Subwords and power free words are not expressible by word equations

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Abstract

We consider several open problems of Karhumäki, Mignosi, and Plandowski, cf. [KMP], concerning the expressibility of languages and relations as solutions of word equations. We show first that the (scattered) subword relation is not expressible. Then, we consider the set of k-power free finite words and solve it negativelly for all nontrivial integer values of k. Finally, we consider the Fibonacci finite words. We do not solve the problem of the expressibility of the set of these words but prove that it cannot be given a negative answer (as believed) using the tools in [KMP].

1 Introduction

Word equations constitute one of the basic topics in combinatorics on words. They can be used to express some properties or relations of words (see [Lo]) and it turned out to be a difficult problem to find examples of languages or relations on words which are not expressible by word equations, that is, which cannot be written as (components of) solutions of word equations. Some tools for solving some problems of this type have been introduced only very recently by Karhumäki, Mignosi, and Plandowski, cf. [KMP]. They have proved some general results in this direction and, as applications, they found a series of languages and relations on words which are not expressible as solutions of word equations, e.g., the primitiveness property, the equal length property, the language of all words over a proper subalphabet.

However, there remained a lot of research to be done in this area. In this paper we are concerned with some problems left open in [KMP]. We show first that the (scattered) subword relation is not expressible as soon as it is not trivial, that is, over at least two-letter alphabets. A second problem we consider concerns the set of k-power free finite words. We give again a negative answer, for all nontrivial integer values of k. Both problems are solved using the tools of [KMP] suitably applied.

The third open problem of [KMP] we consider asks whether or not the set of finite Fibonacci words is expressible. Using the result of [IlPl] proving the expressibility of the set of finite Sturmian words, we prove that it cannot be given a negative answer (as believed) using the tools developed by [KMP].

2 Word equations

We give in this section the basic definition we need, as well as some previous results we shall use. Let Σ be a finite alphabet. We denote by $\mathrm{card}(\Sigma)$ the number of elements of Σ, by Σ^* the set of finite words over Σ, and by 1 the empty word. For $w \in \Sigma^*$, $|w|$ denotes the length of w. For any notions and results of combinatorics on words we refer to [Lo] and [ChKa].

Let Σ and Ξ be two disjoint alphabets, of *constants* and *variables*, respectively. By convention, lower case letters denote constants whereas capitals denote variables. A *word equation* e is a pair of words $\varphi, \psi \in (\Sigma \cup \Xi)^*$, denoted $e : \varphi = \psi$. A solution of e is a morphism $h : (\Sigma \cup \Xi)^* \longrightarrow \Sigma^*$ such that $h(a) = a$, for all $a \in \Sigma$, and $h(\varphi) = h(\psi)$. We say that a language $L \subseteq \Sigma^*$ is *expressible* if there is an equation e and a variable X such that

$$L = \{h(X) \mid h \text{ a solution of } e\}.$$

A relation $\mathcal{R} \subseteq (\Sigma^*)^k$ is *expressible* if there is e and variables $X_i, 1 \leqslant i \leqslant k$, such that

$$\mathcal{R} = \{(h(X_1), h(X_2), \ldots, h(X_k)) \mid h \text{ a solution of } e\}.$$

Example 1. The prefix, factor, and suffix relations, which we shall denote $\leqslant_p$, $\leqslant_f$, $\leqslant_{suf}$, respectively, are expressed by the variables X and Y in the equation $Y = XZ$, $Y = WXZ$, and $Y = ZX$, respectively.

Example 2. The conjugacy relation is expressed by the variables X and Y in the system

$$\begin{cases} X = PQ \\ Y = QP \end{cases}$$

It is known (see, e.g., [ChKa]) that it is then possible to find from this system a single equation expressing the conjugacy relation.

3 Subwords

We consider in this section the subword relation $\leqslant_s$ defined, for $u, v \in \Sigma^*$, by

$$\begin{aligned} u \leqslant_s v \text{ iff } & u = a_1 a_2 \ldots a_n, n \geqslant 0, a_i \in \Sigma, 1 \leqslant i \leqslant n, \\ & v = v_1 a_1 v_2 a_2 \ldots v_n a_n v_{n+1}, v_i \in \Sigma^*, \text{ for } 1 \leqslant i \leqslant n+1. \end{aligned}$$

When Σ has one letter only, the subword relation coincides with the prefix one and it is obviously expressible. We prove in this section that this is not the case in all nontrivial cases.

We need some definitions and a result from [KMP]. Consider $Q \in \Sigma^*$ a primitive word. Then it is well known that each word $w \in \Sigma^*$ can be written uniquely in the form

$$w = w_0 Q^{x_1} w_1 Q^{x_2} w_2 \ldots Q^{x_k} w_k,$$

where, for all $0 \leqslant i \leqslant k$, Q^2 is not a factor of w_i, Q is a proper prefix of w_i, if $i \neq 0$, and Q is a proper suffix of w_i, if $i \neq k$. This leads to the $\mathcal{F}_Q$-*factorization* of w which is $w = w_0.Q^{x_1}.w_1.Q^{x_2}.w_2.\cdots.Q^{x_k}.w_k$. We denote

$$\begin{aligned} & T_Q(w) = \{x \mid Q^x \text{ is an element in the } \mathcal{F}_Q\text{-factorization of } w\}, \\ & \exp_Q(w) = \max(T_Q(w) \cup \{0\}). \end{aligned}$$

Theorem 3 (Karhumäki et all, [KMP]). *Let Q be a primitive word and L an expressible language. Then, there exists k such that, for each $u \in L$ satisfying $\exp_Q(u) > k$, there is a word $v \in L$ with $\exp_Q(v) \leqslant k$ and which is obtained from u by removing some occurrences of Q.*

We can prove now the anounced result.

Theorem 4. *The subword relation $\leqslant_s$ on Σ^* is not expressible by word equations as soon as* $\mathrm{card}(\Sigma) \geqslant 2$.

Proof. Consider an alphabet Σ such that $\mathrm{card}(\Sigma) \geqslant 2$. The set corresponding to the subword relation on Σ^* is

$$\mathcal{R}_{\leqslant_s} = \{(u, v) \mid u, v \in \Sigma^*, u \leqslant_s v\}.$$

We argue by contradiction. Assume that $\leqslant_s$ is expressible and consider a word equation in variables $X, Y, Z_1, Z_2, \ldots, Z_n$, for some $n \geqslant 0$, say

$$e = e(X, Y, Z_1, Z_2, \ldots, Z_n)$$

such that

$$\mathcal{R}_{\leqslant_s} = \{(x, y) \mid x, y \in \Sigma^*, h(X) = x, h(Y) = y, \text{ for } h \text{ solution of } e\}.$$

Consider a new symbol # not in Σ and construct the language

$$L = \{u\#v \mid (u, v) \in \mathcal{R}_{\leqslant_s}\}.$$

Consider also a new variable Z which does not appear in e. Then L is expressed by Z in the system

$$\begin{cases} e(X, Y, Z_1, Z_2, \ldots, Z_n) \\ Z = X\#Y \end{cases}$$

and hence L is expressible by an word equation.

Let $a, b \in \Sigma$ be two different letters. If k is the constant corresponding to the language L and the primitive word ab from Theorem 3, then take the word

$$w = a^{k+3}\#(ab)^{k+3}$$

which belongs to L. As $\exp_{ab}(w) = k + 1 > k$, there is, by Theorem 3, a word $u \in L$ such that $\exp_{ab}(u) \leqslant k$ and u is obtained from w by removing some occurrences of ab. Thus

$$u = a^{k+3}\#(ab)^l,$$

for some $l \leqslant k + 2$. But now $a^{k+3} \not\leqslant_s (ab)^l$, hence $u \notin L$, a contradiction. The theorem is proved. ∎

Remark 5. The same result holds for the Parikh subword relation $\leqslant_\Psi$, defined, for $u, v \in \Sigma^*$, by $u \leqslant_\Psi v$ if and only if $|u|_a \leqslant |v|_a$, for any $a \in \Sigma$. The proof is completely similar with the one of Theorem 4.

4 Power free words

In this section we consider the set of finite k-power free words. We shall consider integer values for k and prove that the associated language is not expressible in all nontrivial cases, that is, (i) cube free words over two-letter alphabets and (ii) square free words over three-letter alphabets.

We define first the notion of synchronizing factorization and recall a result from [KMP]. Consider a procedure of factorizing words, say $\mathcal{F}$. We call the factorization obtained by $\mathcal{F}$, $\mathcal{F}$-*factorization*. $\mathcal{F}$ is called *synchronizing*[1] iff

(i) each word has a unique $\mathcal{F}$-factorization and

(ii) it satisfies the the following *synchronizing condition*. If x and y are two words having the $\mathcal{F}$-factorizations $x = x_1.x_2.\cdots.x_s$ and $y = y_1.y_2.\cdots.y_k$ such that y appears at position i in x (see Figure 1), then

(ii.1) the positions $i + |y_1|$ and $i + |y| - |y_k|$ in x are starting positions of factors, say x_p and x_q, resp.;

(ii.2) $(x_p, x_{p+1}, \ldots, x_{q-1}) = (y_2, y_3, \ldots, y_{k-1})$;

(ii.3) y_1 is a proper suffix of $x_{p-2}x_{p-1}$ (by convention, $x_0 = 1$);

(ii.4) y_k is a proper prefix of $x_q x_{q+1}$ (by convention, $x_{s+1} = 1$).

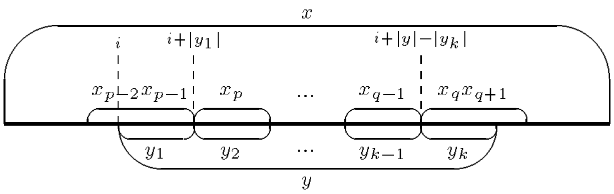

Figure 1: the synchronizing condition

For a word w, the number of different words in its $\mathcal{F}$-factorization is denoted $n_{\mathcal{F}}(w)$. For definitions concerning patters and pattern languages, needed in the statement of the next theorem, we refer to [An] or [JSSY].

Theorem 6 (Karhumäki et all, [KMP]). *Let $\mathcal{F}$ be a synchronizing factorization and L an expressible language. Then, there exists k such that, for each $w \in L$ with $n_{\mathcal{F}}(w) > k$, there is a one variable pattern $p(X)$ satisfying $w \in p(\Sigma^*) \subseteq L$.*

[1] We define the synchronizing factorizations in a simpler way; our definitions means in fact, in terms of [KMP], synchronizing with parameters 1 and 1.

We next define the following factorization which will be essential for our purpose. Let $w \in \Sigma^+$ be an unbordered (nonoverlapping) word. Then, for any $x \in \Sigma^*$, the $\mathcal{F}_w$-*factorization* of x is defined as

$$x = x_0.w.x_1.w.\cdots.x_{n-1}.w.x_n,$$

where no $x_i, 0 \leqslant i \leqslant n$, contains w as a factor. As w is unbordered, the $\mathcal{F}_w$-factorization is well defined and unique. Also, it is clear that it is a synchronizing factorization.

We are now ready to prove the announced results.

Theorem 7. *(i) The set L_1 of all finite cube free words over $\{a, b\}$ is not expressible by word equations.*

(ii) The set L_2 of all finite square free words over $\{a, b, c\}$ is not expressible by word equations.

Proof. (i). Consider the Thue-Morse morphism ([Th], [Mor])

$$h_T : \{a, b\}^* \Longrightarrow \{a, b\}^*$$

given by

$$h_T(a) = ab,$$
$$h_T(b) = ba,$$

and put

$$t_n = h_T^n(a), \text{ for all } n \geqslant 0.$$

Consider also the unbordered word

$$w = aabaabb$$

and construct the words

$$T_n = t_4bwt_5bw \dots t_nbw, \text{ for all } n \geqslant 4,$$
$$\omega = \lim_{n\to\infty} T_n.$$

Notice that none of $t_i, i \geqslant 0$, contains $aabaa$ as a factor. This is because $t_i \in \{ab, ba\}^*$, for any $i \geqslant 1$, and no matter how $aabaa$ is factorized in blocks of length two, aa is obtained as a factor.

We now show that ω is cube free. Assume that ω contains a cube, say

$$\omega = uvvv\omega',$$

$u, v \in \{a,b\}^*, v \neq 1, \omega' \in \{a,b\}^\omega$. We may also assume that $|v| \geqslant |w|$, as the other possibility is immediately ruled out.

If w is a factor of v, then, for some $z \in \{a,b\}^*$, ω has a factor of the form $wzwzw$, which is impossible. So w is not a factor of v.

If w is a factor of v^2, then each of the two possitions in between vs lies inside an occurrence of w. Therefore, there is $n \geqslant 1$ such that $w \leqslant_f t_n$, a contradiction.

Therefore, w cannot be a factor of v^3. (If $w \leqslant_f v^3$, then, since $|w| \leqslant |v|$, we have also $w \leqslant_f v^2$ which is false.) Hence v^3 must be a factor of a word of the form $abaabbt_nbaaba$, for some $n \geqslant 5$. Using the well-known fact that the Thue-Morse infinite word

$$\omega_T = h_T^\infty(a),$$

is 2^+-free (i.e., it does not contain factors of the form $czczc, c \in \{a,b\}, z \in \{a,b\}^*$), we obtain that t_n cannot be longer than twice the length of v. Thus, $2|v| \geqslant |t_n| \geqslant 3|v| - 12$, and so $|v| \leqslant 12$. Consequently, $|t_n| \leqslant 24$ and the only possibility is that $abaabbt_nbaaba$ contains a cube, for some $0 \leqslant n \leqslant 4$. But all these cases are excluded by the definition of ω and so ω is indeed cube free.

Assume now that L_1 is expressible. Consider the $\mathcal{F}_w$-factorization as defined at the beginning of this section. For any $n \geqslant 4$, the $\mathcal{F}_w$-factorization of T_n is

$$T_n = t_4b.w.t_5b.w.\cdots.t_{n-1}b.w.t_nb.w,$$

and so

$$n_{\mathcal{F}_w}(T_n) = n + 2.$$

Consider the constant k from Theorem 6 corresponding to L_1 and $\mathcal{F}_w$. As $T_n \in L_1$, for any $n \geqslant 4$ (ω is cube free), we have, for $n > k-2$, $n_{\mathcal{F}_w}(T_n) > k$, and so, there is a one-variable pattern $p(X)$ such that

$$p(\Sigma^*) \subseteq L_1.$$

But now there are in L_1 words with arbitrarily high repetition, a contradiction. Thus (i) is proved.

(ii). The main idea is the same as at (i), just that here it will be more complicated to use it. We consider here the morphism of Dejean, cf. [De],

$$h_D : \{a,b,c\}^* \longrightarrow \{a,b,c\}^*$$

given by

$$\begin{aligned} h_D(a) &= abc\ acb\ c\underline{ab\ c\ ba}c\ bca\ cba, \\ h_D(b) &= bca\ bac\ abc\ a\ cba\ cab\ acb, \\ h_D(c) &= c\underline{ab\ cba}\ bca\ b\ acb\ \underline{abc\ ba}c. \end{aligned}$$

As well known, the infinite word

$$\omega_D = h_D^{\infty}(a)$$

is $\frac{7}{4}^+$-free (that is, ω_D has no factor of the form uvu, for $u, v \in \{a, b, c\}^*$ with $|u| > 3|v|$).

Put $h_D(a) = ax$ and

$$t_n = h_D^{n-1}(x), \text{ for all } n \geqslant 1.$$

Notice that $\omega_D = at_1t_2t_3\ldots$ and so any $t_n, n \geqslant 1$, is $\frac{7}{4}^+$-free. Consider also the unbordered word

$$w = cab\ acb\ abcba\ c\ abcba$$

and construct

$$\begin{aligned} T_n &= t_1wt_2w\ldots t_nw, \text{ for all } n \geqslant 1, \\ \omega &= \lim_{n\to\infty} T_n. \end{aligned}$$

Our aim is now to show that ω is square free but, for this, we shall need the following two facts.

Fact 1. For any n, m with $n < m$, t_n is not a prefix of t_m.

Indeed, if $t_n \leqslant_p t_m$, that is, $h_D^{n-1}(x) \leqslant_p h_D^{m-1}(x)$, then, as h_D is uniform, we get that $h_D^{n-2}(x) \leqslant_p h_D^{m-2}(x)$. Inductively, we obtain $x \leqslant_p h_D^{m-n}(x)$ and so $x \leqslant_p h_D(b)$, which is false.

Fact 2. The word *abcba c abcba* is not a factor of ω_D.

This is clear since ω_D is $\frac{7}{4}^+$-free and *abcba c abcba* has a repetition of order $\frac{11}{6} > \frac{7}{4}$. Notice that also w is not a factor of ω_D, and so is not a factor of any of the words $T_n, n \geqslant 1$.

Let us prove now that ω is square free. We argue by contradiction. Assume that ω contains a square, say

$$\omega = uvv\omega',$$

for some $u, v \in \{a, b, c\}^*, v \neq 1$, and $\omega' \in \{a, b, c\}^\omega$. We consider the position in between the two vs in ω and distinguish two cases.

Case 1: the position in between the two vs lies inside of or is adjacent to an occurrence of w. Then either (1.1) v has a prefix of the form $v'abcbah_D(b)$ or (1.2) it has a suffix of the form $h_D(a)cab\ acb\ abcbav'$. These are due to the facts that, for any $n \geqslant 1$, $h_D(b) \leqslant_p h_D^n(x)$ (for (1.1)) and $h_D(a) \leqslant_{suf} h_D^n(x)$ (for (1.2)).

Consider the case (1.1). As the only possible occurrences of $abcba$ in an image of a word by h_D are the ones underlined in the definition of h_D, the starting position of the left v in ω_D must lie inside an occurence of w at the same place in w as the position in between the vs. Thus, v has a prefix

$$v'abcbat_m w t_{m+1} w \ldots t_{n-1} w t_n,$$

for some $m \leqslant n$. If $m < n$, then t_{n+1} has w as factor, contradicting Fact 2. If $m = n$, then $t_n \leqslant_p t_{n+1}$, contradicting Fact 1.

In case (1.2), a similar reasoning gives us that v has two suffices

$$t_m cab\ acb\ abcbav' \text{ and } t_n cab\ acb\ abcbav'$$

for some m and n such that $m < n$. Hence, t_n has w as a factor, a contradiction.

Case 2: the position in between the two vs lies strictly inside of some t_n, $n \geqslant 1$. As t_n is square free, the square vv must overlap some occurrence of w.

Assume that w is a factor of v. Then, using Fact 2 and the fact that w is unbordered, we get that v has the form

$$v = v'wt_m w t_{m+1} w \ldots t_{n-1} w v''.$$

If $m \leqslant n-2$, then at least one of t_n and t_{n+1} has w as factor, contradicting Fact 2. If $m = n-1$, then $v = v'wv''$ and $t_n = v''v'$, $v' \leqslant_{suf} t_{n-1}$, $v'' \leqslant_p t_{n+1}$. Hence, v^2 is a factor of $t_{n-1}t_n t_{n+1}$ which, in turn, is a factor of ω_D, contradicting the square freeness of ω_D.

Therefore, w is not a factor of v and so vv is a factor of $c^{-1}wt_n wa^{-1}$. If each of the vs overlaps the respective w, then $n \geqslant 2$ and we get a repetition in t_n of order

$$\frac{|t_n|}{|v|} \geqslant \frac{2|t_n|}{|t_n| + |w| - 2} \geqslant \frac{722}{376} > \frac{7}{4},$$

a contradiction.

Hence, exactly one occurrence of v overlaps w and, if the length of the overlapping part is l, then, by the $\frac{7}{4}^+$-freness of ω_D, the length of v may not exceed $4l$. Now, if only the left v overlaps w, then, from the possibilities of having $abcba$ as factor of ω_D, we obtain $|v| \leqslant 36$; all these possibilities are easily rulled out. For the case when only the right v overlaps w, we similarly get $|v| \leqslant 60$ and these are also excluded by checking.

Consequently, ω is square free. The last part of the reasoning is completely similar with (i) (the factorization $\mathcal{F}_w$ is used). ■

Remark 8. We have not considered above fractional powers. Therefore, the nontrivial cases, given by the Dejean's conjecture, cf. [De], as solved for $k = 4$ in [Pa] and $k = 5, 6, \ldots, 11$ in [Mou], remain still open.

We would like to remark further that our constructions in the proof of Theorem 7 prove actually more than stated in the statement. More precisely, (i) solves the problem for $\frac{8}{3}^+$-free words over two-letter alphabets whereas (ii) solves the problem for $\frac{11}{6}^+$-free words over three-letter alphabets. The arguments are similar with the ones above and left to the reader.

5 Fibonacci

Another open problem in [KMP] asks whether the set of finite Fibonacci words is expressible by word equations. We do not solve this problem here but just show that it cannot be given a negative solution using the tools in [KMP].

The Fibonacci finite words $f_n, n \geqslant 0$, are defined as folows. Consider the Fibonacci morphism

$$h_F : \{a, b\}^* \longrightarrow \{a, b\}^*,$$

given by

$$\begin{aligned} h_F(a) &= ab, \\ h_F(b) &= a. \end{aligned}$$

Then, by definition,

$$f_n = h_F^n(a), \text{ fo any } n \geqslant 0.$$

We shall use the following result from [IlPl]. We do not need to define the *Sturmian* words (see, e.g., [Be], [dLMi]) but just to know that the set of Fibonacci finite words is a subset of the set of the finite Sturmian words.

Theorem 9 (Ilie, Plandowski, [IlPl]). *The set of Sturmian words is expressed by the variable Z in the system*

$$\begin{cases} XbaY = YabX \\ X = WZT \end{cases}$$

It follows by Theorem 9 and the fact that the Fibonacci words are Sturmian words that, for any synchronizing factorization $\mathcal{F}$, the number of different $\mathcal{F}$-factors in the $\mathcal{F}$-factorization of any Fibonacci word is bounded from above by a constant. Therefore, Theorem 6 (as well as the other similar tools in [KMP]) cannot be used for proving the nonexpressibility of the set of Fibonacci finite words.

As, according to [MiPi], the maximal repetition in the Fibonacci finite words is $2+\varphi$, where φ denotes the golden number $\frac{1}{2}(1+\sqrt{5})$, neither Theorem 3 (and the similar ones in [KMP]) can be used for this purpose.

References

[An] Angluin, D., Finding patterns common to a set of strings, *J. Comput. Syst. Sci.* **21** (1980) 46 – 62.

[Be] Berstel, J., Recent results in Sturmian words, *Proc. of 2^{nd} DLT* (Magdeburg, 1995), 13–24, World Sci. Publishing, River Edge, NJ, 1996.

[ChKa] Choffrut, C., Karhumäki, J., Combinatorics of words, in (G. Rozenberg, A. Salomaa, eds.) *Handbook of Formal Languages, Vol. I*, Springer-Verlag, Berlin, Heidelberg, 1997.

[De] Dejean, F., Sur un théorème de Thue, *J. Combin. Theory A* **13** (1972) 90 – 99.

[dLMi] de Luca, A., Mignosi, F., Some combinatorial properties of sturmian words, *Theoret. Comput. Sci.* **136** (1994) 361 – 385.

[IlPl] Ilie, L., Plandowski, W., Languages expressible by two-variable word equations, manuscript.

[JSSY] Jiang, T., Salomaa, A., Salomaa, K., Yu, S., Decision problems for patterns, *J. Comput. Syst. Sci.* **50** (1995) 53 – 63.

[KMP] Karhumäki, J., Mignosi, F., Plandowski, W., The expressibility of languages and relations by word equations, *Proc. of 24^{th} ICALP* (Bologna, 1997), 98–109, Lecture Notes in Comput. Sci., 1256, Springer, Berlin, 1997.

[Lo] Lothaire, M., *Combinatorics on words*, Addison-Wesley, Reading, Mass., 1983.

[MiPi] Mignosi, F., Pirillo, G., Repetitions in the Fibonacci infinite word, *RAIRO Inform. Théor. Appl.* **26** (1992) 199 – 204.

[Mor] Morse, M., Recurrent geodesics on a surface of negative curvature, *Trans. Amer. Math. Soc.* **22** (1921) 84 – 100.

[Mou] Moulin Ollagnier, J., Proof of Dejean's conjecture for alphabets with 5,6,7,8,9,10, and 11 letters, *Theoret. Comput. Sci.* **95** (1992) 187 – 205.

[Pa] Pansiot, J.-J., A propos d'une conjecture de F. Dejean sur les répétitions dans les mots, *Discrete Appl. Math.* **7** (1982) 297 – 311.

[Se] Seibert, S., Quantifier hierarchies and word relations, *Computer science logic* (Berne, 1991), 329–338, Lecture Notes in Comput. Sci., 626, Springer, Berlin, 1992.

[Th] Thue, A., Über unendliche Zeichenreihen, *Kra. Vidensk. Selsk. Skrifter. I. Mat.-Nat. Kl., Cristiana* **7**, 1906.

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