On obtaining minimal variability OWA operator weights

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Abstract

One important issue in the theory of Ordered Weighted Averaging (OWA) operators is the determination of the associated weights. One of the first approaches, suggested by O'Hagan, determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*; algorithmically it is based on the solution of a constrained optimization problem. Another consideration that may be of interest to a decision maker involves the variability associated with a weighting vector. In particular, a decision maker may desire low variability associated with a chosen weighting vector. In this paper, using the Kuhn-Tucker second-order sufficiency conditions for optimality, we shall analytically derive the minimal variablity weighting vector for any level of *orness*.

Keywords: Multiple criteria analysis, Fuzzy sets, OWA operator, Lagrange multiplier

1 Introduction

The process of information aggregation appears in many applications related to the development of intelligent systems. One sees aggregation in neural networks, fuzzy logic controllers, vision systems, expert systems and multi-criteria decision aids. In [6] Yager introduced a new aggregation technique based on the ordered weighted averaging operators.

Definition 1.1. *An OWA operator of dimension n is a mapping $F\colon \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W = (w_1, \dots, w_n)^T$ of having the properties $w_1 + \cdots + w_n = 1$, $0 \leq w_i \leq 1$, $i = 1, \dots, n$, and such that*

$$F(a_1, \dots, a_n) = \sum_{i=1}^{n} w_i b_i,$$

where b_j is the jth largest element of the collection of the aggregated objects $\{a_1, \dots, a_n\}$.

A fundamental aspect of this operator is the re-ordering step, in particular an aggregate a_i is not associated with a particular weight w_i but rather a weight is associated with a particular ordered position of aggregate. When we view the OWA weights as a column vector we shall find it convenient to refer to the weights with the low indices as weights at the top and those with the higher indices with weights at the bottom. It is noted that different OWA operators are distinguished by their weighting function.

In [6], Yager introduced a measure of *orness* associated with the weighting vector W of an OWA operator, defined as

$$\mathrm{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i.$$

and it characterizes the degree to which the aggregation is like an *or* operation. It is clear that $\mathrm{orness}(W) \in [0,1]$ holds for any weighting vector.

It is clear that the actual type of aggregation performed by an OWA operator depends upon the form of the weighting vector [9]. A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation [6, 7], exponential smoothing [3] and learning [10].

Another approach, suggested by O'Hagan [5], determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*. This approach is based on the solution of he following mathematical programming problem:

$$\begin{aligned} &\text{maximize} && \mathrm{disp}(W) = -\sum_{i=1}^{n} w_i \ln w_i \\ &\text{subject to} && \mathrm{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i = \alpha,\ 0 \leq \alpha \leq 1 \qquad (1) \\ & && w_1 + \cdots + w_n = 1,\ 0 \leq w_i,\ i = 1, \dots, n. \end{aligned}$$

In [4] we transfered problem (1) to a polinomial equation which is then was solved to determine the maximal entropy OWA weights.

Another interesting question is to determine the minimal variablity weighting vector under given level of orness [8]. We shall meausure the variance of a given weighting vector as

$$\begin{aligned} D^2(W) &= \sum_{i=1}^{n} \frac{1}{n} \cdot (w_i - E(W))^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} w_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2}. \end{aligned}$$

where $E(W) = (w_1 + \cdots + w_n)/$ stands for the arithmetic mean of weights.

To obtain minimal variablity OWA weights under given level of orness we need to solve the following constrained mathematical programming problem

$$\begin{aligned} &\text{minimize} && D^2(W) = \frac{1}{n} \cdot \sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} \\ &\text{subject to} && \text{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i = \alpha,\ 0 \le \alpha \le 1, \\ & && w_1 + \cdots + w_n = 1,\ 0 \le w_i,\ i = 1, \dots, n. \end{aligned} \tag{2}$$

Using the Kuhn-Tucker second-order sufficiency conditions for optimality, we shall solve problem (2) analytically and derive the exact minimal variablity OWA weights for any level of *orness*.

2 Obtaining minimal variability OWA weights

Let us consider the constrained optimization problem (2). First we note that if $n = 2$ then from $\text{orness}(w_1, w_2) = \alpha$ the optimal weights are uniquely defined as $w_1^* = \alpha$ and $w_2^* = 1 - \alpha$. Furthemore, if $\alpha = 0$ or $\alpha = 1$ then the associated weighting vectors are uniquely defined as $(0, 0, \dots, 0, 1)^T$ and $(1, 0, \dots, 0, 0)^T$, respectively, with variability

$$D^2(1, 0, \dots, 0, 0) = D^2(0, 0, \dots, 0, 1) = \frac{1}{n} - \frac{1}{n^2}.$$

Suppose now that $n \ge 3$ and $0 < \alpha < 1$. Let us

$$L(W, \lambda_1, \lambda_2) = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} + \lambda_1\left(\sum_{i=1}^{n} \frac{n-i}{n-1} w_i - \alpha\right) + \lambda_2\left(\sum_{i=1}^{n} w_i - 1\right).$$

denote the Lagrange function of constrained optimization problem (2), where λ_1 and λ_2 are real numbers. Then the partial derivatives of L are computed as

$$\frac{\partial L}{\partial w_j} = \frac{2w_j}{n} + \lambda_1 + \frac{n-j}{n-1} \cdot \lambda_2 = 0, \; 1 \leq j \leq n, \tag{3}$$
$$\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{n} w_i - 1 = 0,$$
$$\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i - \alpha = 0.$$

We shall suppose that the optimal weighting vector has the following form

$$W = (0, \dots, 0, w_p, \dots, w_q, 0 \dots, 0)^T \tag{4}$$

where $1 \leq p < q \leq n$ and use the notation

$$I_{\{p,q\}} = \{p, p+1, \dots, q-1, q\},$$

for the indexes from p to q. So, $w_j = 0$ if $j \notin I_{\{p,q\}}$ and $w_j \geq 0$ if $j \in I_{\{p,q\}}$.

For $j = p$ we find that

$$\frac{\partial L}{\partial w_p} = \frac{2w_p}{n} + \lambda_1 + \frac{n-p}{n-1} \cdot \lambda_2 = 0$$

and for $j = q$ we get

$$\frac{\partial L}{\partial w_q} = \frac{2w_q}{n} + \lambda_1 + \frac{n-q}{n-1} \cdot \lambda_2 = 0$$

That is,

$$\frac{2(w_p - w_q)}{n} + \frac{q-p}{n-1} \cdot \lambda_2 = 0$$

and therefore, the optimal values of λ_1 and λ_2 (denoted by λ_1^* and λ_2^*) should satisfy the following equations

$$\lambda_1^* = \frac{2}{n}\left[\frac{n-q}{q-p} \cdot w_p - \frac{n-p}{q-p} \cdot w_q\right] \quad \text{and} \quad \lambda_2^* = \frac{n-1}{q-p} \cdot \frac{2}{n} \cdot (w_q - w_p). \tag{5}$$

Substituting λ_1^* for λ_1 and λ_2^* for λ_2 in (3) we get

$$\frac{2}{n} \cdot w_j + \frac{2}{n}\left[\frac{n-q}{q-p} \cdot w_p - \frac{n-p}{q-p} \cdot w_q\right] + \frac{n-j}{n-1} \cdot \frac{n-1}{q-p} \cdot \frac{2}{n} \cdot (w_q - w_p) = 0.$$

That is the jth optimal weight should satisfy the equation

$$w_j^* = \frac{q-j}{q-p} \cdot w_p + \frac{j-p}{q-p} \cdot w_q, \; j \in I_{\{p,q\}}. \tag{6}$$

From representation (4) we get

$$\sum_{i=p}^{q} w_i^* = 1,$$

That is,

$$\sum_{i=p}^{q} \left(\frac{q-i}{q-p} \cdot w_p + \frac{i-p}{q-p} \cdot w_q \right) = 1,$$

i.e.

$$w_p + w_q = \frac{2}{q-p+1}.$$

From the constraint $\mathrm{orness}(w) = \alpha$ we find

$$\sum_{i=p}^{q} \frac{n-i}{n-1} \cdot w_i = \sum_{i=p}^{q} \frac{n-i}{n-1} \cdot \frac{q-i}{q-p} \cdot w_p + \sum_{i=p}^{q} \frac{n-i}{n-1} \cdot \frac{i-p}{q-p} \cdot w_q = \alpha$$

that is,

$$w_p^* = \frac{2(2q+p-2) - 6(n-1)(1-\alpha)}{(q-p+1)(q-p+2)}. \tag{7}$$

and

$$w_q^* = \frac{2}{q-p+1} - w_p^* = \frac{6(n-1)(1-\alpha) - 2(q+2p-4)}{(q-p+1)(q-p+2)}. \tag{8}$$

The optimal weighting vector

$$W^* = (0, \dots, 0, w_p^*, \dots, w_q^*, 0 \dots, 0)^T$$

is feasible if and only if $w_p^*, w_q^* \in [0, 1]$, because according to (6) any other w_j^*, $j \in I_{\{p,q\}}$ is computed as their convex linear combination.

Using formulas (7) and (8) we find

$$w_p^*, w_q^* \in [0,1] \iff \alpha \in \left[1 - \frac{1}{3} \cdot \frac{2q+p-2}{n-1}, 1 - \frac{1}{3} \cdot \frac{q+2p-4}{n-1} \right]$$

The following (disjunctive) partition of the unit interval $(0, 1)$ will be crucial in finding an optimal solution to problem (2):

$$(0,1) = \bigcup_{r=2}^{n-1} J_{r,n} \cup J_{1,n} \cup \bigcup_{s=2}^{n-1} J_{1,s}. \tag{9}$$

where

$$J_{r,n} = \left(1 - \frac{1}{3} \cdot \frac{2n+r-2}{n-1}, 1 - \frac{1}{3} \cdot \frac{2n+r-3}{n-1} \right], \; r = 2, \dots, n-1,$$

$$J_{1,n} = \left(1 - \frac{1}{3} \cdot \frac{2n-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{n-2}{n-1} \right),$$

$$J_{1,s} = \left[1 - \frac{1}{3} \cdot \frac{s-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{n-1} \right), \; s = 2, \dots, n-1.$$

Consider again problem (2) and suppose that $\alpha \in J_{r,s}$ for some r and s from partition (9). Such r and s always exist for any $\alpha \in (0,1)$, furthermore, $r = 1$ or $s = n$ should hold.

Then

$$W^* = (0, \dots, 0, w_r^*, \dots, w_s^*, 0 \dots, 0)^T \tag{10}$$

where

$$\begin{aligned}
w_j^* &= 0, \text{ if } j \notin I_{\{r,s\}}, \\
w_r^* &= \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)}, \\
w_s^* &= \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)}, \\
w_j^* &= \frac{s-j}{s-r} \cdot w_r + \frac{j-r}{s-r} \cdot w_s, \text{ if } j \in I_{\{r+1,s-1\}}.
\end{aligned} \tag{11}$$

and $I_{\{r+1,s-1\}} = \{r+1, \dots, s-1\}$. Furthermore, from the construction of W^* it is clear that

$$\sum_{i=1}^{n} w_i^* = \sum_{i=r}^{s} w_i^* = 1, \; w_i^* \geq 0, \; i = 1, 2, \dots, n,$$

and $\mathrm{orness}(W^*) = \alpha$, that is, W^* is feasible for problem (2).

We will show now that W^*, defined by (10), satisfies the Kuhn-Tucker second-order sufficiency conditions for optimality ([2], page 58). Namely,

(i) There exist $\lambda_1^*, \lambda_2^* \in \mathbb{R}$ and $\mu_1^* \geq 0, \dots, \mu_n^* \geq 0$ such that,

$$\frac{\partial}{\partial w_k}\Bigg(D^2(W) + \lambda_1^* \Bigg[\sum_{i=1}^{n} w_i - 1 \Bigg] + \lambda_2^* \Bigg[\sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha \Bigg] + \sum_{j=1}^{n} \mu_j^*(-w_j) \Bigg) \Bigg|_{W=W^*} = 0$$

for $1 \leq k \leq n$ and $\mu_j^* w_j^* = 0$, $j = 1, \dots, n$.

(ii) W^* is a regular point of the constraints,

(iii) The Hessian matrix,

$$\frac{\partial^2}{\partial W^2}\Bigg(D^2(W) + \lambda_1^* \Bigg[\sum_{i=1}^{n} w_i - 1 \Bigg] + \lambda_2^* \Bigg[\sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha \Bigg] + \sum_{j=1}^{n} \mu_j^*(-w_j) \Bigg) \Bigg|_{W=W^*}$$

is positive definite on

$$\hat{X} = \left\{ y \middle| h_1 y^T = 0, h_2 y^T = 0 \text{ and } g_j y^T = 0 \text{ for all } j \text{ with } \mu_j > 0 \right\}, \tag{12}$$

where

$$h_1 = \left(\frac{n-1}{n-1}, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0 \right)^T, \tag{13}$$

and

$$h_2 = (1, 1, \ldots, 1, 1)^T. \tag{14}$$

are the gradients of linear equality constraints, and

$$g_j = (0, 0, \ldots, 0, \overbrace{-1}^{j\text{th}}, 0, 0, \ldots, 0)^T \tag{15}$$

is the gradient of the jth linear inequality constraint of problem (2).

Proof. **(i)** According to (5) we get

$$\lambda_1^* = \frac{2}{n} \cdot \left[\frac{n-s}{s-r} \cdot w_r^* - \frac{n-r}{s-r} \cdot w_s^* \right] \quad \text{and} \quad \lambda_2^* = \frac{n-1}{s-r} \cdot \frac{2}{n} \cdot (w_s^* - w_r^*)$$

and

$$\frac{2}{n} \cdot w_k^* + \lambda_1^* + \frac{n-k}{n-1} \cdot \lambda_2^* - \mu_k = 0.$$

for $k = 1, \ldots, n$. If $k \in I_{\{r,s\}}$ then

$$\begin{aligned}
\mu_k^* &= \frac{2}{n} \cdot \left[\frac{s-k}{s-r} \cdot w_r^* + \frac{k-r}{s-r} \cdot w_s^* \right] + \frac{2}{n} \cdot \left[\frac{n-s}{s-r} \cdot w_r^* - \frac{n-r}{s-r} \cdot w_s^* \right] \\
&\quad + \frac{n-k}{n-1} \cdot \frac{n-1}{s-r} \cdot \frac{2}{n} \cdot (w_s^* - w_r^*) \\
&= \frac{2}{n} \cdot \frac{1}{s-r} \big[(s-k+n-s-n+k) w_r^* + (k-r-n+r+n-k) w_s^* \big] \\
&= 0.
\end{aligned}$$

If $k \notin I_{\{r,s\}}$ then $w_k^* = 0$. Then from the equality

$$\lambda_1^* + \frac{n-k}{n-1} \cdot \lambda_2^* - \mu_k = 0,$$

we find

$$\begin{aligned}
\mu_k^* &= \lambda_1^* + \frac{n-k}{n-1} \cdot \lambda_2^* \\
&= \frac{2}{n} \cdot \left[\frac{n-s}{s-r} \cdot w_r^* - \frac{n-r}{s-r} \cdot w_s^* \right] + \frac{n-k}{n-1} \cdot \frac{n-1}{s-r} \cdot \frac{2}{n} \cdot (w_s^* - w_r^*) \\
&= \frac{2}{n} \cdot \frac{1}{s-r} \cdot \big[(k-s) w_r^* + (r-k) w_s^* \big].
\end{aligned}$$

We need to show that $\mu_k^* \geq 0$ for $k \notin I_{\{r,s\}}$. That is,

$$(k-s)w_r^* + (r-k)w_s^* = (k-s) \cdot \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)} \\ +(r-k) \cdot \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)} \geq 0. \tag{16}$$

If $r = 1$ and $s = n$ then we get that $\mu_k^* = 0$ for $k = 1, \dots, n$. Suppose now that $r = 1$ and $s < n$. In this case the inequality $k > s > 1$ should hold and (16) leads to the following requirement for α,

$$\alpha \geq 1 - \frac{(s-1)(3k-2s-2)}{3(n-1)(2k-s-1)}.$$

On the other hand, from $\alpha \in J_{1,s}$ and $s < n$ we have

$$\alpha \in \left[1 - \frac{1}{3} \cdot \frac{s-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{n-1}\right),$$

and, therefore,

$$\alpha \geq 1 - \frac{1}{3} \cdot \frac{s-1}{n-1}$$

Finally, from the inequality

$$1 - \frac{1}{3} \cdot \frac{s-1}{n-1} \geq 1 - \frac{(s-1)(3k-2s-2)}{3(n-1)(2k-s-1)}$$

we get that (16) holds. The proof of the remaining case ($r > 1$ and $s = n$) is carried out analogously.

(ii) The gradient vectors of linear equality and inequality constraints are computed by (13), (14) and (15), respectively. If $r = 1$ and $s = n$ then $w_j^* \neq 0$ for all $j = 1, \dots, n$. Then it is easy to see that h_1 and h_2 are linearly independent. If $r = 1$ and $s < n$ then $w_j^* = 0$ for $j = s+1, \dots, n$, and in this case

$$g_j = (0, 0, \dots, 0, \overbrace{-1}^{j\text{th}}, 0, 0, \dots, 0)^T,$$

for $j = s+1, \dots, n$. Consider the matrix

$$G = [h_1^T, h_2^T, g_{s+1}^T, \dots, g_n^T] \in \mathbb{R}^{n \times (n-s+2)}.$$

Then the determinant of the lower-left submatrix of dimension $(n-s+2) \times (n-$

$s+2)$ of G is

$$\begin{vmatrix} \frac{n-s+1}{n-1} & 1 & 0 & \dots & 0 \\ \frac{n-s}{n-1} & 1 & 0 & \dots & 0 \\ \frac{n-s-1}{n-1} & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 1 & 0 & \dots & -1 \end{vmatrix} = (-1)^{n-s} \begin{vmatrix} \frac{n-s+1}{n-1} & 1 \\ \frac{n-s}{n-1} & 1 \end{vmatrix} = \frac{1}{n-1}(-1)^{n-s}$$

which means that the columns of G are linearly independent, and therefore, the system

$$\{h_1, h_2, g_{s+1}, \dots, g_n\},$$

is linearly independent.

If $r > 1$ and $= n$ then $w_j^* = 0$ for $j = 1, \dots, r-1$, and in this case

$$g_j = (0, 0, \dots, 0, \overbrace{-1}^{j\text{th}}, 0, 0, \dots, 0)^T,$$

for $j = 1, \dots, r-1$. Consider the matrix

$$F = [h_1^T, h_2^T, g_1^T, \dots, g_{r-1}^T] \in \mathbb{R}^{n \times (r+1)}.$$

Then the determinant of the upper-left submatrix of dimension $(r+1) \times (r+1)$ of F is

$$\begin{vmatrix} \frac{n-1}{n-1} & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{n-r+1}{n-1} & 1 & 0 & \dots & -1 \\ \frac{n-r}{n-1} & 1 & 0 & \dots & 0 \\ \frac{n-r-1}{n-1} & 1 & 0 & \dots & 0 \end{vmatrix} = (-1)^{r-1} \begin{vmatrix} \frac{n-r}{n-1} & 1 \\ \frac{n-r-1}{n-1} & 1 \end{vmatrix} = \frac{1}{n-1}(-1)^{r-1}$$

which means that the columns of F are linearly independent, and therefore, the system

$$\{h_1, h_2, g_1, \dots, g_{r-1}\},$$

is linearly independent. So W^* is a regular point for problem (2).

(iii) Let us introduce the notation

$$K(W) = D^2(W) + \lambda_1^* \left[\sum_{i=1}^{n} w_i - 1 \right] + \lambda_2^* \left[\sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha \right] + \sum_{j=1}^{n} \mu_j^* (-w_j).$$

The Hessian matrix of K at W^* is

$$\frac{\partial^2}{\partial w_k \partial w_j} K(W)\bigg|_{W=W^*} = \frac{\partial^2}{\partial w_k \partial w_j} D^2(W)\bigg|_{W=W^*} = \frac{2}{n} \cdot \delta_{kj},$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$K''(W^*) = \frac{2}{n} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

which is a positive definite matrix on $\mathbb{R}^n$. $\square$

So, the objective function $D^2(W)$ has a local minimum at point $W = W^*$ on

$$X = \left\{ W \in \mathbb{R}^n \middle| W \geq 0, \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i = \alpha \right\} \tag{17}$$

where X is the set of feasible solutions of problem (2). Taking into consideration that $D^2 \colon \mathbb{R}^n \to \mathbb{R}$ is a strictly convex, bounded and continuous function, and X is a covex and compact subset of $\mathbb{R}^n$, we can conclude that D^2 attains its (unique) global minimum on X at point W^*.

3 Example

In this Section we will determine the minimal variability five-dimensional weighting vector under orness levels $\alpha = 0, 0.1, \dots, 0.9$ and 1.0. First, we construct the corresponding partition as

$$(0,1) = \bigcup_{r=2}^{4} J_{r,5} \cup J_{1,5} \cup \bigcup_{s=2}^{4} J_{1,s}.$$

where

$$J_{r,5} = \left(\frac{1}{3} \cdot \frac{5-r-1}{5-1}, \frac{1}{3} \cdot \frac{5-r}{5-1} \right] = \left(\frac{4-r}{12}, \frac{5-r}{12} \right],$$

for $r = 2, 3, 4$ and

$$J_{1,5} = \left(\frac{1}{3} \cdot \frac{5-2}{5-1}, \frac{1}{3} \cdot \frac{10-1}{5-1} \right) = \left(\frac{3}{12}, \frac{9}{12} \right),$$

and

$$J_{1,s} = \left[1 - \frac{1}{3} \cdot \frac{s-1}{5-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{5-1} \right) = \left[\frac{13-s}{12}, \frac{14-s}{12} \right),$$

for $s = 2, 3, 4$, and, therefore we get,

$$(0,1) = \left(0, \frac{1}{12}\right] \cup \left(\frac{1}{12}, \frac{2}{12}\right] \cup \left(\frac{2}{12}, \frac{3}{12}\right] \cup \left(\frac{3}{12}, \frac{9}{12}\right) \\ \cup \left[\frac{9}{12}, \frac{10}{12}\right) \cup \left[\frac{10}{12}, \frac{11}{12}\right) \cup \left[\frac{11}{12}, \frac{12}{12}\right).$$

Without loss of generality we can assume that $\alpha < 0.5$, because if a weighting vector W is optimal for problem (2) under some given degree of orness, $\alpha < 0.5$, then its reverse, denoted by W^R, and defined as

$$w_i^R = w_{n-i+1}$$

is also optimal for problem (2) under degree of orness $(1 - \alpha)$. Really, as was shown by Yager [7], we find that

$$D^2(W^R) = D^2(W) \text{ and } \mathrm{orness}(W^R) = 1 - \mathrm{orness}(W).$$

Therefore, for any $\alpha > 0.5$, we can solve problem (2) by solving it for level of orness $(1 - \alpha)$ and then taking the reverse of that solution.

Then we obtain the optimal weights from (11) as follows

- if $\alpha = 0$ then $W^*(\alpha) = W^*(0) = (0, 0, \dots, 0, 1)^T$ and, therefore,

  $$W^*(1) = (W^*(0))^R = (1, 0, \dots, 0, 0)^T.$$

- if $\alpha = 0.1$ then

  $$\alpha \in J_{3,5} = \left(\frac{1}{12}, \frac{2}{12}\right],$$

  and the associated minimal variablity weights are

  $$\begin{aligned} w_1^*(0.1) &= 0, \\ w_2^*(0.1) &= 0, \\ w_3^*(0.1) &= \frac{2(10 + 3 - 2) - 6(5 - 1)(1 - 0.1)}{(5 - 3 + 1)(5 - 3 + 2)} = \frac{0.4}{12} = 0.0333, \\ w_5^*(0.1) &= \frac{2}{5 - 3 + 1} - w_3^*(0.1) = 0.6334, \\ w_4^*(0.1) &= \frac{1}{2} \cdot w_3^*(0.1) + \frac{1}{2} \cdot w_5^*(0.1) = 0.3333, \end{aligned}$$

  So,

  $$W^*(\alpha) = W^*(0.1) = (0, 0, 0.033, 0.333, 0.633)^T,$$

  and, consequently,

  $$W^*(0.9) = (W^*(0.1))^R = (0.633, 0.333, 0.033, 0, 0)^T.$$

  with variance $D^2(W^*(0.1)) = 0.0625$.

- if $\alpha = 0.2$ then
$$\alpha \in J_{2,5} = \left(\frac{2}{12}, \frac{3}{12}\right]$$
and in a similar manner we find that the associated minimal variablity weighting vector is
$$W^*(0.2) = (0.0, 0.04, 0.18, 0.32, 0.46)^T,$$
and, therefore,
$$W^*(0.8) = (0.46, 0.32, 0.18, 0.04, 0.0)^T,$$
with variance $D^2(W^*(0.2)) = 0.0296$.

- if $\alpha = 0.3$ then
$$\alpha \in J_{1,5} = \left(\frac{3}{12}, \frac{9}{12}\right]$$
and in a similar manner we find that the associated minimal variablity weighting vector is
$$W^*(0.3) = (0.04, 0.12, 0.20, 0.28, 0.36)^T,$$
and, therefore,
$$W^*(0.7) = (0.36, 0.28, 0.20, 0.12, 0.04)^T,$$
with variance $D^2(W^*(0.3)) = 0.0128$.

- if $\alpha = 0.4$ then
$$\alpha \in J_{1,5} = \left(\frac{3}{12}, \frac{9}{12}\right]$$
and in a similar manner we find that the associated minimal variablity weighting vector is
$$W^*(0.4) = (0.12, 0.16, 0.20, 0.24, 0.28)^T,$$
and, therefore,
$$W^*(0.6) = (0.28, 0.24, 0.20, 0.16, 0.12)^T,$$
with variance $D^2(W^*(0.4)) = 0.0032$.

- if $\alpha = 0.5$ then
$$W^*(0.5) = (0.2, 0.2, 0.2, 0.2, 0.2)^T.$$
with variance $D^2(W^*(0.5)) = 0$.

4 Summary

We have extended the power of decision making with OWA operators by introducing considerations of minimizing variability into the process of selecting the optimal alternative. Of particular significance in this work is the development of a methodology to calculate the minimal variability weights in an analytic way.

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