On the Structure of Rough Approximations

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Abstract

We study rough approximations based on indiscernibility relations which are not necessarily reflexive, symmetric or transitive. For this, we define in a lattice-theoretical setting two maps which mimic the rough approximation operators and note that this setting is suitable also for other operators based on binary relations. Properties of the ordered sets of the upper and the lower approximations of the elements of an atomic Boolean lattice are studied.

1 Introduction

The basic ideas of rough set theory introduced by Pawlak [8] deal with situations in which the objects of a certain universe can be identified only within the limits determined by the knowledge represented by a given indiscernibility relation. The indiscernibility relation enables us to divide objects of the universe U into three disjoint sets with respect to any subset $X \subseteq U$:

(1) the objects, which surely are in X;

(2) the objects, which are surely not in X;

(3) the objects, which possibly are in X.

The objects in class 1 form the lower approximation of X, and the objects of type 1 and 3 form together its upper approximation. The boundary of X consists of the objects in class 3.

Usually indiscernibility relations are supposed to be equivalences. In this work we do not restrict the properties of an indiscernibility relation. Namely, as we will see, it can be argued that neither reflexivity, symmetry nor transitivity are necessary properties of indiscernibility relations.

We start our study by defining formally the upper and the lower approximations of an indiscernibility relation $\approx$ on U. For any $x \in U$, we denote

$$[x]_\approx = \{y \in U \mid x \approx y\}.$$

Thus, $[x]_\approx$ consists of the elements which cannot be discerned from x. For any subset X of U, let

$$X^\blacktriangledown = \{x \in U \mid [x]_\approx \subseteq X\} \quad \text{and} \tag{1.1}$$

$$X^\blacktriangle = \{x \in U \mid X \cap [x]_\approx \neq \emptyset\}. \tag{1.2}$$

The sets $X^\blacktriangledown$ and $X^\blacktriangle$ are called the *lower* and the *upper approximation* of X, respectively. The set $B(X) = X^\blacktriangle - X^\blacktriangledown$ is the *boundary* of X.

The above definitions mean that $x \in X^\blacktriangle$ if there is an element in X to which x is $\approx$-related. Similarly, $x \in X^\blacktriangledown$ if all the elements to which x is $\approx$-related are in X. Furthermore, $x \in B(X)$ if both in X and outside X there are elements to which x is $\approx$-related.

Two sets X and Y are said to be *equivalent*, denoted by $X \equiv Y$, if $X^\blacktriangledown = Y^\blacktriangledown$ and $X^\blacktriangle = Y^\blacktriangle$. The equivalence classes of $\equiv$ are called *rough sets*. The set of all rough sets is denoted by $\mathcal{R}$. The idea is that if subsets of U are observed within the limits given by the knowledge represented by $\approx$, then the sets in the same rough set look the same; $X \equiv Y$ means that exactly the same elements belong certainly to X and to Y, and that exactly the same elements belong possibly to X and to Y.

It seems that there is no natural representative for a rough set. However, this problem can be easily avoided by using Iwiński's [4] approach to rough sets based

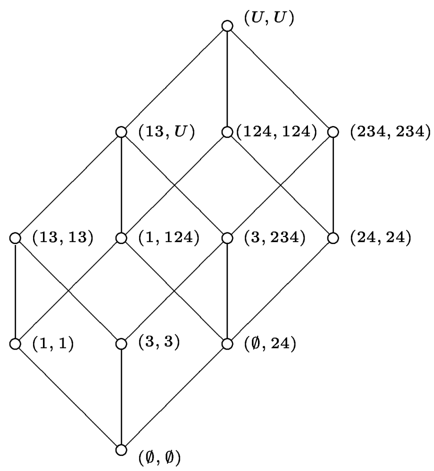

Figure 1: The ordered set $(\mathcal{R}, \leq)$

on the fact that each rough set $\mathcal{S} \in \mathcal{R}$ is uniquely determined by the pair $(X^{\blacktriangledown}, X^{\blacktriangle})$, where X is any member of $\mathcal{S}$. Now there is a natural order relation on $\mathcal{R}$ defined by

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

Example 1.1 Suppose that the object set $U = \{1, 2, 3, 4\}$ consists of four persons called 1, 2, 3, and 4, respectively. In Table 1 is presented some information concerning these persons.

	GENDER	AGE
1	Male	Old
2	Female	Young
3	Male	Middle-aged
4	Female	Young

Table 1: The gender and the age of persons 1, 2, 3, and 4

Let the relation $\approx$ be defined so that two persons are $\approx$-related if and only if their values for the attributes GENDER and AGE are the same. Then obviously $\approx$ is an equivalence which has the $\approx$-classes $\{1\}$, $\{3\}$, and $\{2, 4\}$.

Let us denote the subsets of U, which differ from $\emptyset$ and U, by sequences of their elements. For example, $\{1, 2, 3\}$ is written as 123. It is easy to see that $\mathcal{R}$ consists of twelve rough sets, which are $\{\emptyset\}$, $\{1\}$, $\{3\}$, $\{2, 4\}$, $\{13\}$, $\{12, 14\}$, $\{23, 34\}$, $\{24\}$, $\{123, 134\}$, $\{124\}$, $\{234\}$, and $\{U\}$. The ordered set $(\mathcal{R}, \leq)$ is presented in Figure 1.

For an arbitrary binary relation we can now give the following definition.

Definition 1.2 A binary relation $\approx$ on a nonempty set U is said to be

1. *reflexive*, if $x \approx x$ for all $x \in U$;
2. *symmetric*, if $x \approx y$ implies $y \approx x$ for all $x, y \in U$;
3. *transitive*, if $x \approx y$ and $y \approx z$ imply $x \approx z$ for all $x, y, z \in U$;
4. a *quasi-ordering*, if it is reflexive and transitive;
5. a *tolerance relation*, if it is reflexive and symmetric;
6. an *equivalence relation*, if it is reflexive, symmetric, and transitive.

It is commonly assumed that indiscernibility relations are equivalences. However, in the literature one can find studies in which rough approximation operators are determined by tolerances (see e.g. [6]). Note also that Kortelainen [7] has studied so-called compositional modifiers based on quasi-orderings, which are quite similar to operators (1.1) and (1.2).

Next we will argue that there exist indiscernibility relations which are not reflexive, symmetric, or transitive.

Reflexivity. It may seem reasonable to assume that every object is indiscernible from himself. But in some occasions this is not true, since it is possible that our information is so imprecise. For example, we may discern persons by comparing photographs taken of them. But it may happen that we are unable to recognize that a same person appears in two different photographs.

Symmetry. Usually it is supposed that indiscernibility relations are symmetric, which means that if we cannot discern x from y, then we cannot discern y from x either. But indiscernibility relations may be directional. For example, if a person x speaks English and Finnish, and a person y speaks English, Finnish and German, then x cannot discern y from himself by the property "knowledge of languages" since y can communicate with x in any languages that x speaks. On the other hand, y can discern x from himself by asking a simple question in German, for example.

Transitivity. Transitivity is the least obvious of the three properties usually associated with indiscernibility relations. For example, if we define an indiscernibility relation on a set of human beings in such a way that two person are indiscernible with respect to the property "age" if their time of birth differs by less than two hours. Then there may exist three persons x, y, and z, such that x is born an hour before y and y is born $1\frac{1}{2}$ hours before z. Hence, x is indiscernible from y and y is indiscernible from z, but x and z are not indiscernible.

This work is structured as follows. The following section is devoted to basic notations and general conventions concerning ordered sets. In Section 3 we introduce generalizations of lower and upper approximations in a lattice-theoretical setting and study their properties.

2 Preliminaries

We assume that the reader is familiar with the usual lattice-theoretical notation and conventions, which can be found in [1, 2], for example.

First we recall some definitions concerning properties of maps. Let $\mathcal{P} = (P, \leq)$ be an ordered set. A map $f: P \to P$ is said to be *extensive*, if $x \leq f(x)$ for all $x \in P$. The map f is *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$. Moreover, f is *idempotent* if $f(f(x)) = f(x)$ for all $x \in P$.

A map $c: P \to P$ is said to be a *closure operator* on $\mathcal{P}$, if c is extensive, order-preserving, and idempotent. An element $x \in P$ is *c-closed* if $c(x) = x$. Furthermore, if $i: P \to P$ is a closure operator on $P^{\partial} = (P, \geq)$ then i is an *interior operator* on $\mathcal{P}$.

Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be ordered sets. A map $f: P \to Q$ is an *order-embedding*, if for any $a, b \in P$, $a \leq b$ in $\mathcal{P}$ if and only if $f(a) \leq f(b)$ in $\mathcal{Q}$. Note that an order-embedding is always an injection. An order-embedding f onto Q is called an *order-isomorphism* between $\mathcal{P}$ and $\mathcal{Q}$. When there exists an order-isomorphism between $\mathcal{P}$ and $\mathcal{Q}$, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *order-isomorphic* and write $\mathcal{P} \cong \mathcal{Q}$. If $(P, \leq)$ and $(Q, \geq)$ are order-isomorphic, then $\mathcal{P}$ and $\mathcal{Q}$ are said to be *dually order-isomorphic*.

Next we define dual Galois connections. It is known [6] that the pair of maps which assigns to every set its upper and lower approximations forms a dual Galois connection when the corresponding indiscernibility relation is symmetric. In the next section we will show that an analogous result holds also in our generalized setting.

Definition 2.1 Let $\mathcal{P} = (P, \leq)$ be an ordered set. A pair $(^{\blacktriangleright}, ^{\blacktriangleleft})$ of maps $^{\blacktriangleright}: P \to P$ and $^{\blacktriangleleft}: P \to P$ (which we refer to as the *right map* and the *left map*, respectively) is called a *dual Galois connection* on $\mathcal{P}$ if $^{\blacktriangleright}$ and $^{\blacktriangleleft}$ are order-preserving and $p^{\blacktriangleleft\blacktriangleright} \leq p \leq p^{\blacktriangleright\blacktriangleleft}$ for all $p \in P$.

The following proposition presents some basic properties of dual Galois connections, which follow from the properties of Galois connections (see [2], for example).

Proposition 2.2 *Let* $(^{\blacktriangleright}, ^{\blacktriangleleft})$ *be a dual Galois connection on a complete lattice* $\mathcal{P}$.

(a) *For all* $p \in P$, $p^{\blacktriangleright\blacktriangleleft\blacktriangleright} = p^{\blacktriangleright}$ *and* $p^{\blacktriangleleft\blacktriangleright\blacktriangleleft} = p^{\blacktriangleleft}$.

(b) *The map* $c: P \to P$, $p \mapsto p^{\blacktriangleright\blacktriangleleft}$ *is a closure operator on* $\mathcal{P}$ *and the map* $k: P \to P$, $p \mapsto p^{\blacktriangleleft\blacktriangleright}$ *is an interior operator on* $\mathcal{P}$.

(c) *If c and k are the mappings defined in (b), then restricted to the sets of c-closed elements P_c and k-closed elements P_k, respectively, $^\blacktriangleright$ and $^\blacktriangleleft$ yield a pair $^\blacktriangleright\colon P_c \to P_k$, $^\blacktriangleleft\colon P_k \to P_c$ of mutually inverse order-isomorphisms between the complete lattices $(P_c, \leq)$ and $(P_k, \leq)$.*

Before we consider Boolean lattices, we present the following lemma. Note that condition (a) can be found in [2], for example. Here $\wp(P)$ denotes the *power set* of P, that is, the set of all subsets of P.

Lemma 2.3 *Let $(P, \leq)$ be a complete lattice, $S, T \subseteq P$, and $\{X_i \mid i \in I\} \subseteq \wp(P)$.*

(a) *If $S \subseteq T$, then $\bigvee S \leq \bigvee T$.*

(b) $\bigvee (\bigcup\{X_i \mid i \in I\}) = \bigvee\{\bigvee X_i \mid i \in I\}$.

Proof. (b) For all $i \in I$, $X_i \subseteq \bigcup\{X_i \mid i \in I\}$. This implies by (a) that $\bigvee X_i \leq \bigvee(\bigcup\{X_i \mid i \in I\})$ for all $i \in I$.

Let x be an upper bound of $\{\bigvee X_i \mid i \in I\}$. Then for all $i \in I$ and $y_i \in X_i$, $y_i \leq \bigvee X_i \leq x$. This means that $z \leq x$ for all $z \in \bigcup\{X_i \mid i \in I\}$. Thus, $\bigvee (\bigcup\{X_i \mid i \in I\}) \leq x$, which completes the proof. □

Next we recall some definitions concerning Boolean lattices. They are bounded distributive lattices with a complementation operation.

Definition 2.4 A lattice $\mathcal{L} = (L, \leq)$ is called a *Boolean lattice*, if

(a) L is distributive,

(b) L has a least element 0 and a greatest element 1, and

(c) each $a \in L$ has a *complement* $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

In the following lemma is some properties of Boolean lattices.

Lemma 2.5 *Let $\mathcal{B} = (B, \leq)$ be a Boolean lattice. Then for all $a, b \in B$,*

(a) $0' = 1$ *and* $1' = 0$,

(b) $a'' = a$,

(c) $(a \vee b)' = a' \wedge b'$ *and* $(a \wedge b)' = a' \vee b'$, *and*

(d) $a \wedge b' = 0$ *if and only if* $a \leq b$.

The following definition introduces the dual of a map. In Section 3 we show that the pair of maps which assigns the lower and upper approximations to each element are mutually dual.

Definition 2.6 Let $\mathcal{B} = (B, \leq)$ be a Boolean lattice. Two maps $f \colon B \to B$ and $g \colon B \to B$ are the *duals* of each other if

$$f(x') = g(x)' \text{ and } g(x') = f(x)'$$

for all $x \in B$.

The following obvious lemma shows that the dual of a closure operator is an interior operator.

Lemma 2.7 *Let $\mathcal{B} = (B, \leq)$ be a Boolean lattice and let $f \colon B \to B$ be a closure operator on $\mathcal{B}$. If $g \colon B \to B$ is the dual of f, then g is an interior operator on $\mathcal{B}$.*

By the following lemma, which can be found in [1], the infinite distributive laws hold in a complete Boolean lattice.

Lemma 2.8 *Let $\mathcal{B} = (B, \leq)$ be a complete Boolean lattice. Then for all $\{x_i \mid i \in I\} \subseteq B$ and $y \in B$,*

$$y \wedge (\textstyle\bigvee_{i \in I} x_i) = \textstyle\bigvee_{i \in I}(y \wedge x_i)$$

and

$$y \vee (\textstyle\bigwedge_{i \in I} x_i) = \textstyle\bigwedge_{i \in I}(y \vee x_i).$$

We end this section by introducing atomic Boolean lattices. Let $(P, \leq)$ be an ordered set and $x, y \in P$. We say that x *is covered by* y (or that y *covers* x), and write $x \prec y$, if $x < y$ and there is no element z in P with $x < z < y$.

Definition 2.9 Let $\mathcal{L} = (L, \leq)$ be a lattice with a least element 0. Then $a \in L$ is called an *atom* if $0 \prec a$. The set of atoms of $\mathcal{L}$ is denoted by $\mathcal{A}(\mathcal{L})$. The lattice $\mathcal{L}$ is *atomic* if every element x of L is the supremum of the atoms below it, that is, $x = \bigvee\{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\}$.

It is obvious that in a lattice $\mathcal{L} = (L, \leq)$ with a least element 0,

$$a \wedge x \neq 0 \iff a \leq x \tag{2.1}$$

for all $a \in \mathcal{A}(\mathcal{L})$ and $x \in L$. This implies that $a \wedge b = 0$ for all $a, b \in \mathcal{A}(\mathcal{L})$ such that $a \neq b$. Furthermore, if $\mathcal{L}$ is atomic, then for all $x \neq 0$ there exists an atom $a \in \mathcal{A}(\mathcal{L})$ such that $a \leq x$. Namely, if $\{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\} = \emptyset$, then $x = \bigvee\{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\} = \bigvee \emptyset = 0$.

3 Generalizations of approximations

In this section we study properties of approximations in a more general setting of complete atomic Boolean lattices. We begin with the following definition.

Definition 3.1 Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. We say that a map $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is

extensive, if $x \leq \varphi(x)$ for all $x \in \mathcal{A}(\mathcal{B})$;

symmetric, if $x \leq \varphi(y)$ implies $y \leq \varphi(x)$ for all $x, y \in \mathcal{A}(\mathcal{B})$;

closed, if $y \leq \varphi(x)$ implies $\varphi(y) \leq \varphi(x)$ for all $x, y \in \mathcal{A}(\mathcal{B})$.

Let $\approx$ be a binary relation on a set U. The ordered set $(\wp(U), \subseteq)$ is a complete atomic Boolean lattice. Since the atoms $\{x\}$ ($x \in U$) of $(\wp(U), \subseteq)$ can be identified with the elements of U, the map

$$\varphi\colon U \to \wp(U), x \mapsto [x]_{\approx} \tag{3.1}$$

may be considered to be of the form $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$, where $\mathcal{B} = (B, \leq)$ equals $(\wp(U), \subseteq)$. The following observations are obvious:

1. $\approx$ is reflexive $\iff$ φ is extensive;
2. $\approx$ is symmetric $\iff$ φ is symmetric;
3. $\approx$ is transitive $\iff$ φ is closed.

Next we introduce the generalizations of lower and upper approximations.

Definition 3.2 Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map. For any element $x \in B$, let

$$\begin{aligned} x^{\blacktriangledown} &= \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq x\}, \\ x^{\blacktriangle} &= \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}. \end{aligned}$$

The elements $x^{\blacktriangledown}$ and $x^{\blacktriangle}$ are the *lower* and the *upper approximation of* x with respect to φ, respectively. Two elements x and y are equivalent if they have the same upper and the same lower approximations. The resulting equivalence classes are called *rough sets*.

It is clear that if φ is a map defined in (3.1), then the functions of Definition 3.2 coincide with the operators (1.1) and (1.2). The next lemma is useful in our considerations.

Lemma 3.3 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map. Then for all $a \in \mathcal{A}(\mathcal{B})$ and $x \in B$,*

(a) $a \leq x^{\blacktriangledown} \iff \varphi(a) \leq x$;

(b) $a \leq x^{\blacktriangle} \iff \varphi(a) \wedge x \neq 0$.

Proof. (a) Suppose that $a \leq x^{\blacktriangledown} = \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid \varphi(b) \leq x\}$. If $\varphi(a) \nleq x$, then

$$\begin{aligned} a \wedge x^{\blacktriangledown} &= a \wedge \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid \varphi(b) \leq x\} \\ &= \bigvee\{a \wedge b \mid b \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(b) \leq x\} = 0. \end{aligned}$$

This implies $a \leq (x^{\blacktriangledown})'$, a contradiction! Thus, $\varphi(a) \leq x$. On the other hand, if $\varphi(a) \leq x$, then $a \leq \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid \varphi(b) \leq x\} = x^{\blacktriangledown}$.

Condition (b) can be proved similarly. □

The end of this work is devoted the study of the operators (3.1) and (3.2) in cases when the map φ is extensive, symmetric, or closed. However, we begin by assuming that φ is arbitrary and present some obvious properties of the maps ${}^{\blacktriangledown}\colon P \to P$ and ${}^{\blacktriangle}\colon P \to P$.

Lemma 3.4 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map.*

(a) $0^{\blacktriangle} = 0$ *and* $1^{\blacktriangledown} = 1$*;*

(b) $x \leq y$ *implies* $x^{\blacktriangledown} \leq y^{\blacktriangledown}$ *and* $x^{\blacktriangle} \leq y^{\blacktriangle}$.

Note that Lemma 3.4(b) means that the maps ${}^{\blacktriangledown}$ and ${}^{\blacktriangle}$ are order-preserving. For all $S \subseteq B$, we denote $S^{\blacktriangledown} = \{x^{\blacktriangledown} \mid x \in S\}$ and $S^{\blacktriangle} = \{x^{\blacktriangle} \mid x \in S\}$.

Recall that for a semilattice $\mathcal{P} = (P, \circ)$, an equivalence Θ on P is a *congruence* on $\mathcal{P}$ if $x_1 \Theta x_2$ and $y_1 \Theta y_2$ imply $x_1 \circ y_1 \Theta x_2 \circ x_2$, for all $x_1, x_2, y_1, y_2 \in P$.

Proposition 3.5 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map.*

(a) *The maps ${}^{\blacktriangle}\colon B \to B$ and ${}^{\blacktriangledown}\colon B \to B$ are mutually dual.*

(b) *For all $S \subseteq B$, $\bigvee S^{\blacktriangle} = (\bigvee S)^{\blacktriangle}$.*

(c) *For all $S \subseteq B$, $\bigwedge S^{\blacktriangledown} = (\bigwedge S)^{\blacktriangledown}$.*

(d) *$(B^{\blacktriangle}, \leq)$ is a complete lattice; 0 is the least element and $1^{\blacktriangle}$ is the greatest element of $(B^{\blacktriangle}, \leq)$.*

(e) *$(B^{\blacktriangledown}, \leq)$ is a complete lattice; $0^{\blacktriangledown}$ is the least element and 1 is the greatest element of $(B^{\blacktriangledown}, \leq)$*

(f) *The kernel $\Theta_{\blacktriangledown} = \{(x, y) \mid x^{\blacktriangledown} = y^{\blacktriangledown}\}$ of the map ${}^{\blacktriangledown}\colon B \to B$ is a congruence on the semilattice $(B, \wedge)$ such that the $\Theta_{\blacktriangledown}$-class of any x has a least element.*

(g) *The kernel $\Theta_{\blacktriangle} = \{(x, y) \mid x^{\blacktriangle} = y^{\blacktriangle}\}$ of the map ${}^{\blacktriangle}\colon B \to B$ is a congruence on the semilattice $(B, \vee)$ such that the $\Theta_{\blacktriangle}$-class of any x has a greatest element.*

Proof. (a) Let $a \in \mathcal{A}(\mathcal{B})$. Then

$$a \leq (x^{\blacktriangle})' \text{ iff } a \not\leq x^{\blacktriangle} \text{ iff } \varphi(a) \wedge x = 0 \text{ iff } \varphi(a) \leq x' \text{ iff } a \leq (x')^{\blacktriangledown}.$$

This implies that $(x^{\blacktriangle})' = (x')^{\blacktriangledown}$. The other part can be proved similarly.

(b) Let $S \subseteq B$. The map ${}^{\blacktriangle}\colon B \to B$ is order-preserving, which implies that $\bigvee S^{\blacktriangle} \leq (\bigvee S)^{\blacktriangle}$. Let $a \in \mathcal{A}(\mathcal{B})$ and assume that $\varphi(a) \wedge \bigvee S \neq 0$. Then $0 \neq \varphi(a) \wedge \bigvee S = \bigvee\{\varphi(a) \wedge x \mid x \in S\}$, which implies that $\varphi(a) \wedge x \neq 0$ for some $x \in S$. Thus,

$$\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge \bigvee S \neq 0\} \subseteq \bigcup_{x \in S} \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}$$

and by Lemma 2.3,

$$\begin{aligned}
(\bigvee S)^{\blacktriangle} &= \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge \bigvee S \neq 0\} \\
&\leq \bigvee\left(\bigcup_{x \in S}\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}\right) \\
&= \bigvee_{x \in S}\left\{\bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}\right\} \\
&= \bigvee\{x^{\blacktriangle} \mid x \in S\} = \bigvee S^{\blacktriangle}.
\end{aligned}$$

Hence, $\bigvee S^{\blacktriangle} = (\bigvee S)^{\blacktriangle}$. Claim (c) can be proved similarly. Assertions (d) and (e) follow easily from (b), (c), and Lemma 3.4(a).

(f) It can be easily seen that $\Theta_{\blacktriangledown}$ is an equivalence on P. Let $x_1, x_2, y_1, y_2 \in B$ and assume that $(x_1, y_1), (x_2, y_2) \in \Theta_{\blacktriangledown}$. Then

$$(x_1 \wedge x_2)^{\blacktriangledown} = x_1^{\blacktriangledown} \wedge x_2^{\blacktriangledown} = y_1^{\blacktriangledown} \wedge y_2^{\blacktriangledown} = (y_1 \wedge y_2)^{\blacktriangledown}.$$

Thus, $\Theta_{\blacktriangledown}$ is a congruence on $(B, \wedge)$.

It is clear that $\bigwedge[x]_{\Theta_{\blacktriangledown}}$ is the least element in the congruence class of x since

$$(\bigwedge\{y \mid y^{\blacktriangledown} = x^{\blacktriangledown}\})^{\blacktriangledown} = \bigwedge\{y^{\blacktriangledown} \mid y^{\blacktriangledown} = x^{\blacktriangledown}\} = x^{\blacktriangledown}$$

by (c). Assertion (g) can be proved similarly. □

Let us denote by $c_{\blacktriangle}(x)$ the greatest element in the $\Theta_{\blacktriangle}$-class of any $x \in B$. It is easy to see that the map $x \mapsto c_{\blacktriangle}(x)$ is a closure operator on $\mathcal{B}$. Similarly, if we denote by $c_{\blacktriangledown}(x)$ the least element of the $\Theta_{\blacktriangledown}$-class of x, the the map $x \mapsto c_{\blacktriangledown}(x)$ is an interior operator on $\mathcal{B}$ (see [5] for details).

Next we show that $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are dually order-isomorphic.

Lemma 3.6 $(B^{\blacktriangle}, \leq) \cong (B^{\blacktriangledown}, \geq)$.

Proof. We show that $x^\blacktriangle \mapsto (x')^\blacktriangledown$ is the required dual order-isomorphism. It is obvious that $x^\blacktriangle \mapsto (x')^\blacktriangledown$ is onto $(B^\blacktriangledown, \leq)$.

Suppose that $x^\blacktriangle \leq y^\blacktriangle$. Then for all $a \in \mathcal{A}(\mathcal{B})$, $\varphi(a) \wedge x \neq 0$ implies $\varphi(a) \wedge y \neq 0$. Suppose that $(y')^\blacktriangledown \not\leq (x')^\blacktriangledown$. This means that there exists an $a \in \mathcal{A}(\mathcal{B})$ such that $\varphi(a) \leq y'$ but $\varphi(a) \not\leq x'$. Since $\varphi(a) \not\leq x'$ is equivalent to $\varphi(a) \wedge x \neq 0$, we have that $\varphi(a) \wedge y \neq 0$. But this means that $\varphi(a) \not\leq y'$, a contradiction! Hence, $(y')^\blacktriangledown \leq (x')^\blacktriangledown$.

On the other hand, assume that $(y')^\blacktriangledown \leq (x')^\blacktriangledown$. Let us suppose that $x^\blacktriangle \not\leq y^\blacktriangle$. This means that there exists an $a \in \mathcal{A}(\mathcal{B})$ such that $\varphi(a) \wedge x \neq 0$ and $\varphi(a) \wedge y = 0$. But this implies that $\varphi(a) \leq y'$ and $a \leq (y')^\blacktriangledown$. Hence, $a \leq (x')^\blacktriangledown$ and $\varphi(a) \leq x'$. This is equivalent to $\varphi(a) \wedge x = 0$, a contradiction! Thus, $x^\blacktriangledown \leq y^\blacktriangledown$. □

Next we study the properties of approximations more closely in cases when the corresponding map $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive, symmetric, or closed.

Extensiveness

Here we study the functions ${}^\blacktriangle\colon B \to B$ and ${}^\blacktriangledown\colon B \to B$ defined by an extensive mapping φ. We show that $(B^\blacktriangledown, \leq)$ and $(B^\blacktriangle, \leq)$ are bounded by 0 and 1. Furthermore, each element of B is proved to be between its approximations.

Proposition 3.7 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be an extensive map. Then*

(a) $0^\blacktriangledown = 0$ *and* $1^\blacktriangle = 1$*;*

(b) $x^\blacktriangledown \leq x \leq x^\blacktriangle$ *for all* $x \in \mathcal{B}$.

Proof. (a) Let $a \in \mathcal{A}(\mathcal{B})$. Since $\varphi(a) \geq a$, $\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq 0\} = \emptyset$. Hence, $0^\blacktriangledown = 0$. Similarly, $\{a \in \mathcal{A}(\mathcal{B}) \mid a \wedge 1 \neq 0\} = \mathcal{A}(\mathcal{B})$ and thus $1^\blacktriangle = 1$.

(b) Let $a \in \mathcal{A}(\mathcal{B})$. Assume that $a \leq x$. If $\varphi(a) \leq x'$, then $a \leq \varphi(a) \leq x'$. Hence, $a \leq x \wedge x' = 0$, a contradiction! This means that $\varphi(a) \not\leq x'$ and $\varphi(a) \wedge x \neq 0$. Thus,

$$\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}$$

and

$$x = \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\} \leq \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\} = x^\blacktriangle.$$

On the hand, if $\varphi(a) \leq x$, then $a \leq \varphi(a) \leq x$, and hence

$$\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq x\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\}.$$

This implies that

$$x^\blacktriangledown = \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq x\} \leq \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\} = x.$$

□

Symmetry

In this subsection we assume that φ is symmetric. First we show that the pair $(^{\blacktriangle}, ^{\blacktriangledown})$ is a dual Galois connection.

Proposition 3.8 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a symmetric map. Then the pair $(^{\blacktriangle}, ^{\blacktriangledown})$ is a dual Galois connection on $\mathcal{B}$.*

Proof. As we have noted, the maps $^{\blacktriangledown}\colon B \to B$ and $^{\blacktriangle}\colon B \to B$ are order-preserving. Let $a \in \mathcal{A}(\mathcal{B})$. If $a \leq x^{\blacktriangledown\blacktriangle}$, then $\varphi(x) \wedge x^{\blacktriangledown} \neq 0$. This implies that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq \varphi(a) \wedge x^{\blacktriangledown}$. Thus $b \leq \varphi(a)$ and $b \leq x^{\blacktriangledown}$. Since φ is symmetric, also $a \leq \varphi(b)$ holds. Because $b \leq x^{\blacktriangledown}$ is equivalent to $\varphi(b) \leq x$, we get $a \leq x$ and hence

$$\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x^{\blacktriangledown\blacktriangle}\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\},$$

which implies $x^{\blacktriangledown\blacktriangle} \leq x$.

Let us denote $y = x'$. Then $y^{\blacktriangledown\blacktriangle} \leq y$ implies

$$x = y' \leq (y^{\blacktriangledown\blacktriangle})' = ((y^{\blacktriangledown})')^{\blacktriangledown} = (y')^{\blacktriangle\blacktriangledown} = x^{\blacktriangle\blacktriangledown}.$$

□

By the previous proposition, the pair $(^{\blacktriangle}, ^{\blacktriangledown})$ is a dual Galois connection whenever the map φ is symmetric. This means that these maps have the properties of Proposition 2.2. In particular, the map $x \mapsto x^{\blacktriangle\blacktriangledown}$ is a closure operator and the map $x \mapsto x^{\blacktriangledown\blacktriangle}$ is an interior operator. Let us denote

$$B_g = \{x^{\blacktriangle\blacktriangledown} \mid x \in B\} \text{ and } B_l = \{x^{\blacktriangledown\blacktriangle} \mid x \in B\}.$$

Proposition 3.9 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a symmetric map. Then*

(a) $a^{\blacktriangle} = \varphi(a)$ *for all* $a \in \mathcal{A}(\mathcal{B})$*;*

(b) $x^{\blacktriangle} = \bigvee\{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B})$ *and* $a \leq x\}$ *for every* $x \in B$*;*

(c) $B_g = B^{\blacktriangledown}$ *and* $B_l = B^{\blacktriangle}$*;*

(d) $(B^{\blacktriangle}, \leq) \cong (B^{\blacktriangledown}, \leq)$.

Proof. (a) Let $a \in \mathcal{A}(\mathcal{B})$. It obvious that

$$\{b \in \mathcal{A}(\mathcal{B}) \mid b \leq \varphi(a)\} = \{b \in \mathcal{A}(\mathcal{B}) \mid a \leq \varphi(b)\},$$

since φ is symmetric. Thus,

$$\begin{aligned}\varphi(a) &= \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid b \leq \varphi(a)\} \\ &= \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid a \leq \varphi(b)\} \\ &= \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid a \wedge \varphi(b) \neq 0\} \\ &= a^{\blacktriangle}.\end{aligned}$$

(b) Let $x \in B$. Then

$$x = \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\}.$$

and

$$\begin{aligned}x^{\blacktriangle} &= (\bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\})^{\blacktriangle} \\ &= \bigvee\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } a \leq x\} \\ &= \bigvee\{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } a \leq x\}.\end{aligned}$$

(c) Assume that $x \in B_g$. Then $x = y^{\blacktriangledown}$ for $y = x^{\blacktriangle}$, that is, $x \in B^{\blacktriangledown}$. On the other hand, if $x \in B^{\blacktriangledown}$, then $x = y^{\blacktriangledown}$ for some $y \in B$ and thus $x^{\blacktriangle\blacktriangledown} = y^{\blacktriangledown\blacktriangle\blacktriangledown} = y^{\blacktriangledown} = x$, that is, $x \in B_g$. The other part can be proved similarly.

Claim (d) is obvious, since $(B_g, \leq) \cong (B_l, \leq)$ by Proposition 2.2. □

We denoted by $c_{\blacktriangle}(x)$ the greatest element in the $\Theta_{\blacktriangle}$-class of x. Similarly, $c_{\blacktriangledown}(x)$ denotes the least element in the x's $\Theta_{\blacktriangledown}$-class. We can now write the following corollary.

Corollary 3.10 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a symmetric map. Then*

(a) $c_{\blacktriangle}(x) = x^{\blacktriangle\blacktriangledown}$ *for all* $x \in B$,

(b) $c_{\blacktriangledown}(x) = x^{\blacktriangledown\blacktriangle}$ *for all* $x \in B$,

(c) $(B^{\blacktriangledown}, \leq)$ *is dually isomorphic with itself, and*

(d) $(B^{\blacktriangle}, \leq)$ *is dually isomorphic with itself.*

Proof. (a) Since $x^{\blacktriangle} = x^{\blacktriangle\blacktriangledown\blacktriangle}$, we have $(x, x^{\blacktriangle\blacktriangledown}) \in \Theta_{\blacktriangle}$. Furthermore, $x \leq x^{\blacktriangle\blacktriangledown}$ holds by the definition of dual Galois connections. Equality (b) can be proved similarly.

By Lemma 3.6 and Proposition 3.9(d),

$$(B^{\blacktriangledown}, \leq) \cong (B^{\blacktriangle}, \geq) \cong (B^{\blacktriangledown}, \geq) = (B^{\blacktriangle}, \leq),$$

which proves (c) and (d). □

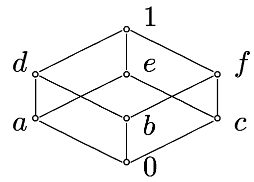

Figure 2: The ordered set $(B, \leq)$

Note that we have now showed that

$$\{x^{\blacktriangle\blacktriangledown} \mid x \in B\} = \{c_{\blacktriangle}(x) \mid x \in B\} = \{x^{\blacktriangledown} \mid x \in B\}$$

and

$$\{x^{\blacktriangledown\blacktriangle} \mid x \in B\} = \{c_{\blacktriangledown}(x) \mid x \in B\} = \{x^{\blacktriangle} \mid x \in B\}.$$

In the following example the map φ is extensive and closed, but not symmetric.

Example 3.11 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 2. The set of atoms of the complete atomic Boolean lattice $\mathcal{B} = (B, \leq)$ is $\{a, b, c\}$. Let the map $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be defined by

$$\varphi(a) = d, \qquad \varphi(b) = b, \qquad \varphi(c) = f.$$

Then obviously φ is extensive and closed. The map φ is not symmetric, since $b \leq \varphi(a)$, but $a \not\leq \varphi(b)$, for example.

x	$x^{\blacktriangledown}$	$x^{\blacktriangle}$
0	0	0
a	0	a
b	b	$a \vee b \vee c = 1$
c	0	c
d	$a \vee b = d$	$a \vee b \vee c = 1$
e	0	$a \vee c = e$
f	$b \vee c = f$	$a \vee b \vee c = 1$
1	$a \vee b \vee c = 1$	$a \vee b \vee c = 1$

Table 2: Lower and upper approximations

In Table 2 is presented the lower and upper approximations of the elements of B. The dually order-isomorphic ordered sets $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are shown in Figure 3. Note that $(B^{\blacktriangle}, \leq) \not\cong (B^{\blacktriangledown}, \leq)$. Furthermore, these ordered sets are not dually isomorphic with themselves.

Closedness

We end this work by studying the case in which φ is closed. First we present the following observation.

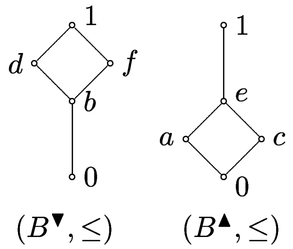

Figure 3: The ordered sets $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$

Lemma 3.12 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a closed map. Then for all $x \in B$,*

(a) $x^{\blacktriangle\blacktriangle} \leq x^{\blacktriangle}$*;*

(b) $x^{\blacktriangledown} \leq x^{\blacktriangledown\blacktriangledown}$.

Proof. (a) Let $a \in \mathcal{A}(\mathcal{B})$. If $a \leq x^{\blacktriangle\blacktriangle}$, then $\varphi(a) \wedge x^{\blacktriangle} \neq 0$. This means that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq \varphi(a)$ and $b \leq x^{\blacktriangle}$. Now $b \leq x^{\blacktriangle}$ is equivalent to $\varphi(b) \wedge x \neq 0$ and since φ is closed, $b \leq \varphi(a)$ implies $\varphi(b) \leq \varphi(a)$. Thus, also $\varphi(a) \wedge x \neq 0$ holds and so $a \leq x^{\blacktriangle}$. Hence,

$$\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x^{\blacktriangle\blacktriangle}\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x^{\blacktriangle}\}$$

and $x^{\blacktriangle\blacktriangle} \leq x^{\blacktriangle}$.

(b) Let us denote $y = x'$. Then $y^{\blacktriangle\blacktriangle} \leq y^{\blacktriangle}$ and

$$x^{\blacktriangledown} = (y')^{\blacktriangledown} = (y^{\blacktriangle})' \leq (y^{\blacktriangle\blacktriangle})' = ((y^{\blacktriangle})')^{\blacktriangledown} = (y')^{\blacktriangledown\blacktriangledown} = x^{\blacktriangledown\blacktriangledown}.$$

□

Now we can write the following proposition.

Proposition 3.13 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be an extensive and closed map.*

(a) *The map* ${}^{\blacktriangle}\colon B \to B$ *is a closure operator.*

(b) *The map* ${}^{\blacktriangledown}\colon B \to B$ *is an interior operator.*

(c) $(B^{\blacktriangledown}, \leq)$ *and* $(B^{\blacktriangle}, \leq)$ *are sublattices of* $(B, \leq)$.

Proof. (a) The map ${}^{\blacktriangle}\colon B \to B$ is extensive because φ is extensive, and it is order-preserving by Lemma 3.4(b). By Lemma 3.12 $x^{\blacktriangle\blacktriangle} \leq x^{\blacktriangle}$, and $x^{\blacktriangle} \leq x^{\blacktriangle\blacktriangle}$ holds since φ is extensive. Claim (b) follows from Lemma 2.7 and Proposition 3.5(a).

(c) Suppose that $x^{\blacktriangledown}, y^{\blacktriangledown} \in B^{\blacktriangledown}$. Then obviously,

$$x^{\blacktriangledown} \wedge y^{\blacktriangledown} = (x \wedge y)^{\blacktriangledown},$$

which implies that $x^{\blacktriangledown} \wedge y^{\blacktriangledown} \in B^{\blacktriangledown}$.

Next we show that

$$x^{\blacktriangledown} \vee y^{\blacktriangledown} = (x^{\blacktriangledown} \vee y^{\blacktriangledown})^{\blacktriangledown}$$

It is clear that

$$x^{\blacktriangledown} \leq x^{\blacktriangledown} \vee y^{\blacktriangledown} \text{ and } x^{\blacktriangledown} = x^{\blacktriangledown\blacktriangledown} \leq (x^{\blacktriangledown} \vee y^{\blacktriangledown})^{\blacktriangledown}.$$

Similarly, we can show that $y^{\blacktriangledown} \leq (x^{\blacktriangledown} \vee y^{\blacktriangledown})^{\blacktriangledown}$. Thus, $(x^{\blacktriangledown} \vee y^{\blacktriangledown})^{\blacktriangledown}$ is an upper bound of $x^{\blacktriangledown}$ and $y^{\blacktriangledown}$. Let $z \in B$ be an upper bound of $x^{\blacktriangledown}$ and $y^{\blacktriangledown}$. Then $x^{\blacktriangledown} \leq z$ and $y^{\blacktriangledown} \leq z$, which implies $x^{\blacktriangledown} \vee y^{\blacktriangledown} \leq z$. Because φ is extensive, $(x^{\blacktriangledown} \vee y^{\blacktriangledown})^{\blacktriangledown} \leq x^{\blacktriangledown} \vee y^{\blacktriangledown} \leq z$. Thus, $x^{\blacktriangledown} \vee y^{\blacktriangledown} = (x^{\blacktriangledown} \vee y^{\blacktriangledown})^{\blacktriangledown}$ and $x^{\blacktriangledown} \vee y^{\blacktriangledown} \in B^{\blacktriangledown}$. The other part can be proved analogously. □

It is known that every sublattice of a distributive lattice is distribute (see [2], for example). Therefore, we can write the following corollary.

Corollary 3.14 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. If $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive and closed map, then $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are distributive.*

Note that it is known that if $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive and symmetric, then $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are not necessarily distributive (see [5] for an example).

Lemma 3.15 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be extensive, symmetric, and closed. Then for all $x \in B$,*

$$x^{\blacktriangledown\blacktriangle} = x^{\blacktriangledown} \textit{ and } x^{\blacktriangle\blacktriangledown} = x^{\blacktriangle}.$$

Proof. Since φ is extensive, $x^{\blacktriangledown} \leq x^{\blacktriangledown\blacktriangle}$ for all $x \in P$. Let $a \in \mathcal{A}(\mathcal{B})$ and suppose that $a \leq x^{\blacktriangledown\blacktriangle}$, which means that $\varphi(a) \wedge x^{\blacktriangledown} \neq 0$. This implies that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq \varphi(a)$ and $b \leq x^{\blacktriangledown}$. Since φ is symmetric, $b \leq \varphi(a)$ implies $a \leq \varphi(b)$. The fact that φ is closed yields $\varphi(b) \leq \varphi(a)$ and $\varphi(a) \leq \varphi(b)$, that is, $\varphi(a) = \varphi(b)$. Because $b \leq x^{\blacktriangledown}$ we obtain $\varphi(a) = \varphi(b) \leq x$. Hence, $a \leq x^{\blacktriangledown}$ and so also $x^{\blacktriangledown\blacktriangle} \leq x^{\blacktriangledown}$ holds.

The other part is now obvious:

$$x^{\blacktriangle} = (x^{\blacktriangle})'' = ((x')^{\blacktriangledown})' = ((x')^{\blacktriangledown\blacktriangle})' = x^{\blacktriangle\blacktriangledown}.$$

□

By Proposition 3.13 and Lemma 3.15 it is clear that if $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is an extensive, symmetric, and closed map, then $B^{\blacktriangledown} = \{x^{\blacktriangledown} \mid x \in B\}$ equals $B^{\blacktriangle} = \{x^{\blacktriangle} \mid x \in B\}$. For simplicity, let us denote $\mathcal{E} = B^{\blacktriangledown} = B^{\blacktriangle}$.

Proposition 3.16 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be an extensive, reflexive, and closed map. The ordered set $(\mathcal{E}, \leq)$ is a complete atomic Boolean sublattice of $\mathcal{B}$.*

Proof. Let $x \in \mathcal{E}$. This means that $x = y^{\blacktriangledown}$ for some $y \in B$. Thus, $x^{\blacktriangle} = y^{\blacktriangledown\blacktriangle} = y^{\blacktriangledown} = x$ and $x^{\blacktriangledown} = y^{\blacktriangledown\blacktriangledown} = y^{\blacktriangledown} = x$.

Let $\{x_i \mid i \in I\} \subseteq \mathcal{E}$. Since $x_i^{\blacktriangle} = x_i$ for all $i \in I$, we obtain

$$\bigvee_B \{x_i \mid i \in I\} = \bigvee \{x_i^{\blacktriangle} \mid i \in I\} = (\bigvee_B \{x_i \mid i \in I\})^{\blacktriangle}.$$

Thus, $\bigvee_B \{x_i \mid i \in I\} \in \mathcal{E}$. Similarly, we can show that $\bigwedge_B \{x_i \mid i \in I\} \in \mathcal{E}$. This means that $(\mathcal{E}, \leq)$ is a complete sublattice of $\mathcal{B}$.

We have already shown that $(\mathcal{E}, \leq)$ is distributive. By Lemma 3.4(a) and Proposition 3.7(a), $0, 1 \in \mathcal{E}$. It is also clear that $x' = (x^{\blacktriangledown})' = (x')^{\blacktriangle} \in \mathcal{E}$ for all $x \in \mathcal{E}$. Thus we have shown that $(\mathcal{E}, \leq)$ is a complete Boolean lattice.

We have still to show that $(\mathcal{E}, \leq)$ is atomic. We show that $\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B})\}$ is the set of atoms of $(\mathcal{E}, \leq)$. Let $x \in \mathcal{E}$. This means that $x^{\blacktriangle} = x$. Let us denote $S_x = \{b \in \mathcal{A}(\mathcal{B}) \mid b \leq x\}$. Then $x = \bigvee S_x$ and

$$x = x^{\blacktriangle} = (\bigvee S_x)^{\blacktriangle} = (\bigvee \{b \mid b \in S_x\})^{\blacktriangle} = \bigvee \{b^{\blacktriangle} \mid b \in S_x\}.$$

Let $a \in \mathcal{A}(\mathcal{B})$ and suppose that $0 < x < a^{\blacktriangle}$ for some $x \in S$. This means that for all $b \in S_x$, $b \leq x \leq a^{\blacktriangle} = \varphi(a)$, which implies that $a \leq \varphi(b) = b^{\blacktriangle}$ for all $b \in S_x$. Hence, $a \leq \bigvee \{b^{\blacktriangle} \mid b \in S_x\} = x$ and $a^{\blacktriangle} \leq x^{\blacktriangle} = x$, a contradiction! Thus, $a^{\blacktriangle}$ is an atom of $(\mathcal{E}, \leq)$. It is also obvious that $\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B})\}$ is the set of atoms.

Because $a \leq x$ iff $a^{\blacktriangle} \leq x$ for all $a \in \mathcal{A}(\mathcal{B})$ and $x \in \mathcal{E}$, $x = x^{\blacktriangle} = \bigvee \{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } a \leq x\}$, which means that every element of $\mathcal{E}$ is the supremum of the atoms below it. Hence, $(\mathcal{E}, \leq)$ is also atomic. □

Remark 3.17 We have proved that if $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive, reflexive, and closed, then $(\mathcal{E}, \leq)$ is a complete atomic Boolean sublattice of $\mathcal{B}$. Furthermore, $\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B})\}$ is the set of atoms of $(\mathcal{E}, \leq)$.

Now we may present a result similar to Theorem 2 of Gehrke and Walker [3]. We state that the pointwise ordered set of all rough sets $\{(x^{\blacktriangledown}, x^{\blacktriangle}) \mid x \in B\}$ is order-isomorphic to $\mathbf{2}^I \times \mathbf{3}^J$, where $I = \{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) = a\}$ and $J = \{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) \neq a\}$.

We end this paper with the following corollary of Proposition 3.16.

Corollary 3.18 *Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. If $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is reflexive and closed, then $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are complete atomic Boolean lattices.*

Proof. Let $A^* = \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \neq 0\}$ and let $B^* = \{\bigvee X \mid X \subseteq A^*\}$. It is easy to see that $(B^*, \leq)$ is a complete atomic Boolean lattice and that A^* is the set of atoms of $(B^*, \leq)$.

Let us denote by φ^* the map φ restricted to A^*. First we show that $\varphi^*(a) \in B^*$ for all $a \in A^*$. Namely, if we assume that $0 \neq \varphi^*(a) \in B - B^*$ for some $a \in A^*$, then there exists a $b \in \mathcal{A}(\mathcal{B}) - A^*$ such that $b \leq \varphi^*(a) = \varphi(a)$. But since φ is symmetric, we obtain $0 \neq a \leq \varphi(b)$ and $b \in A^*$, a contradiction!

Obviously $\varphi^* : A^* \to B^*$ symmetric and closed. We show that φ^* is also extensive, that is, $a \leq \varphi^*(a)$ for $a \in A^*$. Let $a \in A^*$. Then $0 \neq \varphi^*(a) \in B^*$, which implies that there is an atom $b \in A^*$ such that $b \leq \varphi^*(a)$. But since φ^* is symmetric, also $a \leq \varphi^*(b)$ holds. Because φ^* is closed, $b \leq \varphi^*(a)$ implies $\varphi^*(b) \leq \varphi^*(a)$. Hence, $a \leq \varphi^*(b) \leq \varphi^*(a)$ and so φ^* is also extensive.

We can now apply Proposition 3.16, which says that the set of upper approximations $(B^{\vartriangle}, \leq)$ defined by the map $\varphi^* : A^* \to B^*$ is a complete atomic Boolean lattice. For all $x \in B$,

$$\begin{aligned} x^{\blacktriangle} &= \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\} \\ &= \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi^*(a) \wedge x \neq 0\} \\ &= \bigvee\{a \in A^* \mid \varphi^*(a) \wedge x \neq 0\} = x^{\vartriangle}. \end{aligned}$$

This means that $B^{\blacktriangle} = B^{\vartriangle}$. Because $(B^{\vartriangle}, \leq)$ is a complete atomic Boolean lattice, $(B^{\blacktriangle}, \leq)$ is a complete atomic Boolean lattice. Since $(B^{\blacktriangle}, \leq) \cong (B^{\blacktriangledown}, \leq)$, also $(B^{\blacktriangledown}, \leq)$ is a complete atomic Boolean lattice. □

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