Global Asymptotical Stability of The Non-Trivial Steady State of an Epidemic Model

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Abstract

In this paper I prove a conjecture of Diekmann and Heesterbeek about the global asymptotocal stability of the non-trivial steady state of an epidemic model.

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In their discussion about regulation of host populations Diekmann and Heesterbeek [1] considered the SIR model

$$\begin{aligned}
\frac{dS}{dt} &= bS + bR - \mu S - \gamma \frac{SI}{N}, \\
\frac{dI}{dt} &= -\mu I + \gamma \frac{SI}{N} - \alpha I, \\
\frac{dR}{dt} &= -\mu R + f\alpha I.
\end{aligned}$$

Here S denotes susceptible, I infected and R recovered individuals. b is the per capita birth rate (infected individuals are assumed not to give birth), μ is the per capita death rate due to other causes than the disease, γ is the expected number of contacts per unit of time times the probability of transmission given a contact, and α is the removal rate.

When an individual is removed from the I-class it recovers and acquires permanent immunity with probability f and dies from the disease with probability $1-f$.

Introducing the relative quantities

$$y = I/N, \quad z = R/N$$

with $N = S + I + R$ being the total population one obtains the two-dimensional system

$$\begin{aligned}
\dot{y} &= y(\gamma(1-y-z) - \alpha + \alpha(1-f)y + b(y-1)) \equiv F_1(y,z), \\
\dot{z} &= y(f\alpha + (1-f)\alpha z) - bz(1-y) \equiv F_2(y,z)
\end{aligned} \tag{1}$$

supplemented by the scalar equation

$$\frac{dN}{dt} = (b - \mu - (b + \alpha(1-f))y)N.$$

Assume

$$R_{0,\text{relative}} = \frac{\gamma}{b+\alpha} > 1.$$

Then the system (1) has a unique non-trivial steady state $(\bar{y},\ \bar{z})$ $(0 < \bar{y} < 1,\ 0 < \bar{z} < 1,\ 0 < \bar{y} + \bar{z} < 1)$, and it can be shown that $(\bar{y},\ \bar{z})$ is locally asymptotically stable [1, p. 217, Exercise 3.23].

On page 218 of their book, Diekmann and Heesterbeek [1] conjecture that $(\bar{y},\ \bar{z})$ is in fact globally asymtotically stable in the set $D = \{(y,\ z) : y > 0,\ z \geq 0,\ y + z \leq 1\}$. It is the purpose of this note to prove that this is indeed the case.

It is easily shown [1, p. 217] that D is positively invariant and that orbits starting on the boundary of D immediately enter the interior G of D. In particular, every positive semiorbit starting in D is bounded. By the Poincaré–Bendixson Theorem the ω-limit set is therefore either the point $(\bar{y},\ \bar{z})$ or a non-trivial periodic orbit. To prove the conjecture we therefore only have to exclude the existence of a non-trivial periodic orbit in G. We do this by applying the well-known Bendixson–Dulac Theorem [2, Theorem 1.11], which in the present situation (note that G is a simply connected region) says that if there exists a continuously differentiable function $B(y,\ z)$ in G such that

$$\frac{\partial(BF_1)}{\partial y} + \frac{\partial(BF_2)}{\partial z}$$

has constant sign in G, then system (1) does not have any closed trajectory in G.

Define $B(y,\ z) = y^{-1}z^{-1}$ for $(y, z) \in G$. Then

$$\begin{aligned}
&\tfrac{\partial(BF_1)}{\partial y} + \tfrac{\partial(BF_2)}{\partial z} = y^{-1}z^{-1}((\alpha + b - \gamma - \alpha f)y - \gamma z\\
&+(\gamma - \alpha - b) + (\alpha + b - \gamma - \alpha f)y)\\
&-y^{-1}z^{-1}((\alpha + b - \gamma - \alpha f)y - \gamma z + (\gamma - \alpha - b))\\
&+y^{-1}z^{-1}((b + \alpha(1 - f))y - b)\\
&-y^{-1}z^{-2}(f\alpha y + (b + \alpha(1 - f))yz - bz)\\
&= -y^{-1}z^{-1}((\gamma + \alpha f - \alpha - b)y + f\alpha y z^{-1})\\
&< 0
\end{aligned}$$

for all $(y, z) \in G$. This proves the conjecture.

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References

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