^j, X_j) \to \mathcal{A}_j$ be the syntactic morphism of T_j. One can construct a g-morphism $(\lambda_j, \chi_j) : \mathcal{T}(\Sigma, X) \to \mathcal{T}(\Sigma^j, X_j)$ such that $\chi_j \psi_j = \varphi_j$. Then $(a_j]\varphi_j^{-1} = (a_j]\psi_j^{-1}\chi_j^{-1}$, and since $\mathscr{V}$ is closed under inverse g-morphisms, for showing $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ it is enough to show $(a_j]\psi_j^{-1} \in \mathscr{V}(\Sigma^j, X_j)$. It was shown in the proof of Lemma 3.2.10 that $(a_j]\psi_j^{-1} = \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma^j, X_j), P(t) \in T_j\}$ for some $t \in \mathrm{T}(\Sigma^j, X_j)$. Hence, from $T_j \in \mathscr{V}(\Sigma^j, X_j)$ and the fact that $\mathscr{V}$ is closed under inverse translations and positive Boolean operations, it follows that $(a_j]\psi_j^{-1} \in \mathscr{V}(\Sigma^j, X_j)$. Therefore, $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ for all j, thus $T \in \mathscr{V}(\Sigma, X)$. □

The examples of families of recognizable tree languages and classes of finite ordered algebras in the previous sections do not heavily depend on their ranked alphabets. Here we will see that the collection of those varieties for various ranked alphabets form generalized varieties.

Let $\mathbf{Nil} = \{\mathbf{Nil}(\Sigma)\}$ and $\mathbf{coNil} = \{\mathbf{coNil}(\Sigma)\}$ be the class of all nilpotent ordered algebras and conilpotent ordered algebras, where Σ ranges over all ranked alphabets, and $\mathrm{Cof} = \{\mathrm{Cof}(\Sigma, X)\}$ and $\mathrm{Fin} = \{\mathrm{Fin}(\Sigma, X)\}$ be the family of all cofinite and finite tree languages for all ranked alphabets Σ and leaf alphabets X.

Proposition 3.3.6 *Classes* **Nil** *and* **coNil** *are gVFOA, families* Cof *and* Fin *are gPVTL, and* $\mathrm{Cof} = \mathbf{Nil}^t$, $\mathrm{Fin} = \mathbf{coNil}^t$.

Proof. That Cof and Fin are gPVTL can be verified directly: the families are closed under the positive Boolean operations, inverse translations and inverse g-morphisms. Similarly, classes **Nil** and **coNil** can be proved to be gVFOA. Moreover, from Proposition 3.2.13 it follows that $\mathrm{Cof} = \mathbf{Nil}^t$ and $\mathrm{Fin} = \mathbf{coNil}^t$. □

It can be shown that an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is conilpotent iff the ordered algebra $\mathcal{A}^d = (A, \Sigma, \leqslant^{-1})$ is nilpotent. Here we will see that it is not an accident that finite tree languages are characterizable by ordered conilpotent algebras, and cofinite tree languages are characterizable by ordered nilpotent algebras.

Definition 3.3.7 The dual of an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is the structure $\mathcal{A}^d = (A, \Sigma, \leqslant^{-1})$.

It is easy to see that if $\mathcal{A}$ is an ordered algebra then the structure $\mathcal{A}^d$ is an ordered algebra as well.

Lemma 3.3.8 *For any ordered* Σ*-algebras* $\mathcal{A}$ *and* $\mathcal{B}$*,*

(1) $(\mathcal{A}^d)^d = \mathcal{A}$;
(2) $\mathcal{A} \subseteq \mathcal{B} \iff \mathcal{A}^d \subseteq \mathcal{B}^d$;
(3) $\mathcal{A} \leftarrow \mathcal{B} \iff \mathcal{A}^d \leftarrow \mathcal{B}^d$;
(4) $(\mathcal{A} \times \mathcal{B})^d \cong \mathcal{A}^d \times \mathcal{B}^d$. □

The proof is straightforward. So is the proof of the following lemma. Denote the complement $\mathrm{T}(\Sigma, X) \setminus T$ of a subset $T \subseteq \mathrm{T}(\Sigma, X)$ by T^d.

Lemma 3.3.9 *For any ordered algebra* $\mathcal{A} = (A, \Sigma, \leqslant)$ *and any tree language* $T \subseteq \mathrm{T}(\Sigma, X)$*,*

(1) *if* $I \subseteq A$*, then* $I \trianglelefteq \mathcal{A} \iff A \setminus I \trianglelefteq \mathcal{A}^d$;
(2) T *is recognized by* $\mathcal{A} \iff T^d$ *is recognized by* $\mathcal{A}^d$;
(3) $\mathrm{SOA}(T^d) \cong \mathrm{SOA}(T)^d$. □

For a VFOA $\mathbf{K}$, let $\mathbf{K}^d = \{\mathcal{A}^d \mid \mathcal{A} \in \mathbf{K}\}$, and for a PVTL $\mathscr{V}$ define the family $\mathscr{V}^d = \{\mathscr{V}^d(X)\}$ by $\mathscr{V}^d(X) = \{T^d \mid T \in \mathscr{V}(X)\}$. We note that $S^d \cap T^d = (S \cup T)^d$, $S^d \cup T^d = (S \cap T)^d$, $P^{-1}(T^d) = P^{-1}(T)^d$ and $T^d\varphi^{-1} = (T\varphi^{-1})^d$ for any tree languages $S, T \subseteq \mathrm{T}(\Sigma, X)$, any context $P \in \mathrm{C}(\Sigma, X)$ and any morphism $\varphi : \mathcal{T}(\Sigma, Y) \rightarrow \mathcal{T}(\Sigma, X)$. The following proposition immediately follows from the previous lemmas.

Proposition 3.3.10 *For any VFOA* $\mathbf{K}$ *and PVTL* $\mathscr{V}$*, the class* $\mathbf{K}^d$ *is a VFOA and the family* $\mathscr{V}^d$ *is a PVTL.*
Moreover, $(\mathbf{K}^d)^t = (\mathbf{K}^t)^d$ *and* $(\mathscr{V}^d)^a = (\mathscr{V}^a)^d$*.* □

Informally speaking, the proposition states that the operations of inversion and complementation generate new VFOA's and PVTL's respectively, and are compatible with each other. This can also be done and verified for generalized varieties (gVFOA's and gPVTL's) in a very similar way. For example, we have $\mathrm{Fin}^d = \mathrm{Cof}$ and $\mathbf{coNil}^d = \mathbf{Nil}$.

Chapter 4

Definability by monoids

Syntactic monoids of tree languages were introduced by Thomas [58] as a useful structure for studying recognizable tree languages. As an example, aperiodic tree languages were characterized by aperiodic monoids in [58]. Syntactic monoids were further studied by K. Salomaa [48], and a different formalism based on essentially the same concept was studied by Nivat and Podelski [32]. As tree languages with different ranked alphabets may have the same syntactic monoid, one immediately gets the impression that syntactic monoids are weaker structures than syntactic algebras. Here this impression will be explicitly confirmed in the context of variety theory.

In this chapter we prove a Variety Theorem that establishes a bijective correspondence between general varieties of tree languages definable by syntactic monoids and varieties of finite monoids. This solves a relatively long-standing open problem, the most recent references to which are made by Ésik [19] as "No variety theorem is known in the semigroup [monoid] approach" (page 759), and by Steinby [54] as "there are no general criteria for deciding whether or not a given gVTL [general variety of tree languages] can or cannot be defined by syntactic monoids" (page 41). The question was also implicitly mentioned in the last section of Wilke's paper [60]. It was already known that any family of tree languages definable by syntactic monoids is a (generalized) variety of tree languages (see e.g. [54]), though not every variety of tree languages is definable by syntactic monoids; one example is the family of reverse definite tree languages (see [60]).

To establish a correspondence between varieties of tree languages and varieties of finite monoids, we add three more closure properties to the definition of a generalized tree language variety [54].

In Section 4.1 we characterize the classes of algebras which can be defined by translation monoids, and in Section 4.2 a characterization for families of recognizable tree languages definable by syntactic monoids is given. The semigroup variant of the theory is dealt with in Section 4.3.

Here we fix some notation used throughout the chapter. A variety of finite monoids, abbreviated by VFM, is a class of finite monoids closed under submonoids, homomorphic images, and finite direct products. For contexts $P, Q \in \mathrm{C}(\Sigma, X)$ and a tree $t \in \mathrm{T}(\Sigma, X)$, the context $Q \cdot P$, the composite of P and Q, results from P by replacing the special leaf ξ with Q, and the tree $t \cdot P$ results from P by replacing ξ with t. Let us recall that $\mathrm{C}(\Sigma, X)$ is a monoid with respect to the composition operation, and ξ is the unit element. Moreover $t \cdot (Q \cdot P) = (t \cdot Q) \cdot P$ holds for all $P, Q \in \mathrm{C}(\Sigma, X)$, $t \in \mathrm{T}(\Sigma, X)$. Let $\mathcal{A} = (A, \Sigma)$ be an algebra. Every elementary context

$$P = f(a_1, \ldots, \xi, \ldots, a_m) \in \mathrm{C}(\Sigma, A),$$

where $f \in \Sigma_m$ and $a_1, \ldots, a_m \in A$, induces an elementary translation on $\mathcal{A}$ defined by $P^{\mathcal{A}}(a) = f^{\mathcal{A}}(a_1, \ldots, a, \ldots, a_m)$ for each $a \in A$. The functions induced by compositions of such elementary contexts are defined by setting $(Q \cdot P)^{\mathcal{A}}(a) = P^{\mathcal{A}}(Q^{\mathcal{A}}(a))$ for any two contexts P and Q and any $a \in A$. Note that two different contexts may induce the same translation. Recall that the set $\mathrm{Tr}(\mathcal{A})$ of translations of $\mathcal{A}$ is a monoid with composition as the operation, called the *translation monoid* of $\mathcal{A}$, which is also denoted by $\mathrm{Tr}(\mathcal{A})$. We note that $\mathrm{Tr}(\mathcal{A})$ includes the identity translation $\xi^{\mathcal{A}} = 1_A$. The composition of translations p and q is denoted by $q \cdot p$, that is $(q \cdot p)(a) = p(q(a))$ for all $a \in A$ (cf. Section 5 of [54]).

We recall from the Preliminaries that for a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the *syntactic congruence* $\approx^T$ of T is defined by

$$t \approx^T s \iff \forall P \in \mathrm{C}(\Sigma, X)\big(t \cdot P \in T \leftrightarrow s \cdot P \in T\big) \quad (t, s \in \mathrm{T}(\Sigma, X)).$$

The *syntactic algebra* $\mathrm{SA}(T)$ of T is the quotient Σ-algebra $\mathcal{T}(\Sigma, X)/{\approx^T}$ (see Definition 5.9 of [54]). The *syntactic monoid congruence* $\sim^T$ of T on the monoid $\mathrm{C}(\Sigma, X)$ is defined by

$$P \sim^T Q \iff \forall R \in \mathrm{C}(\Sigma, X) \forall t \in \mathrm{T}(\Sigma, X)\big(t \cdot P \cdot R \in T \leftrightarrow t \cdot Q \cdot R \in T\big)$$

for $P, Q \in \mathrm{C}(\Sigma, X)$, and the *syntactic monoid* $\mathrm{SM}(T)$ of T is the quotient monoid $\mathrm{C}(\Sigma, X)/{\sim^T}$ (cf. [58] or Definition 10.1 of [54]).

Remark 4.0.11 It was shown in [48] that the translation monoid of the syntactic algebra of a tree language is isomorphic to the syntactic monoid of the tree language, i.e., $\mathrm{Tr}(\mathrm{SA}(T)) \cong \mathrm{SM}(T)$ for every tree language T. See also Proposition 5.2.2 in Chapter 5 below.

Let Σ and Ω be ranked alphabets, and X and Y be leaf alphabets. A *tree homomorphism* is a mapping $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ determined by some mappings $\varphi_X : X \to \mathrm{T}(\Omega, Y)$, and $\varphi_m : \Sigma_m \to \mathrm{T}(\Omega, Y \cup \{\xi_1, \ldots, \xi_m\})$ for all such $m \geq 0$ that $\Sigma_m \neq \emptyset$, and the ξ_i's are new variables, inductively as follows:

(1) $x\varphi = \varphi_X(x)$ for $x \in X$, $c\varphi = \varphi_0(c)$ for $c \in \Sigma_0$, and

(2) $f(t_1, \ldots, t_n)\varphi = \varphi_n(f)[\xi_1 \leftarrow t_1\varphi, \ldots, \xi_n \leftarrow t_n\varphi]$ for each $f \in \Sigma_n$ $(n \geq 1)$ and any $t_1, \ldots, t_n \in \mathrm{T}(\Sigma, X)$, where each ξ_i is replaced with $t_i\varphi$ (for any $i = 1, \ldots, m$); cf. [54], page 7.

A tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is called *regular* if for every $f \in \Sigma_m$ $(m \geq 1)$, each $\xi_1, \ldots, \xi_m$ appears exactly once in $\varphi_m(f)$.

The unique extension $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ of a regular tree homomorphism φ to contexts is obtained by setting $\varphi_*(\xi) = \xi$ (cf. [54], Proposition 10.3).[1] We note that the identities $(Q \cdot P)\varphi_* = Q\varphi_* \cdot P\varphi_*$ and $(t \cdot Q \cdot P)\varphi = t\varphi \cdot Q\varphi_* \cdot P\varphi_*$ hold for all $P, Q \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$.

4.1 Algebras definable by translation monoids

The notions of *subalgebra*, *homomorphism*, and *direct product* are defined as usual in universal algebra, whereas for their generalizations, *g-subalgebra*, *g-homomorphism*, and *generalized product*, are defined for algebras which are not necessarily of the same type. We recall the following definitions from [54], Definitions 3.1, 3.2, 3.3, 3.14.

Definition 4.1.1 Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Omega)$ be finite algebras.

(i) The algebra $\mathcal{B}$ is a *g-subalgebra* of $\mathcal{A}$, if $B \subseteq A$, $\Omega_m \subseteq \Sigma_m$ for all $m \geq 0$, and for every $g \in \Omega_m$, $g^{\mathcal{B}}$ is the restriction of $g^{\mathcal{A}}$ to B. When $\mathcal{B}$ is a g-subalgebra of $\mathcal{A}$ we write $\mathcal{B} \subseteq_g \mathcal{A}$.

(ii) An *assignment* is a mapping $\kappa : \Sigma \to \Omega$ such that $\kappa(\Sigma_m) \subseteq \Omega_m$ holds for all $m \geq 0$.

A *g-morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair (κ, φ), where $\kappa : \Sigma \to \Omega$ is an assignment and $\varphi : A \to B$ is a mapping satisfying $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = (f\kappa)^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for any $m \geq 0$, $f \in \Sigma_m$, and $a_1, \ldots, a_m \in A$. If both κ and φ are surjective, then (κ, φ) is called a *g-epimorphism*, and in that case we write $\mathcal{B} \leftarrow_g \mathcal{A}$ ($\mathcal{B}$ is a g-epimorphic image of $\mathcal{A}$). When $\mathcal{B}$ is a g-epimorphic image of a g-subalgebra of $\mathcal{A}$, we write $\mathcal{B} \preceq_g \mathcal{A}$. When both κ and φ are bijective, (κ, φ) is called a *g-isomorphism*, and $\mathcal{B} \cong_g \mathcal{A}$ means that $\mathcal{B}$ and $\mathcal{A}$ are g-isomorphic.

(iii) Let $\Sigma^1, \ldots, \Sigma^n, \Gamma$ be ranked alphabets. The product $\Sigma^1 \times \cdots \times \Sigma^n$ is a ranked alphabet such that $(\Sigma^1 \times \cdots \times \Sigma^n)_m = \Sigma^1_m \times \cdots \times \Sigma^n_m$ for every $m \geq 0$. For any assignment $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$, and any algebras $\mathcal{A}_1 = (A_1, \Sigma^1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n)$, the *$\kappa$-product* of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is the algebra $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (A_1 \times \cdots \times A_n, \Gamma)$ defined by
(1) $c^{\kappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)} = (c_1^{\mathcal{A}_1}, \ldots, c_n^{\mathcal{A}_n})$ for $c \in \Gamma_0$, where $c\kappa = (c_1, \ldots, c_n)$,
(2) $f^{\kappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = (f_1^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, f_n^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn}))$

for $f \in \Gamma_m$ $(m > 0)$ and $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in A_1 \times \cdots \times A_n$, where $f\kappa = (f_1, \ldots, f_n)$.
Without specifying the assignment κ, such algebras are called *g-products*.

[1] Indeed any tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ can be extended to a mapping $\widetilde{\varphi} : \mathrm{C}(\Sigma, X) \to \mathrm{T}(\Omega, Y \cup \{\xi\})$ by setting $\xi\widetilde{\varphi} = \xi$, but if φ is not regular, the range of $\widetilde{\varphi}$ may not be $\mathrm{C}(\Omega, Y)$. Hence the regularity of φ is needed for the existence of the extension φ_*, see also Example 4.2.7.

In the notations $\subseteq_g, \leftarrow_g, \preceq_g$, and $\cong_g$, the subscript g is dropped when $\mathcal{A}$ and $\mathcal{B}$ are over the same ranked alphabet Σ, and the assignment $\kappa : \Sigma \to \Sigma$ is the identity map.

The abbreviation gVFA stands for *general variety of finite algebras* which is a class of finite algebras, of all finite types, closed under g-sub-algebras, g-epimorphic images, and g-products (Definition 4.3 of [54]).

It is easy to see that a class $\mathbf{K}$ is a gVFA, if for any $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathbf{K}$, any g-product $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, and any algebra $\mathcal{A}$, if $\mathcal{A} \preceq_g \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ then $\mathcal{A} \in \mathbf{K}$ (cf. Corollary 4.8 of [54]).

Definition 4.1.2 For a VFM $\mathbf{M}$, $\mathbf{M}^a$ is the class of all finite algebras whose translation monoid is in $\mathbf{M}$, i.e., $\mathcal{A} \in \mathbf{M}^a \Leftrightarrow \mathrm{Tr}(\mathcal{A}) \in \mathbf{M}$ for any $\mathcal{A}$.

A class of finite algebras $\mathbf{K}$ is said to be *definable by translation monoids*, if there is a VFM $\mathbf{M}$ such that $\mathbf{M}^a = \mathbf{K}$.

By Proposition 10.8 of [54], a class of finite algebras definable by translation monoids is a gVFA. In fact, any such class can be proved to be a *d-variety of finite algebras* (see page 758 of [19]). An algebraic characterization of the classes of finite algebras definable by translation monoids is given in Proposition 4.1.7 below.

Definition 4.1.3 Let $\mathcal{A}$ be a finite algebra. With each translation p in $\mathrm{Tr}(\mathcal{A})$ we associate a unary function symbol $\overline{p}$. Let $\Lambda_{\mathcal{A}} = \{\overline{p} \mid p \in \mathrm{Tr}(\mathcal{A})\}$ be the unary ranked alphabet formed by these symbols and let the $\Lambda_{\mathcal{A}}$-algebra $\mathcal{A}^{\varrho} = (\mathrm{Tr}(\mathcal{A}), \Lambda_{\mathcal{A}})$ be defined by $\overline{p}^{\mathcal{A}^{\varrho}}(q) = q \cdot p$ for all $p, q \in \mathrm{Tr}(\mathcal{A})$.

The following lemmas are needed for characterizing the definability by translation monoids (cf. [35, 36] for similar results for unary algebras).

Lemma 4.1.4 *For any finite algebra $\mathcal{A}$,* $\mathrm{Tr}(\mathcal{A}) \cong \mathrm{Tr}(\mathcal{A}^{\varrho})$.

Proof. The elementary translations of $\mathcal{A}^{\varrho}$ are of the form $\overline{p}^{\mathcal{A}^{\varrho}}(\xi)$ where $p \in \mathrm{Tr}(\mathcal{A})$, and clearly $\overline{q}^{\mathcal{A}^{\varrho}}(\xi) \cdot \overline{p}^{\mathcal{A}^{\varrho}}(\xi) = \overline{q \cdot p}^{\mathcal{A}^{\varrho}}(\xi)$ for all $q, p \in \mathrm{Tr}(\mathcal{A})$. For the identity translation 1_A of $\mathcal{A}$ the translation $\overline{1_A}^{\mathcal{A}^{\varrho}}(\xi)$ is the identity translation of $\mathcal{A}^{\varrho}$. This means that $\mathrm{Tr}(\mathcal{A}^{\varrho}) = \{\overline{p}^{\mathcal{A}^{\varrho}}(\xi) \mid p \in \mathrm{Tr}(\mathcal{A})\}$. Moreover, $\overline{p}^{\mathcal{A}^{\varrho}}(\xi) \neq \overline{q}^{\mathcal{A}^{\varrho}}(\xi)$ whenever $p \neq q$, since $\overline{p}^{\mathcal{A}^{\varrho}}(\xi) = \overline{q}^{\mathcal{A}^{\varrho}}(\xi)$ implies $p = 1_A \cdot p = \overline{p}^{\mathcal{A}^{\varrho}}(1_A) = \overline{q}^{\mathcal{A}^{\varrho}}(1_A) = 1_A \cdot q = q$. Hence, the mapping $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{A}^{\varrho})$, $p \mapsto \overline{p}^{\mathcal{A}^{\varrho}}(\xi)$ is a monoid isomorphism. □

Lemma 4.1.5 *Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Omega)$ be two finite algebras.*

(1) *If* $\mathrm{Tr}(\mathcal{A}) \preceq \mathrm{Tr}(\mathcal{B})$*, then $\mathcal{A}^{\varrho} \preceq_g \mathcal{B}^{\varrho}$.*
(2) $\mathrm{Tr}(\mathcal{A}) \times \mathrm{Tr}(\mathcal{B}) \cong \mathrm{Tr}(\kappa(\mathcal{A}^{\varrho}, \mathcal{B}^{\varrho}))$ *for some g-product $\kappa(\mathcal{A}^{\varrho}, \mathcal{B}^{\varrho})$.*

Proof. (1) Suppose $\mathrm{Tr}(\mathcal{A}) \leftarrow M \subseteq \mathrm{Tr}(\mathcal{B})$ for some monoid M. Let $\Lambda_M = \{\overline{p} \in \Lambda_{\mathcal{B}} \mid p \in M\}$. Then clearly $\mathcal{M} = (M, \Lambda_M) \subseteq_g \mathcal{B}^\varrho$, where $\mathcal{M}$ is defined by $\overline{p}^{\mathcal{M}}(q) = q \cdot p$ $(p, q \in M)$. Let $\varphi : M \to \mathrm{Tr}(\mathcal{A})$ be a monoid epimorphism. Define the assignment $\kappa : \Lambda_M \to \Lambda_{\mathcal{A}}$ by $\overline{q}\kappa = \overline{q\varphi}$ for all $q \in M$. It is clear that κ is surjective and for all $q, r \in M \subseteq \mathrm{Tr}(\mathcal{B})$, $\overline{q}^{\mathcal{B}^\varrho}(r)\varphi = (r \cdot q)\varphi = r\varphi \cdot q\varphi = \overline{q\varphi}^{\mathcal{A}^\varrho}(r\varphi) = (q\kappa)^{\mathcal{A}^\varrho}(r\varphi)$. Hence $(\kappa, \varphi) : \mathcal{M} \to \mathcal{A}^\varrho$ is a g-epimorphism. Thus $\mathcal{A}^\varrho \leftarrow_g \mathcal{M} \subseteq_g \mathcal{B}^\varrho$.
(2) Let $\Gamma = \{\overline{\langle p, q\rangle} \mid p \in \mathrm{Tr}(\mathcal{A}), q \in \mathrm{Tr}(\mathcal{B})\}$ be a set of unary function symbols, and define the assignment $\kappa : \Gamma \to \Lambda_{\mathcal{A}} \times \Lambda_{\mathcal{B}}$ by $\overline{\langle p, q\rangle}\kappa = (\overline{p}, \overline{q})$. Let $\mathcal{P} = \kappa(\mathcal{A}^\varrho, \mathcal{B}^\varrho)$ be the corresponding g-product of $\mathcal{A}^\varrho$ and $\mathcal{B}^\varrho$. We show that $\mathrm{Tr}(\mathcal{P}) = \{\overline{\langle p, q\rangle}^{\mathcal{P}}(\xi) \mid p \in \mathrm{Tr}(\mathcal{A}), q \in \mathrm{Tr}(\mathcal{B})\}$. Firstly, we note that if 1_A and 1_B are the identity translations of $\mathcal{A}$ and $\mathcal{B}$ respectively, then $\overline{\langle 1_A, 1_B\rangle}^{\mathcal{P}}(\xi)$ is the identity translation of $\mathcal{P}$. Secondly, by the definition of κ-products, for all $p, p' \in \mathrm{Tr}(\mathcal{A})$, $q, q' \in \mathrm{Tr}(\mathcal{B})$, $\overline{\langle p, q\rangle}^{\mathcal{P}}(p', q') = (\overline{p}^{\mathcal{A}^\varrho}(p'), \overline{q}^{\mathcal{B}^\varrho}(q')) = (p' \cdot p, q' \cdot q)$. So, if $\overline{\langle p, q\rangle}^{\mathcal{P}}(\xi) = \overline{\langle p', q'\rangle}^{\mathcal{P}}(\xi)$ then $(p, q) = (1_A \cdot p, 1_B \cdot q) = \overline{\langle p, q\rangle}^{\mathcal{P}}(1_A, 1_B) = \overline{\langle p', q'\rangle}^{\mathcal{P}}(1_A, 1_B) = (1_A \cdot p', 1_B \cdot q') = (p', q')$. So, $\overline{\langle p, q\rangle}^{\mathcal{P}}(\xi) \neq \overline{\langle p', q'\rangle}^{\mathcal{P}}(\xi)$, when $p \neq p'$ or $q \neq q'$. Finally, we show that the set $\{\overline{\langle p, q\rangle}^{\mathcal{P}}(\xi) \mid p \in \mathrm{Tr}(\mathcal{A}), q \in \mathrm{Tr}(\mathcal{B})\}$ is closed under the composition of translations. For all $p, p', p'' \in \mathrm{Tr}(\mathcal{A})$, $q, q', q'' \in \mathrm{Tr}(\mathcal{B})$,

$$\begin{aligned}
\overline{\langle p', q'\rangle}^{\mathcal{P}} \cdot \overline{\langle p, q\rangle}^{\mathcal{P}}(p'', q'') &= \overline{\langle p, q\rangle}^{\mathcal{P}}(p'' \cdot p', q'' \cdot q') \\
&= \big((p'' \cdot p') \cdot p, (q'' \cdot q') \cdot q\big) \\
&= \big(p'' \cdot (p' \cdot p), q'' \cdot (q' \cdot q)\big) \\
&= \overline{\langle p' \cdot p, q' \cdot q\rangle}^{\mathcal{P}}(p'', q'').
\end{aligned}$$

It follows that $\overline{\langle p', q'\rangle}^{\mathcal{P}}(\xi) \cdot \overline{\langle p, q\rangle}^{\mathcal{P}}(\xi) = \overline{\langle p' \cdot p, q' \cdot q\rangle}^{\mathcal{P}}(\xi)$. Thus, the mapping $\mathrm{Tr}(\mathcal{A}) \times \mathrm{Tr}(\mathcal{B}) \to \mathrm{Tr}(\mathcal{P}), (p, q) \mapsto \overline{\langle p, q\rangle}^{\mathcal{P}}(\xi)$, is a monoid isomorphism. □

Since g-products of g-products are g-isomorphic to a g-product of the original algebras (Lemma 4.2 of [54]), Lemma 4.1.5(2) can be generalized as follows.

Lemma 4.1.6 *For any algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ $(n > 0)$ there exists a g-product $\kappa(\mathcal{A}_1^\varrho, \ldots, \mathcal{A}_n^\varrho)$ such that* $\mathrm{Tr}(\mathcal{A}_1) \times \cdots \times \mathrm{Tr}(\mathcal{A}_n) \cong \mathrm{Tr}(\kappa(\mathcal{A}_1^\varrho, \ldots, \mathcal{A}_n^\varrho))$. □

Now we can present our characterization of the classes of finite algebras that are definable by translation monoids.

Proposition 4.1.7 *Any class of finite algebras* $\mathbf{K}$ *is definable by translation monoids iff it is a gVFA such that $\mathcal{A} \in \mathbf{K}$ iff $\mathcal{A}^\varrho \in \mathbf{K}$, for any $\mathcal{A}$.*

Proof. Suppose $\mathbf{K} = \mathbf{M}^a$ for a VFM $\mathbf{M}$. Then by Lemma 4.1.4 for any algebra $\mathcal{A}$, $\mathrm{Tr}(\mathcal{A}) \cong \mathrm{Tr}(\mathcal{A}^\varrho)$, so

$$\mathcal{A} \in \mathbf{K} \Leftrightarrow \mathrm{Tr}(\mathcal{A}) \in \mathbf{M} \Leftrightarrow \mathrm{Tr}(\mathcal{A}^\varrho) \in \mathbf{M} \Leftrightarrow \mathcal{A}^\varrho \in \mathbf{K}.$$

For the converse, suppose the gVFA $\mathbf{K}$ satisfies $\mathcal{A} \in \mathbf{K} \Leftrightarrow \mathcal{A}^\varrho \in \mathbf{K}$ for any finite algebra $\mathcal{A}$. Let $\mathbf{M}$ be the VFM generated by $\{\mathrm{Tr}(\mathcal{A}) \mid \mathcal{A} \in \mathbf{K}\}$. We show that $\mathbf{K} = \mathbf{M}^a$. Obviously $\mathbf{K} \subseteq \mathbf{M}^a$. For the opposite inclusion, let $\mathcal{B} \in \mathbf{M}^a$. So, there are $\mathcal{A}_1, \ldots, \mathcal{A}_m \in \mathbf{K}$ such that $\mathrm{Tr}(\mathcal{B})$ divides the product $\mathrm{Tr}(\mathcal{A}_1) \times \cdots \times \mathrm{Tr}(\mathcal{A}_m)$. By Lemma 4.1.6, $\mathrm{Tr}(\mathcal{B}) \preceq \mathrm{Tr}(\mathcal{P})$, and hence $\mathcal{B}^\varrho \preceq_g \mathcal{P}^\varrho$ by Lemma 4.1.5 (1), for some g-product $\mathcal{P}$ of $\mathcal{A}_1^\varrho, \ldots, \mathcal{A}_m^\varrho$. We have $\mathcal{A}_1^\varrho, \ldots, \mathcal{A}_m^\varrho \in \mathbf{K}$, and so $\mathcal{P} \in \mathbf{K}$, hence $\mathcal{P}^\varrho \in \mathbf{K}$. Since $\mathcal{P}^\varrho \in \mathbf{K}$, also $\mathcal{B}^\varrho \in \mathbf{K}$, which implies that $\mathcal{B} \in \mathbf{K}$. Thus $\mathbf{M}^a \subseteq \mathbf{K}$. □

Remark 4.1.8 The proof of Proposition 4.1.7 also yields the fact that for any gVFA $\mathbf{K}$ definable by translation monoids, the class $\{\mathrm{Tr}(\mathcal{A}) \mid \mathcal{A} \in \mathbf{K}\}$ is a variety of finite monoids.

Another characterization of the classes of finite algebras definable by translation monoids which follows from Lemmas 4.1.4 and 4.1.5 is the following.

Proposition 4.1.9 *Any class of finite algebras* $\mathbf{K}$ *is definable by translation monoids iff it is a gVFA such that for all finite algebras* $\mathcal{A}$ *and* $\mathcal{B}$*, if* $\mathcal{A} \in \mathbf{K}$ *and* $\mathrm{Tr}(\mathcal{A}) \cong \mathrm{Tr}(\mathcal{B})$ *then* $\mathcal{B} \in \mathbf{K}$. □

4.2 Tree languages definable by monoids

A *general variety of tree languages* (abbreviated by gVTL) is a family of recognizable tree languages closed under all Boolean operations, inverse translations, and inverse g-morphisms; see Definition 7.1 of [54], also Definition 3.3.1 in Chapter 3.

Definition 4.2.1 For a VFM $\mathbf{M}$, let $\mathbf{M}^t$ be the family of all recognizable tree languages whose syntactic monoids are in $\mathbf{M}$, that is to say for any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, $T \in \mathbf{M}^t(\Sigma, X) \Leftrightarrow \mathrm{SM}(T) \in \mathbf{M}$ holds.

A family of recognizable tree languages $\mathscr{V}$ is said to be *definable by syntactic monoids* if there is a VFM $\mathbf{M}$ such that $\mathbf{M}^t = \mathscr{V}$.

Steinby has shown that for any VFM $\mathbf{M}$, $\mathbf{M}^t$ is a gVTL ([54], Proposition 10.3). His proof can be applied to show that $\mathbf{M}^t$ is also closed under inverse images of regular tree homomorphisms. The general varieties of tree languages closed under inverse (arbitrary) tree homomorphisms are studied by Ésik [19] who characterized them by their *syntactic theories*. Theorem 14.2 of [19] establishes a correspondence between so-called d-varieties ([19], page 758) of finite algebras and general tree language varieties closed under inverse tree homomorphisms. However, those varieties may not be definable by syntactic monoids.

Example 4.2.2 Let $\mathrm{Def}_1 = \{\mathrm{Def}_1(\Sigma, X)\}$ be the family of 1-definite tree languages, i.e., $T \in \mathrm{Def}_1(\Sigma, X)$ iff for all ΣX-trees t and s, $\mathrm{root}(t) = \mathrm{root}(s)$ and $t \in T$ imply $s \in T$, where $\mathrm{root}(t)$ is the root symbol of t. It is a gVTL ([54]) closed under inverse strict tree homomorphisms (see Subsection 11.1 of [19], and Section 4.3 below). Let $\Sigma = \Sigma_2 = \{f, g\}$, $X = \{x, y\}$, and $T = \{x\} \cup \{f(t_1, t_2) \mid t_1, t_2 \in \mathrm{T}(\Sigma, X)\}$. Clearly $T \in \mathrm{Def}_1(\Sigma, X)$. It can easily be shown that the syntactic monoid of T consists of an identity element and two right zeros. This is also the syntactic monoid of the language T' of the ΣX-trees whose leftmost leaves are x, by Example 10.4 of [54]. Since $T' \notin \mathrm{Def}_1(\Sigma, X)$, then Def_1 is not definable by syntactic monoids.

This actually shows that the gVTL of all definite tree languages is not definable by syntactic monoids, since T' is not k-definite for any $k \geq 1$.

Remark 4.2.3 In [33] it is claimed that the variety of definite tree languages can be characterized by the property that all the non-identity idempotents of their syntactic monoids are right zeros (left zeros in the formalism of [33]). This clearly stands in conflict with the above Example 4.2.2.

Indeed, it can be shown that Theorem 1 of [33] does not hold. When the syntactic semigroup of a tree language is defined as the syntactic monoid with the identity element removed, the authors clearly overlook the possibility that the identity element may be obtained also as the product of some non-identity elements, and the proof of the theorem of [33] holds in just one direction. A concrete example showing that the equality between lines 9 and 10 on page 189 does not necessarily hold, can be obtained by considering the tree language T' of our Example 4.2.2.

It can also be noted that the class of all finite monoids whose non-identity idempotents are right zeros, is not a VFM[2]. Finally, in Example 4.3.5 below we shall see that a more appropriate definition of the syntactic semigroup and omitting trees that in a sense correspond to the empty word, does not save the result of [33].

We shall characterize the general varieties of tree languages that are definable by syntactic monoids by requiring them to satisfy two more conditions in addition to being closed under inverse regular tree homomorphisms.

[2]One can show that the class is not closed under direct products, for example let $M_1 = \{e, f\}$ consist of two right-zero idempotents, and let $M_2 = \{1, a\}$ be the monoid in which 1 is the identity element and $a \cdot a = 1$. Then $(e, 1) \in M_1 \times M_2$ is a non-identity idempotent but is not right-zero, since $(f, a) \cdot (e, 1) = (e, a) \neq (e, 1)$.

Definition 4.2.4 A regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is said to be *full with respect to* a tree language $T \subseteq \mathrm{T}(\Omega, Y)$, if for every context $Q \in \mathrm{C}(\Omega, Y)$ and every tree $s \in \mathrm{T}(\Omega, Y)$, there are $P \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$, such that $Q \sim^T P\varphi_*$ and $s \approx^T t\varphi$ hold.

Recall that for an equivalence relation θ on a set A, the quotient set of A under θ is denoted by A/θ, and $a/\theta = \{b \in A \mid a\ \theta\ b\}$ is the equivalence θ-class containing $a \in A$.

Remark 4.2.5 At first glance it seems that verifying the fullness of φ with respect to T requires checking the existence of $P \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$ for all (infinitely many) $Q \in \mathrm{C}(\Omega, Y)$ and $s \in \mathrm{T}(\Omega, Y)$ such that $Q \sim^T P\varphi_*$ and $s \approx^T t\varphi$ hold. In fact it is decidable for a recognizable T to check whether or not φ is full with respect to T: let $\varphi^T : \mathrm{T}(\Omega, Y) \to \mathrm{T}(\Omega, Y)/{\approx^T}$, $t\varphi^T = t/{\approx^T}$ and $\lambda^T : \mathrm{C}(\Omega, Y) \to \mathrm{C}(\Omega, Y)/{\sim^T}$, $P\lambda^T = P/{\sim^T}$ be the natural morphisms. Then the tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is full with respect to T iff both the mappings $\varphi\varphi^T : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)/{\approx^T}$ and $\varphi_*\lambda^T : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)/{\sim^T}$ are surjections.

Lemma 4.2.6 *If $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is a regular tree homomorphism and $T \subseteq \mathrm{T}(\Omega, Y)$, then $\mathrm{SM}(T\varphi^{-1}) \preceq \mathrm{SM}(T)$, and if φ is full with respect to T, then $\mathrm{SM}(T\varphi^{-1}) \cong \mathrm{SM}(T)$.*

Proof. We note that $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid homomorphism. Let $S = \mathrm{C}(\Sigma, X)\varphi_*$ be the range of φ_*, and let μ be the restriction of $\sim^T$ to S. Then S/μ is a submonoid of $\mathrm{C}(\Omega, Y)/{\sim^T}$. We show that $P\varphi_*\, \mu\, Q\varphi_*$ implies $P \sim^{T\varphi^{-1}} Q$ for all $P, Q \in \mathrm{C}(\Sigma, X)$.
Suppose $P\varphi_*\, \mu\, Q\varphi_*$ and take arbitrary $t \in \mathrm{T}(\Omega, Y)$ and $R \in \mathrm{C}(\Omega, Y)$. Then

$$\begin{aligned} t \cdot P \cdot R \in T\varphi^{-1} \quad &\Leftrightarrow \quad t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T \\ &\Leftrightarrow \quad t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T \\ &\Leftrightarrow \quad t \cdot Q \cdot R \in T\varphi^{-1}, \end{aligned}$$

that is $P \sim^{T\varphi^{-1}} Q$. So the mapping $\psi : S/\mu \to \mathrm{C}(\Sigma, X)/{\sim^{T\varphi^{-1}}}$ defined by $(P\varphi_*/\mu)\psi = P/{\sim^{T\varphi^{-1}}}$ is well-defined and surjective. It is also a monoid homomorphism, since $(P\varphi_*/\mu \cdot Q\varphi_*/\mu)\psi = ((P \cdot Q)\varphi_*/\mu)\psi = (P \cdot Q)/{\sim^{T\varphi^{-1}}} = P/{\sim^{T\varphi^{-1}}} \cdot Q/{\sim^{T\varphi^{-1}}} = (P\varphi_*/\mu)\psi \cdot (Q\varphi_*/\mu)\psi$ for all $P, Q \in \mathrm{C}(\Sigma, X)$. Hence $\mathrm{SM}(T\varphi^{-1}) \leftarrow S/\mu \subseteq \mathrm{SM}(T)$ thus $\mathrm{SM}(T\varphi^{-1}) \preceq \mathrm{SM}(T)$.

Now, suppose φ is full with respect to T. We show $P \sim^{T\varphi^{-1}} Q \iff P\varphi_* \sim^T Q\varphi_*$ for any $P, Q \in \mathrm{C}(\Sigma, X)$. Clearly, $P\varphi_* \sim^T Q\varphi_*$ implies $P \sim^{T\varphi^{-1}} Q$. For the converse, suppose $P \sim^{T\varphi^{-1}} Q$, and take arbitrary $R' \in \mathrm{C}(\Omega, Y)$, and $t' \in \mathrm{T}(\Omega, Y)$. There are $R \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$

such that $R\varphi_* \sim^T R'$ and $t\varphi \approx^T t'$. Hence

$$\begin{array}{rcl} t'\cdot P\varphi_*\cdot R'\in T & \Leftrightarrow & t\varphi\cdot P\varphi_*\cdot R\varphi_*\in T \\ & \Leftrightarrow & (t\cdot P\cdot R)\varphi\in T \\ & \Leftrightarrow & t\cdot P\cdot R\in T\varphi^{-1} \\ & \Leftrightarrow & t\cdot Q\cdot R\in T\varphi^{-1} \\ & \Leftrightarrow & t\varphi\cdot Q\varphi_*\cdot R\varphi_*\in T \\ & \Leftrightarrow & t'\cdot Q\varphi_*\cdot R'\in T, \end{array}$$

which shows that $P\varphi_* \sim^T Q\varphi_*$. Hence $P \sim^{T\varphi^{-1}} Q \iff P\varphi_* \sim^T Q\varphi_*$, and since the function $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid homomorphism, the mapping $\mathrm{C}(\Sigma,X)/{\sim^{T\varphi^{-1}}} \to \mathrm{C}(\Omega,Y)/{\sim^T}$, $P/{\sim^{T\varphi^{-1}}} \mapsto (P\varphi_*)/{\sim^T}$ is a monoid isomorphism between $\mathrm{SM}(T\varphi^{-1})$ and $\mathrm{SM}(T)$. □

In the following example we show that the regularity condition on φ in the previous lemma can not be relaxed.

Example 4.2.7 Define the ranked alphabets $\Omega = \Omega_2 = \{f\}$ and $\Sigma = \Sigma_1 = \{g, h\}$, and the leaf alphabet $X = \{u, v, w\}$. Let $(\mathbb{Z}_3, +)$ be the cyclic group of order 3. Define $\chi : \mathrm{T}(\Omega, X) \to \mathbb{Z}_3$ inductively by $u\chi = 0$, $v\chi = 1$, $w\chi = 2$, and $f(t, s)\chi = t\chi + s\chi$. Let $T = \{0\}\chi^{-1}$. It is easy to see that the syntactic monoid of T consists of the $\sim^T$-classes of the elementary contexts $f(u, \xi), f(v, \xi), f(w, \xi)$, and in fact $\mathrm{SM}(T) \simeq (\mathbb{Z}_3, +)$.

Define the tree homomorphisms $\varphi, \psi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, X)$ by $\varphi_X(x) = \psi_X(x) = x$ for $x \in X$, and $\varphi_1(g) = \psi_1(g) = f(v, \xi_1)$, $\varphi_1(h) = f(\xi_1, \xi_1)$, and $\psi_1(h) = u$. These tree homomorphisms are not regular: ξ_1 appears twice in $\varphi_1(h)$ and does not appear at all in $\psi_1(h)$.

We show that neither $\mathrm{SM}(T\varphi^{-1})$ nor $\mathrm{SM}(T\psi^{-1})$ can divide $\mathrm{SM}(T)$. The following identities can be verified by straightforward computations:

i. $(v \cdot h(\xi) \cdot g(\xi))\varphi\chi = 0$, $(v \cdot g(\xi) \cdot h(\xi))\varphi\chi = 1$, and

ii. $(v \cdot h(\xi) \cdot g(\xi))\psi\chi = 1$, $(v \cdot g(\xi) \cdot h(\xi))\psi\chi = 0$.

So, $(h(\xi)\cdot g(\xi), g(\xi)\cdot h(\xi)) \notin {\sim^{T\varphi^{-1}}}, {\sim^{T\psi^{-1}}}$ which proves that $\mathrm{SM}(T\varphi^{-1})$ and $\mathrm{SM}(T\psi^{-1})$ are not commutative.

Remark 4.2.8 Let **C** be the variety of finite commutative monoids. By Example 4.2.7, the gVTL $\mathbf{C}^t$ is not closed under inverse non-regular tree homomorphisms; cf. Proposition 4.2.14. So, $\mathbf{C}^t$ is not definable by syntactic theories [19]. On the other hand, by Example 4.2.2, the family of definite tree languages is not definable by syntactic monoids, even though it is definable by syntactic theories, cf. [19] Subsection 11.1.

Thus, "definability by syntactic theories" and "definability by syntactic monoids" are not comparable to each other, although both are weaker than "definability by syntactic algebras".

Lemma 4.2.9 *Let $\mathcal{A} = (A, \Sigma)$ be a finite algebra, and X be a leaf alphabet disjoint from A. For any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^{\varrho}$, there exist a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$, and a tree language $T \subseteq \mathrm{T}(\Sigma, X \cup A)$ such that $L = T\varphi^{-1}$, and T can be recognized by a finite power $\mathcal{A}^n$ where $n = |A|$.*

Proof. Let $\alpha : X \to \mathrm{Tr}(\mathcal{A})$ be an initial assignment for $\mathcal{A}^{\varrho}$ and $F \subseteq \mathrm{Tr}(\mathcal{A})$ be a subset such that $L = \{t \in \mathrm{T}(\Lambda_{\mathcal{A}}, X) \mid t\alpha^{\mathcal{A}^{\varrho}} \in F\}$. Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ by $\varphi_X(x) = x$ for all $x \in X$, and for every $p \in \mathrm{Tr}(\mathcal{A})$ choose a $\varphi_1(\overline{p}) \in \mathrm{C}(\Sigma, A)$ such that $\varphi_1(\overline{p})^{\mathcal{A}} = p$. Obviously φ is a regular tree homomorphism. Suppose that $A = \{a_1, \ldots, a_n\}$. Let $F' = \{(p(a_1), \ldots, p(a_n)) \in A^n \mid p \in F\}$, and define the initial assignment $\beta : X \cup A \to A^n$ for $\mathcal{A}^n$ by $x\beta = \big((x\alpha)(a_1), \ldots, (x\alpha)(a_n)\big)$ for all $x \in X$, and $a\beta = (a, \ldots, a) \in A^n$ for all $a \in A$. Let T be the subset of $\mathrm{T}(\Sigma, X \cup A)$ recognized by $(\mathcal{A}^n, \beta, F')$. We show that $L = T\varphi^{-1}$. Every tree w in $\mathrm{T}(\Lambda_{\mathcal{A}}, X)$ is of the form $w = \overline{p_1}\Big(\overline{p_2}\big(\ldots \overline{p_k}(x) \ldots\big)\Big)$ for some $p_1, \ldots, p_k \in \mathrm{Tr}(\mathcal{A})$ $(k \geq 0)$ and $x \in X$. For such a tree w, $w\alpha^{\mathcal{A}^{\varrho}} = x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1$, and

$$(w\varphi)\beta^{\mathcal{A}^n} = (x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a_1), \ldots, x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a_n)).$$

So,
$$\begin{aligned} w\varphi \in T &\Leftrightarrow (w\varphi)\beta^{\mathcal{A}^n} \in F' \\ &\Leftrightarrow \text{for some } p \in F,\ p(a) = x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a) \text{ for all } a \in A \\ &\Leftrightarrow x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1 \in F \\ &\Leftrightarrow w\alpha^{\mathcal{A}^{\varrho}} \in F \\ &\Leftrightarrow w \in L. \end{aligned}$$
□

Lemma 4.2.10 *Let $\mathcal{A} = (A, \Sigma)$ be a finite algebra and X be a leaf alphabet disjoint from $A \cup \Sigma$. For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$ recognized by $\mathcal{A}$ there exists a unary ranked alphabet Λ, and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ such that φ is full with respect to T, and for every $z \in X \cup \Sigma_0$, $T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^{\varrho}$.*

Proof. Let $\mathcal{B} = (B, \Sigma)$ be the syntactic algebra of T. Then $\mathcal{B} \preceq \mathcal{A}$. Suppose $T = \{t \in \mathrm{T}(\Sigma, X) \mid t\beta^{\mathcal{B}} \in F\}$, where $\beta : X \to B$ is an initial assignment and $F \subseteq B$. Since $\mathcal{B}$ is the minimal tree automaton recognizing T, it is generated by $\beta(X)$. The mapping $\beta : X \to B$ can uniquely be extended to a monoid homomorphism $\beta_c : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Sigma, B)$. Since B is generated by $\beta(X)$, the mapping $\mathrm{C}(\Sigma, X) \to \mathrm{Tr}(\mathcal{B})$, $Q \mapsto \beta_c(Q)^{\mathcal{B}}$ is surjective. Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ by $\varphi_X(x) = x$ for all $x \in X \cup \Sigma_0$, and for every $q \in \mathrm{Tr}(\mathcal{B})$ choose a $\varphi_1(\overline{q}) = Q \in \mathrm{C}(\Sigma, X)$ such that $\beta_c(Q)^{\mathcal{B}} = q$. Note that φ is a regular tree homomorphism. It remains to show that φ is full with respect to T and that for every $z \in X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{B}^{\varrho}$.

This will finish the proof since $\mathrm{Tr}(\mathcal{B}) \preceq \mathrm{Tr}(\mathcal{A})$ follows from $\mathcal{B} \preceq \mathcal{A}$ by Lemma 10.7 of [54], and so $\mathcal{B}^\varrho \preceq \mathcal{A}^\varrho$ by Lemma 4.1.5, which implies that also L_z can be recognized by $\mathcal{A}^\varrho$.

First, we show that φ is full with respect to T. If $Q \in \mathrm{C}(\Sigma, X)$, then $\overline{q}(\xi)\varphi_* \sim^T Q$ holds for $q = \beta_{\mathrm{c}}(Q)^{\mathcal{B}} \in \mathrm{Tr}(\mathcal{B})$. By induction on the height of t we show that for any $t \in \mathrm{T}(\Sigma, X)$ there is an $s \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$ such that $t \approx^T s\varphi$. If $t = x \in X \cup \Sigma_0$, then $s\varphi \approx^T t$ for $s = t$. If $t = t' \cdot P$ for some $P \in \mathrm{C}(\Sigma, X)$ and $t' \in \mathrm{T}(\Sigma, X)$ such that the height of t' is less than the height of t, then by the induction hypothesis there is an $s' \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$ such that $t' \approx^T s'\varphi$. Also, for some $p \in \mathrm{Tr}(\mathcal{B})$, $\overline{p}(\xi)\varphi_* \sim^T P$ holds. If $s = \overline{p}(s')$, then $s\varphi = s'\varphi \cdot \overline{p}(\xi)\varphi_* \approx^T t' \cdot P = t$.

Secondly, we show that L_z is recognized by $\mathcal{B}^\varrho$ for a fixed $z \in X \cup \Sigma_0$. Define the initial assignment $\alpha : \{z\} \to \mathrm{Tr}(\mathcal{B})$ for $\mathcal{B}^\varrho$ by $z\alpha = 1_B$, where 1_B is the identity translation of $\mathcal{B}$, and let $F_z = \{q \in \mathrm{Tr}(\mathcal{B}) \mid q(z\beta^{\mathcal{B}}) \in F\}$. We prove that $(\mathcal{B}^\varrho, \alpha, F_z)$ recognizes L_z. Every $w \in \mathrm{T}(\Lambda_{\mathcal{B}}, \{z\})$ is of the form $w = \overline{q_1}\Big(\overline{q_2}\big(\ldots\overline{q_h}(z)\ldots\big)\Big)$ for some $q_1, \ldots, q_h \in \mathrm{Tr}(\mathcal{B})$ $(h \geq 0)$. For such a tree w, $w\alpha^{\mathcal{B}^\varrho} = 1_B \cdot q_h \cdot \ldots \cdot q_2 \cdot q_1$, and $(w\varphi)\beta^{\mathcal{B}} = q_h \cdot \ldots \cdot q_2 \cdot q_1(z\beta^{\mathcal{B}})$. Thus

$$\begin{aligned} w \in L_z \ \Leftrightarrow\ w\varphi \in T \quad &\Leftrightarrow \quad (w\varphi)\beta^{\mathcal{B}} \in F \\ &\Leftrightarrow \quad q_h \cdot \ldots \cdot q_2 \cdot q_1(z\beta^{\mathcal{B}}) \in F \\ &\Leftrightarrow \quad q_h \cdot \ldots \cdot q_2 \cdot q_1 \in F_z \\ &\Leftrightarrow \quad w\alpha^{\mathcal{B}^\varrho} \in F_z. \end{aligned}$$

So, $L_z = \{w \in \mathrm{T}(\Lambda, \{z\}) \mid w\alpha^{\mathcal{B}^\varrho} \in F_z\}$. □

We end the section by proving a Variety Theorem for tree languages and syntactic monoids, and presenting some examples that justify the proposition (another interesting example is presented in [37]). Before presenting the main theorem we note two remarks.

Remark 4.2.11 Let Λ be a unary ranked alphabet. For every leaf alphabet X and every subset $Y \subseteq X$, $\mathrm{C}(\Lambda, Y) = \mathrm{C}(\Lambda, X)$ and the relation $\sim^T$ for a tree language $T \subseteq \mathrm{T}(\Lambda, Y)$ on $\mathrm{C}(\Lambda, Y)$ is the same relation $\sim^T$ on $\mathrm{C}(\Lambda, X)$ when T is viewed as a subset of $\mathrm{T}(\Lambda, X)$.

So, if a family of tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ is definable by syntactic monoids, then for every unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.

For a family of recognizable tree languages $\mathscr{V}$, let $\mathscr{V}^a$ be the gVFA generated by the class $\{\mathrm{SA}(T) \mid T \in \mathscr{V}(\Sigma, X), \text{ for some } \Sigma, X\}$; cf. Definition 3.3.4 in Chapter 3.

Remark 4.2.12 The Generalized Variety Theorem, [54] Proposition 9.15, includes the following statements (cf. Proposition 3.3.5, Chapter 3):

(1) For any gVTL $\mathscr{V}$, the class $\mathscr{V}^a$ satisfies the following equivalence for any tree language $T \subseteq \mathrm{T}(\Sigma, X)$: $T \in \mathscr{V}(\Sigma, X) \Leftrightarrow \mathrm{SA}(T) \in \mathscr{V}^a$.

(2) For any gVFA $\mathbf{K}$ there is a unique gVTL $\mathscr{V}$ such that $\mathscr{V}^a = \mathbf{K}$.

Remark 4.2.13 By Propositions 6.13 and 5.8(b) of [54] it follows that every finite algebra can be represented as a subdirect product of the syntactic algebras of some tree languages that are recognizable by the algebra (see also Lemma 3.2.7 in Chapter 3). This implies that for any gVTL $\mathscr{V}$ and any finite algebra $\mathcal{A}$, if every tree language recognizable by $\mathcal{A}$ belongs to $\mathscr{V}$, then $\mathcal{A} \in \mathscr{V}^a$.

Proposition 4.2.14 *A family of recognizable tree languages $\mathscr{V}$ is definable by syntactic monoids iff $\mathscr{V}$ is a gVTL that is closed under inverse regular tree homomorphisms and satisfies the following conditions:*
(1) *For every unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.*
(2) *For any regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \rightarrow \mathrm{T}(\Omega, Y)$ which is full with respect to a tree language $T \subseteq \mathrm{T}(\Omega, Y)$, if $T\varphi^{-1} \in \mathscr{V}(\Sigma, X)$ then $T \in \mathscr{V}(\Omega, Y)$.*

Proof. For any VFM $\mathbf{M}$, $\mathbf{M}^t$ satisfies the conditions of Proposition 4.2.14 by Lemma 4.2.6, Remark 4.2.11, and the facts mentioned at the beginning of the section. For the converse, suppose the gVTL $\mathscr{V}$ satisfies the conditions of the proposition. We may complete the proof by showing that $\mathscr{V}^a$ satisfies the condition of Proposition 4.1.7. Indeed, Proposition 4.1.7 implies then that there exists a VFM $\mathbf{M}$ such that $\mathscr{V}^a = \mathbf{M}^a$, and then

$$T \in \mathscr{V} \Leftrightarrow \mathrm{SA}(T) \in \mathscr{V}^a \Leftrightarrow \mathrm{Tr}(\mathrm{SA}(T)) \in \mathbf{M} \Leftrightarrow \mathrm{SM}(T) \in \mathbf{M}$$

holds for every tree language T by Remarks 4.2.12 and 4.0.11, which proves that $\mathscr{V} = \mathbf{M}^t$. So, all we have to show is that $\mathcal{A} \in \mathscr{V}^a \iff \mathcal{A}^\varrho \in \mathscr{V}^a$ for any finite algebra $\mathcal{A}$.

Let $\mathcal{A} = (A, \Sigma)$ be in $\mathscr{V}^a$. By Lemma 4.2.9, any subset $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^\varrho$ can be written as $L = T\varphi^{-1}$, where $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \rightarrow \mathrm{T}(\Sigma, X \cup A)$ is a regular tree homomorphism, and T is a tree language recognized by some power $\mathcal{A}^n$ of $\mathcal{A}$. Then $\mathcal{A}^n \in \mathscr{V}^a$ implies that $T \in \mathscr{V}(\Sigma, X \cup A)$, and hence $L = T\varphi^{-1} \in \mathscr{V}(\Lambda_{\mathcal{A}}, X)$. This holds for every tree language L recognizable by $\mathcal{A}^\varrho$, so $\mathcal{A}^\varrho \in \mathscr{V}^a$ by Remark 4.2.13.

Now, suppose $\mathcal{A}^\varrho \in \mathscr{V}^a$ for some algebra $\mathcal{A} = (A, \Sigma)$, and that $T \subseteq \mathrm{T}(\Sigma, X)$ is recognizable by $\mathcal{A}$. By Lemma 4.2.10, there is a unary ranked alphabet Λ and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \rightarrow \mathrm{T}(\Sigma, X)$ full with respect to T such that for every $z \in X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized by $\mathcal{A}^\varrho$ as a subset of $\mathrm{T}(\Lambda, \{z\})$. So, $L_z \in \mathscr{V}(\Lambda, \{z\})$, and thus $L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$. Hence $T\varphi^{-1} = \bigcup_{z \in X \cup \Sigma_0} L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$. Since φ is full with respect to T, then we get $T \in \mathscr{V}(\Sigma, X)$. This holds for every tree language T recognizable by $\mathcal{A}$, hence $\mathcal{A} \in \mathscr{V}^a$ by Remark 4.2.13. □

Example 4.2.15 It was shown in Example 4.2.2 that Def_1 is not definable by syntactic monoids. Here we show that it does not satisfy condition (2) of Proposition 4.2.14. Let Σ, X, T, T' be as in Example 4.2.2. Define the regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Sigma, X)$, by $\varphi_X(x) = x$, $\varphi_X(y) = y$, and $\varphi_2(f) = f(x, f(\xi_1, \xi_2))$, $\varphi_2(g) = g(y, g(\xi_1, \xi_2))$. Now φ is full with respect to T' since for any $t \in \mathrm{T}(\Sigma, X)$, if $t \in T'$ then $f(y, x)\varphi \approx^{T'} t$, and if $t \notin T'$ then $g(y, x)\varphi \approx^{T'} t$. Similarly, for $P \in \mathrm{C}(\Sigma, X)$, if the leftmost leaf of P is x then $f(y, \xi)\varphi_* \sim^{T'} P$, if the leftmost leaf of P is y then $g(y, \xi)\varphi_* \sim^{T'} P$, and if the leftmost leaf of P is ξ then $\xi\varphi_* \sim^{T'} P$. Clearly $T'\varphi^{-1} = T$, since for any $t \in \mathrm{T}(\Sigma, X)$, the leftmost leaf of $t\varphi$ is x iff either $t = x$ or the root of t is f. By Example 4.2.2, $T'\varphi^{-1} = T \in \mathrm{Def}_1$, but $T' \notin \mathrm{Def}_1$.

Example 4.2.16 The family of nilpotent tree languages $\mathrm{Nil} = \{\mathrm{Nil}(\Sigma, X)\}$, which consists of the finite and cofinite tree languages, is a gVFA (see [54], Example 7.5). Let $\Lambda = \Lambda_1 = \{f\}$ be a unary ranked alphabet and $X = \{x, y\}$ be a leaf alphabet. Then $T = \{f(y), f(f(y)), f(f(f(y))), \ldots\}$ belongs to $\mathrm{Nil}(\Lambda, \{y\})$ but $T \notin \mathrm{Nil}(\Lambda, X)$. Hence, Nil does not satisfy condition (1) of Proposition 4.2.14, so it is not definable by syntactic monoids.

Example 4.2.17 Let $\mathrm{Ap} = \{\mathrm{Ap}(\Sigma, X)\}$ be the family of aperiodic tree languages. It was shown to be a gVTL in Example 7.8 of [54]. It is also known that Ap is definable by the variety of aperiodic (syntactic) monoids, see [58]. The argument of Example 7.8 in [54] showing that Ap is closed under inverse g-morphisms can be applied to show that Ap is in fact closed under inverse regular tree homomorphisms. It is also straightforward to see that Ap satisfies condition (1) of Proposition 4.2.14. We show that it also satisfies condition (2). Suppose the mapping $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is a regular tree homomorphism full with respect to some $T \subseteq \mathrm{T}(\Omega, Y)$ such that $T\varphi^{-1} \in \mathrm{Ap}(\Sigma, X)$. There is an n such that for all $t \in \mathrm{T}(\Sigma, X)$ and all $P, Q \in \mathrm{C}(\Sigma, X)$, $t \cdot P^n \cdot Q \in T\varphi^{-1} \Leftrightarrow t \cdot P^{n+1} \cdot Q \in T\varphi^{-1}$. For any $s \in \mathrm{T}(\Omega, Y)$ and any $R, U \in \mathrm{C}(\Omega, Y)$, there are $t \in \mathrm{T}(\Sigma, X)$ and $P, Q \in \mathrm{C}(\Sigma, X)$ such that $t\varphi \approx^T s$, $P\varphi_* \sim^T R$, and $Q\varphi_* \sim^T U$. So,

$$s \cdot R^n \cdot U \in T \Leftrightarrow t\varphi \cdot P^n\varphi_* \cdot Q\varphi_* \in T \Leftrightarrow t \cdot P^n \cdot Q \in T\varphi^{-1} \Leftrightarrow$$
$$\Leftrightarrow t \cdot P^{n+1} \cdot Q \in T\varphi^{-1} \Leftrightarrow t\varphi \cdot P^{n+1}\varphi_* \cdot Q\varphi_* \in T \Leftrightarrow s \cdot R^{n+1} \cdot U \in T,$$

which shows that $T \in \mathrm{Ap}(\Omega, Y)$.

4.3 Definability by semigroups

In this section we show how to modify the above results as to yield characterizations of varieties of finite algebras definable by translation semigroups and of varieties of tree languages definable by syntactic semigroups. We also compare the definability by monoids and definability by semigroups with each other.

The translation semigroup of an algebra is defined to be the smallest set of unary functions on the algebra that contains the elementary translations and is closed under composition. The difference between the translation monoid and the translation semigroup of an algebra is that the latter does not automatically contain the identity translation, although it may be included as an elementary translation or as a composition of some elementary translations.

Denote the translation semigroup of an algebra $\mathcal{A} = (A, \Sigma)$ by $\mathrm{TrS}(\mathcal{A})$ and let $\Lambda_{\mathcal{A}}$ be as in Definition 4.1.3 except that $\mathrm{Tr}(\mathcal{A})$ is replaced with $\mathrm{TrS}(\mathcal{A})$. We associate with $\mathcal{A}$ a new symbol $\mathbf{I}_{\mathcal{A}}$ that does not appear in the set $A \cup \Sigma \cup \mathrm{TrS}(\mathcal{A})$. Define the $\Lambda_{\mathcal{A}}$-algebra $\mathcal{A}^{\varsigma} = (\mathrm{TrS}(\mathcal{A}) \cup \{\mathbf{I}_{\mathcal{A}}\}, \Lambda_{\mathcal{A}})$ by $\overline{p}^{\mathcal{A}^{\varsigma}}(q) = q \cdot p$ and $\overline{p}^{\mathcal{A}^{\varsigma}}(\mathbf{I}_{\mathcal{A}}) = p$ for all $p, q \in \mathrm{TrS}(\mathcal{A})$.

Lemma 4.3.1 *For any finite algebras* $\mathcal{A} = (A, \Sigma)$ *and* $\mathcal{B} = (B, \Omega)$*,*

(1) $\mathrm{TrS}(\mathcal{A}) \cong \mathrm{TrS}(\mathcal{A}^{\varsigma})$*;*

(2) *if* $\mathrm{TrS}(\mathcal{A}) \preceq \mathrm{TrS}(\mathcal{B})$*, then* $\mathcal{A}^{\varsigma} \preceq_g \mathcal{B}^{\varsigma}$*; and*

(3) $\mathrm{TrS}(\mathcal{A}) \times \mathrm{TrS}(\mathcal{B}) \cong \mathrm{Tr}(\kappa(\mathcal{A}^{\varsigma}, \mathcal{B}^{\varsigma}))$ *for some g-product* $\kappa(\mathcal{A}^{\varsigma}, \mathcal{B}^{\varsigma})$*.*

Moreover, for any $k \geq 1$ *and algebras* $\mathcal{A}_1, \ldots, \mathcal{A}_k$*, there is a g-product* $\mathcal{P}$ *of* $\mathcal{A}_1^{\varsigma}, \ldots, \mathcal{A}_k^{\varsigma}$ *such that* $\mathrm{TrS}(\mathcal{A}_1) \times \cdots \times \mathrm{TrS}(\mathcal{A}_k) \cong \mathrm{TrS}(\mathcal{P})$*.*

Proof. The statements (1) and (3) can be proved similarly as their counterparts in Lemmas 4.1.4, 4.1.5, and 4.1.6 just by replacing the identity translation 1_A (and 1_B) with $\mathbf{I}_{\mathcal{A}}$ (with $\mathbf{I}_{\mathcal{B}}$). We prove (2):

Suppose that a semigroup S satisfies $\mathrm{TrS}(\mathcal{A}) \leftarrow S \subseteq \mathrm{TrS}(\mathcal{B})$, and let $\Lambda_S = \{\overline{p} \in \Lambda_{\mathcal{B}} \mid p \in S\}$. Then clearly $\mathcal{S} = (S \cup \{\mathbf{I}_{\mathcal{B}}\}, \Lambda_M) \subseteq_g \mathcal{B}^{\varsigma}$ if the interpretation of $\overline{p} \in \Lambda_S$ in $\mathcal{S}$ is defined by $\overline{p}^{\mathcal{S}}(q) = q \cdot p$ and $\overline{p}^{\mathcal{S}}(\mathbf{I}_{\mathcal{B}}) = p$ for $p, q \in S$. Suppose $\varphi : S \to \mathrm{TrS}(\mathcal{A})$ is a semigroup epimorphism. Define the assignment $\kappa : \Lambda_S \to \Lambda_{\mathcal{A}}$ by $\overline{q}\kappa = \overline{q\varphi}$ for all $q \in S$. It is clear that κ is surjective and for all $q, r \in S \subseteq \mathrm{TrS}(\mathcal{B})$, $\big(\overline{q}^{\mathcal{B}^{\varsigma}}(r)\big)\varphi = (r \cdot q)\varphi = r\varphi \cdot q\varphi = \overline{q\varphi}^{\mathcal{A}^{\varsigma}}(r\varphi) = (q\kappa)^{\mathcal{A}^{\varsigma}}(r\varphi)$. Hence $(\kappa, \widetilde{\varphi}) : \mathcal{S} \to \mathcal{A}^{\varsigma}$ defined by $s\widetilde{\varphi} = s\varphi$ for $s \in S$ and $\mathbf{I}_{\mathcal{B}}\widetilde{\varphi} = \mathbf{I}_{\mathcal{A}}$, is a g-epimorphism. Thus $\mathcal{A}^{\varrho} \leftarrow_g \mathcal{S} \subseteq_g \mathcal{B}^{\varrho}$. □

The following characterization of the class of finite algebras definable by translation semigroups can be proved similarly as Proposition 4.1.7.

Proposition 4.3.2 *A class of finite algebras* $\mathbf{K}$ *is definable by translation semigroups iff it is a gVFA such that* $\mathcal{A} \in \mathbf{K}$ *iff* $\mathcal{A}^{\varsigma} \in \mathbf{K}$ *for any* $\mathcal{A}$*.* □

Recall that we always assume $\Sigma \neq \Sigma_0$. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is called *trivial* if $T \subseteq \Sigma_0 \cup X$. The sets of *non-trivial* ΣX*-trees* and *non-trivial* ΣX*-contexts* are defined by $\mathrm{T}^+(\Sigma, X) = \mathrm{T}(\Sigma, X) \setminus (\Sigma_0 \cup X)$ and $\mathrm{C}^+(\Sigma, X) = \mathrm{C}(\Sigma, X) \setminus \{\xi\}$, respectively. Any subset of $\mathrm{T}^+(\Sigma, X)$ is called a *trivial-free tree language.* For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ the syntactic semigroup of T is, by definition, the quotient semigroup $\mathrm{C}^+(\Sigma, X)/{\sim^T}$ where

$\sim^T$ is restricted to $\mathrm{C}^+(\Sigma, X)$. For a trivial tree language T, the syntactic semigroup of T is the trivial semigroup consisting of a zero element, while its syntactic monoid consists of a zero element and an identity element. Since the trivial semigroup belongs to every variety of finite semigroups, any family of tree languages definable by syntactic semigroups should contain all these trivial tree languages. So, it is reasonable to consider +–varieties of tree languages (cf. [19] Section 11).

A regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is called *strict*, if $\varphi_m(f)$ is not trivial for any $f \in \Sigma_m$ with $m > 0$, and $\varphi_X(X), \varphi_0(\Sigma_0) \subseteq Y \cup \Omega_0$ (cf. Definition 11.1 of [19]). We note that if φ is strict and regular, then $\mathrm{T}^+(\Sigma, X)\varphi^{-1} = \mathrm{T}^+(\Omega, Y)$. A family of recognizable trivial-free tree languages $\{\mathscr{V}(\Sigma, X)\}$, where $\mathscr{V}(\Sigma, X) \subseteq \mathrm{T}^+(\Sigma, X)$ for any Σ and X, is called a +–gVTL if it is closed under Boolean operations, inverse translations and inverse strict regular tree homomorphisms, and moreover satisfies the following conditions:

(1) For every unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.

(2) For any strict regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ full with respect to $T \subseteq \mathrm{T}^+(\Omega, Y)$, if $T\varphi^{-1} \in \mathscr{V}(\Sigma, X)$ then $T \in \mathscr{V}(\Omega, Y)$.

That any variety of trivial-free tree languages definable by syntactic semigroups is a +–gVTL can be proved similarly as the monoid case. We claim also the converse in the following proposition.

Proposition 4.3.3 *A family of trivial-free tree languages is definable by syntactic semigroups iff it is a +–gVTL of tree languages.*

The proof, once we have proved the following semigroup counterparts of Lemmas 4.2.9 and 4.2.10, is very similar to that of Proposition 4.2.14.

Lemma 4.3.4 *Let $\mathcal{A} = (A, \Sigma)$ be a finite algebra, and X be a leaf alphabet disjoint from $A \cup \Sigma$.*

(1) *For any trivial-free tree language $L \subseteq \mathrm{T}^+(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^{\varsigma}$, there are a strict regular tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ and a trivial-free tree language $T \subseteq \mathrm{T}^+(\Sigma, X \cup A)$ such that $L = T\varphi^{-1}$, and T can be recognized by a finite power of $\mathcal{A}$.*

(2) *For any trivial-free tree language $T \subseteq \mathrm{T}^+(\Sigma, X)$ recognized by $\mathcal{A}$ there exist a unary ranked alphabet Λ and a strict regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ such that φ is full with respect to T, and for every $z \in X \cup \Sigma_0$, $T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized by $\mathcal{A}^{\varsigma}$ as a subset of $\mathrm{T}(\Lambda, \{z\})$.*

Proof. (1) Suppose that $L = \{t \in \mathrm{T}(\Lambda_{\mathcal{A}}, X) \mid t\alpha^{\mathcal{A}^{\varrho}} \in F\}$ for an initial assignment $\alpha : X \to \mathrm{Tr}(\mathcal{A}) \cup \{\mathbf{I}_{\mathcal{A}}\}$ and a subset $F \subseteq \mathrm{Tr}(\mathcal{A}) \cup \{\mathbf{I}_{\mathcal{A}}\}$. Since L is trivial-free, we can assume that $F \subseteq \mathrm{Tr}(\mathcal{A})$. Let $Y = \{x \in X \mid x\alpha = \mathbf{I}_{\mathcal{A}}\}$.

Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ by setting $\varphi_X(x) = x$ for all $x \in X$, and for every $p \in \mathrm{Tr}(\mathcal{A})$ choose a $\varphi_1(\overline{p}) \in \mathrm{C}(\Sigma, A)$ such that $\varphi_1(\overline{p})^{\mathcal{A}} = p$. Obviously φ is a strict regular tree homomorphism. Suppose that $A = \{a_1, \ldots, a_m\}$. Let $F' = \{(p(a_1), \ldots, p(a_m)) \in A^m \mid p \in F\}$, and define the initial assignment $\beta : X \cup A \to A^m$ by setting $x\beta = \big((x\alpha)(a_1), \ldots, (x\alpha)(a_m)\big)$ for all $x \in X \setminus Y$, $y\beta = \big(a_1, \ldots, a_m\big)$ for all $y \in Y$, and $a\beta = (a, \ldots, a) \in A^m$ for all $a \in A$. Let T be the subset of $\mathrm{T}(\Sigma, X \cup A)$ recognized by $(\mathcal{A}^m, \beta, F')$. We show $L = T\varphi^{-1}$. Every trivial-free tree w in $\mathrm{T}^+(\Lambda_{\mathcal{A}}, X)$ is of the form $w = \overline{p_1}\Big(\overline{p_2}\big(\ldots\overline{p_k}(x)\ldots\big)\Big)$ for some $p_1, \ldots, p_k \in \mathrm{Tr}(\mathcal{A})$ $(k > 0)$ and $x \in X$. For such a tree w, $w\alpha^{\mathcal{A}^\varrho} = x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1$ if $x \in X \setminus Y$, and $w\alpha^{\mathcal{A}^\varrho} = p_k \cdot \ldots \cdot p_2 \cdot p_1$ if $x \in Y$; also $(w\varphi)\beta^{\mathcal{A}^m} = (x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a_1), \ldots, x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a_m))$ holds. So, for $x \in X \setminus Y$ we have $w\varphi \in T$ iff $(w\varphi)\beta^{\mathcal{A}^m} \in F'$ iff for some $p \in F$, $p(a) = x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a)$, for all $a \in A$ iff $x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1 \in F$ iff $w\alpha^{\mathcal{A}^\varrho} \in F$ iff $w \in L$. Similarly, for $x \in Y$ we have $w\varphi \in T$ iff $(w\varphi)\beta^{\mathcal{A}^m} \in F'$ iff for some $p \in F$, $p(a) = p_k \cdot \ldots \cdot p_2 \cdot p_1(a)$, for all $a \in A$ iff $p_k \cdot \ldots \cdot p_2 \cdot p_1 \in F$ iff $w\alpha^{\mathcal{A}^\varrho} \in F$ iff $w \in L$.
(2) The proof is almost identical to that of Lemma 4.2.10, only 1_A is replaced with $\mathbf{I}_{\mathcal{A}}$. □

It was shown in Example 4.2.2 that the variety of 1-definite tree languages is not definable by syntactic monoids. In the following example we show that likewise the family of trivial-free 1-definite tree languages is not definable by syntactic semigroups.

Example 4.3.5 The syntactic semigroup of the trivial-free 1-definite tree language $T\setminus\{x\}$ where T is defined in Example 4.2.2, consists of two elements both of which are right zeros. Let $\Lambda = \Lambda_1 = \{\alpha, \beta\}$ and $X = \{x, y\}$. Let T'' be the set of all ΛX-trees which either have root label α and leaf label x or have root label β and leaf label y, i.e.,

$$T'' = \{\alpha(p(x)) \mid p \in \mathrm{C}(\Lambda, X)\} \cup \{\beta(p(y)) \mid p \in \mathrm{C}(\Lambda, X)\}.$$

It is easy to see that the syntactic semigroup of T'' consists of two right zero elements, but clearly T'' is not 1-definite. So, the trivial-free 1-definite tree languages are not definable by syntactic semigroups. Indeed, T'' is not k-definite for any $k \geq 1$, thus the trivial-free definite tree languages are not definable by syntactic semigroups (cf. Remark 4.2.3).

In the sequel we show that neither one of the properties "definability by semigroups" and "definability by monoids" implies the other one.

The abbreviation VFS stands for variety of finite semigroups. For a VFS $\mathbf{S}$, let $\mathbf{S}^a$ be the class of all finite algebras whose translation semigroups are in $\mathbf{S}$, and $\mathbf{S}^t$ be the family of all recognizable trivial-free tree languages whose syntactic semigroups are in $\mathbf{S}$ (cf. Definitions 4.1.2 and 4.2.1).

We recall Proposition 10.9 of [54] which can be extended to VFS's as well; see also Lemma 5.2.5 in Chapter 5 below.

Proposition 4.3.6 *For any VFM* $\mathbf{M}$ *and VFS* $\mathbf{S}$*, the identities* $\mathbf{M}^{at} = \mathbf{M}^{t}$, $\mathbf{M}^{ta} = \mathbf{M}^{a}$, $\mathbf{S}^{at} = \mathbf{S}^{t}$ *and* $\mathbf{S}^{ta} = \mathbf{S}^{a}$ *hold.* □

Proposition 4.3.7 (1) *There exists a VFM* $\mathbf{M}$ *such that neither* $\mathbf{M}^{a} = \mathbf{S}^{a}$ *nor* $\mathbf{M}^{t} = \mathbf{S}^{t}$ *holds for any VFS* $\mathbf{S}$*.*
(2) *For some VFS* $\mathbf{S}$*, no VFM* $\mathbf{M}$ *satisfies* $\mathbf{M}^{a} = \mathbf{S}^{a}$ *or* $\mathbf{M}^{t} = \mathbf{S}^{t}$*.*

Proof. (1) Let $\mathbf{M}$ be the class of all finite monoids that satisfy the equation $\mathsf{y} \cdot \mathsf{x} \cdot \mathsf{x} = \mathsf{y}$. Obviously, $\mathbf{M}$ is a VFM. Let $\Sigma = \Sigma_1 = \{f\}$ and let the algebras $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be defined by $A = \{a\}$, $f^{\mathcal{A}}(a) = a$, and $B = \{a, b\}$, $f^{\mathcal{B}}(a) = f^{\mathcal{B}}(b) = a$. Then $\mathrm{Tr}(\mathcal{A}) \cong \mathrm{TrS}(\mathcal{A}) \cong \mathrm{TrS}(\mathcal{B})$ is the trivial semigroup, but the monoid $\mathrm{Tr}(\mathcal{B})$ consists of a zero element (let us denote it by 0) and a unit (denoted by 1). Now, $\mathcal{A} \in \mathbf{M}^{a}$, but $\mathcal{B} \notin \mathbf{M}^{a}$ since $\mathrm{Tr}(\mathcal{B})$ does not satisfy the equation $\mathsf{y} \cdot \mathsf{x} \cdot \mathsf{x} = \mathsf{y}$: $1 \cdot 0 \cdot 0 = 0 \neq 1$. Hence, $\mathbf{M}^{a}$ is not definable by translation semigroups. Now if $\mathbf{M}^{t} = \mathbf{S}^{t}$ hold for a VFS $\mathbf{S}$, then by Theorem 4.3.6 we would have $\mathbf{M}^{a} = \mathbf{M}^{ta} = \mathbf{S}^{ta} = \mathbf{S}^{a}$, contradiction.

(2) Let $\mathbf{S}$ be the variety of finite right zero semigroups, i.e., the class of semigroup that satisfy the equation $\mathsf{x} \cdot \mathsf{y} = \mathsf{y}$. It can be easily seen that if T and T' are the tree languages of Example 4.2.2, then $T \setminus \{x\}$ belongs to $\mathbf{S}^{t}(\Sigma, X)$ since the syntactic semigroup of $T \setminus \{x\}$ has two elements both of which are right zeros. On the other hand, the syntactic semigroup of T' consists of an identity element and two right zeros (like its syntactic monoid). Thus $T' \notin \mathbf{S}^{t}(\Sigma, X)$. This shows that $\mathbf{S}^{t}$ is not definable by syntactic monoids (since $T \setminus \{x\}$ and T' have isomorphic syntactic monoids) whence $\mathbf{M}^{t} = \mathbf{S}^{t}$ does not hold for any VFM $\mathbf{M}$. On the other hand if the identity $\mathbf{M}^{a} = \mathbf{S}^{a}$ hold for some VFM $\mathbf{M}$, then by Proposition 4.3.6 we would have $\mathbf{M}^{t} = \mathbf{M}^{at} = \mathbf{S}^{at} = \mathbf{S}^{t}$, contradiction. □

Propositions 4.3.7 means that the definability by semigroups deserves a separate study.

Chapter 5

Definability by ordered monoids

We already know that the syntactic algebras of tree languages can be ordered. Likewise, the syntactic monoids of tree languages can also be ordered. Due to its richer structure, the ordered syntactic monoid of a tree language reflects more of the combinatorial properties of the language than its syntactic monoid. In Chapter 4 the classes of finite algebras and the families of recognizable tree languages that are definable by (translation or syntactic) monoids were characterized. In this chapter we characterize the classes of finite ordered algebras and the families of tree languages that are definable by ordered monoids. Informally speaking, we prove the ordered version of the results of Chapter 4. By doing so, Propositions 4.1.7 and 4.2.14 become special cases of the results of the present chapter (when the inequality is taken to be the equality).

In Section 5.1 we introduce ordered translation monoids of ordered algebras and give necessary and sufficient conditions for a class of finite ordered algebras to be definable by ordered translation monoids. In Section 5.2, after introducing the syntactic ordered monoid of a tree language, we characterize the families of tree languages that are definable by syntactic ordered monoids. In the last section 5.3 we study semilattice and symbolic tree languages in detail. These provide some instances for the variety theorems of Chapter 4 and the present chapter.

5.1 Ordered algebras vs. ordered monoids

We assume familiarity with ordered algebras (Chapter 3) and translation monoids of algebras (Chapter 4). The translations of ordered algebras can be ordered as follows:

Definition 5.1.1 The *ordered translation monoid* of an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is the structure $\mathrm{OTr}(\mathcal{A}) = (\mathrm{Tr}(\mathcal{A}), \cdot, \lesssim_{\mathcal{A}})$ where $(\mathrm{Tr}(\mathcal{A}), \cdot)$ is the translation monoid of $\mathcal{A}$ and the binary relation $\lesssim_{\mathcal{A}}$ is defined by

$$p \lesssim_{\mathcal{A}} q \iff (\forall a \in A)\big(p(a) \leqslant q(a)\big) \quad (p, q \in \mathrm{Tr}(\mathcal{A})).$$

The relation $\lesssim_{\mathcal{A}}$ is indeed an order on $\mathrm{Tr}(\mathcal{A})$ compatible with the composition of translations: if $p \lesssim_{\mathcal{A}} q$ then $p \cdot r \lesssim_{\mathcal{A}} q \cdot r$ and $r \cdot p \lesssim_{\mathcal{A}} r \cdot q$ for any $p, q, r \in \mathrm{Tr}(\mathcal{A})$.

Recall the notions of $\subseteq_g$, $\leftarrow_g$ and $\cong_g$ from Chapter 3. We note that Proposition 3.1.7 in Chapter 3 can be generalized to g-morphisms:

Proposition 5.1.2 *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ be two ordered algebras, and $(\kappa, \varphi) : \mathcal{A} \to \mathcal{B}$ be an order g-morphism. The mappings κ, φ induce a monoid morphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_{(\kappa,\varphi)}$ such that $p(a)\varphi = p_{(\kappa,\varphi)}(a\varphi)$ for all $a \in A$. Moreover, if (κ, φ) is an order g-epimorphism then the induced map is a monoid epimorphism.*

Proof. For any elementary translation $p = f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ of $\mathcal{A}$ where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$, the unary function $p_{(\kappa,\varphi)}$ on B defined by $b \mapsto (f\kappa)^{\mathcal{B}}(a_1\varphi, \ldots, b, \ldots, a_m\varphi)$ is an elementary translation of $\mathcal{B}$, and if κ and φ are surjective then every elementary translation of $\mathcal{B}$ is of this form. The mapping $p \mapsto p_{(\kappa,\varphi)}$ can be extended inductively to all translations by setting $(1_A)_{(\kappa,\varphi)} = 1_B$ and $(p \cdot q)_{(\kappa,\varphi)} = p_{(\kappa,\varphi)} \cdot q_{(\kappa,\varphi)}$. The identity $p_{(\kappa,\varphi)}(a\varphi) = p(a)\varphi$ is obvious. □

The following proposition is the ordered version of Lemma 10.7 in [54].

Proposition 5.1.3 *For any finite ordered algebras $\mathcal{A}$ and $\mathcal{B}$,*

(1) *if $\mathcal{A} \subseteq_g \mathcal{B}$, then $\mathrm{OTr}(\mathcal{A}) \preceq \mathrm{OTr}(\mathcal{B})$;*
(2) *if $\mathcal{A} \leftarrow_g \mathcal{B}$, then $\mathrm{OTr}(\mathcal{A}) \leftarrow \mathrm{OTr}(\mathcal{B})$;*
(3) *$\mathrm{OTr}(\kappa(\mathcal{A}, \mathcal{B})) \subseteq \mathrm{OTr}(\mathcal{A}) \times \mathrm{OTr}(\mathcal{B})$ for any g-product $\kappa(\mathcal{A}, \mathcal{B})$.*

Proof. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$.

(1) Let $\mathcal{M}$ be the order submonoid of $\mathrm{OTr}(\mathcal{B})$ generated by the elementary translations of the form $f^{\mathcal{B}}(a_1, \ldots, \xi, \ldots, a_m)$ where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$. The mapping

$$f^{\mathcal{B}}(a_1, \ldots, \xi, \ldots, a_m) \mapsto f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$$

can be uniquely extended to an order monoid epimorphism $\mathcal{M} \to \mathrm{OTr}(\mathcal{A})$. Thus $\mathrm{OTr}(\mathcal{A}) \leftarrow \mathcal{M} \subseteq \mathrm{OTr}(\mathcal{B})$.

(2) Let $(\kappa, \varphi) : \mathcal{B} \to \mathcal{A}$ be an order g-epimorphism. By Proposition 5.1.2, the mapping $\mathrm{OTr}(\mathcal{B}) \to \mathrm{OTr}(\mathcal{A})$, $p \mapsto p_{(\kappa,\varphi)}$ is a monoid epimorphism. It also preserves the order of translations, since for any $p, q \in \mathrm{OTr}(\mathcal{B})$,

$p \lesssim_{\mathcal{B}} q \;\Rightarrow\; p(b) \leqslant' q(b)$ for all $b \in B$

$\Rightarrow\ p(b)\varphi \leqslant q(b)\varphi$ for all $b \in B$
$\Rightarrow\ p_{(\kappa,\varphi)}(b\varphi) \leqslant q_{(\kappa,\varphi)}(b\varphi)$ for all $b \in B$
$\Rightarrow\ p_{(\kappa,\varphi)}(a) \leqslant q_{(\kappa,\varphi)}(a)$ for all $a \in A$
$\Rightarrow\ p_{(\kappa,\varphi)} \lesssim_{\mathcal{A}} q_{(\kappa,\varphi)}$.

(3) Let Γ be a ranked alphabet and $\kappa : \Gamma \to \Sigma \times \Omega$ be an assignment. It is easy to verify that the mapping

$$g^{\kappa(\mathcal{A},\mathcal{B})}\big((a_1, b_1), \ldots, \xi, \ldots, (a_m, b_m)\big) \mapsto \Big(f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m), h^{\mathcal{B}}(b_1, \ldots, \xi, \ldots, b_m)\Big)$$

where $a_1, \ldots, a_m \in A, b_1, \ldots, b_m \in B$, $g \in \Gamma_m$ and $g\kappa = (f, h)$, can be extended to a monomorphism $\psi : \mathrm{OTr}(\kappa(\mathcal{A}, \mathcal{B})) \to \mathrm{OTr}(\mathcal{A}) \times \mathrm{OTr}(\mathcal{B})$ which satisfies $p(a, b) = (p\psi_1(a), p\psi_2(b))$ for all $a \in A$, $b \in B$ and $p \in \mathrm{Tr}(\kappa(\mathcal{A}, \mathcal{B}))$, where ψ_1 and ψ_2 are the components of ψ, i.e., $p\psi = (p\psi_1, p\psi_2)$. The mapping ψ is also order preserving, since for $p, q \in \mathrm{Tr}(\kappa(\mathcal{A}, \mathcal{B}))$,

$p \lesssim_{\kappa(\mathcal{A},\mathcal{B})} q \ \Rightarrow\ p(a, b) \leqslant \times \leqslant' q(a, b)$ for all $a \in A, b \in B$
$\Rightarrow\ p\psi_1(a) \leqslant q\psi_1(a)$ & $p\psi_2(b) \leqslant' q\psi_2(b)$ for all $a \in A, b \in B$
$\Rightarrow\ p\psi_1 \lesssim_{\mathcal{A}} q\psi_1$ & $p\psi_2 \lesssim_{\mathcal{B}} q\psi_2$
$\Rightarrow\ (p\psi_1, p\psi_2) \lesssim_{\mathcal{A}} \times \lesssim_{\mathcal{B}} (q\psi_1, q\psi_2)$
$\Rightarrow\ p\psi \lesssim_{\mathcal{A}} \times \lesssim_{\mathcal{B}} q\psi$. □

Definition 5.1.4 A *variety of finite ordered monoids*, in notation VFOM, is a class of finite ordered monoids closed under order submonoids, order epimorphic images and finite direct products.

For a VFOM $\mathbf{M}$, $\mathbf{M}^a$ is the class of all finite ordered algebras whose ordered translation monoids are in $\mathbf{M}$, i.e.,

$$\mathbf{M}^a = \{\mathcal{A} \mid \mathcal{A} \text{ is an ordered algebra such that } \mathrm{OTr}(\mathcal{A}) \in \mathbf{M}\}.$$

A class of finite ordered algebras $\mathbf{K}$ is said to be *definable by ordered translation monoids* if there is a VFOM $\mathbf{M}$ such that $\mathbf{M}^a = \mathbf{K}$.

Corollary 5.1.5 *For any VFOM* $\mathbf{M}$*, the class* $\mathbf{M}^a$ *is a gVFOA.* □

This follows from Proposition 5.1.3. It is known that not every gVFOA is definable by ordered translation monoids; such an example is the gVFOA **Nil** of ordered nilpotent algebras considered in Chapter 3. In this section we give necessary and sufficient conditions for a class of algebras to be of the form $\mathbf{M}^a$ for some VFOA $\mathbf{M}$.

Definition 5.1.6 For any set D, let $\Lambda_D = \{\overline{d} \mid d \in D\}$ be the unary ranked alphabet consisting of a unary function symbol $\overline{d}$ for each $d \in D$. For a finite ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$ the unary ordered algebra $\mathcal{M}^\nu = (M, \Lambda_M, \lesssim)$ is defined by $\overline{m}^{\mathcal{M}^\nu}(a) = a \cdot m$ for all $a, m \in M$.

The structure $\mathcal{M}^\nu$ for a finite ordered monoid $\mathcal{M}$ is indeed an ordered algebra since for any $a, b, m \in M$,

$$a \lesssim b \Rightarrow a \cdot m \lesssim b \cdot m \Rightarrow \overline{m}^{\mathcal{M}^\nu}(a) \lesssim \overline{m}^{\mathcal{M}^\nu}(b).$$

Proposition 5.1.7 *For a finite ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$,*

$$\mathrm{OTr}(\mathcal{M}^\nu) \cong \mathcal{M}.$$

Proof. The elementary translations of $\mathcal{M}^\nu$ are of the form $\overline{m}^{\mathcal{M}^\nu}(\xi)$ where $m \in M$, and clearly $\overline{m}^{\mathcal{M}^\nu}(\xi) \cdot \overline{n}^{\mathcal{M}^\nu}(\xi) = \overline{m \cdot n}^{\mathcal{M}^\nu}(\xi)$ for all $m, n \in M$. For the unit element 1_M of $\mathcal{M}$, the translation $\overline{1_M}^{\mathcal{M}^\nu}(\xi)$ is the identity translation of $\mathcal{M}^\nu$. This means that $\mathrm{Tr}(\mathcal{M}^\nu) = \{\overline{m}^{\mathcal{M}^\nu}(\xi) \mid m \in M\}$. Moreover, $\overline{m}^{\mathcal{M}^\nu}(\xi) \neq \overline{n}^{\mathcal{M}^\nu}(\xi)$ whenever $m \neq n$, since $\overline{m}^{\mathcal{M}^\nu}(\xi) = \overline{n}^{\mathcal{M}^\nu}(\xi)$ implies $m = 1_M \cdot m = \overline{m}^{\mathcal{M}^\nu}(1_M) = \overline{n}^{\mathcal{M}^\nu}(1_M) = 1_M \cdot n = n$. Hence, the mapping $\mathcal{M} \to \mathrm{OTr}(\mathcal{M}^\nu)$, $m \mapsto \overline{m}^{\mathcal{M}^\nu}(\xi)$ is a monoid isomorphism. It is also an order isomorphism. Indeed, for any $m, n \in M$, $m \lesssim n$ iff $a \cdot m \lesssim a \cdot n$ for every $a \in M$, i.e., $\overline{m}^{\mathcal{M}^\nu}(a) \lesssim \overline{n}^{\mathcal{M}^\nu}(a)$ for every $a \in M$, what by definition is equivalent to $\overline{m}^{\mathcal{M}^\nu}(\xi) \lesssim_{\mathcal{M}^\nu} \overline{n}^{\mathcal{M}^\nu}(\xi)$. □

Proposition 5.1.8 *For all finite ordered monoids $\mathcal{M}$ and $\mathcal{P}$,*

(1) *if $\mathcal{M} \subseteq \mathcal{P}$ then $\mathcal{M}^\nu \subseteq_g \mathcal{P}^\nu$;*

(2) *if $\mathcal{M} \leftarrow \mathcal{P}$ then $\mathcal{M}^\nu \leftarrow_g \mathcal{P}^\nu$;*

(3) *$(\mathcal{M} \times \mathcal{P})^\nu \cong_g \kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$ for some g-product $\kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$.*

Proof. Let $\mathcal{M} = (M, \cdot, \lesssim)$ and $\mathcal{P} = (P, \cdot, \lesssim')$. Statement (1) is obvious. For (2) we note that if $\varphi : \mathcal{P} \to \mathcal{M}$ is an order monoid epimorphism, then $(\overline{\varphi}, \varphi) : \mathcal{P}^\nu \to \mathcal{M}^\nu$, where $\overline{\varphi} : \Lambda_P \to \Lambda_M$ is defined by $(\overline{m})\overline{\varphi} = \overline{m\varphi}$, is an order g-epimorphism. For proving (3) let $\kappa : \Lambda_{M\times P} \to \Lambda_M \times \Lambda_P$ be an assignment defined by $\overline{(m,p)}\kappa = (\overline{m}, \overline{p})$ for $m \in M, p \in P$, and let $\kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$ be the corresponding g-product of $\mathcal{M}^\nu$ and $\mathcal{P}^\nu$. It is easy to verify that $(\lambda, \varphi) : (\mathcal{M} \times \mathcal{P})^\nu \to \kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$, where λ is the identity map on $\Lambda_{M\times P}$ and φ is the identity map on $M \times P$, is an order g-isomorphism. □

Clause (3) of Proposition 5.1.8 can be generalized to any finite number of finite ordered monoids $\mathcal{M}_1, \ldots, \mathcal{M}_n$, that is to say, the g-isomorphism

$$(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n)^\nu \cong_g \kappa(\mathcal{M}_1^\nu, \ldots, \mathcal{M}_n^\nu)$$

holds for some g-product $\kappa(\mathcal{M}_1^\nu, \ldots, \mathcal{M}_n^\nu)$.

For a finite ordered algebra $\mathcal{A}$, the unary algebra $\mathcal{A}^\rho$ is defined to be $\mathrm{OTr}(\mathcal{A})^\nu$; cf. Definition 4.1.3 in Chapter 4. The following is an immediate consequence of Proposition 5.1.8.

Corollary 5.1.9 *If $\mathrm{OTr}(\mathcal{A}) \preceq \mathrm{OTr}(\mathcal{A}_1) \times \cdots \times \mathrm{OTr}(\mathcal{A}_n)$ for any finite ordered algebras $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n$, then $\mathcal{A}^\rho \preceq_g \kappa(\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho)$ for some g-product $\kappa(\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho)$.* □

Our characterization of gVFOA's definable by ordered monoids is the following.

Proposition 5.1.10 *For any class* $\mathbf{K}$ *of finite ordered algebras the following conditions are equivalent:*

(1) $\mathbf{K}$ *is definable by ordered translation monoids;*

(2) $\mathbf{K}$ *is a gVFOA such that for all finite ordered algebras* $\mathcal{A}$ *and* $\mathcal{B}$*, if* $\mathrm{OTr}(\mathcal{A}) \cong \mathrm{OTr}(\mathcal{B})$ *and* $\mathcal{A} \in \mathbf{K}$ *then* $\mathcal{B} \in \mathbf{K}$*;*

(3) $\mathbf{K}$ *is a gVFOA such that* $\mathcal{A} \in \mathbf{K} \Longleftrightarrow \mathcal{A}^\rho \in \mathbf{K}$ *for any* $\mathcal{A}$*.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows from Proposition 5.1.7. For (3) $\Rightarrow$ (1), suppose that a gVFOA $\mathbf{K}$ satisfies the equivalence $\mathcal{A} \in \mathbf{K} \Leftrightarrow \mathcal{A}^\rho \in \mathbf{K}$ for any $\mathcal{A}$. Let $\mathbf{M}$ be the VFOM generated by $\{\mathrm{OTr}(\mathcal{A}) \mid \mathcal{A} \in \mathbf{K}\}$. We claim that $\mathbf{K} = \mathbf{M}^a$. Obviously $\mathbf{K} \subseteq \mathbf{M}^a$. For the opposite inclusion let $\mathcal{B} \in \mathbf{M}^a$. So, $\mathrm{OTr}(\mathcal{B}) \preceq \mathrm{OTr}(\mathcal{A}_1) \times \cdots \times \mathrm{OTr}(\mathcal{A}_n)$ for some $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathbf{K}$. By Corollary 5.1.9, $\mathcal{B}^\rho \preceq_g \kappa(\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho)$ for some g-product $\kappa(\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho)$. Now we have $\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho \in \mathbf{K}$ and this implies that $\mathcal{B}^\rho \in \mathbf{K}$ hence $\mathcal{B} \in \mathbf{K}$. Thus $\mathbf{M}^a \subseteq \mathbf{K}$. □

Remark 5.1.11 Proposition 5.1.8 and the proof of Proposition 5.1.10 also yield the fact that for any gVFOA $\mathbf{K}$ definable by ordered translation monoids, the class $\{\mathrm{OTr}(\mathcal{A}) \mid \mathcal{A} \in \mathbf{K}\}$ is a variety of finite ordered monoids.

5.2 Tree languages definable by ordered monoids

Let Σ be a ranked alphabet and X be a leaf alphabet.

Definition 5.2.1 For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the quasi-order $\precsim_T$ on $\mathrm{C}(\Sigma, X)$ is defined by the condition
$P \precsim_T Q \iff (\forall R \in \mathrm{C}(\Sigma, X))(\forall t \in \mathrm{T}(\Sigma, X))\big(t \cdot Q \cdot R \in T \Rightarrow t \cdot P \cdot R \in T\big)$.

The equivalence relation of $\precsim_T$ is the m-congruence of T defined by
$P \sim^T Q \iff (\forall R \in \mathrm{C}(\Sigma, X))(\forall t \in \mathrm{T}(\Sigma, X))\big(t \cdot P \cdot R \in T \Leftrightarrow t \cdot Q \cdot R \in T\big)$.
Recall also that the quotient monoid $(\mathrm{C}(\Sigma, X)/{\sim^T}, \cdot)$ is the syntactic monoid $\mathrm{SM}(T)$ of T.

The *syntactic ordered monoid* of T is $\mathrm{SOM}(T) = (\mathrm{C}(\Sigma, X)/{\approx^T}, \cdot, \lesssim_T)$ where $\lesssim_T$ is the order induced by $\precsim_T$: $(P/{\sim^T}) \lesssim_T (Q/{\sim^T}) \Leftrightarrow P \precsim_T Q$ for $P, Q \in \mathrm{C}(\Sigma, X)$; cf. [54] or [58]. It is easy to verify that $P \precsim_T Q$ implies $R \cdot P \cdot S \precsim_T R \cdot Q \cdot S$ for any $P, Q, R, S \in \mathrm{C}(\Sigma, X)$. Thus the structure $\mathrm{SOM}(T)$ is indeed an ordered monoid.

It is known that the syntactic monoid of a tree language is the translation monoid of the syntactic algebra of the language ([48, 54]). The following is the corresponding proposition for ordered translation monoids and syntactic ordered algebras.

Proposition 5.2.2 *For a tree language* $T \subseteq \mathrm{T}(\Sigma, X)$*,*

$$\mathrm{OTr}(\mathrm{SOA}(T)) \cong \mathrm{SOM}(T).$$

Proof. It is easy to see that the mapping

$$f(t_1, \ldots, \xi, \ldots, t_m) \mapsto f^{\mathrm{SOA}(T)}(t_1/{\approx^T}, \ldots, \xi, \ldots, t_m/{\approx^T})$$

can be extended to a monoid epimorphism $\varphi : \mathrm{C}(\Sigma, X) \to \mathrm{OTr}(\mathrm{SOA}(T))$ which satisfies $P\varphi(t/{\approx^T}) = (t \cdot P)/{\approx^T}$ for all $t \in \mathrm{T}(\Sigma, X), P \in \mathrm{C}(\Sigma, X)$. We show that for any $P, Q \in \mathrm{C}(\Sigma, X)$, $P \precsim_T Q$ iff $P\varphi \lesssim_{\mathrm{SOA}(T)} Q\varphi$. Indeed, $P \precsim_T Q$ means by definition that $t \cdot Q \cdot R \in T$ implies $t \cdot P \cdot R \in T$ for all $t \in \mathrm{T}(\Sigma, X)$, $R \in \mathrm{C}(\Sigma, X)$, i.e., $t \cdot P \preccurlyeq_T t \cdot Q$ for every $t \in \mathrm{T}(\Sigma, X)$, or equivalently $(t \cdot P)/{\approx^T} \leqslant_T (t \cdot Q)/{\approx^T}$ for every $t \in \mathrm{T}(\Sigma, X)$. This is equivalent to the fact that $P\varphi(t/{\approx^T}) \leqslant_T Q\varphi(t/{\approx^T})$ for every $t \in \mathrm{T}(\Sigma, X)$, or in other words, to $P\varphi \lesssim_{\mathrm{SOA}(T)} Q\varphi$. Thus $\varphi \circ \lesssim_{\mathrm{SOA}(T)} \circ \varphi^{-1} = \precsim_T$, and then from Proposition 3.1.4 it follows that $\mathrm{SOM}(T) \cong \mathrm{OTr}(\mathrm{SOA}(T))$. □

Corollary 5.2.3 *For any ranked alphabets* Σ, Ω*, leaf alphabets* X, Y*, context* $P \in \mathrm{C}(\Sigma, X)$*, order g-morphism* $(\kappa, \varphi) : \mathcal{T}(\Omega, Y) \to \mathcal{T}(\Sigma, X)$*, and tree languages* $T, T' \subseteq \mathrm{T}(\Sigma, X)$*,*

(1) $\mathrm{SOM}(T \cap T'), \mathrm{SOM}(T \cup T') \preceq \mathrm{SOM}(T) \times \mathrm{SOM}(T')$*;*

(2) $\mathrm{SOM}(P^{-1}(T)) \leftarrow \mathrm{SOM}(T)$*;*

(3) $\mathrm{SOM}(T\varphi^{-1}) \preceq \mathrm{SOM}(T)$*, and if* (κ, φ) *is a g-epimorphism then* $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$. □

It follows from Corollary 3.2.3 (Chapter 3) and Propositions 5.1.3 and 5.2.2.

Definition 5.2.4 For a VFOM $\mathbf{M}$, let $\mathbf{M}^t = \{\mathbf{M}^t(\Sigma, X)\}$ be the family of all recognizable tree languages whose syntactic ordered monoids are in $\mathbf{M}$, that is to say, $\mathbf{M}^t(\Sigma, X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOM}(T) \in \mathbf{M}\}$.

A family of recognizable tree languages $\mathscr{V}$ is *definable by syntactic ordered monoids* if there is a VFOM $\mathbf{M}$ such that $\mathbf{M}^t = \mathscr{V}$.

By Corollary 5.2.3, the family $\mathbf{M}^t$ for any VFOM $\mathbf{M}$ is a gPVTL. In this section we characterize the gPVTL's that are definable by syntactic ordered monoids.

Lemma 5.2.5 *For any VFOM* $\mathbf{M}$*,* (1) $\mathbf{M}^{at} = \mathbf{M}^t$ *and* (2) $\mathbf{M}^{ta} = \mathbf{M}^a$.

Proof. (1) For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$ by Proposition 5.2.2,

$$T \in \mathbf{M}^{at}(\Sigma, X) \iff \mathrm{SOA}(T) \in \mathbf{M}^a \iff \mathrm{OTr}(\mathrm{SOA}(T)) \in \mathbf{M}$$
$$\iff \mathrm{SOM}(T) \in \mathbf{M} \iff T \in \mathbf{M}^t(\Sigma, X).$$

(2) By (1) and Proposition 3.3.5, $(\mathbf{M}^t)^a = (\mathbf{M}^{at})^a = (\mathbf{M}^a)^{ta} = \mathbf{M}^a$. □

Corollary 5.2.6 (1) *A gPVTL $\mathscr{V}$ is definable by syntactic ordered monoids iff $\mathscr{V}^a$ is a gVFOA definable by ordered translation monoids.*

(2) *A gVFOA* $\mathbf{K}$ *is definable by ordered translation monoids iff* $\mathbf{K}^t$ *is a gPVTL definable by syntactic ordered monoids.* □

Recall the definition of a tree homomorphism and its extension to contexts from Chapters 4. The following lemma is the ordered version of Lemma 4.2.6.

Lemma 5.2.7 *If* $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ *is a regular tree homomorphism and* $T \subseteq \mathrm{T}(\Omega, Y)$ *then* $\mathrm{SOM}(T\varphi^{-1}) \preceq \mathrm{SOM}(T)$*. If* φ *is full with respect to* T *then* $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$.

Proof. We note that $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid homomorphism. Let $S \subseteq \mathrm{C}(\Omega, Y)$ be the image of φ_*, $\precsim$ be the restriction of $\precsim_T$ to S and μ be the equivalence relation of $\precsim$. Then S/μ is a submonoid of $\mathrm{C}(\Omega, Y)/{\sim^T}$. We show that $P\varphi_* \precsim Q\varphi_*$ implies $P \precsim_{T\varphi^{-1}} Q$ for all $P, Q \in \mathrm{C}(\Sigma, X)$.

Suppose $P\varphi_* \precsim Q\varphi_*$ and take arbitrary $t \in \mathrm{T}(\Sigma, X)$ and $R \in \mathrm{C}(\Sigma, X)$. Then $t \cdot Q \cdot R \in T\varphi^{-1}$ implies $t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T$ then $t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T$, and so $t \cdot P \cdot R \in T\varphi^{-1}$ or equivalently $P \precsim_{T\varphi^{-1}} Q$. Hence the mapping $\psi : S/\mu \to \mathrm{C}(\Sigma, X)/\sim^{T\varphi^{-1}}$ defined by $((P\varphi_*)\mu)\psi = P \sim^{T\varphi^{-1}}$ is well-defined, order preserving and surjective. It is also a monoid morphism, since $((P\varphi_*)\mu \cdot (Q\varphi_*)\mu)\psi = ((P \cdot Q)\varphi_*\mu)\psi = (P \cdot Q) \sim^{T\varphi^{-1}} = P \sim^{T\varphi^{-1}} \cdot Q \sim^{T\varphi^{-1}} = ((P\varphi_*)\mu)\psi \cdot ((Q\varphi_*)\mu)\psi$ for all $P, Q \in \mathrm{C}(\Sigma, X)$. Hence, $\mathrm{SOM}(T\varphi^{-1}) \leftarrow S/{\precsim} \subseteq \mathrm{SOM}(T)$ holds and so $\mathrm{SOM}(T\varphi^{-1}) \preceq \mathrm{SOM}(T)$.

Suppose now that φ is full with respect to T. We show that $P \precsim_{T\varphi^{-1}} Q$ iff $P\varphi_* \precsim_T Q\varphi_*$ for any $P, Q \in \mathrm{C}(\Sigma, X)$. It has already been proved that $P\varphi_* \precsim_T Q\varphi_*$ implies $P \precsim_{T\varphi^{-1}} Q$. For the converse, suppose $P \precsim_{T\varphi^{-1}} Q$ and take arbitrary $R' \in \mathrm{C}(\Omega, Y)$ and $t' \in \mathrm{T}(\Omega, Y)$. There are $R \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$ such that $R\varphi_* \sim^T R'$ and $t\varphi \approx^T t'$. Hence, $t' \cdot Q\varphi_* \cdot R' \in T$ implies $t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T$, which is equivalent to $(t \cdot Q \cdot R)\varphi \in T$, or to $t \cdot Q \cdot R \in T\varphi^{-1}$ and hence $t \cdot P \cdot R \in T\varphi^{-1}$. This is equivalent to $t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T$, and so $t' \cdot P\varphi_* \cdot R' \in T$, what shows that $P\varphi_* \precsim_T Q\varphi_*$. Hence $P \precsim_{T\varphi^{-1}} Q$ iff $P\varphi_* \precsim_T Q\varphi_*$, and since the mapping $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid homomorphism then $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$ by Proposition 3.1.4. □

In the following two lemmas, some connections between tree languages recognizable by a finite ordered algebra $\mathcal{A}$, and tree languages recognizable by $\mathcal{A}^\rho$ are presented. Recall that the unary ranked alphabet of the algebra $\mathcal{A}^\rho$ is $\{\overline{p} \mid p \in \mathrm{Tr}(\mathcal{A})\}$; for simplicity we denote this alphabet by $\Lambda_{\mathcal{A}}$; cf. Definition 4.1.3.

Suppose $\mathcal{A} = (A, \Sigma)$ is a finite algebra. Every context in $\mathrm{C}(\Sigma, A)$ corresponds to a translation in $\mathrm{Tr}(\mathcal{A})$ in a natural way: for any $m > 0$, $f \in \Sigma_m$

and $a_1, \ldots, a_m \in A$ the elementary translation $f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ corresponds to the elementary context $f(a_1, \ldots, \xi, \ldots, a_m)$. This correspondence can be extended to a mapping $-^{\mathcal{A}} : \mathrm{C}(\Sigma, A) \to \mathrm{Tr}(\mathcal{A})$ which satisfies $\xi^{\mathcal{A}} = 1_A$ (the identity translation) and $(P \cdot Q)^{\mathcal{A}} = P^{\mathcal{A}} \cdot Q^{\mathcal{A}}$ for all $P, Q \in \mathrm{C}(\Sigma, A)$. We note that for any translation $p \in \mathrm{Tr}(\mathcal{A})$, there is a $P \in \mathrm{C}(\Sigma, A)$ such that $P^{\mathcal{A}} = p$ but this P may not be unique. In other words, $-^{\mathcal{A}}$ is a non-injective monoid epimorphism.

We also note that the mapping $-^{\mathcal{A}} : \mathrm{C}(\Sigma, A) \setminus \{\xi\} \to \mathrm{TrS}(\mathcal{A})$ is a semigroup epimorphism that assigns non-unit contexts of $\mathrm{C}(\Sigma, A)$ to translations of $\mathcal{A}$. Let us recall that $\mathrm{TrS}(\mathcal{A})$ is the translation semigroup of the algebra $\mathcal{A}$; cf. Chapter 4.

Lemma 5.2.8 *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra and X be a leaf alphabet disjoint from A. For any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^\rho$ there exist a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ and a tree language $T \subseteq \mathrm{T}(\Sigma, X \cup A)$ such that $L = T\varphi^{-1}$ and T is recognized by a finite power $\mathcal{A}^n$ where $n = |A|$.*

Proof. Let $\alpha : X \to \mathrm{Tr}(\mathcal{A})$ be an initial assignment for $\mathcal{A}^\rho$ and $F \subseteq \mathrm{Tr}(\mathcal{A})$ be an ideal of $\mathrm{OTr}(\mathcal{A})$ such that $L = \{t \in \mathrm{T}(\Lambda_{\mathcal{A}}, X) \mid t\alpha^{\mathcal{A}^\rho} \in F\}$. Let $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ be a tree homomorphism such that $\varphi_X(x) = x$ for all $x \in X$, and $\varphi_1(\overline{p}) \in \mathrm{C}(\Sigma, A)$ satisfies $\varphi_1(\overline{p})^{\mathcal{A}} = p$ for every $p \in \mathrm{Tr}(\mathcal{A})$. Obviously φ is a regular tree homomorphism. Suppose that $A = \{a_1, \ldots, a_n\}$. Let F' be the ideal of $\mathcal{A}^n$ generated by $\{(p(a_1), \ldots, p(a_n)) \in A^n \mid p \in F\}$, i.e., $(b_1, \ldots, b_m) \in F'$ iff there is a $p \in F$ such that $b_j \leqslant p(a_j)$ for every $j \leq n$. Define the initial assignment $\beta : X \cup A \to A^n$ for $\mathcal{A}^n$ by $a\beta = (a, \ldots, a) \in A^n$ for all $a \in A$ and $x\beta = \big((x\alpha)(a_1), \ldots, (x\alpha)(a_n)\big)$ for all $x \in X$. Let $T = \{t \in \mathrm{T}(\Sigma, X \cup A) \mid t\beta^{\mathcal{A}^n} \in F'\}$ be the tree language recognized by $(\mathcal{A}^n, \beta, F')$. We are proving that $L = T\varphi^{-1}$. Every tree w in $\mathrm{T}(\Lambda_{\mathcal{A}}, X)$ is of the form $w = \overline{p_1}(\overline{p_2}(\ldots \overline{p_k}(x) \ldots))$ where $p_1, \ldots, p_k \in \mathrm{Tr}(\mathcal{A})$ $(k \geq 0)$ and $x \in X$. For such a tree w, $w\alpha^{\mathcal{A}^\varrho} = x\alpha \cdot p_k \cdots p_2 \cdot p_1$ and $(w\varphi)\beta^{\mathcal{A}^n} = (x\alpha \cdot p_k \cdots p_2 \cdot p_1(a_1), \ldots, x\alpha \cdot p_k \cdots p_2 \cdot p_1(a_n))$. Hence, we have $w\varphi \in T$ iff $(w\varphi)\beta^{\mathcal{A}^n} \in F'$ which holds iff there is a $p \in F$ such that $x\alpha \cdot p_k \cdots p_2 \cdot p_1(a) \leqslant p(a)$ for every $a \in A$, or equivalently, $x\alpha \cdot p_k \cdots p_2 \cdot p_1 \lesssim_{\mathcal{A}} p$ for some $p \in F$, which is equivalent to $x\alpha \cdot p_k \cdots p_2 \cdot p_1 \in F$ or in other words $w\alpha^{\mathcal{A}^\varrho} \in F$ which means $w \in L$. □

Lemma 5.2.9 *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra and X be a leaf alphabet disjoint from $A \cup \Sigma$. For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$ recognized by $\mathcal{A}$ there exists a unary ranked alphabet Λ and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ such that φ is full with respect to T, and for every $z \in X \cup \Sigma_0$, $T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^\rho$.*

Proof. If we denote SOA(T) by $\mathcal{B} = (B, \Sigma, \leqslant')$, then $\mathcal{B} \preceq \mathcal{A}$. Suppose $T = \{t \in \mathrm{T}(\Sigma, X) \mid t\beta^{\mathcal{B}} \in F\}$ where $\beta : X \to B$ is an initial assignment for $\mathcal{B}$ and $F \trianglelefteq \mathcal{B}$. Since $\mathcal{B}$ is the least ordered algebra that recognizes T, the algebra $\mathcal{B}$ is generated by $\beta(X)$. The mapping $\beta : X \to B$ can be uniquely extended to a monoid homomorphism $\beta_{\mathrm{c}} : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Sigma, B)$. Since B is generated by $\beta(X)$, the mapping $\mathrm{C}(\Sigma, X) \to \mathrm{Tr}(\mathcal{B})$, $Q \mapsto \beta_c(Q)^{\mathcal{B}}$ is surjective. Let $\varphi : \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ be a tree homomorphism such that $\varphi_X(x) = x$ for all $x \in X \cup \Sigma_0$, and $\varphi_1(\overline{q}) = Q \in \mathrm{C}(\Sigma, X)$ satisfies $\beta_{\mathrm{c}}(Q)^{\mathcal{B}} = q$ for every $q \in \mathrm{Tr}(\mathcal{B})$. Note that φ is a regular tree homomorphism. It remains to show that φ is full with respect to T and that for every $z \in X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized by $\mathcal{B}^\rho$ as a subset of $\mathrm{T}(\Lambda, \{z\})$. This will finish the proof since $\mathrm{OTr}(\mathcal{B}) \preceq \mathrm{OTr}(\mathcal{A})$ follows from $\mathcal{B} \preceq \mathcal{A}$ by Proposition 5.1.3, and so $\mathcal{B}^\rho \preceq \mathcal{A}^\rho$ by Proposition 5.1.8, which implies that L_z can also be recognized by $\mathcal{A}^\rho$.

First, we show that φ is full with respect to T. For any $Q \in \mathrm{C}(\Sigma, X)$, $\overline{q}(\xi)\varphi_* \sim^T Q$ holds, where $q = \beta_{\mathrm{c}}(Q)^{\mathcal{B}} \in \mathrm{Tr}(\mathcal{B})$. By induction on the height of t we show that for any $t \in \mathrm{T}(\Sigma, X)$ there is an $s \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$ such that $t \approx^T s\varphi$. If $t = x \in X \cup \Sigma_0$, then $s\varphi \approx^T t$ for $s = t$. If $t = t' \cdot P$ for some $P \in \mathrm{C}(\Sigma, X)$ and $t' \in \mathrm{T}(\Sigma, X)$ such that the height of t' is less than the height of t, then by the induction hypothesis there is an $s' \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$ such that $t' \approx^T s'\varphi$. Also, for some $p \in \mathrm{Tr}(\mathcal{B})$, $\overline{p}(\xi)\varphi_* \sim^T P$ holds. Let $s = \overline{p}(s')$. Then $s\varphi = s'\varphi \cdot \overline{p}(\xi)\varphi_* \approx^T t' \cdot P = t$.

Second, we are showing that L_z can be recognized by $\mathcal{B}^\rho$ for a fixed $z \in X \cup \Sigma_0$. Let 1_B be the identity translation of $\mathcal{B}$. Define the initial assignment $\alpha : \{z\} \to \mathrm{Tr}(\mathcal{B})$ for $\mathcal{B}^\rho$ by $z\alpha = 1_B$, and let $F_z = \{q \in \mathrm{Tr}(\mathcal{B}) \mid q(z\beta^{\mathcal{B}}) \in F\}$. We show that $F_z \trianglelefteq \mathcal{B}^\rho$ and that L_z is recognized by $(\mathcal{B}^\rho, \alpha, F_z)$. For $p, q \in \mathrm{Tr}(\mathcal{B})$, if $p \lesssim_{\mathcal{B}} q \in F_z$ then $p(z\beta^{\mathcal{B}}) \leqslant' q(z\beta^{\mathcal{B}}) \in F$, so $p(z\beta^{\mathcal{B}}) \in F$, and thus $p \in F_z$. Hence $F_z \trianglelefteq \mathcal{B}^\rho$. Every $w \in \mathrm{T}(\Lambda_{\mathcal{B}}, \{z\})$ can be written in the form $w = \overline{q_1}(\overline{q_2}(\ldots \overline{q_h}(z) \ldots))$ for some $q_1, \ldots, q_h \in \mathrm{Tr}(\mathcal{B})$ $(h \geq 0)$. For such a tree w, $w\alpha^{\mathcal{B}^\rho} = 1_B \cdot q_h \cdots q_2 \cdot q_1$ and $(w\varphi)\beta^{\mathcal{B}} = q_h \cdots q_2 \cdot q_1(z\beta^{\mathcal{B}})$. Thus, $w \in L_z$ iff $w\varphi \in T$, or equivalently $(w\varphi)\beta^{\mathcal{B}} \in F$, which means $q_h \cdots q_2 \cdot q_1(z\beta^{\mathcal{B}}) \in F$. This is equivalent to $q_h \cdots q_2 \cdot q_1 \in F_z$, that is $w\alpha^{\mathcal{B}^\rho} \in F_z$. Hence we showed that $L_z = \{w \in \mathrm{T}(\Lambda, \{z\}) \mid w\alpha^{\mathcal{B}^\rho} \in F_z\}$. □

Now, we are almost ready to characterize the gPVTL's definable by syntactic ordered monoids. Before that we make a remark.

Remark 5.2.10 Let Λ be a unary ranked alphabet. For every leaf alphabet X and every subset $Y \subseteq X$, $\mathrm{C}(\Lambda, Y) = \mathrm{C}(\Lambda, X)$, and the quasi-order $\precsim_T$ for a tree language $T \subseteq \mathrm{T}(\Lambda, Y)$ on $\mathrm{C}(\Lambda, Y)$ is the same relation $\precsim_T$ on $\mathrm{C}(\Lambda, X)$ when T is viewed as a subset of $\mathrm{T}(\Lambda, X)$. Therefore, if a family of tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ is definable by syntactic ordered monoids, then for any unary ranked alphabet Λ and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$ (cf. Remark 4.2.11).

Proposition 5.2.11 *A family of recognizable tree languages $\mathscr{V}$ is definable by syntactic ordered monoids if and only if $\mathscr{V}$ is a gPVTL that satisfies the following conditions:*

(1) *The family $\mathscr{V}$ is closed under inverse regular tree homomorphisms.*
(2) *For every unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.*
(3) *For any regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ which is full with respect to a tree language $T \subseteq \mathrm{T}(\Omega, Y)$, if $T\varphi^{-1} \in \mathscr{V}(\Sigma, X)$ then $T \in \mathscr{V}(\Omega, Y)$.*

Proof. The fact that for any VFOM $\mathbf{M}$, $\mathbf{M}^t$ is a gPVTL follows from Corollary 5.2.3, that it satisfies the conditions (1) and (3) follows from Proposition 5.2.7, and that it satisfies the condition (2) follows from Remark 5.2.10.

For the converse, suppose that a gPVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ satisfies the conditions of the proposition. By Corollary 5.2.6 it is enough to show that $\mathscr{V}^a$ satisfies the condition of Proposition 5.1.10.

Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra in $\mathscr{V}^a$. By Lemma 5.2.8, any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^\rho$ can be written as $L = T\varphi^{-1}$ where $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ is a regular tree homomorphism and T is a tree language recognized by some power $\mathcal{A}^n$ of $\mathcal{A}$. Then $\mathcal{A}^n \in \mathscr{V}^a$ implies that $T \in \mathscr{V}(\Sigma, X \cup A)$, and hence $L = T\varphi^{-1} \in \mathscr{V}(\Lambda_{\mathcal{A}}, X)$ by (1). This holds for every tree language L recognizable by $\mathcal{A}^\rho$, so $\mathcal{A}^\rho \in \mathscr{V}^a$ by Corollary 3.2.8(2).

Now, suppose $\mathcal{A}^\rho \in \mathscr{V}^a$ for a finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$. Let $T \subseteq \mathrm{T}(\Sigma, X)$ be a tree language recognizable by $\mathcal{A}$. By Lemma 5.2.9 there exists a unary ranked alphabet Λ and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ that is full with respect to T such that for every z in $X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ is recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^\rho$. So, $L_z \in \mathscr{V}(\Lambda, \{z\})$, and thus $L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$ by (2). Hence, $T\varphi^{-1} = \bigcup_{z \in X \cup \Sigma_0} L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$. Since φ is full with respect to T, then $T \in \mathscr{V}(\Sigma, X)$ by (3). This holds for every tree language T recognizable by $\mathcal{A}$, so by Corollary 3.2.8(2) we have $\mathcal{A} \in \mathscr{V}^a$. □

We end the section with some further remarks and examples. Recall from Chapter 3 that the families Fin and Cof of finite and cofinite tree languages are gPVTL's. Here we show that neither of them is definable by syntactic ordered monoids. Let $\Omega = \Omega_1 = \{g\}$, $\Sigma = \Sigma_1 = \{f, g\}$, $X = \{x\}$, and let $T = \{x, f(x), f(f(x)), \ldots\} \subseteq \mathrm{T}(\Sigma, X)$. Define the tree homomorphism $\varphi : \mathrm{T}(\Omega, X) \to \mathrm{T}(\Sigma, X)$ by $\varphi_1(g) = g(\xi_1)$ and $\varphi_X(x) = x$. The tree homomorphism φ is full with respect to T since $f(\xi) \sim^T \xi = \xi\varphi_*$. Now, we have $T\varphi^{-1} = \{x\} \in \mathrm{Fin}(\Omega, X)$ but $T \notin \mathrm{Fin}(\Sigma, X)$. Also, if $T^d = \mathrm{T}(\Sigma, X) \setminus T$ is the complement of T, then $T^d\varphi^{-1} = (T\varphi^{-1})^d \in \mathrm{Cof}(\Omega, X)$

but $T^d \notin \mathrm{Cof}(\Sigma, X)$. Thus, neither Fin nor Cof satisfies condition (3) of Proposition 5.2.11. Note that φ is also full with respect to T^d.

One can define the *ordered translation semigroup* of an ordered algebra and the the *syntactic ordered semigroup* of a tree language as in Definition 5.1.1 and (the notes after) Definition 5.2.1. Below we show that the definability by ordered semigroups is not comparable to the definability by ordered monoids (cf. Proposition 4.3.7). For a variety of ordered semigroups $\mathbf{S}$, let $\mathbf{S}^a$ be the class of ordered algebras whose ordered translation semigroups are in $\mathbf{S}$, and $\mathbf{S}^t$ be the family of tree languages whose syntactic ordered semigroups are in $\mathbf{S}$. It can be shown that for such an $\mathbf{S}$, the class $\mathbf{S}^a$ is a gVFOA, and the family $\mathbf{S}^t$ is a gPVTL.

(**I**) Let $\mathbf{M}$ be the variety of finite ordered monoids defined by the equation $\mathsf{x} \cdot \mathsf{y} \cdot \mathsf{y} = \mathsf{x}$. Let $\Sigma = \Sigma_1 = \{f\}$, and let the ordered algebras $\mathcal{A} = (\{a\}, \Sigma, \leqslant)$ and $\mathcal{B} = (\{a, b\}, \Sigma, \leqslant')$ be defined by

(1) $\leqslant = \{(a, a)\}$, $f^{\mathcal{A}}(a) = a$;

(2) $\leqslant' = \{(a, a), (a, b), (b, b)\}$, $f^{\mathcal{B}}(a) = f^{\mathcal{B}}(b) = a$.

It is straightforward to see that the ordered translation semigroup of $\mathcal{A}$ consists of unit element $f^{\mathcal{A}}(\xi) = 1_A$ like its ordered translation monoid. This is also the case for the ordered translation semigroup of $\mathcal{B}$, while the ordered translation monoid of $\mathcal{B}$ consists of two elements $f^{\mathcal{B}}(\xi)$ and 1_B, where $f^{\mathcal{B}}(\xi) \lneqq_{\mathcal{B}} 1_B$. So, $\mathcal{A} \in \mathbf{M}^a$ but $\mathcal{B} \notin \mathbf{M}^a$ which shows that $\mathbf{M}^a$ is not definable by ordered translation semigroups. Similarly, the positive variety $\mathbf{M}^t$ is not definable by syntactic ordered semigroups.

(**II**) Let $\mathbf{S}$ be the the variety of right zero ordered semigroups, i.e., $\mathbf{S}$ is defined by the equation $\mathsf{x} \cdot \mathsf{y} = \mathsf{y}$. Let $\Sigma = \Sigma_1 = \{f, g, h\}$ be a ranked alphabet. We define two ordered algebras $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (A, \Sigma, \leqslant)$, where $A = \{a, b\}$ and $\leqslant = \{(a, a), (b, a), (b, b)\}$, as follows:

(1) $f^{\mathcal{A}}(a) = h^{\mathcal{A}}(a) = f^{\mathcal{A}}(b) = h^{\mathcal{A}}(b) = b$, $g^{\mathcal{A}}(a) = g^{\mathcal{A}}(b) = a$;

(2) $f^{\mathcal{B}}(a) = f^{\mathcal{B}}(b) = h^{\mathcal{B}}(b) = b$, $g^{\mathcal{B}}(a) = g^{\mathcal{B}}(b) = h^{\mathcal{B}}(a) = a$.

It can be directly verified that the translation semigroup of $\mathcal{A}$ consists of two elements $\{f^{\mathcal{A}}(\xi), g^{\mathcal{A}}(\xi)\}$ and the translation semigroup of $\mathcal{B}$ consists of three elements $\{f^{\mathcal{B}}(\xi), g^{\mathcal{B}}(\xi), h^{\mathcal{B}}(\xi)\}$ in which $h^{\mathcal{B}}(\xi)$ is the unit element. The orders are as follows:

(i) $h^{\mathcal{A}}(\xi) = f^{\mathcal{A}}(\xi) \lneqq_{\mathcal{A}} 1_A \lneqq_{\mathcal{A}} g^{\mathcal{A}}(\xi)$;

(ii) $f^{\mathcal{B}}(\xi) \lneqq_{\mathcal{B}} h^{\mathcal{B}}(\xi) = 1_B \lneqq_{\mathcal{B}} g^{\mathcal{B}}(\xi)$.

Thus $\mathcal{A}$ and $\mathcal{B}$ have isomorphic ordered translation monoids while their ordered translation semigroups are not isomorphic. Moreover, $\mathcal{A} \in \mathbf{S}^a$ and $\mathcal{B} \notin \mathbf{S}^a$. So, $\mathbf{S}^a$ is not definable by ordered translation monoids. Similarly, the positive variety $\mathbf{S}^t$ is not definable by syntactic ordered monoids.

Recall the operations $-^d$ from Chapter 3. It is easy to show that $\mathrm{OTr}(\mathcal{A}^d) \cong \mathrm{OTr}(\mathcal{A})^d$ for any finite ordered algebra $\mathcal{A}$. It follows that $\mathrm{SOM}(T^d) \cong \mathrm{SOM}(T)^d$ for any tree language T. Hence, $(\mathbf{M}^a)^d = (\mathbf{M}^d)^a$ and $(\mathbf{M}^t)^d = (\mathbf{M}^d)^t$ for any VFOM $\mathbf{M}$. Thus, any gVFOA $\mathbf{K}$ is definable by ordered translation monoids exactly in case $\mathbf{K}^d$ is. Similarly, any gPVTL $\mathscr{V}$ is definable by syntactic ordered monoids iff $\mathscr{V}^d$ is so.

5.3 Examples of varieties

Here we characterize the varieties of finite algebras and tree languages that are definable by semilattice monoids. This is an instance of the variety theorem, Proposition 4.2.14, in Chapter 4 (cf. Theorem 6.3 of [39]). We also introduce symbolic finite ordered algebras and symbolic tree languages, and show that they are definable by semilattice ordered monoids in which the unit is the greatest element (called symbolic ordered monoids here). This is an instance of Propositions 5.1.10 and 5.2.11.

A monoid $(M, \cdot)$ is a *semilattice monoid* if it is commutative and idempotent, i.e., for every $a, b \in M$, $a \cdot a = a$ and $a \cdot b = b \cdot a$ hold. Sequences of elements of a given set D are denoted by bold face lower case letters. For example $\mathbf{d}$ is a (possibly empty) sequence $\langle d_1, \ldots, d_m \rangle$ where $d_1, \ldots, d_m \in D$. For simplicity we write $\mathbf{d} \in D$ when all components of the sequence $\mathbf{d}$ belong to D. In that case for a function symbol $f \in \Sigma_{m+1}$, $f(d, \mathbf{d})$ stands for $f(d, d_1, \ldots, d_m)$. We assume that the lengths of the sequences always add up correctly.

Definition 5.3.1 An algebra $\mathcal{A} = (A, \Sigma)$ is a *semilattice algebra* if it satisfies the following two identities for every $f, g \in \Sigma$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, a \in A$:

$$f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{b}) = f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b});$$
$$f^{\mathcal{A}}(\mathbf{a}, g^{\mathcal{A}}(\mathbf{c}, a, \mathbf{d}), \mathbf{b}) = g^{\mathcal{A}}(\mathbf{c}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{d}).$$

The class of all semilattice Σ-algebras is denoted by $\mathbf{SL}(\Sigma)$.

Lemma 5.3.2 *An algebra is semilattice if and only if its translation monoid is a semilattice; and the class* $\mathbf{SL}(\Sigma)$ *is a variety of finite algebras.* □

The proof is straightforward.

Lemma 5.3.3 *Let* $\mathcal{A} = (A, \Sigma)$ *be a semilattice algebra. For any* $a, b \in A$ *and translations* $p, q \in \mathrm{Tr}(\mathcal{A})$ *the following hold:*

(1) *if* $p(q(a)) = a$ *then* $p(a) = q(a) = a$*;*
(2) *if* $p(a) = b$ *and* $a = q(b)$ *then* $a = b$*.*

Proof. Claim (2) is an immediate corollary of (1). Let us prove (1). Suppose $p, q \in \mathrm{Tr}(\mathcal{A})$. Since $q \cdot q = q$, $p \cdot p = p$ and $q \cdot p = p \cdot q$, we have $q(a) = q(p(q(a))) = q(q(p(a))) = q(p(a)) = p(q(a)) = a$, and similarly $p(a) = p(p(q(a))) = p(q(a)) = a$. □

The following 6 lemmas are some identities of semilattice algebras that will be used later. Let $\mathcal{A} = (A, \Sigma)$ be a semilattice algebra.

Lemma 5.3.4 For any $\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b \in A$ and $f \in \Sigma$,
(s1) $f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})$.

Proof. If $p = f^{\mathcal{A}}(\mathbf{a}, \xi, \mathbf{b}, b, \mathbf{c})$, then
$f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}), \mathbf{b}, b, \mathbf{c})$
$= f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, b, \mathbf{c}), \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}), \mathbf{c}) =$
$= f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}), \mathbf{b}, b, \mathbf{c}) = p(f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}))$.
By swapping a and b the identity $f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}) = q(f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}))$ can be proved for some $q \in \mathrm{Tr}(\mathcal{A})$. Now Lemma 5.3.3(2) implies that $f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})$. □

Lemma 5.3.5 For any $\mathbf{a}, a, b \in A$ and $f \in \Sigma$,
(s2) $f^{\mathcal{A}}(a, a, b, \mathbf{a}) = f^{\mathcal{A}}(a, b, b, \mathbf{a})$.

Proof. By (s1) Lemma 5.3.4, we have:
$f^{\mathcal{A}}(a, a, b, \mathbf{a}) = f^{\mathcal{A}}(f^{\mathcal{A}}(a, a, b, \mathbf{a}), a, b, \mathbf{a})$
$= f^{\mathcal{A}}(a, a, f^{\mathcal{A}}(b, a, b, \mathbf{a}), \mathbf{a}) = f^{\mathcal{A}}(f^{\mathcal{A}}(a, b, b, \mathbf{a}), a, a, \mathbf{a}) = p(f^{\mathcal{A}}(a, b, b, \mathbf{a}))$,
where $p = f^{\mathcal{A}}(\xi, a, a, \mathbf{a})$. By the same argument and swapping a and b it can be proved that $f^{\mathcal{A}}(a, b, b, \mathbf{a}) = q(f^{\mathcal{A}}(a, a, b, \mathbf{a}))$ for some $q \in \mathrm{Tr}(\mathcal{A})$. Hence, $f^{\mathcal{A}}(a, a, b, \mathbf{a}) = f^{\mathcal{A}}(a, b, b, \mathbf{a})$ by Lemma 5.3.3(2). □

Lemma 5.3.6 For any $\mathbf{a}, \mathbf{b}, a, b \in A$ and $f, g \in \Sigma$,
(s3) $f^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{a}), b, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(b, \mathbf{a}), a, \mathbf{b})$.

Proof. The second equality follows from (s1) Lemma 5.3.4:
$f^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{a}), b, \mathbf{b}) = g^{\mathcal{A}}(f^{\mathcal{A}}(a, b, \mathbf{b}), \mathbf{a}) = g^{\mathcal{A}}(f^{\mathcal{A}}(b, a, \mathbf{b}), \mathbf{a})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(b, \mathbf{a}), a, \mathbf{b})$. □

Lemma 5.3.7 For any $a_1, a_2, \cdots, a_m \in A$ and $f \in \Sigma_m$,
(s4) $f^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \cdots, a_1), a_2, \cdots, a_m) = f^{\mathcal{A}}(a_1, a_2, \cdots, a_m)$.

Proof. The third equality is implied by (s2) in Lemma 5.3.5:
$f^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \cdots, a_1), a_2, \cdots, a_m) = f^{\mathcal{A}}(a_1, \cdots, a_1, f^{\mathcal{A}}(a_1, a_2, \cdots, a_m))$
$= f^{\mathcal{A}}(a_1, f^{\mathcal{A}}(a_1, \cdots, a_1, a_1, a_2), a_3, \cdots, a_m)$
$= f^{\mathcal{A}}(a_1, f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, a_2), a_3, \cdots, a_m)$
$= f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m))$.
Now, we show for any $j < m$,
$f^{\mathcal{A}}(a_1, \cdots, a_1, a_1, a_2, \ldots, a_j, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m))$
$= f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, a_{j+1}, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m))$,
as follows, by using Lemmas 5.3.4 (s1), and 5.3.5 (s2),
$f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m))$

$= f^{\mathcal{A}}(a_1, a_2, \cdots, a_j, f^{\mathcal{A}}(a_1, \cdots, a_1, a_1, a_2, \ldots, a_j, a_{j+1}), a_{j+2}, \cdots, a_m)$
$= f^{\mathcal{A}}(a_1, a_2, \cdots, a_j, f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, a_{j+1}, a_{j+1}), a_{j+2}, \cdots, a_m)$
$= f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, a_{j+1}, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m))$.

By repeating this argument $m-1$ times, we get

$f^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \cdots, a_1), a_2, \cdots, a_m)$
$= f^{\mathcal{A}}(a_1, \cdots, a_{m-1}, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)) = f^{\mathcal{A}}(a_1, a_2, \cdots, a_m)$. □

Lemma 5.3.8 For any $a, b, \mathbf{a}, \mathbf{b} \in A$ and $f \in \Sigma$,
(s5) $f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), b, \mathbf{b})$.

Proof. We distinguish two cases.
(1) If the sequence $\mathbf{a}$ is empty, then by using identities (s4), (s3), (s1), (s3), (s3) and (s4) consecutively, we get

$f^{\mathcal{A}}(g^{\mathcal{A}}(a, b), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, g^{\mathcal{A}}(b, b)), a, \mathbf{b})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(b, g^{\mathcal{A}}(a, b)), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, b), b), a, \mathbf{b})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, b), a), b, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, a), b), b, \mathbf{b})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b), b, \mathbf{b})$.

(2) If the sequence $\mathbf{a}$ is not empty, then write $\mathbf{a} = (c, \mathbf{c})$ and use identities (s3), (s1), (s2) and (s3) consecutively

$f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, c, \mathbf{c}), a, \mathbf{b})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, a, \mathbf{c}), c, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, a, b, \mathbf{c}), c, \mathbf{b})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, b, \mathbf{c}), c, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, c, \mathbf{c}), b, \mathbf{b})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), b, \mathbf{b})$. □

Lemma 5.3.9 For any $f \in \Sigma_m$ and $g \in \Sigma_n$ where $m \leq n$ and $n \geq 2$, and any $a, b, \mathbf{a}, \mathbf{b}, \mathbf{c} \in A$ where the sequence $\bar{b}$ consists of $n-m$ times b,
(s6) $f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), \mathbf{b}), \mathbf{c}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{b}, \bar{b}), b, \mathbf{a}), \mathbf{c})$.

Proof. Use identities (s1), (s3) and (s4) alternatively:

$f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), \mathbf{b}), \mathbf{c}) = f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a, g^{\mathcal{A}}(b, \ldots, b), \mathbf{a}), \mathbf{b}), \mathbf{c})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(b, \ldots, b), \mathbf{b}), a, \mathbf{a}), \mathbf{c})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(b, \ldots, b), \bar{b}, \mathbf{b}), a, \mathbf{a}), \mathbf{c})$
$= g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(b, \ldots, b), \bar{b}, \mathbf{b}), \mathbf{c}), a, \mathbf{a})$
$= g^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(f^{\mathcal{A}}(b, \ldots, b), \mathbf{c}), \bar{b}, \mathbf{b}), a, \mathbf{a})$
$= g^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(b, \mathbf{c}), \bar{b}, \mathbf{b}), a, \mathbf{c}) = g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(b, \bar{b}, \mathbf{b}), \mathbf{c}), a, \mathbf{a})$
$= f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(b, \bar{b}, \mathbf{b}), a, \mathbf{a}), \mathbf{c}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{b}, \bar{b}), b, \mathbf{a}), \mathbf{c})$. □

We note that the identity corresponding to (s6) for $m = n = 1$ also holds, i.e., $f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a))) = f^{\mathcal{A}}(g^{\mathcal{A}}(a)) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a)))$.

We can assume that the leaf alphabets X are always disjoint from the ranked alphabet Σ.

Definition 5.3.10 For a tree $t \in \mathrm{T}(\Sigma, X)$, the *contents* $\mathrm{c}(t)$ of t is the set of symbols from $\Sigma \cup X$ which appear in t. It can be defined inductively as:

(1) $\mathrm{c}(x) = \{x\}$ for $x \in \Sigma_0 \cup X$;

(2) $\mathrm{c}(f(t_1, \ldots, t_m)) = \{f\} \cup \mathrm{c}(t_1) \cup \cdots \cup \mathrm{c}(t_m)$ for $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$ and $f \in \Sigma_m$.

For any subset $Z \subseteq \Sigma \cup X$ let $C(Z) = \{t \in \mathrm{T}(\Sigma, X) \mid Z = \mathrm{c}(t)\}$. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *semilattice* if it is a union of tree languages of the form $C(Z)$ for some subsets $Z \subseteq \Sigma \cup X$. The family of all semilattice ΣX-tree languages is denoted by $\mathrm{SL}(\Sigma, X)$, and $\mathrm{SL}_\Sigma = \{\mathrm{SL}(\Sigma, X)\}$ is the family of semilattice tree languages.

Lemma 5.3.11 *For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ the following properties are equivalent:*

(1) *T is semilattice;*

(2) *for all trees $t, t' \in \mathrm{T}(\Sigma, X)$, $\mathrm{c}(t) = \mathrm{c}(t')$ and $t \in T$ imply $t' \in T$;*

(3) $T = \bigcup_{t \in T} C(\mathrm{c}(t))$.

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are straightforward. Let us verify the implication (2) $\Rightarrow$ (3). The inclusion $T \subseteq \bigcup_{t \in T} C(\mathrm{c}(t))$ always holds. Suppose $t' \in C(\mathrm{c}(t))$ for some $t \in T$. Then $\mathrm{c}(t) = \mathrm{c}(t')$, and so $t' \in T$, whence $\bigcup_{t \in T} C(\mathrm{c}(t)) \subseteq T$. □

The rest of this subsection is devoted to proving the fact that semilattice tree languages are definable by semilattice algebras, i.e., $\mathrm{SL}_\Sigma = \mathbf{SL}(\Sigma)^t$.

Fix a ranked alphabet Σ and a leaf alphabet X. The sequences of trees are denoted by bold face letters, e.g., $\mathbf{t}$ is a sequence like $(t_1, \ldots, t_m)$ for some trees $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$.

Let σ be a congruence on $\mathcal{T}(\Sigma, X)$ such that $\mathcal{T}(\Sigma, X)/\sigma$ is a semilattice algebra, i.e., it satisfies the following relations for all symbols $f, g \in \Sigma$ and trees $\mathbf{t}, \mathbf{r}, \mathbf{u}, \mathbf{v}, t \in \mathrm{T}(\Sigma, X)$:

(d1) $f(\mathbf{t}, f(\mathbf{t}, t, \mathbf{r}), \mathbf{r}) \;\sigma\; f(\mathbf{t}, t, \mathbf{r})$

(d2) $f(\mathbf{t}, g(\mathbf{u}, t, \mathbf{v}), \mathbf{r}) \;\sigma\; g(\mathbf{u}, f(\mathbf{t}, t, \mathbf{r}), \mathbf{v})$.

The following lemma is implied by Lemmas 5.3.4, 5.3.5, 5.3.6, 5.3.7, 5.3.8 and 5.3.9.

Lemma 5.3.12 The following relations hold for any $f \in \Sigma_m$, $g \in \Sigma_n$, and any ΣX-trees $t, s, \mathbf{r}, \mathbf{t}, \mathbf{s}$:

(s1) $f(\mathbf{t}, t, \mathbf{r}, r, \mathbf{u}) \;\sigma\; f(\mathbf{t}, r, \mathbf{r}, t, \mathbf{u})$;

(s2) $f(t, t, r, \mathbf{t}) \;\sigma\; f(t, r, r, \mathbf{t})$;

(s3) $f(g(t, \mathbf{t}), r, \mathbf{r}) \;\sigma\; f(g(r, \mathbf{t}), t, \mathbf{r})$;

(s4) $f(f(t, \cdots, t), \mathbf{t}) \;\sigma\; f(t, \mathbf{t})$;

(s5) $f(g(t, r, \mathbf{t}), t, \mathbf{r}) \;\sigma\; f(g(t, r, \mathbf{t}), r, \mathbf{r})$;

(s6) $f(f(g(t, s, \mathbf{t}), \mathbf{r}), \mathbf{u}) \;\sigma\; f(g(g(t, \mathbf{r}, \overline{r}), r, \mathbf{t}), \mathbf{u})$

where $m \leq n$ and the sequence $\overline{r}$ consists of $n - m$ times r. □

The family of Σ-congruences on $\mathcal{T}(\Sigma, X)$ satisfying (d1) and (d2) is closed under intersections and contains the universal relation $\nabla_{\mathrm{T}(\Sigma,X)} = \mathrm{T}(\Sigma, X) \times \mathrm{T}(\Sigma, X)$, and so it has a smallest element τ. Our aim is to prove that τ is determined by $t_1 \,\tau\, t_2 \iff \mathrm{c}(t_1) = \mathrm{c}(t_2)$ for any trees t_1, t_2.

Suppose that the elements of $\Sigma \setminus \Sigma_0$ are linearly ordered in such a way that function symbols with smaller arity are smaller than function symbols with greater arity. Assume also that the leaves $X \cup \Sigma_0$ are linearly ordered.

Let $\mathrm{c}_\Sigma(t) = (\Sigma \setminus \Sigma_0) \cap \mathrm{c}(t)$ be the set of nodes of a tree $t \in \mathrm{T}(\Sigma, X)$ and $\mathrm{c}_X(t) = (X \cup \Sigma_0) \cap \mathrm{c}(t)$ be its set of leaves.

A tree t is in the *canonical form* if:

(1) either $t \in X \cup \Sigma_0$, or
(2) $t = f(t_1, x_2, \ldots, x_m)$ where
   (a) t_1 is in the canonical form and $x_2 \leq \ldots \leq x_m \in \Sigma_0 \cup X$,
   (b) f is the smallest in $\mathrm{c}_\Sigma(t)$,
   (c) either $f \notin \mathrm{c}_\Sigma(t_1)$ or $\mathrm{c}_\Sigma(t_1) = \{f\}$ and then $|\mathrm{c}_X(t_1)| > 1$,
   (d) if $|\mathrm{c}_X(t)| > m - 1$ then $x_2 \lneqq \ldots \lneqq x_m$ are the smallest $m - 1$ elements in $\mathrm{c}_X(t)$, and
   (e) otherwise if $\mathrm{c}_X(t) = \{x_2, \ldots, x_n\}$ with $n \leq m$, then $x_2 \lneqq \ldots \lneqq x_n$, $x_{n+1} = \ldots = x_m = x_n$ and $\mathrm{c}_X(t_1) = \{x_n\}$.

In other words, a tree is in the canonical form if on each its level only the leftmost node may be from $\Sigma \setminus \Sigma_0$, all the others are leaves from $\Sigma_0 \cup X$, nodes grow from the root downwards and leaves grow from left to right and from top to down. As soon as the set of nodes or leaves is exhausted, the last symbol from the exhausted set is repeated as long as there are still symbols in the other set to be used.

Let us fix σ to be any congruence on $\mathcal{T}(\Sigma, X)$ satisfying (d1) and (d2). Our aim is to show that every tree t is σ-equivalent to a tree t' in the canonical form where $\mathrm{c}(t) = \mathrm{c}(t')$. A tree is called *leftmost branching* if its every subtree is either a leaf or of the form $f(t, \mathbf{x})$ where t is a tree and $\mathbf{x}$ is a sequence of leaves (from $X \cup \Sigma_0$). For a tree t, the root of t, in notation $\mathrm{root}(t)$, is its topmost symbol. Transformation of a tree into a σ-equivalent tree in the canonical form consists of the following steps.

Step 1. *Shaping the tree into a leftmost branching tree while arranging the nodes in the increasing order from top to down.*

We show that this can be done by induction on the number of nodes and leaves in t. The claim clearly holds for $t \in \Sigma_0 \cup X$. Suppose that $t = f(t_1, t_2, \ldots, t_n)$ where $t_1, \ldots, t_m$ have the shape of a leftmost branching tree and the nodes are in increasing order. Let $g = \min\{\mathrm{root}(t_1), \ldots, \mathrm{root}(t_m)\}$. Without loosing generality, by (s1), we can assume that $g = \mathrm{root}(t_1)$, and let $t_1 = g(t_1', x_2, \ldots, x_n)$. We distinguish two cases:

If $g \leq f$ then by (d2),

$$\begin{aligned} t = \; & f(g(t_1', x_2, \ldots, x_n), t_2, \ldots, t_m) \; \sigma \\ \sigma \; & g(f(t_1', t_2, \ldots, t_m), x_2, \ldots, x_n), \end{aligned}$$

and now we can apply the induction hypotheses to $f(t_1', t_2, \ldots, t_m)$.

If $f < g$ then $m \leq n$ and by (s3) we have

$$\begin{aligned} t = \; & f(g(t_1', x_2, \ldots, x_n), t_2, \ldots, t_m) \; \sigma \\ \sigma \; & f(g(t_1', t_2, \ldots, t_m, x_2, \ldots, x_{n-m+1}), x_{n-m+2}, \ldots, x_n), \end{aligned}$$

and then we can continue by induction.

We get a tree of the desired shape with nodes increasing from top to down, but there may be repetitions of same nodes.

Step 2. *Removing repetitions of nodes different from the greatest node.*

The clause (s6) of Lemma 5.3.4 provides a transformation that pushes repetitions, i.e., if $f \leq g$ and ffg is a subsequence of the sequence of nodes, then the transformation will replace an extra copy of f by a copy of g. Namely, let $f_1, \ldots, f_{i-1}, f_i, \ldots, f_i, f_{i+1}, \ldots, f_k$, $k \in \mathbb{N}$, be the sequence of nodes read from the root downwards after Step 1, and assume that f_i is the first repeated symbol. By applying (s6) from Lemma 5.3.4, the last copy of f_i is replaced by a new copy of f_{i+1}. This is repeated as long as there is more than one f_i in the sequence. Thus all repetitions of f_i are replaced by repetitions of f_{i+1}. After that, the last copy of f_{i+1} is replaced by a new copy of f_{i+2}, etc. Finally, only the last symbol f_k may have multiple copies, all the others appear only once.

After these transformations we get a tree σ-equivalent to t, branching only in the leftmost node and with increasing nodes where only the greatest node is possibly repeated. The tree is still not in the canonical form since leaves are not necessarily already arranged.

Step 3. *Arranging leaves into increasing order.*

The sequence of leaves is read starting from left to right and from top downwards. This sequence can be sorted using standard algorithms for sorting sequences what assumes comparing the first symbol with the rest one by one and when a smaller one appears swap them and continue comparing the new first symbol with the rest of the sequence. After this the smallest leaf is on the first place. Repeat the same with the second one and the rest of the sequence, etc. We note that this swapping is supported by σ, since places of leaves on the same level can be changed by (s1), and if they are on different levels then (s3) can be applied.

After this, leaves will be in increasing order, but there are possibly repetitions of those leaves which are not the greatest.

Step 4. *Removing repetitions of leaves different from the greatest leaf.*

The idea is the same as in Step 2, the repetition of a smaller leaf is replaced by a repetition of the next greater leaf, so that repetitions are pushed trough the sequence and finally only the greatest leaf may be repeated. In other words, if $x < y$ then the subsequence of leaves of the form xxy is replaced by xyy. We distinguish four cases.

First, xxy appears on the same level, i.e., as the components of the same node. This case is solved by applying (s2).

Second, the first x is on one level and the second x and y are both on the next. This is solved easily by applying first (s1), then (s5) and so changing the first x into y, then applying (s3) to swap x and outer y, and finally once more (s1):

$$\begin{aligned} & f(g(t,x,y,\mathbf{x}),\mathbf{y},x) \;\sigma\; f(g(t,x,y,\mathbf{x}),x,\mathbf{y}) \;\sigma \\ \sigma\;\; & f(g(t,x,y,\mathbf{x}),y,\mathbf{y}) \;\sigma\; f(g(t,y,y,\mathbf{x}),x,\mathbf{y}) \;\sigma\; f(g(t,y,y,\mathbf{x}),\mathbf{y},x). \end{aligned}$$

Third, both x's are on the upper level and y is on the lower. We proceed as

$$\begin{aligned} & f(g(t,y,\mathbf{x}),\mathbf{y},x,x) \;\sigma\; f(g(x,y,\mathbf{x}),\mathbf{y},x,t) \;\sigma \\ \sigma\;\; & f(g(x,y,\mathbf{x}),\mathbf{y},y,t) \;\sigma\; f(g(t,y,\mathbf{x}),\mathbf{y},y,x) \;\sigma\; f(g(t,y,\mathbf{x}),\mathbf{y},x,y). \end{aligned}$$

Note that t is needed here and the existence of such a symbol follows from the fact that $f \le g$ and thus the arity of g is at least 2.

Fourth, all three leaves appear on different levels. The tree is of the form $f(g(h(t,y,\mathbf{z}),x),x)$ where $f,g \in \Sigma_2$, and so the arity of h is at least two. The first x should be changed into y. The transformation is as follows:

$$\begin{aligned} & f(g(h(t,y,\mathbf{z}),x),x) \;\sigma\; f(g(h(x,y,\mathbf{z}),t),x) \;\sigma \\ \sigma\;\; & f(g(h(x,y,\mathbf{z}),x),t) \;\sigma\; f(g(h(x,y,\mathbf{z}),y),t) \;\sigma\; f(g(h(x,y,\mathbf{z}),t),y) \;\sigma \\ \sigma\;\; & f(g(h(t,y,\mathbf{z}),x),y) \;\sigma\; f(g(h(t,y,\mathbf{z}),y),x). \end{aligned}$$

After this, our tree has almost the canonical form, the only disturbing thing may be the existence of a too long subtree at the end having only the greatest symbol from $c_\Sigma(t)$ as nodes and the greatest element from $c_X(t)$ as leaves.

Step 5. *Fold the unnecessary part.*

Applying (s4) as many times as needed the tree is folded into one without repetitions of the greatest symbol from $c_\Sigma(t)$, or with its repetitions but not with only the greatest element of $c_X(t)$ as leaves on the deepest level.

This finishes the procedure.

Clearly, the procedure results in a unique tree in the canonical form which is σ-equivalent to a given tree.

For example, suppose $h \in \Sigma_3$, $f,g \in \Sigma_2$, $c \in \Sigma_0$, $x \in X$, and the orders of symbols are $f < g < h$ and $x < c$. Let $t = h(g(x,f(x,c)),x,g(x,c))$. Then by applying the above steps we get the tree r_j in the j-th step as follows:

$$\begin{array}{rl} t\ \sigma & r_1 = f(g(g(h(x,x,x),c),x),c) \\ \sigma & r_2 = f(g(h(h(x,c,x),x,x),x),c) \\ \sigma & r_3 = f(g(h(h(c,c,x),x,x),x),x) \\ \sigma & r_4 = f(g(h(h(c,c,c),c,c),c),x) \\ \sigma & r_5 = f(g(h(c,c,c),c),x). \end{array}$$

It can be noticed that the canonical form tree corresponding to a given tree t is determined by $c(t)$ and can be constructed directly from this set. The procedure can roughly be described as follows:

1. put the smallest node in the root of the tree, draw the necessary branches, put the next smallest symbol from $c_\Sigma(t)$ in the left most node, continue doing this as long as $c_\Sigma(t)$ is not exhausted;
2. put the smallest leaf in the topmost leftmost free place, choose the next smallest and put in the next place, etc., as long as there are free places in the tree or the set $c_X(t)$ of leaves is not empty;
3. if not all $c_X(t)$ is used, continue building the tree by shifting all symbols on the last level by one place to the right, return the last leaf to $c_X(t)$, put the greatest element of $c_\Sigma(t)$ to the leftmost place, add its arity new branches, fill them with remaining symbols from $c_X(t)$ as in Step 2, and repeat this step until the whole $c_X(t)$ is used;
4. if there are still free places put the greatest symbol from $c_X(t)$ there.

Recall that τ denotes the smallest congruence satisfying (d1) and (d2).

Lemma 5.3.13 *For any trees t_1 and t_2, $t_1 \tau t_2 \iff \mathrm{c}(t_1) = \mathrm{c}(t_2)$.*

Proof. Define τ' by $t_1\tau' t_2$ iff $\mathrm{c}(t_1) = \mathrm{c}(t_2)$. Obviously τ' satisfies (d1) and (d2). Let σ be any congruence satisfying (d1) and (d2). We are proving $\tau' \subseteq \sigma$. Assume $t_1\tau' t_2$. There are trees t_1' and t_2' in canonical form such that $t_1\sigma t_1'$ and $t_2\sigma t_2'$. Then $\mathrm{c}(t_1') = \mathrm{c}(t_1) = \mathrm{c}(t_2) = \mathrm{c}(t_2')$ and since the canonical tree is uniquely determined by its contents, it follows that $t_1' = t_2'$ which immediately implies that $t_1\sigma t_2$. Therefore, τ' is the smallest congruence satisfying (d1) and (d2), and thus $\tau = \tau'$. □

Proposition 5.3.14 *A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is semilattice if and only if it is recognizable by a finite semilattice algebra.*

Proof. Since semillatice algebras form a variety of finite algebras, it suffices to prove that a tree language is semilattice iff its syntactic algebra is semilattice. By Lemma 5.3.13, T is a semilattice tree language iff $\tau \subseteq \approx^T$ iff the syntactic algebra of T is a semilattice algebra. □

Corollary 5.3.15 *Family* SL_Σ *is a variety of tree languages and the identity* $\mathrm{SL}_\Sigma = \mathbf{SL}(\Sigma)^t$ *holds.* □

Let $\mathbf{SL} = \{\mathbf{SL}(\Sigma)\}$ the class of all semilattice finite algebras, and $\mathrm{SL} = \{\mathrm{SL}(\Sigma, X)\}$ be the family of all semilattice tree languages. Recall the notions of gVFA and gVTL from Chapter 4.

Proposition 5.3.16 *Class* $\mathbf{SL}$ *is a gVFA, family* SL *is a gVTL, and moreover* $\mathrm{SL} = \mathbf{SL}^t$. □

We call an ordered monoid *symbolic* if it is a semilattice monoid and its unit is the greatest element.

Definition 5.3.17 An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *symbolic*, if it is a semilattice algebra and $f^{\mathcal{A}}(a_1, \cdots, a_m) \leqslant a_j$ holds for every $f \in \Sigma_m$ $(m > 0)$, $j \leq m$, and $a_1, \cdots, a_m \in A$.

For a subset $Z \subseteq \Sigma \cup X$ let $T(Z) = \{t \in \mathrm{T}(\Sigma, X) \mid Z \subseteq \mathrm{c}(t)\}$. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *symbolic*, if it is a union of tree languages of the form $T(Z)$ for some subsets $Z \subseteq \Sigma \cup X$.

Lemma 5.3.18 (i) *An ordered algebra* $\mathcal{A} = (A, \Sigma, \leqslant)$ *is symbolic, iff it is a semilattice algebra and* $p(a) \leqslant a$ *holds for every* $p \in \mathrm{Tr}(\mathcal{A})$ *and* $a \in A$*, iff its ordered translation monoid is symbolic.*

(ii) *A tree language* $T \subseteq \mathrm{T}(\Sigma, X)$ *is symbolic, iff for all trees* t, t' *in* $\mathrm{T}(\Sigma, X)$, $\mathrm{c}(t) \subseteq \mathrm{c}(t')$ *and* $t \in T$ *imply* $t' \in T$. □

We denote the class of symbolic ordered Σ-algebras by $\mathbf{Sym}(\Sigma)$ and $\mathrm{Sym}_\Sigma = \{\mathrm{Sym}(\Sigma, X)\}$ denotes the family of symbolic tree languages. It can be directly verified that $\mathbf{Sym}(\Sigma)$ is a gVOA and Sym_Σ is a gPVTL. We note that the complement of a symbolic tree languages is not necessarily symbolic. The contents $\mathrm{c}(P)$ of a context $P \in \mathrm{C}(\Sigma, X)$ is the set of symbols from $\Sigma \cup X$ which appear in P. We note that $\mathrm{c}(P(t)) = \mathrm{c}(P) \cup \mathrm{c}(t)$ holds for any context $P \in \mathrm{C}(\Sigma, X)$ and tree $t \in \mathrm{T}(\Sigma, X)$.

Proposition 5.3.19 *A tree language* $T \subseteq \mathrm{T}(\Sigma, X)$ *is symbolic if and only if it is recognizable by a finite symbolic ordered algebra.*

Proof. Every symbolic tree language is also a semilattice tree language. So, if T is symbolic then the syntactic algebra of T is semilattice. On the other hand, since $\mathrm{c}(t) \subseteq \mathrm{c}(P(t))$ for all $t \in \mathrm{T}(\Sigma, X)$ and $P \in \mathrm{C}(\Sigma, X)$, then $P(t) \preccurlyeq_T t$ always holds. This shows that $\mathrm{SOA}(T)$ is a symbolic ordered algebra. Conversely, if $\mathrm{SOA}(T)$ is a symbolic ordered algebra, then $\tau \subseteq \approx^T$ and $P(t) \preccurlyeq_T t$. Suppose that $\mathrm{c}(t) \subseteq \mathrm{c}(t')$ for some $t \in \mathrm{T}(\Sigma, X)$, and $t \in T$. Then there exists a context P such that $\mathrm{c}(t') = \mathrm{c}(P(t))$. By Lemma 5.3.13,

$t' \ \tau \ P(t)$, and so $t' \ \approx^T \ P(t)$ holds. On the other hand $P(t) \preccurlyeq_T t$ implies $t' \preccurlyeq_T t$, and this by $t \in T$ implies $t' \in T$. Hence, T is a symbolic tree language by Lemma 5.3.11. □

Let $\mathbf{Sym} = \{\mathbf{Sym}(\Sigma)\}$ the class of all finite symbolic ordered algebras, and $\mathrm{SL} = \{\mathrm{SL}(\Sigma, X)\}$ be the family of all symbolic tree languages.

Proposition 5.3.20 *Class* $\mathbf{Sym}$ *is a gVFOA, family* Sym *is a gPVTL, and* $\mathrm{Sym} = \mathbf{Sym}^t$. □

Another characterization of symbolic tree languages is given below. We will show that they are exactly those semilattice languages recognized by so-called translation closed subsets of semilattice algebras.

Proposition 5.3.21 *For a semilattice algebra* $\mathcal{A} = (A, \Sigma)$ *the structure* $\mathcal{A}_s = (A, \Sigma, \leqslant)$, *where* $\leqslant$ *is defined by*

$$a \leqslant b \iff a = p(b) \text{ for some } p \in \mathrm{Tr}(\mathcal{A})$$

for all $a, b \in A$, *is a symbolic ordered algebra.*

Proof. It is clear that the relation $\leqslant$ is reflexive and transitive, and it is anti-symmetric by Lemma 5.3.3. It is also compatible with Σ since for any $a, b \in A$ such that $a \leqslant b$, it follows that $a = p(b)$ for some $p \in \mathrm{Tr}(\mathcal{A})$. Hence $q(a) = q(p(b)) = p(q(b))$ for every $q \in \mathrm{Tr}(\mathcal{A})$, thus $q(a) \leqslant q(b)$ for every $q \in \mathrm{Tr}(\mathcal{A})$. Obviously $p(a) \leqslant a$ for every $a \in A$ and $p \in \mathrm{Tr}(\mathcal{A})$, which implies that $\mathcal{A}_s$ is a symbolic ordered algebra by Lemma 5.3.18 (i). □

Definition 5.3.22 For a semilattice algebra (A, Σ), a subset $D \subseteq A$ is *translation closed* if $d \in D$ implies $p(d) \in D$ for any $p \in \mathrm{Tr}(\mathcal{A})$.

Translation closed subsets are known as *ideals* of algebras, but we chose a different name since the term "idea" has already a different meaning here. Note that any ideal of a symbolic ordered algebra is translation closed.

Lemma 5.3.23 *A subset* $D \subseteq A$ *of a semilattice algebra* $\mathcal{A} = (A, \Sigma)$ *is translation closed if and only if* D *is an ideal of the symbolic ordered algebra* $\mathcal{A}_s$ *defined in Proposition* 5.3.21. □

It was proved in Proposition 5.3.14 that semilattice tree languages are recognized by finite semilattice algebras. By Proposition 5.3.21 and Lemma 5.3.23 it follows that symbolic tree languages are exactly those semilattice tree languages that are recognized by semilattice algebras with translation closed sets of final states.

Proposition 5.3.24 *A tree language* $T \subseteq \mathrm{T}(\Sigma, X)$ *is a symbolic tree language if and only if there exist a finite semilattice algebra* $\mathcal{A} = (A, \Sigma)$, *a morphism* $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ *and a translation closed subset* $F \subseteq A$ *such that* $T = F\varphi^{-1}$. □

Chapter 6

Tree algebras

Another syntactic structure for recognizable tree languages is introduced by Wilke [60]. This formalism considers only binary trees (in which every node is either a leaf or two-branching) whose nodes and leaves are labeled by symbols of a finite alphabet.

In this formalism trees are represented by terms over a signature Γ consisting of six operation symbols involving the three sorts **label**, **tree** and **context**. A tree algebra is a Γ-algebra satisfying certain identities which identify some pairs of Γ-terms that represent the same tree. The syntactic tree algebra of a tree language T is defined in a natural way. Its component of sort **tree** is the syntactic algebra of T while its **context**-component is the syntactic semigroup of T.

"The present algebraic framework based on three-sorted tree algebras can be described as a combination of the semigroup approach and the universal algebra approach that have been used, respectively, by Nivat and Podelski and by Steinby to characterize classes of regular tree languages. It is observed that the three-sorted tree algebras are more suitable when considering the class of frontier testable [i.e., reverse definite] tree languages. Frontier testable tree languages cannot be characterized by syntactic semigroups and there is no known finite characterization for frontier testability in the latter universal algebra framework." [49].

This formalism for binary tree languages is studied further in this chapter. A completeness theorem for the axiomatization of tree algebras with respect to representations of binary trees, and a variety theorem for families of binary tree languages and varieties of tree algebras are proved.

In Section 6.1 we consider the canonical homomorphism from the Γ-term algebra generated by a given label alphabet A. Its **tree**-component yields the binary trees from their representations as Γ-terms, and by showing that its kernel is the fully invariant congruence relation generated by Wilke's axioms of tree algebras, we get the soundness and completeness of the axiom system with respect to binary representations, i.e., any two Γ-terms representing the same tree are proved to be equal in the axiom system. For proving this, a complete term rewriting system is constructed for Wilke's axiom system. It follows that the derivability of equations in the axiom system is decidable.

In Section 6.2 we study the variety theory of the formalism. As a particular case of many-sorted variety theorem (Proposition 2.3.14 in Chapter 2) we get a variety theorem for families of recognizable triple subsets of sort **label-tree-context** and varieties of finite tree algebras. But it turns out that there is no such a variety theorem for families of binary tree languages and varieties of finite tree algebras. However, by restricting ourselves to special kind of tree algebras (called reduced algebras) we get a sort of a variety theorem for reduced tree algebras and binary tree languages. This answers a question mentioned by several authors [19, 20, 54, 60]. Finally, we link the varieties of recognizable binary tree languages to generalized varieties of tree languages [54]. It turns out that the binary fragment of any gVTL is a variety of recognizable binary tree languages. So, nilpotency, definiteness, reverse definiteness, locally testability, etc. [54] are canonically translated to binary tree languages.

In Section 6.3 we discuss some algebraic properties of free tree algebras and term algebras. We list without proofs the main results of [43, 44] stating that Wilke's functions generate all the functions involving labels, trees and contexts that preserve the syntactic tree algebra congruences (cf. [60]) of all binary tree languages, if the alphabet contains at least seven labels. Also, all the congruence preserving functions of term algebras over ranked alphabets with at least seven constant symbols can be proved to be term functions.

We assume some familiarity with the basic notions of term rewriting systems, see e.g. [5, 25].

6.1 Binary trees and tree algebras

In what follows, we shall consider especially *binary trees* in which both the inner nodes and the leaves are labelled with symbols from a given finite alphabet A. Such trees can also be defined as terms like in the previous chapters. With every $a \in A$ we associate a constant symbol c_a and a binary function symbol f_a. The ranked alphabet $\Sigma^A = \Sigma_0^A \cup \Sigma_2^A$ is associated with A, where $\Sigma_0^A = \{c_a \mid a \in A\}$ and $\Sigma_2^A = \{f_a \mid a \in A\}$.

The sets T_A and C_A of *A-trees* and *A-contexts*, respectively, are defined inductively as follows:

(1) $c_a \in T_A$ for all $a \in A$, and $\xi \in C_A$;

(2) $f_a(s,t) \in T_A$ and $f_a(p,t), f_a(t,p) \in C_A$ for all $a \in A$, $s,t \in T_A$ and $p \in C_A$.

Moreover, let $C_A^+ = C_A \setminus \{\xi\}$ be the set of non-unit A-contexts.

The Σ^A-*algebra of A-trees* $\mathcal{T}_{\Sigma,A} = (T_A, \Sigma^A)$ is defined as follows:

(1) $c_a^{\mathcal{T}_{\Sigma,A}} = c_a$ for every $a \in A$, and

(2) $f_a^{\mathcal{T}_{\Sigma,A}}(s,t) = f_a(s,t)$ for every $a \in A$ and every $s, t \in T_A$.

Clearly, $\mathcal{T}_{\Sigma,A}$ is the Σ^A-term algebra (over the empty leaf alphabet), and hence for each Σ^A-algebra $\mathcal{D} = (D, \Sigma^A)$, there is a unique homomorphism $h_{\mathcal{D}} : \mathcal{T}_{\Sigma,A} \to \mathcal{D}$ defined inductively by:

(1) $h_{\mathcal{D}}(c_a) = c_a^{\mathcal{D}}$ for $a \in A$;

(2) $h_{\mathcal{D}}(f_a(s,t)) = f_a^{\mathcal{D}}(h_{\mathcal{D}}(s), h_{\mathcal{D}}(t))$ for $a \in A$ and $s, t \in T_A$.

Let us now introduce Wilke's [60] formalism for representing binary trees over a given alphabet A by terms over a many-sorted ranked alphabet Γ. This alphabet Γ contains operators by which A-trees and A-contexts can be constructed starting from the label alphabet A.

The set of *sorts* is $S = \{\mathbf{label}, \mathbf{tree}, \mathbf{context}\}$. For sort names we use the abbreviations $\mathbf{l} = \mathbf{label}$, $\mathbf{t} = \mathbf{tree}$ and $\mathbf{c} = \mathbf{context}$. The *types* (see [59]) of the symbols in the S-sorted ranked alphabet $\Gamma = \{\iota, \kappa, \lambda, \rho, \eta, \sigma\}$ are as follows:

• $\iota : \mathbf{l} \to \mathbf{t}$, • $\lambda : \mathbf{lt} \to \mathbf{c}$, • $\eta : \mathbf{ct} \to \mathbf{t}$

• $\kappa : \mathbf{ltt} \to \mathbf{t}$, • $\rho : \mathbf{lt} \to \mathbf{c}$, • $\sigma : \mathbf{cc} \to \mathbf{c}$.

For defining Γ-terms, let $X = \langle X_{\mathbf{l}}, X_{\mathbf{t}}, X_{\mathbf{c}} \rangle$, where $X_{\mathbf{l}}$ is a set of variables of sort **label**, $X_{\mathbf{t}}$ is a set of variables of sort **tree**, and $X_{\mathbf{c}}$ is a set of variables of sort **context**. The sets $T_\Gamma(X)_{\mathbf{l}}$, $T_\Gamma(X)_{\mathbf{t}}$ and $T_\Gamma(X)_{\mathbf{c}}$ of Γ-*terms over* X of sort **label**, **tree** and **context**, respectively, are defined inductively:

- $X_{\mathbf{l}} \subseteq T_\Gamma(X)_{\mathbf{l}}$, $X_{\mathbf{t}} \subseteq T_\Gamma(X)_{\mathbf{t}}$, $X_{\mathbf{c}} \subseteq T_\Gamma(X)_{\mathbf{c}}$;
- if $a \in T_\Gamma(X)_{\mathbf{l}}$, then $\iota(a) \in T_\Gamma(X)_{\mathbf{t}}$;
- if $a \in T_\Gamma(X)_{\mathbf{l}}$ and $s, t \in T_\Gamma(X)_{\mathbf{t}}$, then $\kappa(a, s, t) \in T_\Gamma(X)_{\mathbf{t}}$;
- if $a \in T_\Gamma(X)_{\mathbf{l}}$ and $t \in T_\Gamma(X)_{\mathbf{t}}$, then $\lambda(a,t), \rho(a,t) \in T_\Gamma(X)_{\mathbf{c}}$;
- if $p \in T_\Gamma(X)_{\mathbf{c}}$ and $t \in T_\Gamma(X)_{\mathbf{t}}$, then $\eta(p,t) \in T_\Gamma(X)_{\mathbf{t}}$;
- if $p, q \in T_\Gamma(X)_{\mathbf{c}}$, then $\sigma(p,q) \in T_\Gamma(X)_{\mathbf{c}}$.

For the particular choice $X = \langle A, \emptyset, \emptyset \rangle$, where A is a given alphabet of labels, we get $T_\Gamma(X)_{\mathbf{l}} = A$, and write $T_\Gamma(X)_{\mathbf{t}} = T_\Gamma(A)$ and $T_\Gamma(X)_{\mathbf{c}} = C_\Gamma(A)^+$. Elements of the sets $T_\Gamma(A)$ and $C_\Gamma(A) = C_\Gamma(A)^+ \cup \{\xi\}$ are called ΓA-*terms* and ΓA-*contexts*, respectively. Note that $C_\Gamma(A)^+$ does not include the unit context ξ.

Binary A-trees and A-contexts are represented by ΓA-terms and ΓA-contexts, as follows. If $s, t \in T_\Gamma(A)$ represent the A-trees $\hat{s}$ and $\hat{t}$, and $p, q \in C_\Gamma(A)$ represent the A-contexts $\hat{p}$ and $\hat{q}$, respectively, then for any label $a \in A$,

- $\iota(a)$ represents the A-tree c_a,
- $\kappa(a, s, t)$ represents the A-tree $f_a(\hat{s}, \hat{t})$,

- $\lambda(a,t)$ represents the A-context $f_a(\xi,\hat{t})$,
- $\rho(a,t)$ represents the A-context $f_a(\hat{t},\xi)$,
- $\eta(p,t)$ represents the A-tree $\hat{p}(\hat{t})$, and
- $\sigma(p,q)$ represents the A-context $\hat{p}(\hat{q})$.

Any A-tree or A-context is, in general, represented by several ΓA-terms or ΓA-contexts, respectively. For example, the $\{a,b\}$-tree $f_b(f_a(c_b,c_a),c_a)$ can be represented by the $\Gamma\{a,b\}$-terms

$$\kappa(b,\kappa(a,\iota(b),\iota(a)),\iota(a)) \quad \text{or} \quad \eta(\lambda(b,\iota(a)),\kappa(a,\iota(b),\iota(a))).$$

Later we shall formulate this representation relation as a homomorphism. A Γ-*algebra* $\mathcal{M} = (\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\rangle, \Gamma)$ consists of

- a nonempty set $M_{\mathbf{l}}$ of elements of sort **label**,
- a nonempty set $M_{\mathbf{t}}$ of elements of sort **tree**, and
- a nonempty set $M_{\mathbf{c}}$ of elements of sort **context**,

and operations

- $\iota^{\mathcal{M}} : M_{\mathbf{l}} \to M_{\mathbf{t}}$,
- $\kappa^{\mathcal{M}} : M_{\mathbf{l}} \times M_{\mathbf{t}} \times M_{\mathbf{t}} \to M_{\mathbf{t}}$,
- $\lambda^{\mathcal{M}}, \rho^{\mathcal{M}} : M_{\mathbf{l}} \times M_{\mathbf{t}} \to M_{\mathbf{c}}$,
- $\eta^{\mathcal{M}} : M_{\mathbf{c}} \times M_{\mathbf{t}} \to M_{\mathbf{t}}$, and
- $\sigma^{\mathcal{M}} : M_{\mathbf{c}} \times M_{\mathbf{c}} \to M_{\mathbf{c}}$,

defined as realizations of the symbols in Γ.

If $X = \langle X_{\mathbf{l}}, X_{\mathbf{t}}, X_{\mathbf{c}}\rangle$ is a triple of sets of variables as above, then the operations of the Γ-*term algebra over* X,

$$\mathcal{T}_\Gamma(X) = (\langle T_\Gamma(X)_{\mathbf{l}}, T_\Gamma(X)_{\mathbf{t}}, T_\Gamma(X)_{\mathbf{c}}\rangle, \Gamma)$$

are defined by setting the following for all $a \in T_\Gamma(X)_{\mathbf{l}}$, $s,t \in T_\Gamma(X)_{\mathbf{t}}$ and $p,q \in T_\Gamma(X)_{\mathbf{c}}$,

- $\iota^{\mathcal{T}_\Gamma(X)}(a) = \iota(a)$,
- $\kappa^{\mathcal{T}_\Gamma(X)}(a,s,t) = \kappa(a,s,t)$,
- $\lambda^{\mathcal{T}_\Gamma(X)}(a,t) = \lambda(a,t)$,
- $\rho^{\mathcal{T}_\Gamma(X)}(a,t) = \rho(a,t)$,
- $\eta^{\mathcal{T}_\Gamma(X)}(p,t) = \eta(p,t)$, and
- $\sigma^{\mathcal{T}_\Gamma(X)}(p,q) = \sigma(p,q)$.

In particular, for an alphabet A, the Γ-*algebra of* ΓA-*terms*

$$\mathcal{T}_\Gamma(A) = (\langle A, T_\Gamma(A), C_\Gamma(A)^+\rangle, \Gamma)$$

is obtained as a special case of the algebra $\mathcal{T}_\Gamma(X)$ by putting $X = \langle A, \emptyset, \emptyset\rangle$.

A further Γ-algebra of special interest is the Γ-*algebra of* A-*trees*

$$\mathcal{F}_{\mathrm{TA}}(A) = (\langle A, T_A, C_A^+\rangle, \Gamma)$$

defined as follows: for any $a \in A$, $s,t \in T_A$ and $p,q \in C_A^+$,

- $\iota^{\mathcal{F}_{\mathrm{TA}}(A)}(a) = c_a$,
- $\kappa^{\mathcal{F}_{\mathrm{TA}}(A)}(a, s, t) = f_a(s, t)$,
- $\lambda^{\mathcal{F}_{\mathrm{TA}}(A)}(a, t) = f_a(\xi, t)$,
- $\rho^{\mathcal{F}_{\mathrm{TA}}(A)}(a, t) = f_a(t, \xi)$,
- $\eta^{\mathcal{F}_{\mathrm{TA}}(A)}(p, t) = p(t)$, and
- $\sigma^{\mathcal{F}_{\mathrm{TA}}(A)}(p, q) = p(q)$,

where $p(t)$ and $p(q)$ are the A-tree and the A-context obtained from p by replacing the ξ in it with t and q, respectively.

As noted in [60], for any alphabet A, the algebra $\mathcal{F}_{\mathrm{TA}}(A)$ satisfies the following identities:

(TA1) $\sigma(\sigma(p, q), r)) \approx \sigma(p, \sigma(q, r))$;
(TA2) $\eta(\sigma(p, q), t) \approx \eta(p, \eta(q, t))$;
(TA3) $\eta(\lambda(a, s), t) \approx \kappa(a, t, s)$;
(TA4) $\eta(\rho(a, s), t) \approx \kappa(a, s, t)$.

Here, a is a variable of sort **label**, s and t are variables of sort **tree**, and p, q and r are variables of sort **context**. Let TA denote the set of these identities. Moreover, let $\equiv^{TA}$ be the fully invariant congruence on $\mathcal{F}_{\mathrm{TA}}(A)$ generated by TA, i.e., the equational theory in variables $\langle A, \emptyset, \emptyset\rangle$ axiomatized by TA.

Following [60], we call a Γ-algebra a *tree algebra*, if it satisfies the identities TA. In particular, $\mathcal{F}_{\mathrm{TA}}(A)$ is the *free tree algebra* generated by $\langle A, \emptyset, \emptyset\rangle$ ([60], Proposition 1). This means that if $\mathcal{M} = (\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\rangle, \Gamma)$ is a tree algebra, then any mapping $\psi : A \to M_{\mathbf{l}}$ can be extended in a unique way to a homomorphism $\varphi = (\varphi_{\mathbf{l}}, \varphi_{\mathbf{t}}, \varphi_{\mathbf{c}})$ of Γ-algebras from $\mathcal{F}_{\mathrm{TA}}(A)$ to $\mathcal{M}$. We note that $\varphi_{\mathbf{l}} = \psi$.

For any finite alphabet A, the Γ-algebras $\mathcal{T}_\Gamma(A)$ and $\mathcal{F}_{\mathrm{TA}}(A)$ are both generated by $\langle A, \emptyset, \emptyset\rangle$. The identity mapping $1_A : A \to A$ can be extended uniquely to a homomorphism $\nu : \mathcal{T}_\Gamma(A) \to \mathcal{F}_{\mathrm{TA}}(A)$ of Γ-algebras that we call the *canonical A-homomorphism*. It is the triple of mappings

$$\langle \nu_{\mathbf{l}} : A \to A, \nu_{\mathbf{t}} : T_\Gamma(A) \to T_A, \nu_{\mathbf{c}} : C_\Gamma(A)^+ \to C_A^+\rangle$$

defined inductively for all $a \in T_\Gamma(X)_{\mathbf{l}}$, $s, t \in T_\Gamma(X)_{\mathbf{t}}$ and $p, q \in T_\Gamma(X)_{\mathbf{c}}$ by:

- $\nu_{\mathbf{l}}(a) = a$,
- $\nu_{\mathbf{t}}(\iota(a)) = c_a$,
- $\nu_{\mathbf{t}}(\kappa(a, s, t)) = f_a(\nu_{\mathbf{t}}(s), \nu_{\mathbf{t}}(t))$,
- $\nu_{\mathbf{c}}(\lambda(a, t)) = f_a(\xi, \nu_{\mathbf{t}}(t))$,
- $\nu_{\mathbf{c}}(\rho(a, t)) = f_a(\nu_{\mathbf{t}}(t), \xi)$,
- $\nu_{\mathbf{t}}(\eta(p, t)) = \nu_{\mathbf{c}}(p)(\nu_{\mathbf{t}}(t))$, and
- $\nu_{\mathbf{c}}(\sigma(p, q)) = \nu_{\mathbf{c}}(p)(\nu_{\mathbf{c}}(q))$.

It is clear that the A-tree represented by any $t \in T_\Gamma(A)$ is precisely $\nu_{\mathbf{t}}(t)$. Similarly, $\nu_{\mathbf{c}}(p)$ is always the A-context represented by $p \in C_\Gamma(A)^+$.

We shall now establish some basic properties of the kernel of the canonical homomorphism $\nu : \mathcal{T}_\Gamma(A) \to \mathcal{F}_{\mathrm{TA}}(A)$ and the algebra $\mathcal{F}_{\mathrm{TA}}(A)$ by converting the identities of *TA* into a convergent (i.e., terminating and confluent) term rewriting system.

Definition 6.1.1 Let $\mathcal{R}$ be the term rewriting system consisting of the following four rules:

(R1) $\sigma(\sigma(p,q),r)) \to \sigma(p,\sigma(q,r))$;
(R2) $\eta(\sigma(p,q),t) \to \eta(p,\eta(q,t))$;
(R3) $\eta(\lambda(a,s),t) \to \kappa(a,t,s)$;
(R4) $\eta(\rho(a,s),t) \to \kappa(a,s,t)$.

Proposition 6.1.2 *The system* $\mathcal{R}$ *is convergent.*

Proof. It is clear that $\mathcal{R}$ is compatible with the lexicographic path ordering (see e.g. [3] or [4]) induced by any order on Γ such that $\eta > \kappa$. Hence, $\mathcal{R}$ is terminating. There are just two critical pairs. The pair

$$\langle \eta(\sigma(p,\sigma(q,r)),t),\ \eta(\sigma(p,q),\eta(r,t))\rangle$$

produced by (R1) and (R2) converges to $\eta(p,\eta(q,\eta(r,t))$ by applications of (R2), and the other critical pair

$$\langle \sigma(\sigma(p,\sigma(q,r)),r'),\ \sigma(\sigma(p,q),\sigma(r,r'))\rangle$$

obtained by overlapping (R1) with itself, converges to

$$\sigma(p,\sigma(q,\sigma(r,r')))$$

by further applications of (R1). Hence, $\mathcal{R}$ is confluent as well. □

Let $IRR_{\mathbf{l}}(\mathcal{R}) \subseteq A$, $IRR_{\mathbf{t}}(\mathcal{R}) \subseteq T_\Gamma(A)$, and $IRR_{\mathbf{c}}(\mathcal{R}) \subseteq C_\Gamma(A)^+$ denote the sets of Γ-terms over $\langle A,\emptyset,\emptyset\rangle$ irreducible by $\mathcal{R}$ of sort **label**, **tree** and **context**, respectively. Clearly, $IRR_{\mathbf{l}}(\mathcal{R}) = A$. The other two sets are described in the following proposition.

Proposition 6.1.3 *The sets of* $\mathcal{R}$*-irreducible* ΓA*-terms and* ΓA*-contexts are obtained as follows.*

I. *A* ΓA*-term is irreducible iff it contains the operators* ι *and* κ *only, that is to say,* $IRR_{\mathbf{t}}(\mathcal{R})$ *is the smallest subset of* $T_\Gamma(A)$ *satisfying the following two conditions:*

(1) $\iota(a) \in IRR_{\mathbf{t}}(\mathcal{R})$ *for every* $a \in A$;
(2) *if* $a \in A$ *and* $s,t \in IRR_{\mathbf{t}}(\mathcal{R})$, *then* $\kappa(a,s,t) \in IRR_{\mathbf{t}}(\mathcal{R})$.

II. $IRR_{\mathbf{c}}(\mathcal{R})$ *is the smallest subset of* $C_\Gamma(A)^+$ *satisfying the following two conditions:*

(1') $\lambda(a,t), \rho(a,t) \in IRR_{\mathbf{c}}(\mathcal{R})$ *for all* $a \in A$ *and* $t \in IRR_{\mathbf{t}}(\mathcal{R})$;
(2') $\sigma(\lambda(a,t),p) \in IRR_{\mathbf{c}}(\mathcal{R})$ *and* $\sigma(\rho(a,t),p) \in IRR_{\mathbf{c}}(\mathcal{R})$ *whenever* $a \in A$, $t \in IRR_{\mathbf{t}}(\mathcal{R})$ *and* $p \in IRR_{\mathbf{c}}(\mathcal{R})$.

Proof. By considering the rules of $\mathcal{R}$ we see immediately that clauses (1) and (2) define a set of irreducible ΓA-terms. On the other hand, any term with a subterm of the form $\eta(p,t)$ is reducible as the ΓA-context p must begin with λ, ρ or σ. Hence, all irreducible ΓA-terms are obtained by (1) and (2) using only letters of A and the operators ι and κ.

It is again clear that no rule of $\mathcal{R}$ applies to any ΓA-context obtained by rules (1') and (2'). The converse claim, that (1') and (2') yield all the irreducible ΓA-contexts, can be easily verified by induction on the *ξ-depth* $d_\xi(p)$ of $p \in IRR_{\mathbf{c}}(\mathcal{R})$, i.e., the distance of the ξ-labelled node from the root of $\nu_{\mathbf{c}}(p)$. Indeed, if $d_\xi(p) = 1$, then p must be a ΓA-context given by clause (1'). If $d_\xi(p) > 1$, then $p = \sigma(q,r)$ for some $q, r \in IRR_{\mathbf{c}}(\mathcal{R})$, and because of rule (R1), q must be of the form $\lambda(a,t)$ or $\rho(a,t)$ with $t \in IRR_{\mathbf{t}}(\mathcal{R})$. Since the induction assumption applies to r, then p also is of the required type. □

Any two $\equiv^{TA}$-congruent ΓA-terms represent the same A-tree, and similarly, for any $p, q \in C_\Gamma(A)$, $p \equiv^{TA}_{\mathbf{c}} q$ implies $\nu_{\mathbf{c}}(p) = \nu_{\mathbf{c}}(q)$. It follows from Propositions 6.1.2 and 6.1.3 that any A-tree is represented by a ΓA-term in which only the operators ι and κ are used. Also, every A-context is represented by a ΓA-context of the form

$$\sigma(p_1, (\sigma(p_2, \ldots \sigma(p_{n-1}, p_n) \ldots),$$

where $n \geq 1$, and each p_i is of the form $\lambda(a,t)$ or $\rho(a,t)$ with $a \in A$ and $t \in IRR_{\mathbf{t}}(\mathcal{R})$. The following proposition completes the picture. The proof is straightforward.

Proposition 6.1.4 *If $s, t \in IRR_{\mathbf{t}}(\mathcal{R})$ and $s \neq t$, then $\nu_{\mathbf{t}}(s) \neq \nu_{\mathbf{t}}(t)$. Hence each A-tree is represented by a unique $\mathcal{R}$-irreducible ΓA-term. Similarly, if $p, q \in IRR_{\mathbf{c}}(\mathcal{R})$ and $p \neq q$, then $\nu_{\mathbf{c}}(p) \neq \nu_{\mathbf{c}}(q)$, and hence each A-context is represented by a unique $\mathcal{R}$-irreducible ΓA-context.* □

The definition of $\mathcal{R}$ implies directly that the equivalence closure $\Leftrightarrow^*_{\mathcal{R}}$ of $\Rightarrow_{\mathcal{R}}$ equals the fully invariant congruence $\equiv^{TA}$ on $\mathcal{T}_\Gamma(A)$ generated by the identities *TA* ([3, 4]). Since $\mathcal{F}_{\mathrm{TA}}(A)$ satisfies the identities TA, the inclusion $\equiv^{TA} \subseteq \ker\,\nu$ holds. On the other hand, $\ker\,\nu \subseteq \Leftrightarrow^*_{\mathcal{R}}$ by Proposition 6.1.4. Hence, $\equiv^{TA} = \ker\,\nu$. By the homomorphism theorem (Proposition 2.1.1 in Chapter 2) we know that $\mathcal{F}_{\mathrm{TA}}(A) \cong \mathcal{T}_\Gamma(A)/\ker\,\nu$ (as ν is an epimorphism) and that $\mathcal{T}_\Gamma(A)/{\equiv^{TA}}$ is the free tree algebra generated by $\langle A, \emptyset, \emptyset\rangle$. These observations yield the soundness and completeness of Wilke's axiomatization of tree algebras for binary tree or context representations, namely that every two Γ-terms represent the same tree or context iff they are TA-provably equal. This can be formalized more precisely in the following corollary.

Corollary 6.1.5 *The kernel of the homomorphism $\nu : \mathcal{T}_\Gamma(A) \to \mathcal{F}_{\mathrm{TA}}(A)$ equals the fully invariant congruence $\equiv^{TA}$ on $\mathcal{T}_\Gamma(A)$ generated by the identities TA, and $\mathcal{F}_{\mathrm{TA}}(A)$ is the free tree algebra generated by $\langle A, \emptyset, \emptyset\rangle$.* □

That $\mathcal{F}_{\mathrm{TA}}(A)$ is the free tree algebra generated by $\langle A, \emptyset, \emptyset\rangle$ is actually Wilke's Proposition 1 in [60]. From this result, on can derive the identity $\equiv^{TA} = \ker\nu$ directly (without using any term rewriting system). To see this, we note that since $\mathcal{T}_\Gamma(A)/{\equiv^{TA}}$ is also the free tree algebra generated by $\langle A, \emptyset, \emptyset\rangle$, then there exists a morphism $\jmath : \mathcal{F}_{\mathrm{TA}}(A) \to \mathcal{T}_\Gamma(A)/{\equiv^{TA}}$ such that $(u)\,\nu\jmath = (u/{\equiv^{TA}})$ for all $u \in \mathcal{T}_\Gamma(A)$. So, $\ker\nu \subseteq \equiv^{TA}$ follows; the inverse inclusion $\equiv^{TA} \subseteq \ker\nu$ is even more obvious.

Proposition 6.1.2 implies that the equational theory $\equiv^{TA}$ is decidable: to decide whether any two given ΓA-terms, or any two given ΓA-contexts, are $\equiv^{TA}$-equivalent, it suffices to compute and compare their $\mathcal{R}$-normal forms. Since $\equiv^{TA} = \ker\nu$, the question "$s \equiv^{TA}_{\mathbf{t}} t$?" can also be decided for any $s, t \in \mathcal{T}_\Gamma(A)$ simply by forming the A-trees represented by s and t: $s \equiv^{TA}_{\mathbf{t}} t$ if and only if $\nu_{\mathbf{t}}(s) = \nu_{\mathbf{t}}(t)$. Similarly, for any ΓA-contexts p and q, $p \equiv^{TA}_{\mathbf{c}} q$ if and only if $\nu_{\mathbf{c}}(p) = \nu_{\mathbf{c}}(q)$.

6.2 Varieties of binary tree languages

A *binary tree language* is any subset $T \subseteq T_A$ for a finite alphabet A. For such a binary tree language T, the triple $\langle T\rangle = \langle\emptyset, T, \emptyset\rangle$ is a subset of $\mathcal{F}_{\mathrm{TA}}(A)$. We know from Chapter 2 that for any subset $L = \langle L_{\mathbf{l}}, L_{\mathbf{t}}, L_{\mathbf{c}}\rangle \subseteq \langle A, T_A, C_A^+\rangle$, the syntactic congruence $\approx^L = \langle\approx^L_{\mathbf{l}}, \approx^L_{\mathbf{t}}, \approx^L_{\mathbf{c}}\rangle$ of L is a congruence relation on $\mathcal{F}_{\mathrm{TA}}(A)$. For a binary tree language $T \subseteq T_A$, *the syntactic tree algebra congruence of* T is defined to be the syntactic congruence of $\langle T\rangle = \langle\emptyset, T, \emptyset\rangle$ and is denoted by $\approx^T$, i.e., $\approx^T = \approx^{\langle T\rangle}$. The following lemma simplifies this definition (cf. [60]).

Lemma 6.2.1 *The syntactic tree algebra congruence relation $\approx^T$ of a binary tree language $T \subseteq T_A$ satisfies the following identities:*

- $\approx^T_{\mathbf{l}} = \{(a, a') \in A \times A \mid \forall p \in C_A\big(p(c_a) \in T \leftrightarrow p(c_{a'}) \in T\big)\ \&$
  $\forall p \in C_A \forall t, t' \in T_A\big(p(f_a(t, t')) \in T \leftrightarrow p(f_{a'}(t, t')) \in T\big)\}$,
- $\approx^T_{\mathbf{t}} = \{(t, t') \in T_A \times T_A \mid \forall p \in C_A\big(p(t) \in T \leftrightarrow p(t') \in T\big)\}$,
- $\approx^T_{\mathbf{c}} = \{(p, p') \in C_A \times C_A \mid \forall q \in C_A \forall t \in T_A\big(q(p(t)) \in T \leftrightarrow q(p'(t)) \in T\big)\}$.

□

For a context $p \in C_A$ and a subset $T \subseteq T_A$, $p^{-1}(T)$ is defined to be the set $\{t \in T_A \mid p(t) \in T\}$.

Corollary 6.2.2 *For a binary tree language $T \subseteq T_A$ and a tree $t \in T_A$,*

$$\{s \in T_A \mid s \approx^T_{\mathbf{t}} t\} = \bigcap_{p(t)\in T} p^{-1}(T) \setminus \bigcup_{p(t)\notin T} p^{-1}(T)$$

where p ranges over C_A.

The *syntactic tree algebra* $\mathrm{STA}(T)$ of a binary tree language $T \subseteq T_A$ is defined as the quotient algebra $\mathcal{F}_{\mathrm{TA}}(A)/{\approx^T}$. Hence, $\mathrm{STA}(T)$ is the syntactic algebra of $\langle T \rangle$ in the sense of Chapter 2. The following lemma follows from Proposition 2.2.12 in Chapter 2.

Lemma 6.2.3 *Let A and B be two alphabets and $T, T' \subseteq T_A$ be binary tree languages. Then*

(1) $\mathrm{STA}(T) = \mathrm{STA}(T_A \setminus T)$ *and* $\mathrm{STA}(T \cap T') \preceq \mathrm{STA}(T) \times \mathrm{STA}(T')$.
(2) $\mathrm{STA}(p^{-1}(T)) \preceq \mathrm{STA}(T)$ *for any* $p \in C_A$.
(3) $\mathrm{STA}(T\varphi^{-1}) \preceq \mathrm{STA}(T)$ *for any morphism* $\varphi : \mathcal{F}_{\mathrm{TA}}(B) \to \mathcal{F}_{\mathrm{TA}}(A)$. □

Proposition 2.3.14 in Chapter 2 results in a bijective correspondence between varieties of finite tree algebras and varieties of recognizable many-sorted subsets of the form $L = \langle L_{\mathbf{l}}, L_{\mathbf{t}}, L_{\mathbf{c}} \rangle \subseteq \langle A, T_A, C_A^+ \rangle$. Here we are interested in families of binary tree languages rather than sorted subsets. A *family of binary tree languages* is a mapping $\mathscr{V} = \{\mathscr{V}(A)\}$ which assigns a collection $\mathscr{V}(A)$ of binary tree languages ($\subseteq T_A$) to any finite alphabet A.

First we show that no variety theorem for families of binary tree languages and varieties of finite tree algebras can be proved. This was mentioned as an open question by Ésik in [19] page 759, and in [20] Remark 8; and by Steinby in [54] page 3 (see also [60] page 105).

For any class of finite tree algebras $\mathbf{K}$, let $\mathbf{K}^t = \{\mathbf{K}^t(A)\}$ be the family of binary tree languages defined by $\mathbf{K}^t(A) = \{T \subseteq T_A \mid \mathrm{STA}(T) \in \mathbf{K}\}$ for any finite alphabet A; and for any family of binary tree languages $\mathscr{V}$, let $\mathscr{V}^a$ be the variety of finite tree algebras generated by $\{\mathrm{STA}(T) \mid T \in \mathscr{V}(A)\}$ where A ranges over finite alphabets.

In what follows we shall show that there exists a variety of finite tree algebras $\mathbf{K}$ for which the identity $\mathbf{K}^{ta} = \mathbf{K}$ does not hold. For a tree algebra $\mathcal{M} = (\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}} \rangle, \Gamma)$, let $\mathcal{M}'$ be the smallest subalgebra of $\mathcal{M}$ which contains $\langle M_{\mathbf{l}}, \emptyset, \emptyset \rangle$; we call it the $\mathbf{l}$-*subalgebra* of $\mathcal{M}$. The algebra $\mathcal{M}$ is called $\mathbf{l}$-*generated*, if $\mathcal{M} = \mathcal{M}'$. These were called A-generated tree algebras in [60] (Remark 6). We immediately observe the following facts.

Lemma 6.2.4 (i) *For any alphabet A, the algebra $\mathcal{F}_{\mathrm{TA}}(A)$ is $\mathbf{l}$-generated.*
(ii) *Any homomorphic image of an $\mathbf{l}$-generated tree algebra is $\mathbf{l}$-generated.*
(iii) *The syntactic tree algebra of any binary tree language is $\mathbf{l}$-generated.*
Suppose $\mathcal{N}, \mathcal{M}, \mathcal{M}_1, \ldots, \mathcal{M}_k$ are tree algebras.
(iv) *If $\mathcal{N} \subseteq \mathcal{M}$, then $\mathcal{N}' \subseteq \mathcal{M}'$.*
(v) *If $\mathcal{N} \leftarrow \mathcal{M}$, then $\mathcal{N}' \leftarrow \mathcal{M}'$.*
(vi) *If $\mathcal{N} \preceq \mathcal{M}$, then $\mathcal{N}' \preceq \mathcal{M}'$.*
(vii) $(\mathcal{M}_1 \times \cdots \times \mathcal{M}_k)' \subseteq \mathcal{M}_1' \times \cdots \times \mathcal{M}_k'$.
(viii) *If $\mathcal{N} \preceq \mathcal{M}_1 \times \cdots \times \mathcal{M}_k$, then $\mathcal{N}' \preceq \mathcal{M}_1' \times \cdots \times \mathcal{M}_k'$.* □

Now we construct a variety $\mathbf{K}$ of finite tree algebras such that $\mathbf{K}^{ta} = \mathbf{K}$ does not hold.

Definition 6.2.5 Let $A = \{a, b\}$, $A' = \{a', b'\}$, and let $\vee$ be the Boolean sum on $\{0, 1\}$, i.e., $0 \vee 0 = 0$ and $1 \vee 0 = 0 \vee 1 = 1 \vee 1 = 1$. Let $\mathcal{D} = (\langle A, A \times \{0,1\}, A' \times \{0,1\}\rangle, \Gamma)$ be the structure defined by the following for $x, y, z \in A$, $x', y' \in A'$, and $i, j \in \{0, 1\}$:
$\iota^{\mathcal{D}}(x) = (x, 0)$;
$\kappa^{\mathcal{D}}(x, (y, i), (z, j)) = (x, i \vee j)$;
$\lambda^{\mathcal{D}}(a, (x, i)) = \rho^{\mathcal{D}}(a, (x, i)) = (a', i)$,
$\lambda^{\mathcal{D}}(b, (x, i)) = \rho^{\mathcal{D}}(b, (x, i)) = (b', i)$;
$\sigma^{\mathcal{D}}((x', i), (y', j)) = (x', i \vee j)$;
$\eta^{\mathcal{D}}((a', i), (x, j)) = (a, i \vee j)$, $\eta^{\mathcal{D}}((b', i), (x, j)) = (b, i \vee j)$.

Lemma 6.2.6 *The structure $\mathcal{D}$ is a tree algebra (satisfies Wilke's axioms TA) and the* $\mathbf{l}$*-subalgebra of $\mathcal{D}$ is the tree algebra*
$$\mathcal{D}' = (\langle A, A \times \{0\}, A' \times \{0\}\rangle, \Gamma)$$
which satisfies $\sigma^{\mathcal{D}'}(p, q) = p$ for every $p, q \in A' \times \{0\}$.

The proof is straightforward. Let $\mathbf{K}$ be the variety of finite tree algebras generated by $\mathcal{D}$. So, a tree algebra $\mathcal{N}$ belongs to $\mathbf{K}$ iff $\mathcal{N}$ divides a power of $\mathcal{D}$, i.e., $\mathcal{N} \preceq \mathcal{D} \times \cdots \times \mathcal{D}$.

Lemma 6.2.7 *Suppose T is a binary tree language. If $\mathrm{STA}(T) \in \mathbf{K}$, then $\sigma^{\mathrm{STA}(T)}(p, q) = p$ holds for every $p, q \in \mathrm{STA}(T)_{\mathbf{c}}$.*

Proof. Let $\mathcal{N} = \mathrm{STA}(T)$. If $\mathcal{N} \in \mathbf{K}$, then $\mathcal{N} \preceq \mathcal{D} \times \cdots \times \mathcal{D}$, and then by Lemma 6.2.4, $\mathcal{N} = \mathcal{N}' \preceq \mathcal{D}' \times \cdots \times \mathcal{D}'$. Lemma 6.2.6 now implies that $\sigma^{\mathcal{N}}(p, q) = p$ holds for every p, q. □

Lemma 6.2.8 $\mathcal{D} \notin \mathbf{K}^{ta}$.

Proof. Assume $\mathcal{D} \in \mathbf{K}^{ta}$. Then there are finite alphabets $A_1, \ldots, A_n$ and binary tree languages $T_j \in \mathbf{K}^t(A_j)$ $(j = 1, \ldots, n)$ such that
$$\mathcal{D} \preceq \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n).$$
For any $j \in \{1, \ldots, n\}$, from $T_j \in \mathbf{K}^t(A_j)$ it follows that $\mathrm{STA}(T_j) \in \mathbf{K}$ and thus by Lemma 6.2.7, $\sigma^{\mathrm{STA}(T_j)}(p, q) = p$ holds for every $p, q \in \mathrm{STA}(T_j)_{\mathbf{c}}$. So, the identity $\sigma^{\mathcal{D}}(p, q) = p$ must hold for every $p, q \in A' \times \{0, 1\}$, but this is a contradiction, since for example $\sigma^{\mathcal{D}}((a', 0), (b', 1)) = (a', 1) \neq (a', 0)$. □

Summarizing, we have shown the following.

Proposition 6.2.9 *There exists a variety of finite tree algebras $\mathbf{K}$ such that $\mathbf{K} \not\subseteq \mathbf{K}^{ta}$.*

However, we note that the inclusion $\mathbf{K}^{ta} \subseteq \mathbf{K}$ and the identity $\mathscr{V}^{at} = \mathscr{V}$ always hold for any variety $\mathbf{K}$ of finite tree algebras and any variety $\mathscr{V}$ of binary tree languages (to be defined later). We do not need this fact here.

Wilke [60] anticipated a variety theorem for families of binary tree languages and special classes of $\mathbf{l}$-generated tree algebras. In the sequel we introduce reduced algebras which are more restricted than $\mathbf{l}$-generated algebras, and prove a variety theorem for families of recognizable binary tree languages and varieties of finite reduced tree algebras.

Definition 6.2.10 A tree algebra $\mathcal{M} = (\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\rangle, \Gamma)$ is called *reduced*, if it is $\mathbf{l}$-generated and

(1) for every $a, b \in M_{\mathbf{l}}$, if $\iota^{\mathcal{M}}(a) = \iota^{\mathcal{M}}(b)$ and $\kappa^{\mathcal{M}}(a,t,s) = \kappa^{\mathcal{M}}(b,t,s)$ hold for all $t, s \in M_{\mathbf{t}}$, then $a = b$;

(2) for every $p, q \in M_{\mathbf{c}}$, if $\eta^{\mathcal{M}}(p,t) = \eta^{\mathcal{M}}(q,t)$ for all $t \in M_{\mathbf{t}}$, then $p = q$.

Every tree algebra $\mathcal{M}$ can be reduced as follows. Take $\mathcal{M}'$ be the $\mathbf{l}$-subalgebra of $\mathcal{M}$, and let θ be the following relation on $\mathcal{M}'$:

$\theta_{\mathbf{l}} = \{(a,b) \in M'_{\mathbf{l}} \mid \iota^{\mathcal{M}'}(a) = \iota^{\mathcal{M}'}(b)\ \&$
$(\forall t, s \in M'_{\mathbf{t}})(\kappa^{\mathcal{M}'}(a,t,s) = \kappa^{\mathcal{M}'}(b,t,s))\}$;
$\theta_{\mathbf{t}} = \{(t,s) \in M'_{\mathbf{t}} \mid t = s\}$;
$\theta_{\mathbf{c}} = \{(p,q) \in M'_{\mathbf{c}} \mid (\forall t \in M'_{\mathbf{t}})(\eta^{\mathcal{M}'}(p,t) = \eta^{\mathcal{M}'}(q,t))\}$.

It can be shown that θ is a congruence on $\mathcal{M}'$ and $\mathcal{M}'/\theta$ is a reduced tree algebra; moreover $\mathcal{M}'/\theta \preceq \mathcal{M}$.

We also note that the syntactic tree algebra of any binary tree language is a reduced tree algebra by Lemma 6.2.1. The most important property of reduced tree algebras is the following.

Lemma 6.2.11 *For any reduced tree algebra $\mathcal{M}$ there are binary tree languages $T_1, \ldots, T_n \subseteq T_A$ for an alphabet A such that* $\mathrm{STA}(T_j) \preceq \mathcal{M}$ *for $j = 1, \ldots, n$, and $\mathcal{M} \subseteq \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$.*

Proof. Let $A = M_{\mathbf{l}}$. Since $\mathcal{M}$ is $\mathbf{l}$-generated, there exists an epimorphism $\varphi : \mathcal{F}_{\mathrm{TA}}(A) \to \mathcal{M}$. Suppose $M_{\mathbf{t}} = \{m_1, \ldots, m_n\}$ and for any $m_j \in M_{\mathbf{t}}$ let $T_j = \{m_j\}\varphi_{\mathbf{t}}^{-1} \subseteq T_A$. By Proposition 2.2.12(4) we have $\mathcal{F}_{\mathrm{TA}}(A)/\langle T_j\rangle \cong \mathcal{M}/\langle\{m_j\}\rangle$, so $\mathrm{STA}(T_j) \preceq \mathcal{M}$ for $j = 1, \ldots, n$.

We show $\mathcal{M} \subseteq \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$ for which it is enough to show

$$\mathcal{M} \subseteq \mathcal{M}/\langle\{m_1\}\rangle \times \cdots \times \mathcal{M}/\langle\{m_n\}\rangle.$$

Define the mapping

$$\imath : \mathcal{M} \to \mathcal{M}/\langle\{m_1\}\rangle \times \cdots \times \mathcal{M}/\langle\{m_n\}\rangle$$

by $\imath(u) = \big(u/\langle\{m_1\}\rangle, \ldots u/\langle\{m_n\}\rangle\big)$ for $u \in \langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\rangle$. Obviously, $\imath$ is a homomorphism. It remains to show that it is a monomorphism.

If $\imath_{\mathbf{l}}(a) = \imath_{\mathbf{l}}(b)$ for some $a, b \in M_{\mathbf{l}}$, then $a \approx_{\mathbf{l}}^{\langle\{m_j\}\rangle} b$ for every $m_j \in M_{\mathbf{t}}$. In particular, $a \approx_{\mathbf{l}}^{\langle\{\iota^{\mathcal{M}}(a)\}\rangle} b$ which implies that $\iota^{\mathcal{M}}(a) = \iota^{\mathcal{M}}(b)$. Also, for every $t, s \in M_{\mathbf{t}}$, $a \approx_{\mathbf{l}}^{\langle\{\kappa^{\mathcal{M}}(a,t,s)\}\rangle} b$ and so $\kappa^{\mathcal{M}}(a,t,s) = \kappa^{\mathcal{M}}(b,t,s)$. Since $\mathcal{M}$ is reduced, it follows that $a = b$.

Similarly, if $\imath_{\mathbf{t}}(t) = \imath_{\mathbf{t}}(s)$ for some $t, s \in M_{\mathbf{t}}$, then $t \approx_{\mathbf{t}}^{\langle\{t\}\rangle} s$, and hence $t = s$.

Finally, if $\imath_{\mathbf{c}}(p) = \imath_{\mathbf{c}}(q)$ for some $p, q \in M_{\mathbf{c}}$, then for every $t \in M_{\mathbf{t}}$, $p \approx_{\mathbf{c}}^{\langle\{\eta^{\mathcal{M}}(p,t)\}\rangle} q$, and so $\eta^{\mathcal{M}}(p,t) = \eta^{\mathcal{M}}(q,t)$ for every $t \in M_{\mathbf{t}}$. Since $\mathcal{M}$ is reduced, it follows that $p = q$.

All in all we showed that $\imath$ is injective which finishes the proof. □

We note that by Lemma 6.2.6, the tree algebra $\mathcal{D}$ of Definition 6.2.5 is not reduced ($\mathcal{D} \neq \mathcal{D}'$) and the proof of Lemma 6.2.8 shows that Lemma 6.2.11 does not hold for $\mathcal{D}$.

Definition 6.2.12 An *r-variety* is a class $\mathbf{K}$ of finite reduced tree algebras such that for any $\mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathbf{K}$ and any reduced tree algebra $\mathcal{N}$, if $\mathcal{N} \preceq \mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ then $\mathcal{N} \in \mathbf{K}$.

A *b-variety* is a family $\mathscr{V} = \{\mathscr{V}(A)\}$ of binary tree languages such that for any finite alphabets A and B,

(1) if $T, T' \in \mathscr{V}(A)$, then $T_A \setminus T, T \cap T' \in \mathscr{V}(A)$,
(2) if $T \in \mathscr{V}(A)$ and $p \in C_A$, then $p^{-1}(T) \in \mathscr{V}(A)$, and
(3) if $T \in \mathscr{V}(A)$ and $\varphi : \mathcal{F}_{\mathrm{TA}}(B) \to \mathcal{F}_{\mathrm{TA}}(A)$ is a homomorphism, then $T\varphi^{-1} \in \mathscr{V}(B)$.

We note that the collection of all the r-varieties of finite reduced tree algebras and the collection of all the b-varieties of recognizable binary tree languages are complete lattices with respect to the inclusion relation. So, we may speak about the r-variety generated by a collection of finite reduced tree algebras.

Definition 6.2.13 For any class of finite reduced tree algebras $\mathbf{K}$, let $\mathbf{K}^t = \{\mathbf{K}^t(A)\}$ be the family of recognizable binary tree languages defined by

$$\mathbf{K}^t(A) = \{T \subseteq T_A \mid \mathrm{STA}(T) \in \mathbf{K}\}$$

for any finite alphabet A.

For any family of recognizable binary tree languages $\mathscr{V} = \{\mathscr{V}(A)\}$, let $\mathscr{V}^a$ be the r-variety of finite reduced tree algebras generated by the collection

$$\{\mathrm{STA}(T) \mid T \in \mathscr{V}(A)\}$$

where A ranges over finite alphabets.

By Lemma 6.2.3, the family $\mathbf{K}^t$ is a b-variety for any r-variety $\mathbf{K}$. Thus the operations $\mathbf{K} \mapsto \mathbf{K}^t$ and $\mathscr{V} \mapsto \mathscr{V}^a$ map r-varieties to b-varieties and the other way round. It is also easy to see that the operations preserve inclusion, i.e., if $\mathbf{K}_1 \subseteq \mathbf{K}_2$ and $\mathscr{V}_1 \subseteq \mathscr{V}_2$, then $\mathbf{K}_1^t \subseteq \mathbf{K}_2^t$ and $\mathscr{V}_1^a \subseteq \mathscr{V}_2^a$. In the variety theorem below, we will show that they are bijective correspondences between r-varieties and b-varieties which are the inverses of each other, i.e., $\mathbf{K}^{ta} = \mathbf{K}$ and $\mathscr{V}^{at} = \mathscr{V}$ for any r-variety $\mathbf{K}$ and b-variety $\mathscr{V}$.

Proposition 6.2.14 *For any r-variety* $\mathbf{K}$, $\mathbf{K}^{ta} = \mathbf{K}$.

Proof. The inclusion $\mathbf{K}^{ta} \subseteq \mathbf{K}$ is obvious. For the inverse inclusion, suppose $\mathcal{M} \in \mathbf{K}$. Since $\mathcal{M}$ is a reduced tree algebra, there are by Lemma 6.2.11 for some alphabet A, recognizable binary tree languages $T_1, \ldots, T_n \subseteq T_A$ such that $\mathrm{STA}(T_j) \preceq \mathcal{M}$ for $j = 1, \ldots, n$, and $\mathcal{M} \subseteq \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$. Then $\mathrm{STA}(T_j) \in \mathbf{K}$ and thus $T_j \in \mathbf{K}^t(A)$ for $j = 1, \ldots, n$. Now the relation $\mathcal{M} \subseteq \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$ implies that $\mathcal{M} \in \mathbf{K}^{ta}$. □

Proposition 6.2.15 *For any b-variety* $\mathscr{V}$, $\mathscr{V}^{at} = \mathscr{V}$.

Proof. The inclusion $\mathscr{V} \subseteq \mathscr{V}^{at}$ is obvious. We show $\mathscr{V}^{at} \subseteq \mathscr{V}$. Suppose that $T \in \mathscr{V}^{at}(A)$ for some alphabet A. Then $\mathrm{STA}(T) \in \mathscr{V}^a$ implies that $\mathrm{STA}(T) \preceq \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$ for some $n \geq 1$, some finite alphabets A_i, and some binary tree languages $T_i \in \mathscr{V}(A_i)$ $(i = 1, \ldots, n)$. For each $i = 1, \ldots, n$, let φ^i denote the syntactic homomorphisms $\varphi^{T_i} \colon \mathcal{F}_{\mathrm{TA}}(A_i) \to \mathrm{STA}(T_i)$. Then there is a homomorphism

$$\eta \colon \mathcal{F}_{\mathrm{TA}}(A_1) \times \cdots \times \mathcal{F}_{\mathrm{TA}}(A_n) \longrightarrow \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$$

such that for every $i = 1, \ldots, n$, $\eta\pi^i = \varphi^i\tau^i$, where

$$\pi^i \colon \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n) \longrightarrow \mathrm{STA}(T_i),$$

and

$$\tau^i \colon \mathcal{F}_{\mathrm{TA}}(A_1) \times \cdots \times \mathcal{F}_{\mathrm{TA}}(A_n) \longrightarrow \mathcal{F}_{\mathrm{TA}}(A_i)$$

are the respective projection functions. By Lemma 2.2.11 there exist a homomorphism $\varphi \colon \mathcal{F}_{\mathrm{TA}}(A) \to \mathrm{STA}(T_1) \times \cdots \times \mathrm{STA}(T_n)$ and a subset H of $\mathrm{STA}(T_1)_{\mathbf{t}} \times \cdots \times \mathrm{STA}(T_n)_{\mathbf{t}}$ such that $T = H\varphi_{\mathbf{t}}^{-1}$. Since η is an epimorphism, there is a homomorphism $\psi \colon \mathcal{F}_{\mathrm{TA}}(A) \to \mathcal{F}_{\mathrm{TA}}(T_1) \times \cdots \times \mathcal{F}_{\mathrm{TA}}(A_n)$ such that $\psi\eta = \varphi$. Because H is finite, $T = \bigcup_{u \in H} u\varphi_{\mathbf{t}}^{-1}$ is the union of finitely many sets $u\varphi_{\mathbf{t}}^{-1}$ with $u = (u_1, \ldots, u_k) \in \mathrm{STA}(T_1)_{\mathbf{t}} \times \cdots \times \mathrm{STA}(T_n)_{\mathbf{t}}$. Noting that $\psi\tau^i : \mathcal{F}_{\mathrm{TA}}(A) \to \mathcal{F}_{\mathrm{TA}}(A_i)$ for any $1 \leq i \leq n$, for each such $u \in H$, $u\varphi_{\mathbf{t}}^{-1} = \bigcap\{u_i(\varphi_{\mathbf{t}}\pi^i_{\mathbf{t}})^{-1} \mid 1 \leq i \leq n\} = \bigcap\{u_i(\varphi^i_{\mathbf{t}})^{-1}(\psi\tau^i)_{\mathbf{t}}^{-1} \mid 1 \leq i \leq n\}$. Now, by Corollary 6.2.2, $u_i(\varphi^i_{\mathbf{t}})^{-1} \in \mathscr{V}(T_i)$ for each $1 \leq i \leq n$, and thus we have $T \in \mathscr{V}(A)$. □

Summing up, we proved the following variety theorem for b-varieties of recognizable binary tree languages and r-varieties of finite reduced tree algebras.

Proposition 6.2.16 *The operations* $\mathbf{K} \mapsto \mathbf{K}^t$ *and* $\mathscr{V} \mapsto \mathscr{V}^a$ *form a pair of lattice isomorphism between the class of all r-varieties and the class of all b-varieties that are inverses of each other, i.e.,* $\mathbf{K}^{ta} = \mathbf{K}$ *and* $\mathscr{V}^{at} = \mathscr{V}$ *for each r-variety* $\mathbf{K}$ *and each b-variety* $\mathscr{V}$.

Finally, we note that every generalized variety of tree languages yields in a natural way a variety of binary tree languages.

Proposition 6.2.17 *For any gVTL* $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ *the family* $\mathscr{V}^* = \{\mathscr{V}^*(A)\}$ *where* $\mathscr{V}^*(A) = \mathcal{V}(\Sigma^A, \emptyset)$ *for each finite alphabet* A, *is a b-variety.*

Proof. It is clear that $\mathscr{V}^*$ is closed under all the finite Boolean operations and inverse translations. To show that it is also closed under inverse homomorphisms, we note that for any such homomorphism $\varphi : \mathcal{F}_{\mathrm{TA}}(B) \rightarrow \mathcal{F}_{\mathrm{TA}}(A)$ where A and B are finite alphabets, the mapping $\zeta : \Sigma^B \rightarrow \Sigma^A$ defined by $c_b\zeta = c_{b\varphi_1}$ and $f_b\zeta = f_{b\varphi_1}$ for all $b \in B$, is an assignment and the pair of mappings $(\zeta, \varphi_{\mathbf{t}}) : (T_B, \Sigma^B) \rightarrow (T_A, \Sigma^A)$ is a g-morphism (see Definition 4.1.1 in Chapter 4). □

It follows from this proposition that, for the generalized varieties of *nilpotent*, *definite*, *reverse definite*, *generalized definite*, *locally testable* tree languages, etc (Examples 7.5–7.7 of [54]) we have b-varieties of the corresponding kind. The following example shows that not every b-variety can be obtained in such a way.

Example 6.2.18 Define the family $\mathscr{V} = \{\mathscr{V}(A)\}$ by

$$\mathscr{V}(A) = \{T \subseteq T_A \mid (\forall a \in A)(\forall t \in T_A)\big(f_a(c_a, t) \approx^T t\big)\}$$

for any finite alphabet A.

It is easy to see that $\mathscr{V}$ is a b-variety. We show that there is no gVTL $\mathcal{W} = \{\mathcal{W}(\Sigma, X)\}$ such that $\mathscr{V}(A) = \mathcal{W}(\Sigma^A, \emptyset)$ for all finite alphabets A. Assume there is such a $\mathcal{W}$. Let $A = \{a, b\}$, and let T be the smallest subset of T_A that contains c_a and satisfies $t \in T \Rightarrow f_a(c_a, t), f_b(c_b, t) \in T$ for every $t \in T_A$. Then $T \in \mathscr{V}(A)$. Let the assignment $\varsigma : \Sigma^A \rightarrow \Sigma^A$ be defined by $c_a\varsigma = c_a$, $c_b\varsigma = c_b$, $f_a\varsigma = f_b$, and $f_b\varsigma = f_a$. It can be easily seen that $(\varsigma, 1_{T_A}) : (T_A, \Sigma^A) \rightarrow (T_A, \Sigma^A)$ is a g-morphism, where $1_{T_A} : T_A \rightarrow T_A$ is the identity map. However, $T(\varsigma, 1_{T_A})^{-1}$ is not in $\mathscr{V}(A)$, since $t = f_b(c_a, c_a) \in T(\varsigma, 1_{T_A})^{-1}$ as $t(\varsigma, 1_{T_A}) = f_a(c_a, c_a) \in T$, but $f_a(c_a, t) \notin T(\varsigma, 1_{T_A})^{-1}$ as $f_a(c_a, t)(\varsigma, 1_{T_A}) = f_b(c_a, f_a(c_a, c_a)) \notin T$. So, $\mathcal{W}$ is not closed under inverse g-morphisms, a contradiction.

6.3 Some algebraic properties of tree algebras

In this last section we study some other algebraic properties of term algebras and free tree algebras which are not directly related to the variety theory of tree languages. Our motivation for this research was the wish to understand the significance of the particular choice of Wilke's functions for the signature of tree algebras.

A completeness property of Wilke's functions is that they generate all the functions involving labels, trees and contexts that preserve the syntactic

TA-congruences (Lemma 6.2.1) of all binary tree languages, provided that the alphabet is large enough (contains at least seven labels). Translating this into an algebraic terminology it reads as "the free tree algebra over an alphabet A is affine-complete if $|A| \geq 7$". An algebra is called *affine-complete* [15] if every of its congruence preserving function is a polynomial (i.e., can be constructed from the fundamental operations of the algebra, the projections, and the constant operations corresponding to the elements of the algebra). We note that every polynomial function in any algebra is congruence preserving. We also realized that term algebras are also affine-complete if their ranked alphabet contains at least seven constant symbols. Here we formulate these results without proofs. Interested readers are invited to consult [43, 44] for full presentations and technical details.

Let Σ be a ranked alphabet with the property that $\Sigma \neq \Sigma_0$ and let X be a leaf alphabet. For technical reasons and convenience we sometimes write $\mathrm{T}(\Sigma, X)$ as $T_\Sigma(X)$. When $X = \emptyset$, T_Σ is a shorthand for $\mathrm{T}(\Sigma, \emptyset)$ or $T_\Sigma(\emptyset)$.

Definition 6.3.1 For an algebra $\mathcal{A} = (A, \Sigma)$, a function $F : A^n \to A$ is called *congruence preserving*, if for every congruence relation θ on $\mathcal{A}$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, $F(a_1, \ldots, a_n)\ \theta\ F(b_1, \ldots, b_n)$ whenever $a_1\ \theta\ b_1, \ldots, a_n\ \theta\ b_n$.

Remark 6.3.2 It is known that every congruence relation over an algebra is the intersection of some syntactic congruence relations (see Remark 2.12 of [1] or Lemma 6.2 of [53].) So, a function preserves all congruence relations of an algebra iff it preserves the syntactic congruence relations of all subsets of the algebra. Thus, an equivalent condition for a function $F : A^n \to A$ to be congruence preserving is that for all $L \subseteq A$ and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, if $a_1 \approx^L b_1, \ldots, a_n \approx^L b_n$, then $F(a_1, \ldots, a_n) \approx^L F(b_1, \ldots, b_n)$.

Remark 6.3.3 A congruence preserving function $F : A^n \to A$ induces, for any congruence θ on $\mathcal{A}$, a well-defined function $F_\theta : (\mathcal{A}/\theta)^n \to \mathcal{A}/\theta$ on any quotient algebra, defined by $F_\theta(a_1/\theta, \ldots, a_n/\theta) = F(a_1, \ldots, a_n)/\theta$.

In the sequel we will be interested in the congruence preserving functions on term algebras $\mathcal{T}(\Sigma, X)$. Let $u_1, \ldots, u_n \in T_\Sigma(X)$ be some terms and let $t \in T_\Sigma(X \cup \{x_1, \ldots, x_n\})$ where $x_1, \ldots, x_n \notin X$. The term $t[u_1, \ldots, u_n]$ in $T_\Sigma(X)$ is resulted from t by replacing all the occurrences of x_i with u_i for $i = 1, \ldots, n$. The induced function $(T_\Sigma)^n \to T_\Sigma(X)$ defined by $(u_1, \ldots, u_n) \mapsto t[u_1, \ldots, u_n]$ for all $u_1, \ldots, u_n \in T_\Sigma$, is called the *term function* defined by t. It is also called the *tree substitution operation*, see e.g. [21]. It is easy to show that every term function is a congruence preserving function (on $\mathcal{T}(\Sigma, X)$). The following proposition is proved in [43].

Proposition 6.3.4 *If $|\Sigma_0| \geq 7$, then every congruence preserving function $F : (T_\Sigma)^n \to T_\Sigma$, for every $n \geq 0$, is a term function.*

We note that the proposition dose not hold for $|\Sigma_0| = 1$: Let $\Sigma = \Sigma_0 \cup \Sigma_1$ be a ranked alphabet with $\Sigma_1 = \{f\}$ and $\Sigma_0 = \{c\}$. The term algebra (T_Σ, Σ) is isomorphic to $(\mathbb{N}, \mathbf{0}, \mathbf{S})$, where $\mathbf{0}$ is the constant zero and $\mathbf{S}$ is the successor function. Let $F : \mathbb{N} \to \mathbb{N}$ be defined by $F(n) = 2n$. It is easy to see that F is congruence preserving: for every congruence relation θ, if $n\ \theta\ m$ then $\mathbf{S}n\ \theta\ \mathbf{S}m$ and by repeating the same argument n times we get $\mathbf{S}^n n\ \theta\ \mathbf{S}^n m$ or $2n\ \theta\ n+m$. Similarly $\mathbf{S}^m n\ \theta\ \mathbf{S}^m m$, so $m+n\ \theta\ 2m$, and hence $2m\ \theta\ 2n$ that is $F(n)\ \theta\ F(m)$. But F is not a term function, since all term functions are of the form $n \mapsto \mathbf{S}^{\mathbf{k}} n = \mathbf{k} + n$ for a fixed $\mathbf{k} \in \mathbb{N}$. It is not clear at the moment whether Proposition 6.3.4 holds for $2 \le |\Sigma_0| \le 6$.

An algebra is called *congruence-primal* or *hemi-primal*, if all its congruence preserving functions are term functions, and *affine-complete*, if all its congruence preserving functions are polynomials, see e.g. [26]. We note that since in term algebras with empty leaf alphabets polynomials coincide with term functions, a term algebra with empty leaf alphabet is affine-complete iff it is congruence-primal.

Proposition 6.3.4 above can be generalized to contexts. The set of ΣX-contexts is denoted by $\mathrm{C}(\Sigma, X)$. Again for technical reasons and convenience we write it as $C_\Sigma(X)$. For empty $X = \emptyset$, C_Σ is a shorthand for $\mathrm{C}(\Sigma, \emptyset)$. Recall the notation $\sim^L$ from Chapter 4.

Definition 6.3.5 A functions $F : (C_\Sigma)^m \times (T_\Sigma)^n \to T_\Sigma$ is called *congruence preserving*, if for all $p_1, q_1, \ldots, p_m, q_m \in C_\Sigma$, $t_1, s_1, \ldots, t_n, s_n \in T_\Sigma$ and every subset $L \subseteq T_\Sigma$,
whenever $p_1 \sim^L q_1, \ldots, p_m \sim^L q_m$ and $t_1 \approx^L s_1, \ldots, t_n \approx^L s_n$, then
$F(p_1, \ldots, p_m, t_1, \ldots, t_n) \approx^L F(q_1, \ldots, q_m, s_1, \ldots, s_n)$.
Likewise, $F : (C_\Sigma)^m \times (T_\Sigma)^n \to C_\Sigma$ is called *congruence preserving*, if for all $p_1, q_1, \ldots, p_m, q_m \in C_\Sigma$, $t_1, s_1, \ldots, t_n, s_n \in T_\Sigma$ and every subset $L \subseteq T_\Sigma$,
whenever $p_1 \sim^L q_1, \ldots, p_m \sim^L q_m$ and $t_1 \approx^L s_1, \ldots, t_n \approx^L s_n$, then
$F(p_1, \ldots, p_m, t_1, \ldots, t_n) \sim^L F(q_1, \ldots, q_m, s_1, \ldots, s_n)$.

In the following definition term functions are generalized to substitution functions including contexts. Let $\{\varrho_1, \varrho_2, \varrho_3, \ldots\}$ be a set of unary function symbols disjoint from Σ, and for each $m \ge 1$, $\Sigma\{\varrho_1, \ldots, \varrho_m\}$ be the signature Σ augmented by $\{\varrho_1, \ldots, \varrho_m\}$.

Definition 6.3.6 Let $r \in T_{\Sigma\{\varrho_1,\ldots,\varrho_m\}}$ be a term. We write r as $r[\varrho_1, \ldots, \varrho_m]$ to emphasize the possible presence of ϱ_i's. For contexts $p_1, \ldots, p_m \in C_\Sigma$, the term $r[p_1, \ldots, p_m] \in T_\Sigma$ is obtained from r by replacing all the occurrences of $\varrho_i(t)$, for any $t \in T_{\Sigma\{\varrho_1,\ldots,\varrho_m\}}$, with $p_i(t)$ for all $i = 1, \ldots, m\}$.

We call the function $(C_\Sigma)^m \to T_\Sigma$ defined by $(p_1, \ldots, p_m) \mapsto r[p_1, \ldots, p_m]$ for all $p_1, \ldots, p_m \in C_\Sigma$, a *substitution function* defined by $r[\varrho_1, \ldots, \varrho_m]$.

For a set of variables $\{x_1, \ldots, x_n\}$, the term

$$t \in T\big(\Sigma\{\varrho_1, \ldots, \varrho_m\}, \{x_1, \ldots, x_n\}\big)$$

is also written as $t[x_1,\ldots,x_n,\varrho_1,\ldots,\varrho_m]$. The term $t[s_1,\ldots,s_n,p_1,\ldots,p_m]$, for terms $s_1,\ldots,s_n$ and contexts $p_1,\ldots,p_m$, is obtained from t by replacing each x_i with the corresponding s_i and each ϱ_j with p_j for all i,j. This induces a function

$$(T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma,\ (s_1,\ldots,s_n,p_1,\ldots,p_m) \mapsto t[s_1,\ldots,s_n,p_1,\ldots,p_m]$$

which is also called a *substitution function* defined by t.

Similarly, the substitution function $(T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ defined by a context $q[x_1,\ldots,x_n,\varrho_1,\ldots,\varrho_m]$ maps a tuple $(s_1,\ldots,s_n,p_1,\ldots,p_m)$ to the context $q[s_1,\ldots,s_n,p_1,\ldots,p_m]$ for all trees $s_1,\ldots,s_n \in T_\Sigma$ and contexts $p_1,\ldots,p_m \in C_\Sigma$.

Example 6.3.7 The composition of contexts $C_\Sigma \times C_\Sigma \to C_\Sigma$, $(p_1,p_2) \mapsto p_1(p_2)$ is a substitution function defined by $\varrho_1(\varrho_2(\xi)) \in C_{\Sigma\{\varrho_1,\varrho_2\}}$. Also, the function $T_\Sigma \times C_\Sigma \to T_\Sigma$, $(t,p) \mapsto p(t)$, is a substitution function defined by $\varrho_1(x_1) \in T_{\Sigma\{\varrho_1\}}(\{x_1\})$.

Again it is easy to see that all the substitution functions are congruence preserving. Less trivial is the following proposition proved in [44].

Proposition 6.3.8 *For a ranked alphabet Σ such that $\Sigma = \Sigma_0 \cup \Sigma_2$ and $|\Sigma_0|, |\Sigma_2| \geq 7$, every congruence preserving function $(T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma$ and $(T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ is a substitution function.*

We showed by an example that when $\Sigma = \Sigma_0 \cup \Sigma_1$ with $|\Sigma_0| = |\Sigma_1| = 1$, there is a congruence preserving function $T_\Sigma \to T_\Sigma$ which is not a substitution function. So, some lower bound must be set on $|\Sigma_0|$ in Proposition 6.3.8, but it is not yet known whether the bound 7 is the best possible. Here we show that Proposition 6.3.8 does not hold for $\Sigma = \Sigma_0 \cup \Sigma_1$, with $|\Sigma_1| = 1$ and any non-empty Σ_0. For such a Σ suppose $\Sigma_1 = \{\alpha\}$ (note that no condition is set on $|\Sigma_0|$). So, $C_\Sigma = \{\alpha^n(\xi) \mid n \geq 0\}$, and $T_{\Sigma\{\varrho_1\}} = \{\alpha^{n_1}\varrho^{m_1}\cdots\alpha^{n_k}\varrho^{m_k}(c) \mid n_1,m_1,\ldots,n_k,m_k \geq 0,\ c \in \Sigma_0\}$. Hence, all the substitution functions $C_\Sigma \to T_\Sigma$ are of the form $\alpha^m(\xi) \mapsto \alpha^{\mathbf{k}m+\mathbf{n}}(\mathbf{c})$ for some fixed $\mathbf{k},\mathbf{n} \geq 0$ and $\mathbf{c} \in \Sigma_0$. Let, for a fixed $c_0 \in \Sigma_0$, $F : C_\Sigma \to T_\Sigma$ be defined by $F(\alpha^m(\xi)) = \alpha^{m^2}(c_0)$ for all $m \geq 0$. Obviously F is not a substitution function, but we show that it is congruence preserving. For any subset $L \subseteq T_\Sigma$ and any $m,n \geq 0$, if $\alpha^m(\xi) \sim^L \alpha^n(\xi)$, then by induction on j it can be shown that $\alpha^{j+m}(c_0) \approx^L \alpha^{j+n}(c_0)$. By once putting $j = m$ and another time $j = n$, we can conclude that $\alpha^{2m}(c_0) \approx^L \alpha^{2n}(c_0)$. From this and $\alpha^m(\xi) \sim^L \alpha^n(\xi)$ we infer that $\alpha^m(\alpha^{2m}(c_0)) \approx^L \alpha^n(\alpha^{2n}(c_0))$, and so on, by induction on j, it can be shown that $\alpha^{jm}(c_0) \approx^L \alpha^{jn}(c_0)$. Again by putting once $j = m$ and once $j = n$, we can infer that $\alpha^{m^2}(c_0) \approx^L \alpha^{n^2}(c_0)$, or in other words, $F(\alpha^m(\xi)) \approx^L F(\alpha^n(\xi))$.

Finally, we observe that if A contains at least seven labels, then Wilke's functions over A generate all the congruence preserving functions that involve labels from A, A-trees and A-contexts. This implies that the free tree algebra $\mathcal{F}_{\mathrm{TA}}(A)$ over A is affine-complete.

For simplicity we write Wilke's functions on the free tree algebra $\mathcal{F}_{\mathrm{TA}}(A)$ as $\iota^{\mathcal{F}_{\mathrm{TA}}(A)} = \iota^A$, $\kappa^{\mathcal{F}_{\mathrm{TA}}(A)} = \kappa^A$, etc (see Section 6.1).

Definition 6.3.9 A function $F : A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to X$, where X is either A, T_{A} or C_{A}, is called *congruence preserving*, if for every tree language $L \subseteq T_{\mathrm{A}}$ and for all $a_1, b_1, \ldots, a_n, b_n \in A$, $t_1, s_1, \ldots, t_m, s_m \in T_{\mathrm{A}}$ and $p_1, q_1, \ldots, p_k, q_k \in C_{\mathrm{A}}$, whenever $a_1 \approx_{\mathbf{l}}^L b_1, \ldots, a_n \approx_{\mathbf{l}}^L b_n$, $t_1 \approx_{\mathbf{t}}^L s_1, \ldots, t_m \approx_{\mathbf{t}}^L s_m$, and $p_1 \approx_{\mathbf{c}}^L q_1, \ldots, p_k \approx_{\mathbf{c}}^L q_k$, then $F(a_1, \ldots, a_n, p_1, \ldots, p_k, t_1, \ldots, t_m) \approx_{\mathbf{x}}^L F(b_1, \ldots, b_n, q_1, \ldots, q_k, s_1, \ldots, s_m)$, where $\mathbf{x}$ is $\mathbf{l}$, $\mathbf{t}$, or $\mathbf{c}$, if $X = A$, $X = T_{\mathrm{A}}$, or $X = C_{\mathrm{A}}$, respectively.

We borrow the notion of a "clone" from universal algebra (cf. e.g. [26]) and introduce the notion of a "Pclone" for our purpose.

Definition 6.3.10 The *projection functions* $\pi_j^n : B_1 \times \cdots \times B_n \to B_j$ $(j = 1, \cdots, n)$ for sets $B_1, \ldots, B_n$, are defined by $\pi_j^n(b_1, \ldots, b_n) = b_j$. Each element $b \in B_j$ determines the *constant function* $B_1 \times \cdots \times B_n \to B_j$, $(b_1, \ldots, b_n) \mapsto b$.

A *Pclone* over a collection of sets is a collection of functions over the sets of the collection which includes the constant and projection functions and is closed under composition.

Let $\mathcal{B}$ be a collection of sets, and let C be a collection of functions of the form $B_1 \times \cdots \times B_n \to B$ for any $B_1, \ldots, B_n, B \in \mathcal{B}$. The *Pclone generated by* C is the smallest class $\mathrm{Pclone}\langle C \rangle$, of functions of the form $B_1 \times \cdots \times B_n \to B$ for some $B_1, \ldots, B_n, B \in \mathcal{B}$, that contains C, the projection and constant functions, and is closed under the composition of functions (cf. the definition of clone in [26].)

It is easy to see that all functions in the Pclone generated by Wilke's functions are congruence preserving. The main result of [44] states that for an alphabet A which contains at least seven labels, every congruence preserving function over A is in the Pclone generated by Wilke's functions. More precisely, the following can be shown.

Proposition 6.3.11 *If* $|A| \geq 3$*, then every congruence preserving function* $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to A$ $(n, m, k \geq 0)$ *is in* $\mathrm{Pclone}\langle \emptyset \rangle$*, i.e., it is either a constant function or a projection to* A*.*

Proposition 6.3.12 *If* $|A| \geq 7$*, then every congruence preserving function* $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to T_{\mathrm{A}}$ $(n, m, k \geq 0)$ *is in* $\mathrm{Pclone}\langle \{\iota^A, \kappa^A, \eta^A\} \rangle$*.*

Proposition 6.3.13 *If* $|A| \geq 7$*, then every congruence preserving function* $A^n \times C_{\rm A}^k \times T_{\rm A}^m \to C_{\rm A}$ $(n, m, k \geq 0)$ *is in* $\mathrm{Pclone}\langle\{\iota^A, \kappa^A, \eta^A, \lambda^A, \rho^A, \sigma^A\}\rangle$.

We note that the condition $|A| \geq 3$ in Proposition 6.3.11 can not be improved: for $A = \{a, b\}$ the function $F : A \to A$ defined by $F(a) = b$ and $F(b) = a$ is obviously congruence preserving but not a constant or projection function. We close the dissertation with some examples.

Example 6.3.14 Let $A = \{a, b\}$. The function $F : A \times T_{\rm A} \times C_{\rm A} \to C_{\rm A}$ defined by $F(a_1, t_1, p_1) = f_a\big(f_{a_1}\big(f_b(c_a, c_a), \xi\big), p_1\big(f_b(t_1, c_{a_1})\big)\big)$ for $a_1 \in A$, $t_1 \in T_{\rm A}$ and $p_1 \in C_{\rm A}$, is congruence preserving and moreover it belongs to $\mathrm{Pclone}\langle\{\iota^A, \kappa^A, \eta^A, \lambda^A, \rho^A, \sigma^A\}\rangle$, since

$$F(a_1, t_1, p_1) = \sigma^A\big(\lambda^A\big(a, \eta^A\big(p_1, \kappa^A(b, t_1, \iota^A(a_1))\big)\big), \rho^A\big(a_1, \kappa^A(b, \iota^A(a), \iota^A(a))\big)\big).$$

On the other hand, the root function root: $T_A \to A$, which maps a tree to its root label, is not congruence preserving: for $L = \{f_a(c_b, c_b)\}$ we have $f_a(c_a, c_a) \approx^L_{\mathbf{t}} f_b(c_a, c_a)$, but since $f_a(c_b, c_b) \in L$ and $f_b(c_b, c_b) \notin L$, then $\mathrm{root}\big(f_a(c_a, c_a)\big) = a \not\approx^L_{\mathbf{l}} b = \mathrm{root}\big(f_b(c_a, c_a)\big)$.

Epilogue

In this thesis, families of tree languages were characterized by various syntactic structures: (i) syntactic monoids, (ii) syntactic ordered algebras, (iii) syntactic ordered monoids, and (iv) syntactic tree algebras (for binary trees). Also, the classical variety theorem [53] was extended to the many-sorted case, and an algebraic property of free tree algebras and term algebras was investigated. A detailed case study of semilattice and symbolic tree languages was presented.

There are some other aspects of the variety theory of tree automata and tree languages which are not touched here. One of them is (ultimately) definability by equations or inequations. It is known that varieties of (ordered) algebras can be characterized by sequences of (in)equalities. It would be nice to determine those sequences which characterize varieties of (ordered) algebras that are definable by (ordered) monoids.

Another aspect is the link between logic and tree languages, and decidability: is it decidable to determine if a given family of tree languages is definable by (ordered) monoids? Decidability of tree languages is treated e.g. in the recent PhD thesis of Mikołaj Bojańczyk, "Decidable properties of tree languages", Warsaw University, 2004. Logical characterizations of families of tree languages which are definable by (ordered) monoids is another domain to explore.

One line of extension of our results is proving variety theorems for families of tree languages which are less demanding than (positive) varieties. It seems that when a richer syntactic structure is considered for characterizing families of tree languages, fewer conditions are put upon them. For example, positive varieties are less restricted than varieties, while ordered algebras are more restricted than algebras. Characterizing families of tree languages by richer structures (e.g. relational or first-order structures) could be an interesting subject for research in future.

The last open problems we would like to mention here are: whether Propositions 6.3.12 and 6.3.13 hold, when $3 \leq |A| \leq 6$; and similarly, whether Proposition 6.3.4 holds for $2 \leq |\Sigma_0| \leq 6$, and Proposition 6.3.8 holds for $2 \leq |\Sigma_0|, |\Sigma_2| \leq 6$.

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Addendum

No variety theorem can be proved for
the classes of **l**-generated tree algebras.

Let $\mathcal{A} = \big(\{a, b\}, \{t\}, \{p\}, \Gamma\big)$ where
$\iota^{\mathcal{A}}(a) = \iota^{\mathcal{A}}(b) = t;$
$\kappa^{\mathcal{A}}(a, t, t) = \kappa^{\mathcal{A}}(b, t, t) = t;$
$\lambda^{\mathcal{A}}(a, t) = \lambda^{\mathcal{A}}(b, t) = p;$
$\rho^{\mathcal{A}}(a, t) = \rho^{\mathcal{A}}(b, t) = p;$
$\eta^{\mathcal{A}}(p, t) = t;\ \sigma^{\mathcal{A}}(p, p) = p.$
The algebra $\mathcal{A}$ is **l**-generated.
Let $\mathbf{K}$ be the variety of tree algebras generated by $\mathcal{A}$. Every member of $\mathbf{K}$ is **l**-generated, and $\mathbf{K}^t(A) = \{\emptyset, T_A\}$ for any alphabet A. Thus,

$$\mathbf{K}^{ta} = \{\text{The Trivial Tree Algebra}\}.$$

Since $\mathcal{A} \notin \mathbf{K}^{ta}$, then $\mathbf{K} \not\subseteq \mathbf{K}^{ta}$. Q.E.D

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