Sorting permutations on two parallel stacks

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Abstract

We give a new combinatorial proof, based on stack graphs, for the characterization by forbidden subsequences of the permutations which are sortable on two parallel stacks. Some remarks about the permutations sortable on parallel queues are also made.

1. Introduction

Permutations that are sortable on two parallel stacks have been characterized by Even and Itai [5] using union graphs and circle graphs, see also Golumbic [7], and Rosenstiehl and Tarjan [11]. By [5], the circle graphs are very close to the permutations sortable on stacks. In Bouchet [3] the circle graphs are characterized by forbidden subgraphs with respect to local equivalence. For a characterization using intersection graphs, see Fraysseix [6]. For permutations avoiding certain short sequences, we refer to Bóna [2] and Stankova [10].

Another characterization for these permutations, by forbidden subsequences, has been started by Tarjan [12], who gave some partial results on this topic, and completed by Pratt [9].

In this article we give a new proof for characterizing these permutations by forbidden subsequences. Our proof is totally different from the one by Pratt. It is based on a close investigation of the stack graphs and we believe that it gives insight to the problem from a different point of view.

We show that a permutation π is sortable on two parallel stacks if and only if π avoids (that is, does not contain a subsequence order isomorphic to) the permutations

$$\pi_k = 2(2k+1)41\ldots(2i)(2i-3)\ldots(2k+2)(2k-1)$$

and

$$\sigma_k = (2k+3)2(2k+5)41\ldots(2i)(2i-3)\ldots(2k+4)(2k+1)$$

for all odd integers $k \geq 1$. The permutation π_k is formed by shuffling the sequence $2, 4, \ldots, (2k+2)$ of even integers with the sequence $(2k+1), 1, 3, \ldots, (2k-1)$ of odd integers. The permutation σ_k is obtained similarly by shuffling the sequences $2, 4, \ldots, (2k+4)$ and $(2k+3), (2k+5), 1, 3, \ldots, (2k+1)$. For instance, $\pi_1 = 2341$, $\pi_3 = 27416385$, and $\sigma_1 = 5274163$, $\sigma_3 = 92(11)416385(10)7$.

2. Definitions

Let $\pi = (i_1, i_2, \ldots, i_n)$ be a permutation on $[1, n] = \{1, 2, \ldots, n\}$. It will be written more conveniently as $\pi = i_1 i_2 \ldots i_n$. For a $k \geq 1$, we say that π is *sortable on k parallel stacks*[1] if there is a way to obtain from π the identity

[1]This should not be confused with the two-stack sortable permutation of West [13] and Zeilberger [14]; the main difference is that here the stacks are used in parallel whereas in [13, 14] they are used sequentially.

$12\ldots n$ by the following procedure (see also Figure 1). We consider one by one the elements of the input π from left to right and two kinds of moves are allowed: (i) from the left of the input to the top of a stack and (ii) from the top of a stack to the right of the output.

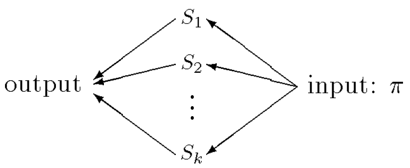

Figure 1: sorting on k parallel stacks

As an example, we sort the permutation $\pi = 2431$ on two parallel stacks. We describe the sorting procedure by quadruples (output, $-$stack$_1$, $-$stack$_2$, input), where "$-$" marks the top of the respective stack. We have $(, -, -, 2431) \to (, -2, -, 431) \to (, -2, -4, 31) \to (, -2, -34, 1) \to (, -12, -34,) \to (1, -2, -34,) \to (12, -, -34,) \to (123, -, -4,) \to (1234, -, -,)$. Thus, π can be sorted on two parallel stacks.

Let π be a permutation on $[1, n]$. We shall write $i \prec j$, if $\pi^{-1}(i) < \pi^{-1}(j)$, that is, if the element i comes before j in π. A sequence $j_1 j_2 \ldots j_k$ is a *subsequence* of π if $j_1 \prec j_2 \prec \ldots \prec j_k$ in π. A *factor* of π is a subsequence of consecutive elements, that is, a subsequence $i_k i_{k+1} \ldots i_{k+m}$ for some k and m. We write $i \in \sigma$, if the element i is in the subsequence σ.

A permutation σ on $[1, m]$ is *contained in* π, if π has a subsequence that is order isomorphic to σ. Also, π *avoids* a set of permutations Σ, if π does not contain any element of Σ. For instance, the permutation $\pi = 524163$ contains $\sigma = 3142$ in three ways; any of the subsequences 5263, 5163, and 4163 of π is order isomorphic to σ. It is easy to see that π avoids 2341.

It is known (see Knuth [8]) that a permutation π on $[1, n]$ avoids $\sigma = 231$ if and only if π is sortable on one stack.

For a permutation π on $[1, n]$, let $S[\pi]$ be the undirected graph on $[1, n]$ such that ij is an edge for $i < j$ if and only if there exists an $m < i$ with $i \prec j \prec m$. We call $S[\pi]$ the *stack graph* of π.

Let $\chi(G)$ be the chromatic number of an undirected graph G, that is, the smallest number of colours for the vertices of G such that no adjacent vertices get the same colour.

Denote by C_k a cycle of k vertices and edges without chords, that is, $C_k = (x_1, x_2, \ldots, x_k)$, where the edges are the pairs $x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1$. Now $\chi(C_k) = 2$, if k is even, and $\chi(C_k) = 3$, if $k \geq 3$ is odd. A cycle C_3 is called a *triangle*. A graph G is *triangle free*, if it has no triangles as subgraphs.

The following theorem is a "straightforward exercise" by Golumbic [7, p.236]; we agree with him.

Theorem 1 *The chromatic number $\chi(S[\pi])$ is the smallest nonnegative number k such that π can be sorted on k parallel stacks.*

3. Permutations sortable on two parallel stacks

As a corollary to Theorem 1, we have

Theorem 2 *A permutation π is sortable on two parallel stacks if and only if its stack graph $S[\pi]$ is bipartite.*

In particular, for the permutation $\pi_1 = 2341$, the graph $S[\pi_1]$ consists of an isolated vertex 1 and of a triangle on the three vertices $2, 3$, and 4. It is also clear that if a stack graph $S[\pi]$ contains a triangle, then π contains 2341, and therefore

Lemma 1 *A permutation π avoids* 2341 *if and only if $S[\pi]$ is triangle free. In particular, if π is sortable on two parallel stacks, then it avoids* 2341*.*

We notice that this lemma generalizes to: the maximal clique of $S[\pi]$ is of order at most k if and only if π avoids $23\ldots(k+2)1$.

We remark at this point that the latter claim in Lemma 1 is not true in converse: the permutation 5274163 is the shortest example that avoids 2341 but is not sortable on two parallel stacks. Indeed, it can be shown that every permutation π that avoids 2341 is sortable on five parallel stacks, but there exists such a π, which is not sortable on four parallel stacks. These assertions follow from results on circle graphs, see Ageev [1]. A graph G is a *circle graph* if it is (isomorphic to) the intersection graph of a set of chords of a (geometric) circle. By a theorem due to Even and Itai [5], a graph G is a circle graph if and only if it is isomorphic to a stack graph $S[\pi]$, when, in both, the isolated vertices are removed. It is worthwhile to mention that the circle graphs, and thus the stack graphs, satisfy the perfect graph conjecture, see Buckingham and Golumbic [4]. According to this result, the chromatic number $\chi(G)$ equals $\omega(G)$, the maximum size of a clique in G, for all induced subgraphs G of $S[\pi]$ unless $S[\pi]$ contains a cycle C_k or its complement $\overline{C}_k$ for an odd $k \geq 5$. The (sub)graph $\overline{C}_k$ contains a triangle when $k \geq 7$, and $\overline{C}_5$ is isomorphic to C_5, and therefore much of our considerations reduce to the exixtence of odd cycles C_k in the stack graphs.

Let $\pi = i_1 i_2 \ldots i_n$ be a permutation on $[1,n]$. A permutation $\pi - i$ is obtained from π by *removing* an element $i \in [1,n]$, if i is removed, and each remaining element $j > i$ is replaced by $j-1$. An element $i \in [1,n]$ is *removable*, if $\pi - i$ is not sortable on two parallel stacks. The permutation π is said to be *critical*, if it is not sortable on two parallel stacks, and it contains no removable elements.

Lemma 2 *For any critical permutation π, the stack graph $S[\pi]$ consists of a unique (induced) cycle $C = C_k$ for an odd $k \geq 3$, and of a set I of isolated vertices. Moreover, the isolated elements form an increasing subsequence $x_1 < x_2 < \ldots < x_t$ of π such that $x_{i+1} \geq x_i + 2$.*

Proof. By Theorem 2, $S[\pi]$ is not bipartite, and, as it is well known, this is equivalent to saying that S contains a cycle C_k for some odd $k \geq 3$. Let $C = C_k$ be such a cycle of $S[\pi]$ of the smallest length k. It is clear that C is an induced subgraph. Let E be the set of those elements m of $[1, n]$ such that there exist $i, j \in C$ with $i \prec j \prec m$ and $m < i < j$ (possibly $C \cap E \neq \emptyset$). Now $C \cup E = [1, n]$, since the subgraph induced by $C \cup E$ contains the odd cycle C as an induced subgraph, and π is critical by assumption (and elements not in $C \cup E$ are removable).

Let $i \notin C$, and assume on the contrary that there exists an edge ij in $S[\pi]$. Then necessarily there exists an $m < i$ such that either $i \prec j \prec m$ or $j \prec i \prec m$. Now, however, for any $x \prec y \prec i$ with $i < x < y$, also $x \prec y \prec m$ with $m < x < y$, and therefore the element i is removable, which contradicts the fact that π is critical. This shows that $[1, n] \setminus C$ consists of the isolated vertices in $S[\pi]$.

For two isolated elements i and j with $i \prec j$, if $i > j$, then, as in the above, i is removable; contradiction. Finally, if both i and $i+1$ were isolated for some i, then, by the above, $i \prec i+1$. Clearly, i would be then removable. □

Lemma 3 *If π is critical, then the isolated elements of π are exactly the elements i that satisfy the following* minimum condition*: for all j with $i \prec j$, $i < j$. Moreover, if $i \in I$, then $(i+1) \prec i$.*

Proof. This first claim is clear from Lemma 2, since no element from C can satisfy the minimum condition. The second claim, $(i+1) \prec i$, follows from this and Lemma 2. □

Lemma 4 *If a critical π avoids 2341, then it contains a factor $2a41$ for some $a \geq 5$.*

Proof. The element 1 is necessarily isolated, and therefore Lemma 3 implies that $2 \prec 1$ and $2 \in C$. Since the degree of 2 in $S[\pi]$ is two, there are exactly two elements a and b such that $2 \prec a \prec b \prec 1$ in π, and these are adjacent with 2, that is, $2a$ and $2b$ are edges of $S[\pi]$. By the assumption, $S[\pi]$ is triangle free, and hence ab is not an edge. Therefore $a > b$, and π has a factor $2ab1$.

We prove next that $4 \prec 1$. Assume the converse. By Lemma 2, either 3 or 4 is in C and we claim that 4 is in C whereas 3 is isolated. Indeed, if 3 is in C, then, necessarily, $3 \prec 1$. If $3 \prec 2$, then a cycle $(3, a, 2, b)$ of length four is created in $S[\pi]$, contradicting Lemma 2, and if $b = 3$, then 3 has degree one, which is a contradiction.

Therefore, we must have, using again Lemma 2, $\pi = u2ab1v4cd3w$, where $d < c < a$ and $d < b$. Now, there must be a chain in $S[\pi]$ from d to a or b, say $(d, d_1, d_2, \ldots, d_r), r \geq 2$ such that $d_i \notin \{4, c\}$, for any $1 \leq i \leq r$ and $d_r \in \{a, b\}$. If $d_1 \prec d$, then $d_1 < d < c$ implies that $d_1 c$ is an edge and a cycle of length four, $(4, c, d_1, d)$, is obtained, contradicting Lemma 2. Thus necessarily $d_1 \in w$ and if $s \geq 2$ is minimal such that $d_s \notin w$, then $d_i < c$, for any $1 \leq i \leq s - 1$, otherwise $d_i c$ is an edge. Therefore, since $d_s \neq c$, $d_s \in u2ab1v$ with $d_s < d_{s-1} < c$. A chain from d to c is obtained, thus contradicting Lemma 2. Consequently, $4 \prec 1$.

On the other hand, it is not the case that $4 \prec 2$, since then $4 < b < a$ ($b \neq 3$ as 3 cannot have degree one) and both $4a$ and $4b$ would be edges in $S[\pi]$, thus creating a cycle of length four in $S[\pi]$. Consequently, $4 \in \{a, b\}$ and, since $a > b$ and $b \neq 3$, we get $b = 4$. $\square$

Theorem 3 *A permutation π on $[1, n]$ is sortable on two parallel stacks if and only if it avoids the permutations*

$$\pi_k = 2(2k+1)41\ldots(2i)(2i-3)\ldots(2k+2)(2k-1) \quad (1)$$

$$\sigma_k = (2k+3)2(2k+5)41\ldots(2i)(2i-3)\ldots(2k+4)(2k+1) \quad (2)$$

for all odd integers $k \geq 1$.

Proof. It is straightforward to verify that the stack graph $S[\pi_k]$ contains the cycle $C_{k+2} = (2, 4, \ldots, 2(k+1), 2k+1)$ and $S[\sigma_k]$ contains the cycle $C_{k+4} = (2, 4, \ldots, 2(k+2), 2k+3, 2k+5)$. Hence, if k is odd, then these cycles are of odd length, and accordingly $S[\pi_k]$ and $S[\sigma_k]$ are not bipartite. Therefore π_k and σ_k are not sortable on two stacks.

Suppose then that π is a critical permutation that avoids $\pi_1 = 2341$. By Lemma 4, π has a factor $2a41$, where $a \geq 5$, and thus 24 and $2a$ are edges of $S[\pi]$. Let the unique cycle C of $S[\pi]$ from Lemma 2 be of length k:

$$(b_1, b_2, \ldots, b_{k-1}, a),$$

where $b_1 = 2$, $b_2 = 4$, and the last vertex a has an edge to b_1.

Let $b_1, b_2, \ldots, b_m$ be the longest initial part such that $b_1 < b_2 < \ldots < b_m$. Here $m \geq 2$. We have then

$$\pi = ub_1ab_2w_2b_3\ldots b_iw_ib_{i+1}\ldots b_{m-1}w_{m-1}b_mw_m$$

for some factors u and w_i.

Let x_i be the minimum element in π after $b_i, 1 \le i \le m$. Therefore $x_1 = x_2 = 1$, and $x_i < b_{i-1}$ for all $i \ge 2$ by the definition of stack graphs. Now for each $2 \le i \le m$,

$$x_i \in w_i. \tag{3}$$

Indeed, it is clear that $x_m \in w_m$. For $i < m$, if $b_{i+1} \prec x_i$, then $x_i < b_{i-1} < b_{i+1}$ would imply that $b_{i-1}b_{i+1}$ is an edge of $S[\pi]$, and thus there would be a triangle in $S[\pi]$ consisting of b_{i-1}, b_i and b_{i+1}.

Similarly, for $2 \le i \le m-1$,

$$y \in w_i \implies y < b_i, \tag{4}$$

since, otherwise, $b_i y$ is an edge of $S[\pi]$, and the degree of b_i would be at least three.

Claim. For any $2 \le i \le m-1$, the factor w_i contains no elements from C and, therefore, $w_i = x_i$.

Proof of Claim. Consider an $i, 2 \le i \le m-1$, and assume that there exists an element of C in w_i. If b_{k-1} is in w_i, then $a < b_{k-1}$ and there exists a $z < a$ such that $b_{k-1} \prec z$. But then $z < a < b_{k-1} < b_i$ and $b_i \prec z$, which implies that ab_i is an edge, a contradiction since $i \le k-2$. Therefore, there exists a $b_t \in w_i$ such that $b_{t+1} \notin w_i$. If $b_{t+1} \prec b_t$, then, $b_{t+1} < b_t$ and there exists an element $z < b_{t+1}$ such that $b_t \prec z$. By (4), $b_{t+1} < b_t < b_i$, and so the pair $b_{t+1}b_i$ is an edge of $S[\pi]$; a contradiction. Therefore $b_t \prec b_{i+1} \prec b_{t+1}$. Take an element z such that $b_{t+1} \prec z$ and $z < b_t$. Using (4), we obtain $b_t < b_i < b_{i+1}$, which implies that $b_t b_{i+1}$ is an edge, and thus $i = m-1$, $t = m+1$. Moreover, we have deduced that $b_{m+1} \in w_{m-1}$ and $b_{m+2} \in w_m$. Now $b_{m+2} < b_m$, since $b_m b_{m+2}$ is not an edge of $S[\pi]$ and $b_{m+1}b_{m+2}$ is. (In fact, we have even that $b_{m+2} < b_{m-1}$.)

Now, in order to complete the cycle C from b_{m+2} towards a, there necessarily exists an edge $b_s b_{s+1}$ connecting an element $b_s \in w_m$ with an element in the initial part $b_{s+1} \in u2a$. Indeed, if $b_s \in w_m$ and $b_{s+1} \in w_i$, for some $2 \le i \le m-1$, then $b_{s+1}b_m$ is an edge, a contradiction since $s+1 \ge m+2$. Therefore there is an x such that $b_s \prec x$ and $x < b_{s+1} < b_s$. Here $b_m < b_s$; for, otherwise, $x < b_{s+1} < b_m$ and thus $b_{s+1}b_m$ would be an edge but this is not possible, since $s+1 > m+1$.

Consequently, there are elements in the sequence $b_{m+3}b_{m+4}\ldots b_{k-1}$ which are larger than b_m. Also, the first such element, say b_r, belongs to w_m and so do all the elements preceding it in the sequence, that is, $b_{m+3}, b_{m+4}, \ldots, b_{r-1}$. Since $b_{r-1}b_r$ is an edge of $S[\pi]$, $z < b_{r-1} < b_m < b_r$ for some z with $b_r \prec z$. But this means that $b_m b_r$ is an edge of $S[\pi]$ which is a contradiction. Therefore $w_i = x_i$, for any $2 \le i \le m-1$. The Claim is proved.

We conclude from the Claim, that

$$\pi = u2a4x_2b_3x_3b_4x_4\dots b_{m-1}x_{m-1}b_mw_m.$$

In particular, $b_{m+1} \in u2a$.

If $b_{m+1} = a$, then we have completed the cycle C, and, in this case, $m = k-1$, u is empty, $w_m = x_m$, and $b_{k-2} < a < b_{k-1}$. Accordingly, $\pi = \pi_{k-2}$, since both sequences x_i and b_i are increasing.

Suppose then that $b_{m+1} \neq a$. In this case, $b_{m+1} \in u$ and $b_{m+1} < b_m$. If $a < b_m$, then, as ab_m is not an edge, we have also $a < x_m$ and there is $i, 1 \leq i \leq m-1$, such that $x_i < a < x_{i+1}$. As ab_i is not an edge, it follows that $a > b_i$ and from $b_i > x_{i+1}$ we get $a > x_{i+1}$, a contradiction. Therefore, $a > b_m$ and hence $b_{m+1}a$ is an egde and the cycle is thus completed: $m = k-2, u = b_{m+1}$, and again $w_m = x_m$. In this case, using again Lemma 3, $\pi = \sigma_{k-4}$. $\square$

4. Queues and stacks

We consider in this section permutations sortable on parallel queues. Their definition is similar with the one for parallel stacks, just that the stacks are replaced by queues and it is also allowed to send elements directly to the output (this is implicitely possible in the case of stacks), see also Figure 2.

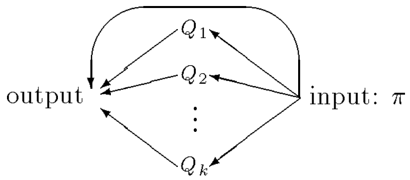

Figure 2: sorting on k parallel queues

A permutation π is sortable on k parallel queues if and only if the longest decreasing subsequence in π has the length $k+2$, see Golumbic [7, p. 166][2]. Therefore π is sortable on k parallel queues if and only if it avoids the permutation

$$\tau_k = (k+2)(k+1)k\dots 1.$$

We note that in the permutations π_{i+2} and σ_i, for $i \geq 1$, the length of the longest decreasing subsequence is three (they do contain $\tau_1 = 321$), and therefore these permutations are sortable by two parallel queues, but not on a single queue. The remaining permutation $\pi_1 = 2341$ is sortable on one queue. We have thus

[2]Notice that in [7] direct output of elements from the input is not allowed in which case the characterization holds with the length of the longest decreasing subsequence $k+1$.

Theorem 4 *A permutation π is sortable on one queue and on two parallel stacks if and only if π avoids $\tau_1 = 321$ and $\pi_1 = 2341$.*

Also, it was shown by Stankova [10], that there are equally many permutations of any given length n that avoid 1234 and 4123, from which it follows that there are equally many permutations on $[1, n]$ that avoid 4321 and 2341. We therefore get

Theorem 5 *For any $n \geq 7$, the number of permutations of length n which are sortable on two parallel queues is strictly larger than the number of permutations of the same length which are sortable on two parallel stacks; that is, two queues are more powerful than two stacks.*

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