Generalized factors of words

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Abstract

We introduce and study relations on words which generalize the factor relation, being restrictions of the subword relation. We give an equivalent condition for the finite basis property for these relations which generalizes the well-known theorem of Higman. Some language-theoretic gaps for infinite antichains are also presented.

1 Introduction

Consider a finite alphabet Σ. Two of the most common relations on words are the factor and the subword partial orders, see [ChKa], [Lo], defined, respectively, for two words $u, v \in \Sigma^*$, by

$$\begin{array}{lll} u \leq_f v & \text{iff} & v = xuy, \text{ for some } x, y \in \Sigma^*, \\ u \leq_s v & \text{iff} & v = v_0 u_1 v_1 u_2 v_2 \ldots u_k v_k, \\ & & u = u_1 u_2 \ldots u_k, \\ & & k \geq 1, v_0, u_i, v_i \in \Sigma^*, 1 \leq i \leq k. \end{array}$$

Roughly speaking, a word is a factor of another one it the former appears inside the latter exactly as it is whereas a subword may appear scattered as several factors. Therefore, the idea of in-between cases naturally arises, that is, of imposing different restrictions on the number of factors as which a word is allowed to be scattered. Following this idea, we define in the next section a class of (uncountably many) relations which are extensions of the factor partial order $\leq_f$ and restrictions of the subword partial order $\leq_s$. We then study in our sections 3 and 4 the finite basis property for these relations and give a characterization of those having this property. This result generalizes the well-known theorem of Higman, cf. [Hi]. In the last section, we consider some language-theoretic properties of the infinite antichains of some of the relations introduced.

2 Generalized factor relations

The first step in defining our generalized factor relations is to consider, for any positive integer k, the relation $\leq_{f,k}$ on Σ^*, defined by

$$\begin{array}{lll} u \leq_{f,k} v & \text{iff} & v = v_0 u_1 v_1 u_2 v_2 \ldots u_k v_k, \\ & & u = u_1 u_2 \ldots u_k, \\ & & v_0, u_i, v_i \in \Sigma^*, 1 \leq i \leq k. \end{array}$$

It is easy to see that, for any $k \geq 1$, $\leq_{f,k}$ is reflexive, antisymmetric, but not transitive.

We have obtained an infinite strict hierarchy of relations between the factor and the subword relation.

Proposition 2.1. $\leq_f = \leq_{f,1} \subset \leq_{f,2} \subset \leq_{f,3} \subset \cdots \subset \leq_s$, *all inclusions being strict.*

Proof. Obviously, for any $k \geq 1$, $\leq_{f,k} \subseteq \leq_{f,k+1}$. We have also $(ab)^k a \leq_{f,k+1} (ab^2)^k a$ but $(ab)^k a \not\leq_{f,k} (ab^2)^k a$ since $(ab)^k a$ appears in $(ab^2)^k a$ scattered in at least $k+1$ factors, as none of them can contain two a's. The strictness of the inclusion follows. ∎

We now define the finite basis property for relations exactly as it was defined for quasi orders, see [ErRa], [Hi]. Consider an arbitrary relation $\leq$ on Σ^*. A set $L \subseteq \Sigma^*$ is an *antichain* of $\leq$ if all elements of L are pairwise incomparable with respect to $\leq$, that is, for any $u, v \in L$, neither $u \leq v$ nor $v \leq u$. The relation $\leq$ has the *finite basis property* if (i) any antichain of it is finite and (ii) there is no infinite descending sequence $w_1 \leq^{-1} w_2 \leq^{-1} w_3 \cdots$ such that, for no $i \geq 1$, $w_i \leq w_{i+1}$, where $\leq^{-1}$ denotes the inverse of $\leq$.

Remark. Since we work only with relations which are included in $\leq_s$, the condition (ii) will always be fulfilled, so we shall consider explicitely the condition (i) only.

Proposition 2.2. *For any $k \geq 1$, the relation $\leq_{f,k}$ does not have the finite basis property.*

Proof. Take $k \geq 1$ and consider the language

$$L_k = \{(ab^n)^k a \mid n \geq 1\}.$$

We observe that L_k is infinite and an antichain of $\leq_{f,k}$. Indeed, if $(ab^n)^k a \leq_{f,k'} (ab^m)^k a$, for some $n < m$, then $(ab^n)^k a$ must be scattered in such a way that no factor contains two a's. Thus $k' \geq k+1$ so $(ab^n)^k \not\leq_{f,k} (ab^m)^k a$. ∎

We generalize further the above idea as follows. Consider the following set of natural functions

$$\mathcal{F} = \{g : \mathbb{N} \longrightarrow \mathbb{N} \mid g(0) = 0 \text{ and } 1 \leq g(n) \leq n, \text{ for any } n \geq 1\}.$$

For any function $g \in \mathcal{F}$, we define the relation $\leq_{f,g}$ on Σ^*, for $u, v \in \Sigma^*$, by

$$u \leq_{f,g} v \quad \text{iff} \quad \begin{array}{l} v = v_0 u_1 v_1 u_2 v_2 \ldots u_{g(|u|)} v_{g(|u|)}, \\ u = u_1 u_2 \ldots u_{g(|u|)}, \\ v_0, u_i, v_i \in \Sigma^*, 1 \leq i \leq g(|u|). \end{array}$$

Notice that

$$\leq_f = \leq_{f,\mathbf{1}},$$

where $\mathbf{1} : \mathbb{N} \longrightarrow \mathbb{N}, \mathbf{1}(0) = 0$, $\mathbf{1}(n) = 1$, for any $n \geq 1$, and

$$\leq_s = \leq_{f,\mathbf{1}_{\mathbb{N}}},$$

where $\mathbf{1}_{\mathbb{N}} : \mathbb{N} \longrightarrow \mathbb{N}, \mathbf{1}_{\mathbb{N}}(n) = n$, for any $n \geq 0$.

As seen in the next proposition, we have now uncountably many relations between $\leq_f$ and $\leq_s$.

Proposition 2.3. *(i). For any $g \in \mathfrak{F}$, $\leq_f \subseteq \leq_{f,g}$ and $\leq_f = \leq_{f,g}$ if and only if $g \equiv \mathbf{1}$.*

(ii). For any $g_1, g_2 \in \mathfrak{F}$ such that $g_1(n) \leq g_2(n)$, for any $n \geq 1$, we have $\leq_{f,g_1} \subseteq \leq_{f,g_2}$. If $g_1 \not\equiv g_2$, then the inclusion is strict.

(iii). For any $g \in \mathfrak{F}, \leq_{f,g} \subseteq \leq_s$ and $\leq_{f,g} = \leq_s$ if and only if $g \equiv \mathbf{1}_{\mathbb{N}}$.

Proof. (i). We prove only that $\leq_f = \leq_{f,g}$ implies $g \equiv \mathbf{1}$. Indeed, if, for some $n_0 \geq 1, g(n_0) \geq 2$, then $a^{n_0} \leq_{f,g} aba^{n_0-1}$ but $a^{n_0} \not\leq_f aba^{n_0-1}$.

(ii). To prove the strictness of the inclusion, suppose that, for some $n_0 \geq 1$, $g_1(n_0) < g_2(n_0)$. Then $a^{n_0} \leq_{f,g_2} (ab)^{g_2(n_0)-1}a$ but $a^{n_0} \not\leq_{f,g_1} (ab)^{g_2(n_0)-1}a$.

(iii). The first part is obviuos and the second is a special case of (ii). ∎

3 Infinite antichains

We present in this section a class of relations $\leq_{f,g}$ which allow infinite antichains. It will turn out in the next section that these are actually all relations in the family $(\leq_{f,g})_{g\in\mathfrak{F}}$ with this property.

Theorem 3.1. *For any $g \in \mathfrak{F}$ such that*

$$\overline{\lim}_{n\to\infty} \frac{n}{g(n)} = \infty,$$

the relation $\leq_{f,g}$ allows infinite antichains and, therefore, it does not have the finite basis property.

Proof. Take $g \in \mathfrak{F}$ such that

$$\overline{\lim}_{n\to\infty} \frac{n}{g(n)} = \infty.$$

We can find a subsequence $(k_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{k_n}{g(k_n)} = \infty.$$

We construct the following natural functions. First, define $k : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$k(n) = \min\{k_m \in \mathbb{N} \mid k_m \geq 1 \text{ and } g(k_m) \leq \frac{k_m}{2(n+1)}\},$$

for any $n \geq 0$. For any fixed $n \geq 1$, since $\lim_{m\to\infty} \dfrac{k_m}{g(k_m)} = \infty$, there is an $m \in \mathbb{N}$ such that

$$\frac{k_m}{g(k_m)} \geq 2(n+1)$$

and so

$$g(k_m) \leq \frac{k_m}{2(n+1)}.$$

Therefore, there is also a minimal such m and the function k is well-defined.

The second function is $h : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$h(n) = \max\{m \in \mathbb{N} \mid m(n+1)+1 \leq k(n)\},$$

for any $n \geq 0$. For any fixed $n \geq 0$, $k(n)$ is a finite number and the set of those $m \in \mathbb{N}$ with $m(n+1)+1 \leq k(n)$ is finite and non-empty. Thus, the above maximum always exists and h is well-defined. Notice also that both h and k are increasing functions.

The last function we use is $r : \mathbb{N} \longrightarrow \mathbb{N}$, defined by

$$r(n) = k(n) - h(n)(n+1) - 1,$$

for any $n \geq 0$. Notice that, from the definition of h, it follows that $k(n) \geq h(n)(n+1)+1$, for any $n \geq 0$, so r is well-defined.

Consider now the set

$$L_g = \{(ab^n)^{h(n)} ab^{r(n)} \mid n \geq 1\}.$$

Obviously L_g is infinite, and we claim that L_g is an antichain of $\leq_{f,g}$. To prove this, suppose that, for some $n, m \geq 1, n < m$, and $k \geq 1$, we have

$$(ab^n)^{h(n)} ab^{r(n)} \leq_{f,k} (ab^m)^{h(m)} ab^{r(m)}.$$

It follows that $(ab^n)^{h(n)} ab^{r(n)}$ must be scattered in at least $h(n)+1$ factors, thus

$$k \geq h(n) + 1. \tag{1}$$

Using the definitions of the functions k, h, and r, we can write

$$g(|(ab^n)^{h(n)} ab^{r(n)}|) = g(h(n)(n+1) + r(n) + 1) = g(k(n)) \leq \frac{k(n)}{2(n+1)} \tag{2}$$

and

$$k(n) \leq (h(n)+1)(n+1) \leq 2h(n)(n+1), \tag{3}$$

since $h(n) \geq 1$, for any $n \geq 1$. From (2) and (3) we get

$$g(|(ab^n)^{h(n)} ab^{r(n)}|) \leq h(n)$$

thus, using (1), we have

$$(ab^n)^{h(n)}ab^{r(n)} \not\leq_{f,g} (ab^m)^{h(m)}ab^{r(m)}.$$

Therefore, indeed L_g is an antichain of $\leq_{f,g}$. The proof is concluded. ■

We give below two examples of the construction in Theorem 3.1.

Example 3.2. Consider the function $g_1 : \mathbb{N} \longrightarrow \mathbb{N}$ given by

$$g_1(n) = \begin{cases} [\sqrt{n}] & \text{if } n \in \{4(k+1)^2 \mid k \geq 1\}, \\ n & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the integer part of x. Then $g_1 \in \mathcal{F}$ and it verifies the condition in Theorem 3.1 as

$$\overline{\lim}_{n\to\infty} \frac{n}{g_1(n)} = \infty,$$

a subsequence realizing this being $k_n = 4(n+1)^2, n \geq 1$. We find, as in the proof of Theorem 3.1, an infinite antichain of $\leq_{f,g_1}$. Compute first the three functions there:

$$\begin{aligned} k(n) &= \min\{k_m \mid k_m \geq 1 \text{ and } g_1(k_m) \leq \frac{k_m}{2(n+1)}\} = \\ &= \min\{4(m+1)^2 \mid 2(m+1) \leq \frac{4(m+1)^2}{2(n+1)}\} = 4(n+1)^2; \\ h(n) &= \max\{m \in \mathbb{N} \mid m(n+1)+1 \leq k(n)\} = \\ &= \max\{m \in \mathbb{N} \mid m(n+1)+1 \leq 4(n+1)^2\} = 4(n+1)-1; \\ r(n) &= 4(n+1)^2 - (4(n+1)-1)(n+1) - 1 = n. \end{aligned}$$

The obtained infinite antichain of $\leq_{f,g_1}$ is

$$L_{g_1} = \{(ab^n)^{4(n+1)-1}ab^n \mid n \geq 1\}.$$

Another example, where the antichin obtained is much more complicated, is given below.

Example 3.3. Take the function $g_2 : \mathbb{N} \longrightarrow \mathbb{N}$ given by

$$g_2(n) = \begin{cases} \left[\frac{n}{\log_2 \log_2 n}\right] & \text{if } n \in \{2^{2^k} \mid k \geq 1\}, \\ n & \text{otherwise.} \end{cases}$$

Then $g_2 \in \mathcal{F}$ and it verifies the condition on the superior limit in the theorem, namely

$$\overline{\lim}_{n\to\infty} \frac{n}{g_2(n)} = \infty.$$

A subsequence that realizes this limit is $k_n = 2^{2^n}, n \geq 1$. The three functions are:

$$\begin{aligned}
k(n) &= \min\{2^{2^m} \mid \left[\frac{2^{2^m}}{m}\right] \leq \frac{2^{2^m}}{2(n+1)}\} = 2^{2^{2(n+1)}}, \\
h(n) &= \max\{m \in \mathbb{N} \mid m(n+1)+1 \leq 2^{2^{2(n+1)}}\} = \left[\frac{2^{2^{2(n+2)}}-1}{n+1}\right], \\
r(n) &= 2^{2^{2(n+1)}} - \left[\frac{2^{2^{2(n+2)}}-1}{n+1}\right](n+1) - 1.
\end{aligned}$$

In this case, we obtain the following infinite antichain of $\leq_{f,g_2}$:

$$L_{g_2} = \{(ab^n)^{\left[\frac{2^{2^{2(n+2)}}-1}{n+1}\right]} ab^{2^{2^{2(n+1)}} - \left[\frac{2^{2^{2(n+2)}}-1}{n+1}\right](n+1)-1} \mid n \geq 1\}.$$

4 A generalization of Higman's theorem

We now show that also the converse of Theorem 3.1 holds true, thus obtaining a complete characterization of the relations in the family $(\leq_{f,g})_{g\in\mathcal{F}}$ having the finite basis property. This result generalizes Higman's theorem in its particular form shown in Theorem 4.1 below. (This form of Higman's theorem has been discovered independently by Haines, cf. [Ha]. In fact, the concept of finite basis property or, equivalently, well quasi order, has been frequently discovered; see, for instance, [ErRa], [Hi], [Kr1], [Ha]. A complete account in this matter is given by [Kr2].)

Theorem 4.1 (Higman [Hi]). *The subword partial order $\leq_s$ has the finite basis property.*

Our proof has a basic idea similar with the one for the subword relation $\leq_s$ by Conway [Co], cf. also [Lo]. (According to [Lo], the basic idea is due to Nash-Williams [NW].)

Theorem 4.2. *For any $g \in \mathcal{F}$ such that*

$$\overline{\lim}_{n\to\infty} \frac{n}{g(n)} < \infty,$$

the relation $\leq_{f,g}$ has the finite basis property.

Proof. Take $g \in \mathfrak{F}$ such that

$$\overline{\lim}_{n\to\infty} \frac{n}{g(n)} < \infty.$$

Then, there is a positive integer $c \geq 1$ such that

$$\frac{n}{g(n)} \leq c, \text{ for any } n \geq 1.$$

So $g(n) \geq \dfrac{n}{c}$. Consider the function $h : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$h(n) = \begin{cases} 0 & \text{if } n = 0, \\ \max\left(1, \left[\dfrac{n}{c}\right]\right) & \text{if } n \geq 1. \end{cases}$$

Obviously, $h \in \mathfrak{F}$ and, since

$$h(n) = \max\left(1, \left[\frac{n}{c}\right]\right) \leq \max\left(1, \frac{n}{c}\right) \leq g(n),$$

for any $n \geq 1$, it follows that if $\leq_{f,h}$ has the finite basis property, then so has $\leq_{f,g}$. Therefore, it is enough to prove that $\leq_{f,h}$ has the finite basis property. We argue by contradiction. Suppose that $\leq_{f,h}$ does not have the finite basis property and consider an infinite antichain of $\leq_{f,h}$, say

$$\{w_1, w_2, w_3, \ldots\},$$

where $w_i \in \Sigma^*$, for any $i \geq 1$. It follows that, in particular, we have

$$\text{if } i < j, \text{ then } w_i \not\leq_{f,h} w_j. \tag{4}$$

Consider now the sequence $w_1, w_2, w_3, \ldots$ built as follows: w_1 is a shortest word beginning a sequence satisfying (4), w_2 is a shortest word such that w_1, w_2 begin a sequence satisfying (4), and so on and so forth. Thus, there is no sequence $v_1, v_2, v_3, \ldots$ satisfying (4) such that, for some $i \geq 1$, $|v_i| < |w_i|$.

Now, as Σ is finite, there must be a word w such that $|w| = c$ and infinitely many words of the sequence $(w_i)_{i\geq 1}$ have w as a prefix. Suppose that all those beginning with w are $(w_{i_k})_{k\geq 1}$ and put $w_{i_k} = wu_k$, for any $k \geq 1$.

Consider the sequence

$$w_1, w_2, \ldots, w_{i_1-1}, u_1, u_2, \ldots. \tag{5}$$

We claim that this sequence satisfies (4). We notice first that $w_i \not\leq_{f,h} w_j$, for all $1 \leq i < j \leq i_1 - 1$, and $w_i \not\leq_{f,h} u_j$, for any $1 \leq i \leq i_1 - 1, j \geq 1$.

If, for some $p, q, p < q$ and $u_p \leq_{f,h} u_q$, then, by definition,

$$u_q = u_{q,0}u_{p,1}u_{q,1}u_{p,2}u_{q,2}\dots u_{p,h(|u_p|)}u_{q,h(|u_p|)},$$

where $u_{q,0}, u_{p,i}, u_{q,i} \in \Sigma^*, 1 \leq i \leq h(|u_p|)$, and

$$u_p = u_{p,1}u_{p,2}\dots u_{p,h(|u_p|)}.$$

But now, since

$$\begin{aligned} w_q &= wu_{q,0}u_{p,1}u_{q,1}u_{p,2}u_{q,2}\dots u_{p,h(|u_p|)}u_{q,h(|u_p|)}, \\ w_p &= wu_{p,1}u_{p,2}\dots u_{p,h(|u_p|)}, \end{aligned}$$

and

$$h(|w_p|) = h(|u_p| + c) = \left\lceil \frac{|u_p| + c}{c} \right\rceil = \left\lceil \frac{|u_p|}{c} \right\rceil + 1 = h(|u_p|) + 1,$$

we obtain that $w_p \leq_{f,h} w_q$, contradicting the fact that $(w_i)_{i\geq 1}$ satisfies (4). Therefore, the sequence (5) satisfies the condition (4).

Since $|u_1| < |w_{i_1}|$, this contradicts our assumption on $(w_i)_{i\geq 1}$ as being the "earliest" sequence satisfying (4). Consequently, $\leq_{f,h}$ has the finite basis property and, as noticed above, so has $\leq_{f,g}$. The proof is concluded. ■

Remark. The subword relation $\leq_s$ is obtained for $g = \mathbf{1}_{\mathbb{N}}$ (see above). Since

$$\lim_{n\to\infty} \frac{n}{\mathbf{1}_{\mathbb{N}}(n)} = 1 < \infty,$$

Higman's theorem is obtained as a consequence of our Theorem 4.2 for $g \equiv \mathbf{1}_{\mathbb{N}}$.

From Theorems 3.1 and 4.2, we obtain a complete characterization of those relations in the family $(\leq_{f,g})_{g\in\mathcal{F}}$ which have the finite basis property.

Theorem 4.3. *For any $g \in \mathcal{F}$, the relation $\leq_{f,g}$ has the finite basis property if and only if*

$$\overline{\lim}_{n\to\infty} \frac{n}{g(n)} < \infty.$$

5 Language-theoretic gaps for antichains

In this section, we consider infinite antichains for some of the relations mentioned in the previous sections and investigate their regularity and context-freeness.

The first result concerns the relations which are restrictions of $\leq_f$.

Theorem 5.1. *There exist infinite regular antichains for any relation $\leq$ which is a restriction of the factor relation $\leq_f$.*

Proof. It is enough to prove the statement for $\leq_f$ since any antichain of $\leq_f$ is an antichain of $\leq$ too. We can take then

$$L = \{ab^n a \mid n \geq 0\}.$$

Indeed, L is an antichain of $\leq_f$ and it is an infinite regular language. ∎

A natural question arises: are there relations which do not have the finite basis property but for which there is no infinite regular antichain ? The answer is positive, as proved in the next result. Moreover, the result holds also in the context-free case, as we show in a moment.

Theorem 5.2. *The relation $\leq_{f,2}$ has no infinite regular antichain. There are infinite context-free antichains for $\leq_{f,2}$.*

Proof. For the second part of the statement, it is enough to consider the language

$$L = \{ab^n ab^n a \mid n \geq 1\}.$$

Obviuosly, L is an antichain of $\leq_{f,2}$ and it is an infinite context-free (even linear) language.

For the first part, we argue by contradiction. Suppose that there is an infinite regular antichain R of $\leq_{f,2}$. By the pumping lemma for regular languages, there is a constant $p \geq 1$ such that any word $w \in R$ with $|w| \geq p$ can be written as $w = uvx$, with $v \neq \lambda$ and $uv^i x \in R$, for any $i \geq 0$. Consider such a $w \in R$ (there is such a w because R is infinite) and also its decomposition as above. We have $uvx, uvvx \in R$ and $uvx \leq_{f,2} uvvx$. Since $uvx \neq uvvx$, a contradiction is obtained. The theorem is proved. ∎

Theorem 5.3. *The relation $\leq_{f,3}$ has no infinite context-free antichains. There are infinite antichains for $\leq_{f,2}$ which are intersections of two context-free languages.*

Proof. The second part is proved by the infinite language

$$L = \{ab^n ab^n ab^n a \mid n \geq 0\}$$

which is an antichain of $\leq_{f,3}$ and is the intersection of the context-free (linear) languages $\{ab^n ab^n ab^m a \mid n, m \geq 0\}$ and $\{ab^m ab^n ab^n a \mid n, m \geq 0\}$.

The proof of the first assertion is similar to the one in the previous theorem. Arguing by contradiction, we suppose that P is an infinite context-free language which is an antichain of $\leq_{f,3}$. By the pumping lemma for context-free languages, there is a constant $n \geq 1$ such that, for any $z \in P$ with $|z| \geq n$, z can be decomposed as $z = uvwxy$, where $vx \neq \lambda$ and $uv^iwx^iy \in P$, for any $i \geq 0$. Take such a z and its decomposition as above. It follows that $uvwxy$ and uv^2wx^2y are both in P and different. Since $uvwxy \leq_{f,3} uv^2wx^2y$, a contradiction is obtained and the result is proved. ∎

Problem. What about the same problem in general for infinite antichains of those relations in the family $(\leq_{f,g})_{g\in\mathcal{F}}$ which no not have the finite basis property?

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