On a Geometric Problem of Zig-Zags

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Abstract

Using a topological argument, we give a simpler proof for the geometrical argument in the original proof of Isbell's Zig-Zag Theorem and an algorithm which finds effectively the required "zig-zag" path. Both the proof and the algorithm remain valid for a more general form of the problem.

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1 Partitioned Rectangles

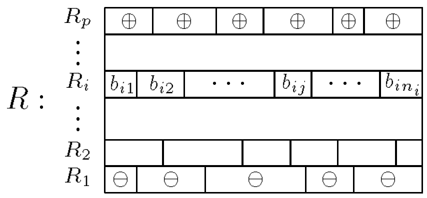
The proof of the well known Isbell's Zig-Zag Theorem in semigroup theory, proved by J. R. Isbell in [Is] (1966) uses the existence of a certain "zig-zag" simple path between the opposite sides of a partitioned rectangle. For the Zig-Zag Theorem we refer to [Ho] (Theorem 8.3.5) where also references are given for more algebraic proofs of this result.

Informally speaking, a rectangle partitioned in horizontally aligned rectangle boxes is given, each box being marked by a plus or by a minus such that all top boxes are marked by pluses and all bottom ones by minuses. The problem is the existence of a simple path connecting the lateral frontiers of the given rectangle such that the path goes all the time between differently marked boxes.

In [Is], Isbell only outlines the proof of the existence of the respective simple path. A formal proof was given by J. M. Philip in [Ph].

Using a topological argument, we give here a simpler proof for the existence of the respective simple path and we provide an algorithm which finds effectively such a path. Our argument allow us to handle this problem in a more general form (which generalizes also the generalization given by [Ph]).

To be more formal, the geometrical situation can be described as follows: a rectangle R having left and right vertical frontiers and top and bottom horizontal frontiers is given. R is partitioned firstly by horizontal lines ending on the two vertical frontiers of R into $p \geq 2$ rectangles $R_1, R_2, \ldots, R_p$ and, secondly, each $R_i, 1 \leq i \leq p$, is divided by vertical lines ending on its horizontal frontiers into $n_i \geq 1$ rectangular boxes $b_{ij}, 1 \leq j \leq n_i$. We have also a function $\sigma : \{b_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq n_i\} \longrightarrow \{\oplus, \ominus\}$ such that $\sigma(b_{1j}) = \ominus$, for any $1 \leq j \leq n_1$, and $\sigma(b_{pj}) = \oplus$, for any $1 \leq j \leq n_p$. For more clarity, see the picture below.



We introduce now several formal notions useful later on. First of all, throughout of this paper a *segment* will be a continuous nonempty line, that is, a (continuous) line which is not reduced to one point. (This line will be straight in section 2 and of any form in the generalization given in section 3.)

Any box b_{ij} has four edges and four vertices at its corners. If two boxes b_{ij} and b_{rs} are adjacent, that is, there is a segment which is common to an edge of b_{ij} and to one of b_{rs}, and $\sigma(b_{ij}) \neq \sigma(b_{rs})$, then we call any such common segment a *valid segment*. A *path* in R is a (continuous) line consisting of edges or parts of edges of boxes b_{ij}. A path is called

– *valid* if it consists only of valid segments,

– *simple* if it does not autointersect, and

– *LR-path* if it begins at the left frontier of R and ends at the right frontier of R.

Here is an example where all valid segments are emphasized.

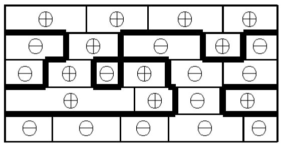


Figure 1

2 How to Find a Path

We can state now the theorem (which is already in this form more general than the one given by Isbell – Theorem 2.2 in [Is] – since we impose no condition on the number or on the vertical alignement of the boxes b_{ij} or other conditions on the mapping σ excepting the ones for the two extremal rows of R).

Theorem 1 *If R is partitioned as above, then there exists a simple valid LR-path in R.*

Proof. First, we notice that it is enough to prove that there exists a valid LR-path in R because by eliminating all cycles (if any) of such a path, a simple valid LR-path is obtained.

We argue by contradiction. Suppose that there is no valid path as required and denote by G the graph consisting of all valid segments in R. It follows that we can draw inside R a (continuous) line, say L, beginning at the bottom side of R and ending at the top side of R such that L does not intersect G. We may suppose also that L does not intersect any vertex in R since if L intersects one, say v, then v is not a vertex in G and so no edge incident with v belongs to G. We can now modify L not to intersect v. Let us denote the sequence of boxes intersected (in order) by L by $b_{r_1s_1}, b_{r_2s_2}, \ldots, b_{r_ks_k}$, where

$k \geq 2, r_1 = 1, r_k = p$, $1 \leq s_l \leq n_{r_l}$ for any $1 \leq l \leq k$, and $|r_m - r_{m+1}| \leq 1$ for any $1 \leq m \leq k-1$. Because for any $1 \leq l \leq k-1$, L crosses the border between $b_{r_l s_l}$ and $b_{r_{l+1} s_{l+1}}$, it follows that there are two edges, one belonging to $b_{r_l s_l}$ and the other belonging to $b_{r_{l+1} s_{l+1}}$, which have in common a segment which is not valid. Thus $\sigma(b_{r_l s_l}) = \sigma(b_{r_{l+1} s_{l+1}})$ and, since $\ominus = \sigma(b_{1 s_1}) \neq \sigma(b_{p s_k}) = \oplus$, a contradiction is obtained.

□

In what follows, we present an algorithm for finding a simple valid LR-path in a rectangle R partitioned as above. Before this, we need one more notion. For a simple LR-path P, we say that a box b_{ij} is *at the left (right) of* P if b_{ij} is adjacent to P (that is, there is a segment which is common to an edge of b_{ij} and to P) and when walking on P starting from the left frontier of R, b_{ij} is on the left (right, resp.) frontier of this line viewed from the direction of the line.

Algorithm

Input: A rectangle R partitioned as above.

Output: A simple valid LR-path in R.

(1) Start with the simple LR-path L consisting of all bottom edges of all top boxes $b_{pi}, 1 \leq i \leq p$, (that is, the bottom edge of R_p) as the current path. Now, all boxes at the left of L are marked by $\oplus$. We will modify this path in a certain manner until a simple valid LR-path will be found.

(2) Check whether or not the current path L is valid. If it is, then **output** L, else go to step (3).

(3) At this point, the current path L is a simple LR-path in R which is not valid. Since, as it will be seen, throughout our algorithm all boxes at the left of the current path L are marked by $\oplus$, it follows that there is a box at the right of L which is marked by $\oplus$ (because L is not valid). Consider the first such box b_{ij} when the order is from left to right w.r.t. L. We modify now L such that it remains an LR-path and b_{ij} is at the left of L while all the other boxes remain at the same side as before. Because there is exactly one possibility to do this modification, the new L is well defined.

(4) After performing the modification in step (3), two situations can appear: (i) L intersects the left/right frontier of R more than once, (ii) L contains a cycle. For instance, (i) appears in the algorithm for the rectangle in Fig. 1 after three modifications at step (3); see Fig. 2 below (L is emphasized):

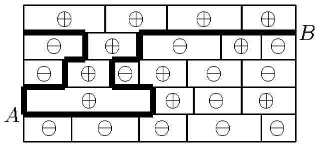


Figure 2

In case (i), we modify L to be the LR-path starting at the last point common to the left frontier of R and ending at the first point common to the right frontier of R (considering again the order from left to right w.r.t. L). In our example in Fig. 2, this means that the new L will be the part between the vertices A and B.

The situation (ii) is obtained for instance after modifying three times (one time by step (4) and two times by step (3)) the path in Fig. 2. We have then

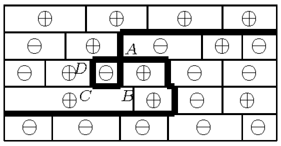


Figure 3

In this case, we just eliminate the cycle appeared in L. In our example in Fig. 3, the cycle $ABCDA$ will be eliminated.

Let us remark that, at this point of the algorithm, L is a simple LR-path such that all boxes at the left of L are marked by $\oplus$.

(5) Go to step (2) to check whether or not the current path L is valid.

The solution obtained by the algorithm for our example is the emphasized simple valid LR-path in Fig. 4.

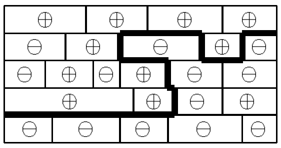


Figure 4

Correctness. The algorithm will stop when all boxes at the right of L are marked by $\ominus$ (and this will certainly happen since all boxes at the bottom of R are marked so). As L is a simple LR-path and all boxes at the left of L are marked by $\oplus$, it follows that L is a simple valid LR-path, as required.

Complexity. Suppose that the total number of boxes is $n \geq 2$. Because any modification in steps (3) or (4) implies an irreversible elimination of at least one box and because the validity checking at step (2) is linear, it follows that our algorithm works in time $\mathcal{O}(n^2)$.

3 Arbitrary Figures

We can settle the result in Theorem 1 in a more general form. Consider a convex figure R such that:

(i) R is partitioned into a finite number of smaller figures $f_i, 1 \leq i \leq n$, each of which being closed, that is, it contains its boundaries.

(ii) The frontier of R is partitioned by four distinct vertices A, B, C, D (that is, each of A, B, C, D belongs to the frontier of R and to the frontiers of at least two f_i's) into four disjoint (excepting the ends) segments AB, BC, CD, and DA.

(iii) A function $\sigma : \{f_i \mid 1 \leq i \leq n\} \longrightarrow \{\oplus, \ominus\}$ is given such that for any f_i whose frontier has a common segment with AB, $\sigma(f_i) = \oplus$, and for any f_j whose frontier has a common segment with CD, $\sigma(f_j) = \ominus$.

As above, a valid segment is a frontier segment which belongs to the frontiers of two adjacent figures which have different images by σ and a valid path is a path consisting only of valid segments.

The following generalization can be proved just as in Theorem 1.

Theorem 2 (Generalization) *If R is a figure partitioned as above, then there exists a simple valid path beginning at the segment DA of the frontier of R and ending at the segment BC of the frontier of R. [The algorithm above, adapted to this case, computes, for any such figure R, a simple valid path as required.]*

Notice that the result in Theorem 2 is more general than the generalization given by [Ph] in the sense that the rectangular form of R and $f_i, 1 \leq i \leq n$, and the parallelism between the frontiers of each f_i and the frontiers of R are not imposed here.

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