On weighted possibilistic mean and variance of fuzzy numbers

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Abstract

Dubois and Prade defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. Carlsson and Fullér defined an interval-valued mean value of fuzzy numbers, viewing them as possibility distributions. In this paper we shall introduce the notation of weighted interval-valued possibilistic mean value of fuzzy numbers and investigate its relationship to the interval-valued probabilistic mean. We shall also introduce the notations of crisp weighted possibilistic mean value, variance and covariance of fuzzy numbers, which are consistent with the extension principle. Furthermore, we show that the weighted variance of linear combination of fuzzy numbers can be computed in a similar manner as in probability theory.

Keywords: Fuzzy number; Possibilistic mean value; Possibilistic variance

1 Weighted possibilistic mean values

In 1987 Dubois and Prade [2] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers.

In this paper introducing a weighting function measuring the importance of γ-level sets of fuzzy numbers we shall define the *weighted* lower possibilistic and upper possibilistic mean values, crisp possibilistic mean value and variance of fuzzy numbers, which are consistent with the extension principle and with the well-known defintions of expectation and variance in probability theory. We shall also show that the weighted interval-valued possibilistic mean is always a subset (moreover a proper subset excluding some special cases) of the interval-valued probabilistic mean for any weighting function.

The theory developed in this paper is fully motivated by the principles introduced in [2, 4, 6] and by the possibilistic interpretation of the ordering introduced in [5].

A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy number A is defined by $[A]^\gamma = \{t \in \mathbb{R} | A(t) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = \mathrm{cl}\{t \in \mathbb{R} | A(t) > 0\}$ (the closure of the support of A) if $\gamma = 0$. It is well-known that if A is a fuzzy number then $[A]^\gamma$ is a compact subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$.

Definition 1.1. *Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. A function $f\colon [0, 1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition*

$$\int_0^1 f(\gamma) d\gamma = 1. \tag{1}$$

We define the f-weighted possibilistic mean (or expected) value of fuzzy number A as

$$\bar{M}_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma. \tag{2}$$

It should be noted that if $f(\gamma) = 2\gamma$, $\gamma \in [0, 1]$ then

$$\bar{M}_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 [a_1(\gamma) + a_2(\gamma)] \gamma d\gamma = \bar{M}(A).$$

That is the f-weighted possibilistic mean value defined by (2) can be considered as a generalization of possibilistic mean value introduced in [1]. From the definition of a weighting function it can be seen that $f(\gamma)$ might be zero for certain (unimportant) γ-level sets of A. So by introducing different weighting functions we can give different (case-dependent) importances to γ-levels sets of fuzzy numbers.

Remark 1.1. *The f-weighted possibilistic mean value of fuzzy number A coincides with the* value *of A (with respect to reducing function f) introduced by Delgado, Vila and Woxman in ([4], page 127). In fact, their paper has inspired us to introduce the notation of f-weighted possibilistic mean.*

In the following we will consider two weighting functions f_1 and f_2 to be equal if the integral of their absolut difference $|f_1 - f_2|$ is zero, that is

$$\int_0^1 |f_1(\gamma) - f_2(\gamma)| d\gamma = 0.$$

Let us introduce a family of weighting function $\mathbf{1}\colon [0,1] \to \mathbb{R}$ defined as

$$\mathbf{1}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in (0,1] \\ a & \text{if } \gamma = 0 \end{cases}$$

where $a \in [0,1]$ is an arbitrary real number. It is clear that $\mathbf{1}$ is a weighting function and

$$\bar{M}_{\mathbf{1}}(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} \mathbf{1}(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma. \tag{3}$$

Remark 1.2. *We note here that the* $\mathbf{1}$*-weighted possibilistic mean value defined by (3) coincides with the generative expectation of fuzzy numbers introduced by Chanas and M. Nowakowski in ([3], page 47).*

Definition 1.2. *Let f be a weighting function and let A be a fuzzy number. Then we define the f-weighted interval-valued possibilistic mean of A as*

$$M_f(A) = [M_f^-(A), M_f^+(A)],$$

where

$$\begin{aligned} M_f^-(A) &= \int_0^1 a_1(\gamma) f(\gamma) d\gamma = \frac{\int_0^1 a_1(\gamma) f(\gamma) d\gamma}{\int_0^1 f(\gamma) d\gamma} \\ &= \frac{\int_0^1 a_1(\gamma) f(\mathrm{Pos}[A \le a_1(\gamma)]) d\gamma}{\int_0^1 f(\mathrm{Pos}[A \le a_1(\gamma)]) d\gamma} \\ &= \frac{\int_0^1 \min[A]^\gamma f(\mathrm{Pos}[A \le a_1(\gamma)]) d\gamma}{\int_0^1 f(\mathrm{Pos}[A \le a_1(\gamma)]) d\gamma}, \end{aligned}$$

and

$$\begin{aligned} M_f^+(A) &= \int_0^1 a_2(\gamma) f(\gamma) d\gamma = \frac{\int_0^1 a_2(\gamma) f(\gamma) d\gamma}{\int_0^1 f(\gamma) d\gamma} \\ &= \frac{\int_0^1 a_2(\gamma) f(\mathrm{Pos}[A \ge a_2(\gamma)]) d\gamma}{\int_0^1 f(\mathrm{Pos}[A \ge a_2(\gamma)]) d\gamma} \\ &= \frac{\int_0^1 \min[A]^\gamma f(\mathrm{Pos}[A \ge a_2(\gamma)]) d\gamma}{\int_0^1 f(\mathrm{Pos}[A \ge a_2(\gamma)]) d\gamma}. \end{aligned}$$

Here we used the relationships

$$\mathrm{Pos}[A \leq a_1(\gamma)] = \sup_{u \leq a_1(\gamma)} A(u) = \gamma,$$

and

$$\mathrm{Pos}[A \geq a_2(\gamma)] = \sup_{u \geq a_2(\gamma)} A(u) = \gamma.$$

So $M_f^-(A)$ is the f-weighted average of the minimum of the γ-cuts and it is why we call it the f-weighted lower possibilistic mean value of fuzzy number A. Similarly, $M_f^+(A)$ is the f-weighted average of the maximum of the γ-cuts and it is why we call it the f-weighted upper possibilistic mean value of fuzzy number A.

The following two theorems can directly be proved using the definition of f-weighted interval-valued possibilistic mean.

Theorem 1.1. *Let $A, B \in \mathcal{F}$ and let f be a weighting function, and let λ be a real number. Then*

$$M_f(A+B) = M_f(A) + M_f(B), \quad M_f(\lambda A) = \lambda M_f(A).$$

Remark 1.3. *The f-weighted possibilistic mean of A, defined by (2), is the arithmetic mean of its f-weighted lower and upper possibilistic mean values, i.e.*

$$\bar{M}_f(A) = \frac{M_f{}^-(A) + M_f^+(A)}{2}. \tag{4}$$

Theorem 1.2. *Let A and B be two fuzzy numbers, and let $\lambda \in \mathbb{R}$. Then we have*

$$\bar{M}_f(A+B) = \bar{M}_f(A) + \bar{M}_f(B), \quad \bar{M}_f(\lambda A) = \lambda \bar{M}_f(A).$$

2 Relation to upper and lower probability mean values

In this Section we will show an important relationship between the interval-valued expectation $E(A) = [E_*(A), E^*(A)]$ introduced by Dubois and Parde in [2] and the f-weighted interval-valued possibilistic mean $M_f(A) = [M_f^-(A), M_f^+(A)]$ for any fuzzy number with strictly decreasing shape functions.

An LR-type fuzzy number A can be described with the following membership function:

$$A(u) = \begin{cases} L\left(\dfrac{q_- - u}{\alpha}\right) & \text{if } q_- - \alpha \leq u \leq q_- \\ 1 & \text{if } u \in [q_-, q_+] \\ R\left(\dfrac{u - q_+}{\beta}\right) & \text{if } q_+ \leq u \leq q_+ + \beta \\ 0 & \text{otherwise} \end{cases}$$

where $[q_-, q_+]$ is the peak of fuzzy number A; q_- and q_+ are the lower and upper modal values; $L, R : [0,1] \to [0,1]$ with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ are non-increasing, continuous functions. We will use the notation $A = (q_-, q_+, \alpha, \beta)_{LR}$. Hence, the closure of the support of A is exactly $[q_- - \alpha, q_+ + \beta]$. If L and R are strictly decreasing functions then the γ-level sets of A can easily be computed as

$$[A]^\gamma = [q_- - \alpha L^{-1}(\gamma), q_+ + \beta R^{-1}(\gamma)], \gamma \in [0,1].$$

The lower and upper probability mean values of the fuzzy number A are computed by [2]

$$E_*(A) = q_- - \alpha \int_0^1 L(u)du, \quad E^*(A) = q_+ + \beta \int_0^1 R(u)du. \tag{5}$$

The f-weighted lower and upper possibilistic mean values are computed by

$$\begin{aligned} M_f^-(A) &= \int_0^1 \big(q_- - \alpha L^{-1}(\gamma)\big) f(\gamma) d\gamma \\ &= \int_0^1 q_- f(\gamma) d\gamma - \int_0^1 \alpha L^{-1}(\gamma) f(\gamma) d\gamma \\ &= q_- - \alpha \int_0^1 L^{-1}(\gamma) f(\gamma) d\gamma, \\ M_f^+(A) &= \int_0^1 \big(q_+ + \beta R^{-1}(\gamma)\big) f(\gamma) d\gamma \\ &= \int_0^1 q_+ f(\gamma) d\gamma + \int_0^1 \beta R^{-1}(\gamma) f(\gamma) d\gamma \\ &= q_+ + \beta \int_0^1 R^{-1}(\gamma) f(\gamma) d\gamma. \end{aligned} \tag{6}$$

We can state the following theorem.

Theorem 2.1. *Let f be a weighting function and let A be a fuzzy number of type LR with strictly decreasing and continuous shape functions. Then, the f-weighted interval-valued possibilistic mean value of A is a subset of the interval-valued probabilistic mean value, i.e.*

$$M_f(A) \subseteq E(A).$$

Furthermore, $M_f(A)$ is a proper subset of $E(A)$ whenever $f \neq \mathbf{1}$.

Proof. From (5) and (6) we can see that it is sufficient to prove that

$$\int_0^1 L^{-1}(\gamma) f(\gamma) d\gamma \leq \int_0^1 L(u)du, \quad \int_0^1 R^{-1}(\gamma) f(\gamma) d\gamma \leq \int_0^1 R(u)du$$

inequatilies hold. Since L and R are continuous, strictly decreasing functions with $L(0) = R(0) = 1$, $L(1) = R(1) = 0$ the following equations hold,

$$\int_0^1 L(u)du = \int_0^1 L^{-1}(\gamma)d\gamma, \quad \int_0^1 R(u)du = \int_0^1 R^{-1}(\gamma)d\gamma.$$

Hence, it is sufficient to prove that

$$\int_0^1 L^{-1}(\gamma)f(\gamma)d\gamma \leq \int_0^1 L^{-1}(\gamma)d\gamma, \tag{7}$$

and

$$\int_0^1 R^{-1}(\gamma)f(\gamma)d\gamma \leq \int_0^1 R^{-1}(\gamma)d\gamma. \tag{8}$$

We will prove only (7) becasue (8) can be proved in a similar manner. First we note that inequality (7) holds for the constant weighting function **1** since

$$\int_0^1 L^{-1}(\gamma)\mathbf{1}(\gamma)d\gamma = \int_0^1 L^{-1}(\gamma)\, d\gamma.$$

Let $n \in \mathbb{N}$ be an integer number and let $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, n$ be real numbers. Then we find,

$$\begin{aligned}
\sum_{1\leq i<j\leq n} \frac{x_i - x_j}{n} \cdot \frac{y_i - y_j}{n} &= \frac{1}{n^2} \sum_{1\leq i<j\leq n} (x_i - x_j)(y_i - y_j) \\
&= \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)(y_i - y_j) \\
&= \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_i + x_j y_j - x_i y_j - x_j y_i) \\
&= \frac{1}{n^2} \left(n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \right) \\
&= \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \left(\frac{1}{n} \sum_{j=1}^{n} y_j \right).
\end{aligned}$$

That is,

$$\sum_{1\leq i<j\leq n} \frac{x_i - x_j}{n} \cdot \frac{y_i - y_j}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} y_i \right), \tag{9}$$

Let us now consider an equidistant partition of interval $[0, 1]$

$$\{0 = z_0 < z_1 < z_2 < \cdots < z_n = 1\} \tag{10}$$

where $z_i - z_{i-1} = 1/n$ for $i = 1, 2, \ldots, n$, and let us introduce the following notations

$$x_i = f(\xi_i), \quad y_i = L^{-1}(\xi_i),$$

where $\xi_i \in (z_{i-1}, z_i)$, for $i = 1, 2, \ldots, n$. Since f is non-decreasing and L^{-1} is strictly decreasing on $[0,1]$ we get $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 > y_2 > \cdots > y_n$. Furthermore, from $x_i - x_j \leq 0$ and $y_i - y_j > 0$, $i < j$ and (9) it follows that

$$\sum_{1 \leq i < j \leq n} \frac{x_i - x_j}{n} \cdot \frac{y_i - y_j}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} y_i \right) \leq 0.$$

Since $x_i = f(\xi_i)$, $y_i = L^{-1}(\xi_i)$, $i = 1, 2, \ldots, n$, in the limit case we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i y_i = \int_0^1 f(\gamma) L^{-1}(\gamma) d\gamma,$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = \int_0^1 f(\gamma) d\gamma,$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} y_i = \int_0^1 L^{-1}(\gamma) d\gamma.$$

Therefore, we get

$$\int_0^1 f(\gamma) L^{-1}(\gamma) d\gamma - \int_0^1 f(\gamma) d\gamma \cdot \int_0^1 L^{-1}(\gamma) d\gamma \leq 0.$$

Finally, from the normality condition of the integral of the weighted function f (1) we get (7).

Now, we will show that $M_f(A)$ is a proper subset of $E(A)$ whenever $f \neq \mathbf{1}$. If $f \neq \mathbf{1}$ then there exist $\alpha, \beta \in (0,1)$ such that $\alpha < \beta$ and $f(\alpha) < f(\beta)$. Using the notation $\varepsilon = \min\{f(\beta) - f(\alpha), L^{-1}(\alpha) - L^{-1}(\beta)\}$ we get that $\varepsilon > 0$ (since $f(\beta) - f(\alpha) > 0$ and $L^{-1}(\alpha) - L^{-1}(\beta) > 0$).

We will now show that the inequalities

$$f(\gamma_1) - f(\gamma_2) \leq -\varepsilon, \quad L^{-1}(\gamma_1) - L^{-1}(\gamma_2) \geq \varepsilon$$

hold for any $\gamma_1 \in [0, \alpha]$ and $\gamma_2 \in [\beta, 1]$. Indeed, from the monotonicity conditions we get

$$f(\gamma_1) - f(\gamma_2) \leq f(\alpha) - f(\beta) \leq -\varepsilon,$$

and

$$L^{-1}(\gamma_1) - L^{-1}(\gamma_2) \geq L^{-1}(\alpha) - L^{-1}(\beta) \geq \varepsilon.$$

Multiplying these inequalities we get

$$(f(\gamma_1) - f(\gamma_2))\big(L^{-1}(\gamma_1) - L^{-1}(\gamma_2)\big) \leq -\varepsilon^2$$

for all $\gamma_1 \in [0, \alpha]$ and $\gamma_2 \in [\beta, 1]$. Consider again the equidistant partition (10) and let us introduce the notations $x_i = f(\xi_i)$ and $y_i = L^{-1}(\xi_i)$, where $\xi_i \in (z_{i-1}, z_i)$, for $i = 1, 2, \ldots, n$. Denote the number of subintervals properly belonging to $[0, \alpha]$ and $[\beta, 1]$ by k and l, respectively. It can easily be seen that $k > n\alpha - 1$ and $l > n(1 - \beta) - 1$. Now, let

$$n \geq \max\{1/\alpha, 1/(1 - \beta)\}.$$

From the inequality $(x_i - x_j)(y_i - y_j) \leq 0$, $(i < j)$ we find

$$\begin{aligned}
\sum_{1 \leq i < j \leq n} \frac{x_i - x_j}{n} \cdot \frac{y_i - y_j}{n} &\leq \frac{1}{n^2} \sum_{i=1}^{k} \sum_{j=n-l+1}^{n} (x_i - x_j)(y_i - y_j) \\
&\leq \frac{1}{n^2} \sum_{i=1}^{k} \sum_{j=n-l+1}^{n} (-\varepsilon^2) \\
&= -\frac{\varepsilon^2}{n^2} \cdot k \cdot l \\
&< -\frac{\varepsilon^2}{n^2} (n\alpha - 1)\big(n(1 - \beta) - 1\big) \\
&= -\varepsilon^2 \left(\alpha - \frac{1}{n}\right) \left((1 - \beta) - \frac{1}{n}\right).
\end{aligned}$$

From $\alpha, \beta \in (0, 1)$ and $\varepsilon > 0$ and (9) it follows that

$$\begin{aligned}
\int_0^1 f(\gamma) L^{-1}(\gamma) d\gamma - \int_0^1 f(\gamma) d\gamma \cdot \int_0^1 L^{-1}(\gamma) d\gamma &= \int_0^1 f(\gamma) L^{-1}(\gamma) d\gamma \\
&\quad - \int_0^1 L^{-1}(\gamma) d\gamma \leq -\alpha(1 - \beta)\varepsilon^2 < 0.
\end{aligned}$$

Which proves the theorem. □

Example 2.1. *Let $f(\gamma) = (n + 1)\gamma^n$ and let $A = (a, \alpha, \beta)$ be a triangular fuzzy number with center a, left-width $\alpha > 0$ and right-width $\beta > 0$ then a γ-level of A is computed by*

$$[A]^\gamma = [a - (1 - \gamma)\alpha, a + (1 - \gamma)\beta], \ \forall \gamma \in [0, 1].$$

Then the power-weighted lower and upper possibilistic mean values of A are computed by

$$\begin{aligned}
M_f^-(A) &= \int_0^1 [a - (1 - \gamma)\alpha](n + 1)\gamma^n d\gamma \\
&= a(n + 1) \int_0^1 \gamma^n d\gamma - \alpha(n + 1) \int_0^1 (1 - \gamma)\gamma^n d\gamma \\
&= a - \frac{\alpha}{n + 2},
\end{aligned}$$

and,

$$M_f^+(A) = \int_0^1 [a + (1-\gamma)\beta](n+1)\gamma^n d\gamma$$
$$= a(n+1)\int_0^1 \gamma^n d\gamma + \beta(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma$$
$$= a + \frac{\beta}{n+2},$$

and therefore,

$$M_f(A) = \left[a - \frac{\alpha}{n+2}, a + \frac{\beta}{n+2}\right].$$

That is,

$$\bar{M}_f(A) = \frac{1}{2}\left(a - \frac{\alpha}{n+2} + a + \frac{\beta}{n+2}\right) = a + \frac{\beta - \alpha}{2(n+2)}.$$

So,

$$\lim_{n\to\infty} \bar{M}_f(A) = \lim_{n\to\infty}\left(a + \frac{\beta-\alpha}{2(n+2)}\right) = a.$$

Example 2.2. *Let $A = (a, b, \alpha, \beta)$ be a fuzzy number of trapezoidal form with peak $[a, b]$, left-width $\alpha > 0$ and right-width $\beta > 0$, and let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$. A γ-level of A is computed by*

$$[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta],\ \forall\gamma \in [0, 1],$$

then the power-weighted lower and upper possibilistic mean values of A are computed by

$$M_f^-(A) = \int_0^1 [a - (1-\gamma)\alpha](n+1)\gamma^n d\gamma$$
$$= a(n+1)\int_0^1 \gamma^n d\gamma - \alpha(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma$$
$$= a - \frac{\alpha}{n+2},$$

and,

$$M_f^+(A) = \int_0^1 [b + (1-\gamma)\beta](n+1)\gamma^n d\gamma$$
$$= b(n+1)\int_0^1 \gamma^n d\gamma + \beta(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma$$
$$= b + \frac{\beta}{n+2},$$

and therefore,

$$M_f(A) = \left[a - \frac{\alpha}{n+2}, b + \frac{\beta}{n+2}\right]$$

That is,

$$\bar{M}_f(A) = \frac{1}{2}\left(a - \frac{\alpha}{n+2} + b + \frac{\beta}{n+2}\right) = \frac{a+b}{2} + \frac{\beta - \alpha}{2(n+2)}.$$

So,

$$\lim_{n\to\infty} \bar{M}_f(A) = \lim_{n\to\infty}\left(\frac{a+b}{2} + \frac{\beta-\alpha}{2(n+2)}\right) = \frac{a+b}{2}.$$

Example 2.3. *Let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$ and let $A = (a, \alpha, \beta)$ be a triangular fuzzy number with center a, left-width $\alpha > 0$ and right-width $\beta > 0$ then*

$$M_f(A) = \left[a - \frac{\alpha}{n+2}, a + \frac{\beta}{n+2}\right] \subset E(A) = \left[a - \frac{\alpha}{2}, a + \frac{\beta}{2}\right]$$

and for $n > 0$ we have

$$\bar{M}_f(A) = a + \frac{\beta - \alpha}{2(n+2)} \neq \bar{E}(A) = a + \frac{\beta - \alpha}{4}.$$

Example 2.4. *Let $A = (a, b, \alpha, \beta)$ be a fuzzy number of trapezoidal form and let*

$$f(\gamma) = (n-1)\left(\frac{1}{\sqrt[n]{1-\gamma}} - 1\right),$$

where $n \geq 2$. It is clear that f is a weighting function with $f(0) = 0$ and

$$\lim_{\gamma\to 1-0} f(\gamma) = \infty.$$

Then the f-weighted lower and upper possibilistic mean values of A are computed by

$$\begin{aligned}
M_f^-(A) &= \int_0^1 [a - (1-\gamma)\alpha](n-1)\left[\frac{1}{\sqrt[n]{1-\gamma}} - 1\right] d\gamma \\
&= a - \alpha(n-1)\int_0^1 \left[(1-\gamma)^{1-1/n} - (1-\gamma)\right] d\gamma \\
&= a - \alpha(n-1)\left(\frac{1}{2-1/n} - \frac{1}{2}\right) \\
&= a - \frac{(n-1)\alpha}{2(2n-1)},
\end{aligned}$$

and

$$
\begin{aligned}
M_f^+(A) &= \int_0^1 [b+(1-\gamma)\beta](n-1)\left[\frac{1}{\sqrt[n]{1-\gamma}}-1\right]d\gamma \\
&= b+\beta(n-1)\int_0^1 \Big[(1-\gamma)^{1-1/n}-(1-\gamma)\Big]\,d\gamma \\
&= b+\beta(n-1)\left(\frac{1}{2-1/n}-\frac{1}{2}\right) \\
&= b+\frac{(n-1)\beta}{2(2n-1)},
\end{aligned}
$$

and therefore

$$
M_f(A) = \left[a-\frac{(n-1)\alpha}{2(2n-1)}, b+\frac{(n-1)\beta}{2(2n-1)}\right].
$$

That is,

$$
\bar{M}_f(A) = \frac{1}{2}\left(a-\frac{(n-1)\alpha}{2(2n-1)}+b+\frac{(n-1)\beta}{2(2n-1)}\right) = \frac{a+b}{2}+\frac{(n-1)(\beta-\alpha)}{4(2n-1)}.
$$

So,

$$
\lim_{n\to\infty}\bar{M}_f(A) = \lim_{n\to\infty}\left(\frac{a+b}{2}+\frac{(n-1)(\beta-\alpha)}{4(2n-1)}\right) = \frac{a+b}{2}+\frac{\beta-\alpha}{8}.
$$

Remark 2.1. *When A is a symmetric fuzzy number then the equation $\bar{M}_f(A) = \bar{E}(A)$ holds for any weighting function f. In the limit case, when $A = (a, b, 0, 0)$ is the characteristic function of interval $[a, b]$, the f-weighted possibilistic and probabilistic interval-valued means are equal, $E(A) = M_f(A) = [a, b]$.*

3 Weighted possibilistic variance

Definition 3.1. *Let A and B be fuzzy numbers and let f be a weighting function. We define the f-weighted possibilistic variance of A by*

$$
\mathrm{Var}_f(A) = \int_0^1 \left(\frac{a_2(\gamma)-a_1(\gamma)}{2}\right)^2 f(\gamma)d\gamma, \tag{11}
$$

and the f-weighted covariance of A and B is defined as

$$
\mathrm{Cov}_f(A,B) = \int_0^1 \frac{a_2(\gamma)-a_1(\gamma)}{2}\cdot\frac{b_2(\gamma)-b_1(\gamma)}{2} f(\gamma)d\gamma. \tag{12}
$$

It should be noted that if $f(\gamma) = 2\gamma$, $\gamma \in [0,1]$ then

$$\begin{aligned}\mathrm{Var}_f(A) &= \int_0^1 \left(\frac{a_2(\gamma) - a_1(\gamma)}{2}\right)^2 2\gamma d\gamma \\ &= \frac{1}{2}\int_0^1 [a_2(\gamma) - a_1(\gamma)]^2 \gamma d\gamma = \mathrm{Var}(A),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}_f(A,B) &= \int_0^1 \frac{a_2(\gamma) - a_1(\gamma)}{2} \cdot \frac{b_2(\gamma) - b_1(\gamma)}{2} f(\gamma) d\gamma \\ &= \frac{1}{2}\int_0^1 (a_2(\gamma) - a_1(\gamma)) \cdot (b_2(\gamma) - b_1(\gamma))\, 2\gamma d\gamma = \mathrm{Cov}(A,B).\end{aligned}$$

That is the f-weighted possibilistic variance and covariance defined by (11) and (12) can be considered as a generalization of possibilistic variance and covariance introduced in [1]. It can easily be verified that the weighted covariance is a symmetrical bilinear operator.

Example 3.1. *Let $A = (a, b, \alpha, \beta)$ be a trapezoidal fuzzy number and let $f(\gamma) = (n+1)\gamma^n$ be a weighting function. Then,*

$$\begin{aligned}\mathrm{Var}_f(A) &= (n+1)\int_0^1 \left[\frac{a_2(\gamma) - a_1(\gamma)}{2}\right]^2 \gamma^n d\gamma \\ &= \frac{n+1}{4}\int_0^1 [(b-a) + (\alpha+\beta)(1-\gamma)]^2 \gamma^n d\gamma \\ &= \frac{n+1}{4}\Big[(b-a)^2 \int_0^1 \gamma^n d\gamma + 2(b-a)(\alpha+\beta)\int_0^1 (1-\gamma)\gamma^n d\gamma \\ &\quad + (\alpha+\beta)^2 \int_0^1 (1-\gamma)^2\gamma^n d\gamma\Big] \\ &= \frac{n+1}{4}\left[\frac{(b-a)^2}{n+1} + \frac{2(b-a)(\alpha+\beta)}{(n+1)(n+2)} + \frac{2(\alpha+\beta)^2}{(n+1)(n+2)(n+3)}\right] \\ &= \frac{(b-a)^2}{4} + \frac{(b-a)(\alpha+\beta)}{2(n+2)} + \frac{(\alpha+\beta)^2}{2(n+2)(n+3)} \\ &= \left[\frac{b-a}{2} + \frac{\alpha+\beta}{2(n+2)}\right]^2 + \frac{(n+1)(\alpha+\beta)^2}{4(n+2)^2(n+3)}.\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Var}_f(A) = \lim_{n\to\infty}\Bigg(&\left[\frac{b-a}{2} + \frac{\alpha+\beta}{2(n+2)}\right]^2 \\ &+ \frac{(n+1)(\alpha+\beta)^2}{4(n+2)^2(n+3)}\Bigg) = \frac{b-a}{2}.\end{aligned}$$

The following theorem can be proved in a similar way as Theorem 4.1 from [1].

Theorem 3.1. *Let f and g be two weighting functions, let A, B and C be fuzzy numbers and let x and y be real numbers. Then the following properties hold,*

$$\begin{aligned}
\mathrm{Cov}_f(A,A) &= \mathrm{Var}_f(A), \mathrm{Cov}_f(A,B) = \mathrm{Cov}_f(B,A), \\
\mathrm{Var}_f(\bar{x}) &= 0, \mathrm{Cov}_f(\bar{x},A) = 0, \\
\mathrm{Cov}_f(xA+yB,C) &= |x|\mathrm{Cov}_f(A,C) + |y|\mathrm{Cov}_f(B,C), \\
\mathrm{Var}_f(xA+yB) &= x^2\mathrm{Var}_f(A) + y^2\mathrm{Var}_f(B) + 2|x||y|\mathrm{Cov}_f(A,B),
\end{aligned}$$

where $\bar{x}$ and $\bar{y}$ denote the characteristic functions of the sets $\{x\}$ and $\{y\}$, respectively.

Example 3.2. *Let $A = (a,b,\alpha,\beta)$ and $B = (a',b',\alpha',\beta')$ be fuzzy numbers of trapezoidal form. Let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$, be a weighting function then the power-weighted covariance between A and B is computed by*

$$\begin{aligned}
&\mathrm{Cov}_f(A,B) \\
&= \int_0^1 \frac{(b-a)+(1-\gamma)(\alpha+\beta)}{2} \cdot \frac{(b'-a')+(1-\gamma)(\alpha'+\beta')}{2}(n+1)\gamma^n d\gamma \\
&= \frac{n+1}{4}\Bigg[(b-a)(b'-a')\int_0^1 \gamma^n d\gamma \\
&\quad + \big[(b-a)(\alpha'+\beta')+(b'-a')(\alpha+\beta)\big]\int_0^1 (1-\gamma)\gamma^n d\gamma \\
&\quad + (\alpha+\beta)(\alpha'+\beta')\int_0^1 (1-\gamma)^2\gamma^n d\gamma\Bigg] \\
&= \frac{n+1}{4}\Bigg[\frac{(b-a)(b'-a')}{n+1} + \frac{(b-a)(\alpha'+\beta')+(b'-a')(\alpha+\beta)}{(n+1)(n+2)} \\
&\quad + \frac{2(\alpha+\beta)(\alpha'+\beta')}{(n+1)(n+2)(n+3)}\Bigg] \\
&= \left[\frac{b-a}{2}+\frac{\alpha+\beta}{2(n+2)}\right]\left[\frac{b'-a'}{2}+\frac{\alpha'+\beta'}{2(n+2)}\right] + \frac{(n+1)(\alpha+\beta)(\alpha'+\beta')}{4(n+2)^2(n+3)}.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{n\to\infty}\mathrm{Cov}_f(A,B) = \lim_{n\to\infty}\Bigg(&\left[\frac{b-a}{2}+\frac{\alpha+\beta}{2(n+2)}\right]\left[\frac{b'-a'}{2}+\frac{\alpha'+\beta'}{2(n+2)}\right] \\
&+\frac{(n+1)(\alpha+\beta)(\alpha'+\beta')}{4(n+2)^2(n+3)}\Bigg) = \frac{b-a}{2}\cdot\frac{b'-a'}{2}.
\end{aligned}$$

If $a = b$ and $a' = b'$, i.e. we have two triangular fuzzy numbers, then their covariance becomes

$$\mathrm{Cov}_f(A, B) = \frac{(\alpha + \beta)(\alpha' + \beta')}{2(n+2)(n+3)}.$$

4 Summary

In this paper we have introduced the notations of the weighted lower possibilistic and upper possibilistic mean values, crisp possibilistic mean value and variance of fuzzy numbers, which are consistent with the extension principle and with the well-known defintions of expectation and variance in probability theory. We have also showed that the weighted interval-valued possibilistic mean is always a subset (moreover a proper subset excluding some special cases) of the interval-valued probabilistic mean for any weighting function.

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