On possibilistic Cauchy-Schwarz inequality

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Abstract

The goal of this paper is to prove the possibilistic analog of the probabilistic Cauchy-Schwarz inequality.

Keywords: Possibility distributions; Possibilistic variance, Possibilistic covariance; Cauchy-Schwarz inequality

1 Introduction

If X and Y are random variables with finite variances σ_X and σ_Y then the probabilistic Cauchy-Schwarz inequality can be stated as

$$[\mathrm{Cov}(X,Y)]^2 \leq \sigma_X \sigma_Y, \tag{1}$$

where $\mathrm{Cov}(X,Y)$ denotes the covariance between X and Y. Furthermore, the correlation coefficient between X and Y is defined by

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\sigma_X \sigma_Y}},$$

and it is clear that $-1 \leq \rho(X,Y) \leq 1$.

A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy set A in $\mathbb{R}^n$ is defined by $[A]^\gamma = \{t \in \mathbb{R}^n | A(t) \geq \gamma\}$ if $\gamma \geq 0$ and $[A]^\gamma = \mathrm{cl}\{\mathrm{t} \in \mathrm{R}^n | \mathrm{A}(\mathrm{t}) > 0\}$ (the closure of the support of A) if $\gamma = 0$.

In the following we shall recall the definition and some basic properties of joint possibility distributions [4].

Definition 1.1. *Let $A_1, \dots, A_m \in \mathcal{F}$ be fuzzy numbers. A fuzzy set B in $\mathbb{R}^n$ is called a joint possibility distribution of A_i, $i = 1, \dots, n$ if it satisfies the relationship*

$$\max_{x_j \in \mathbb{R}, j \neq i} B(x_1, \dots, x_n) = A_i(x_i),$$

for all $x_i \in \mathbb{R}$ and $i = 1, \dots, n$. A_i is called the i-th marginal possibility distribution of B.

Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'.

The biggest (in the sense of subsethood of fuzzy sets) joint possibility distribution defines the concept of independence of fuzzy numbers.

Definition 1.2. *Fuzzy numbers A_i, $i = 1, \dots, n$ are said to be independent if their joint possibility distribution, B, is given as,*

$$B(x_1, \dots, x_n) = \min\{A_1(x_1), \dots, A_n(x_n)\}$$

or, equivalently, $[B]^\gamma = [A_1]^\gamma \times \cdots \times [A_n]^\gamma$, for all $x_1, \dots, x_n \in \mathbb{R}$ and $\gamma \in [0,1]$.

Now we shall recall definition of central value and measure of possibilistic dependency as has been introduced in [2].

Definition 1.3. *[2] Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$. The central value of $[A]^\gamma$ is defined by*

$$\mathcal{C}([A]^\gamma) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx.$$

It is easy to see that the central value of $[A]^\gamma$ is computed as

$$\mathcal{C}([A]^\gamma) = \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x dx = \frac{a_1(\gamma) + a_2(\gamma)}{2}.$$

Definition 1.4. *Let $A_1, \dots, A_n \in \mathcal{F}$ be fuzzy numbers, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then, $g(A_1, \dots, A_n)$ is defined by the sup–min extension principle [3] as follows*

$$g(A_1, \dots, A_n)(y) = \sup_{g(x_1, \dots, x_n) = y} \min\{A_1(x_1), \dots, A_n(x_n)\}.$$

Definition 1.5. *[2] Let $A_1, \dots, A_n \in \mathcal{F}$ be fuzzy numbers, let B be their joint possibility distribution and let $\gamma \in [0,1]$. The central value of the γ-level set of $g(A_1, \dots, A_n)$ with respect to their joint possibility distribution B is defined by*

$$\mathcal{C}_B\left([g(A_1, \dots, A_n)]^\gamma\right) = \frac{1}{\int_{[B]^\gamma} dx} \int_{[B]^\gamma} g(x) dx,$$

where $g(x) = g(x_1, \dots, x_n)$.

Definition 1.6. *[2] Let $A, B \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $\gamma \in [0,1]$. The dependency relation between the γ-level sets of A and B is defined by*

$$\mathrm{Rel}_C([A]^\gamma, [B]^\gamma) = \mathcal{C}_C\big(\big[(A - \mathcal{C}_C([A]^\gamma))(B - \mathcal{C}_C([B]^\gamma))\big]^\gamma\big),$$

Using the definition of central value we have

$$\begin{aligned}
\mathrm{Rel}_C([A]^\gamma, [B]^\gamma) &= \frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} (x - \mathcal{C}_C([A]^\gamma) \cdot (y - \mathcal{C}_C([B]^\gamma))dxdy \\
&= \frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} xydxdy - \mathcal{C}_C([B]^\gamma) \cdot \frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} xdxdy \\
&\quad - \mathcal{C}_C([A]^\gamma) \cdot \frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} ydxdy + \mathcal{C}_C([A]^\gamma) \cdot \mathcal{C}_C([B]^\gamma) \\
&= \mathcal{C}_C([AB]^\gamma) - \mathcal{C}_C([A]^\gamma) \cdot \mathcal{C}_C([B]^\gamma),
\end{aligned}$$

for all $\gamma \in [0,1]$. We recall here that

$$\mathcal{C}_C([A]^\gamma) = \frac{1}{\int_{[C]^\gamma} dx} \int_{[C]^\gamma} x dx,$$

and

$$\mathcal{C}_C([B]^\gamma) = \frac{1}{\int_{[C]^\gamma} dy} \int_{[C]^\gamma} y dy.$$

Definition 1.7. *[1] A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition*

$$\int_0^1 f(\gamma) d\gamma = 1.$$

The covariance of A and B with respect to a weighting function f is defined as [2]

$$\begin{aligned}\mathrm{Cov}_f(A,B) &= \int_0^1 \mathrm{Rel}_C([A]^\gamma, [B]^\gamma) f(\gamma) d\gamma \\ &= \int_0^1 \left[\mathcal{C}_C([AB]^\gamma) - \mathcal{C}_C([A]^\gamma) \cdot \mathcal{C}_C([B]^\gamma)\right] f(\gamma) d\gamma.\end{aligned}$$

In [2] we proved that if $A, B \in \mathcal{F}$ are independent then $\mathrm{Cov}_f(A,B) = 0$. The variance of a fuzzy number A is defined as [2]

$$\mathrm{Var}_f(A) = \mathrm{Cov}_f(A,A) = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

In [2] we proved the following theorem.

Theorem 1.1. *Let A and B be fuzzy numbers, and let $\lambda, \mu \in \mathbb{R}$. Then*

$$\mathrm{Var}_f(\lambda A + \mu B) = \lambda^2 \mathrm{Var}_f(A) + \mu^2 \mathrm{Var}_f(B) + 2\lambda\mu \mathrm{Cov}_f(A,B).$$

and if A and B are independent then $\mathrm{Var}(A+B) = \mathrm{Var}(A) + \mathrm{Var}(B)$. That is $\mathrm{Var}(A+B)$ can be computed using the marginal possibility distributions A and B.

We shall introduce the following notation. Let $A_1, \ldots, A_n \in \mathcal{F}$ be fuzzy numbers, let B be their joint possibility distribution, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then $\mathcal{C}([g(B)]^\gamma)$ will stand for $\mathcal{C}_B\left([g(A_1, \ldots, A_n)]^\gamma\right)$.

The following theorem is an extension of our results above.

Theorem 1.2. *Let $A_1, \ldots, A_n \in \mathcal{F}$ be fuzzy numbers with joint possibility distribution B, let $C_1, \ldots, C_m \in \mathcal{F}$ be fuzzy numbers with joint possibility distribution D, let E be the joint possibility distribution of B and D, and let $g\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R}^m \to \mathbb{R}$ be continuous functions, and let $\gamma \in [0,1]$. Then,*

$$\mathrm{Rel}_E\left([g(B)]^\gamma, [h(D)]^\gamma\right) = \mathcal{C}_E\left([g(B) \cdot h(D)]^\gamma\right) - \mathcal{C}_E([g(B)]^\gamma) \cdot \mathcal{C}_E([h(D)]^\gamma).$$

Remark 1.1. *The dependency relation between γ-level sets of $g(A_1,\ldots,A_n) = g(B)$ and $h(C_1,\ldots,C_m) = h(D)$ can be computed by*

$$\begin{aligned}\mathrm{Rel}_E\left([g(B)]^\gamma,[h(D)]^\gamma\right) &= \frac{1}{\int_{[E]^\gamma} dxdy}\int_{[E]^\gamma} g(x)h(y)dxdy \\ &- \frac{1}{\int_{[E]^\gamma} dxdy}\int_{[E]^\gamma} g(x)dxdy \times \frac{1}{\int_{[E]^\gamma} dxdy}\int_{[E]^\gamma} h(y)dxdy,\end{aligned}$$

for all $\gamma \in [0,1]$. Obviously,

$$\mathrm{Rel}_E\left([g(B)]^\gamma,[h(D)]^\gamma\right) = \mathrm{Rel}_E\left([h(D)]^\gamma,[g(B)]^\gamma\right)$$

holds for all $\gamma \in [0,1]$, i.e. the dependency relation operator is symmetrical.

The following theorem says that the relation operator is bilinear operator as well.

Theorem 1.3. *Let $A_i, B_j \in \mathcal{F}$, $i = 1,\ldots,n, j = 1,\ldots,m$, let C be their joint possibility distribution, and let $\lambda_i, \mu_j \in \mathbb{R}$, $i = 1,\ldots,n, j = 1,\ldots,m$ be real numbers, and let $\gamma \in [0,1]$. Then,*

$$\mathrm{Rel}_C\left(\left[\sum_{i=1}^n \lambda_i A_i\right]^\gamma, \left[\sum_{j=1}^m \mu_j B_j\right]^\gamma\right) = \sum_{i=1}^n\sum_{j=1}^m \lambda_i\mu_j \mathrm{Rel}_C\left([A_i]^\gamma,[B_j]^\gamma\right),$$

Proof 1.1. *We refer to Remark 1.1 with $g(x) = \lambda_1 x_1 + \ldots + \lambda_n x_n$ and $h(y) = \mu_1 y_1 + \ldots + \mu_m y_m$ to obtain the statement of the theorem.* □

Theorem 1.4. *Let $A_1,\ldots,A_n \in \mathcal{F}$ be fuzzy numbers with joint possibility distribution B, and let $C_1,\ldots,C_m \in \mathcal{F}$ be fuzzy numbers with joint possibility distribution D, and let $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^m \to \mathbb{R}$ be continuous functions. If B and D are independent, i.e, their joint possibility distribution E satisfies*

$$[E]^\gamma = [B]^\gamma \times [D]^\gamma$$

for all $\gamma \in [0,1]$ (which implicitly includes that A_i and C_j are independent for $i = 1,\ldots,n, j = 1,\ldots,m$) then

$$\mathrm{Rel}_E\left([g(B)]^\gamma,[h(D)]^\gamma\right) = 0$$

holds for all $\gamma \in [0,1]$.

Proof 1.2. *From Remark 1.1 we have*

$$\begin{aligned}\mathrm{Rel}_E\left([g(B)]^\gamma,[h(D)]^\gamma\right) &= \frac{1}{\int_{[B\times D]^\gamma}dxdy}\int_{[B\times D]^\gamma} g(x)h(y)dxdy \\ &- \frac{1}{\int_{[B\times D]^\gamma}dxdy}\int_{[B\times D]^\gamma} g(x)dxdy\cdot\frac{1}{\int_{[B\times D]^\gamma}dxdy}\int_{[B\times D]^\gamma} h(y)dxdy \\ &= \frac{1}{\int_{[B]^\gamma}dx\int_{[D]^\gamma}dy}\int_{[B]^\gamma} g(x)dx\int_{[D]^\gamma}h(y)dy \\ &- \frac{1}{\int_{[B]^\gamma}dx}\int_{[B]^\gamma} g(x)dx\times\frac{1}{\int_{[D]^\gamma}dy}\int_{[D]^\gamma}h(y)dy = 0.\end{aligned}$$

Definition 1.8. *Let $A\in\mathcal{F}$ be a fuzzy number, let B be the joint possibility distribution of A and $\mathcal{C}([A]^\gamma)$, and let $\gamma\in[0,1]$. The self-relation of the γ-level set of A is defined by*

$$\mathrm{Rel}([A]^\gamma,[A]^\gamma)=\mathcal{C}([(A-\mathcal{C}([A]^\gamma))\cdot(A-\mathcal{C}([A]^\gamma))]^\gamma)=\mathcal{C}([(A-\mathcal{C}([A]^\gamma))^2]^\gamma).$$

The following theorems can be proved by applying Theorem 1.2.

Theorem 1.5. *Let $A\in\mathcal{F}$ be a fuzzy number, and let $\gamma\in[0,1]$. Then,*

$$\mathrm{Rel}\left([A]^\gamma,[A]^\gamma\right)=\mathcal{C}([A^2]^\gamma)-\left(\mathcal{C}([A]^\gamma)\right)^2.$$

Theorem 1.6. *Let $A_1,\ldots,A_n\in\mathcal{F}$ be fuzzy numbers with joint possibility distribution B, let $g:\mathbb{R}^n\to\mathbb{R}$ be a continuous function, and let $\gamma\in[0,1]$. Then,*

$$\mathrm{Rel}([g(B)]^\gamma,[g(B)]^\gamma)=\mathcal{C}([(g(B))^2]^\gamma)-(\mathcal{C}([g(B)]^\gamma))^2.$$

Remark 1.2. *The self-relation of the γ-level set of $g(A_1,\ldots,A_n)=g(B)$ can be computed by*

$$\mathrm{Rel}\left([g(B)]^\gamma,[g(B)]^\gamma\right)=\frac{1}{\int_{[B]^\gamma}dx}\int_{[B]^\gamma}g^2(x)dx-\left(\frac{1}{\int_{[B]^\gamma}dx}\int_{[B]^\gamma}g(x)dx\right)^2,$$

and it is clear that,

$$\mathrm{Rel}\left([g(B)]^\gamma,[g(B)]^\gamma\right)\geq 0$$

for any $\gamma\in[0,1]$.

2 The weak form of the possibilistic Cauchy-Schwarz inequality

Let $A_1,\ldots,A_n\in\mathcal{F}$ be fuzzy numbers with joint possibility distribution B, and let $C_1,\ldots,C_m\in\mathcal{F}$ with joint possibility distribution D, and let E be the joint distribution of B and D, and let $g\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R}^m \to \mathbb{R}$ be continuous functions, and let $\lambda \in \mathbb{R}$. Then applying Remark 1.2 we have that the inequality

$$\mathrm{Rel}\left([g(B) + \lambda h(D)]^\gamma, [g(B) + \lambda h(D)]^\gamma\right) \geq 0,$$

holds for all $\gamma \in [0, 1]$.

Using the linearity and symmetricity properties of the dependency relation operator we have

$$\begin{aligned}\mathrm{Rel}_E([g(B)]^\gamma, [g(B)]^\gamma) + 2\lambda \mathrm{Rel}_E([g(B)]^\gamma, [h(D)]^\gamma)\\ + \lambda^2 \mathrm{Rel}_E([h(D)]^\gamma, [h(D)]^\gamma) \geq 0,\end{aligned} \tag{2}$$

for any $\lambda \in \mathbb{R}$. Furthermore, the discriminant of (2) should satisfy the following inequality,

$$[\mathrm{Rel}_E([g(B)]^\gamma, [h(D)]^\gamma)]^2 - \mathrm{Rel}_E([g(B)]^\gamma, [g(B)]^\gamma)\mathrm{Rel}_E([h(D)]^\gamma, [h(D)]^\gamma) \leq 0,$$

for any $\gamma \in [0, 1]$. Hence, we can state the following theorem.

Theorem 2.1. *Let $A_1, \dots, A_n \in \mathcal{F}$ be fuzzy numbers with joint possibility distribution B, let $C_1, \dots, C_m \in \mathcal{F}$ be fuzzy numbers with joint distribution D, let E be the joint distribution of B and D, and let $g : \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R}^m \to \mathbb{R}$ be continuous functions. Then,*

$$\left(\mathrm{Rel}_E\left([g(B)]^\gamma, [h(D)]^\gamma\right)\right)^2 \leq \mathrm{Rel}_E([g(B)]^\gamma, [g(B)]^\gamma) \times \mathrm{Rel}_E([h(D)]^\gamma, [h(D)]^\gamma)$$

holds for all $\gamma \in [0, 1]$.

Especially, if $g(x) = x$ and $h(y) = y$ then Theorem 2.1 turns into the following theorem, which we will call the weak form of Cauchy-Schwarz inequality for the γ-level sets of the marginal possibility distributions.

Theorem 2.2. *Let $A, B \in \mathcal{F}$ be fuzzy numbers, and let C be their joint possibility distribution. Then,*

$$\left(\mathrm{Rel}_C\left([A]^\gamma, [B]^\gamma\right)\right)^2 \leq \mathrm{Rel}_C([A]^\gamma, [A]^\gamma) \cdot \mathrm{Rel}_C([B]^\gamma, [B]^\gamma),$$

for any $\gamma \in [0, 1]$.

3 The strong form of the possibilistic Cauchy-Schwarz inequality

Let $A, B \in \mathcal{F}$ be fuzzy numbers, and let C be their joint possibility distribution and let

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 | x \in [u, v], y \in [w_1(x), w_2(x)]\},$$

denote the parametrized representation of $[C]^\gamma$. Then applying the Fubini theorem we have

$$\int_{[C]^\gamma} dxdy = \int_u^v \int_{w_1(x)}^{w_2(x)} dydx.$$

Lemma 3.1. *Let*

$$F(x) = \int_{w_1(x)}^{w_2(x)} dy = w_2(x) - w_1(x),\ x \in [u, v].$$

If $[C]^\gamma$ *is a convex subset of* $\mathbb{R}^2$ *then* F *is a concave function.*

Proof 3.1. *If* F *were not concave then there would exist* $x_1, x_2 \in [u, v]$*,* $x_1 < x_2$ *and* $\lambda \in [0, 1]$ *such that for* $x^* = \lambda x_1 + (1 - \lambda)x_2$ *the following inequality would hold*

$$F(x^*) < \lambda F(x_1) + (1 - \lambda)F(x_2),$$

that is,

$$w_2(x^*) - w_1(x^*) < \lambda\left[w_2(x_1) - w_1(x_1)\right] + (1 - \lambda)\left[w_2(x_2) - w_1(x_2)\right] \qquad (3)$$

would hold. Let T *be the convex hull of the points* $(x_i, w_1(x_i)), (x_i, w_2(x_i))$*,* $i = 1, 2$*, i.e.*

$$\begin{aligned} T &= \text{conv}\left\{(x_1, w_1(x_1)), (x_1, w_2(x_1)), (x_2, w_1(x_2)), (x_2, w_2(x_2))\right\} \\ &= \big\{x \in \mathbb{R}^2 | x = \mu_1(x_1, w_1(x_1)) + \mu_2(x_1, w_2(x_1)) \\ &\quad + \mu_3(x_2, w_1(x_2)) + \mu_4(x_2, w_2(x_2)),\ \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1, \\ &\quad \mu_1, \mu_2, \mu_3, \mu_4 \geq 0\big\}. \end{aligned}$$

Since $[C]^\gamma$ *is convex,* $T \subseteq [C]^\gamma$ *holds, and therefore*

$$\{y \in \mathbb{R} | (x, y) \in T\} \subseteq \{y \in \mathbb{R} | (x, y) \in [C]^\gamma\}$$

also holds for all $x \in [x_1, x_2]$*. Applying this inequality relation to* $x = x^*$ *we find that*

$$\begin{aligned} w_2(x^*) - w_1(x^*) &\geq [\lambda w_2(x_1) + (1 - \lambda)w_2(x_2)] - [\lambda w_1(x_1) + (1 - \lambda)w_1(x_2)] \\ &= \lambda\left[w_2(x_1) - w_1(x_1)\right] + (1 - \lambda)\left[w_2(x_2) - w_1(x_2)\right], \end{aligned}$$

which contradicts to (3). □

It should be noted that since F is concave, it is continuous.

Theorem 3.1. *Let* $A, B \in \mathcal{F}$ *be fuzzy numbers, and let* C *be their joint possibility distribution. If* $[C]^\gamma$ *is a convex subset of* $\mathbb{R}^2$ *for any* $\gamma \in [0, 1]$ *then the inequality*

$$\text{Rel}_C([A]^\gamma, [A]^\gamma) \leq \text{Rel}([A]^\gamma, [A]^\gamma) \qquad (4)$$

holds for all $\gamma \in [0, 1]$.

Remark 3.1. *From Theorem 3.1 we can easily prove that*

$$\begin{aligned}\mathrm{Rel}_C([A]^\gamma,[A]^\gamma) &= \int_u^v x^2G(x)dx - \Big(\int_u^v xG(x)dx\Big)^2 \\ &\le \frac{1}{v-u}\int_u^v x^2dx - \left(\frac{1}{v-u}\int_u^v xdx\right)^2 = \mathrm{Rel}([A]^\gamma,[A]^\gamma),\end{aligned}$$

and the equality holds if and only if $G \equiv 1/(v-u)$*, which means that in this case the parametrization of* $[C]^\gamma$ *should be*

$$[C]^\gamma = \{(x,y) \in \mathbb{R}^2 | x \in [u,v], y \in [w_1,w_2]\},$$

that is,

$$[C]^\gamma = \mathrm{Proj}_x([C]^\gamma) \times \mathrm{Proj}_y([C]^\gamma) = [A]^\gamma \times [B]^\gamma.$$

We find that (4) holds with equality for all $\gamma \in [0,1]$ *if and only if* A *and* B *(the marginal possibility distributions of* C*) are independent.*

Now we are in the position to state the strong form of the possibilistic Cauchy-Schwarz inequality for the γ-level sets of marginal possibility distributions.

Theorem 3.2. *Let* $A, B \in \mathcal{F}$ *be fuzzy numbers, and let* C *be their joint possibility distribution. If* $[C]^\gamma$ *is a convex subset of* $\mathbb{R}^2$ *for any* $\gamma \in [0,1]$ *then the inequality*

$$(\mathrm{Rel}_C([A]^\gamma,[B]^\gamma))^2 \le \mathrm{Rel}([A]^\gamma,[A]^\gamma) \cdot \mathrm{Rel}([B]^\gamma,[B]^\gamma)$$

holds for all $\gamma \in [0,1]$.

Proof 3.2. *From Theorem 3.1 we have that the inequalities*

$$\begin{aligned}\mathrm{Rel}_C([A]^\gamma,[A]^\gamma) &\le \mathrm{Rel}([A]^\gamma,[A]^\gamma),\\ \mathrm{Rel}_C([B]^\gamma,[B]^\gamma) &\le \mathrm{Rel}([B]^\gamma,[B]^\gamma),\end{aligned}$$

hold for all $\gamma \in [0,1]$. *From Theorem 2.2 we get*

$$\begin{aligned}(\mathrm{Rel}([A]^\gamma,[B]^\gamma))^2 \le& \mathrm{Rel}_C([A]^\gamma,[A]^\gamma) \cdot \mathrm{Rel}_C([B]^\gamma,[B]^\gamma)\\ \le& \mathrm{Rel}([A]^\gamma,[A]^\gamma) \cdot \mathrm{Rel}([B]^\gamma,[B]^\gamma),\end{aligned}$$

which ends the proof. □

The following theorem states the possibilistic Cauchy-Schwarz inequality.

Theorem 3.3. *Let* $A, B \in \mathcal{F}$ *be fuzzy numbers, and let* C *be their joint possibility distribution and let* f *be a weighting function. If* $[C]^\gamma$ *is a convex subset of* $\mathbb{R}^2$ *for any* $\gamma \in [0,1]$ *then the inequality*

$$(\mathrm{Cov}_f(A,B))^2 \le \mathrm{Var}_f(A) \cdot \mathrm{Var}_f(B),$$

holds for any weighting function f.

Summarizing our findings in this section we define the concept of correlation between fuzzy numbers as follows.

Definition 3.1. *Let $A, B \in \mathcal{F}$ be fuzzy numbers (with $\mathrm{Var}_f(A) \neq 0$ and $\mathrm{Var}_f(B) \neq 0$) with joint possibility distribution C. Then, the correlation coefficient between A and B (with respect to weighting function f) is defined by*

$$\rho_f(A, B) = \frac{\mathrm{Cov}_f(A, B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}.$$

From Theorem 3.3 we find that if $[C]^\gamma$ is a convex subset of $\mathbb{R}^2$ for any $\gamma \in [0, 1]$ then the inequality $-1 \leq \rho_f(A, B) \leq 1$, holds for any $A, B \in \mathcal{F}$.

4 Illustration

We illustrate three important cases of correlation coefficient. In [2] we proved that if A and B are independent, that is, their joint possibility distribution is $A \times B$ then $\rho_f(A, B) = 0$ for any weighting function f (Fig. 1).

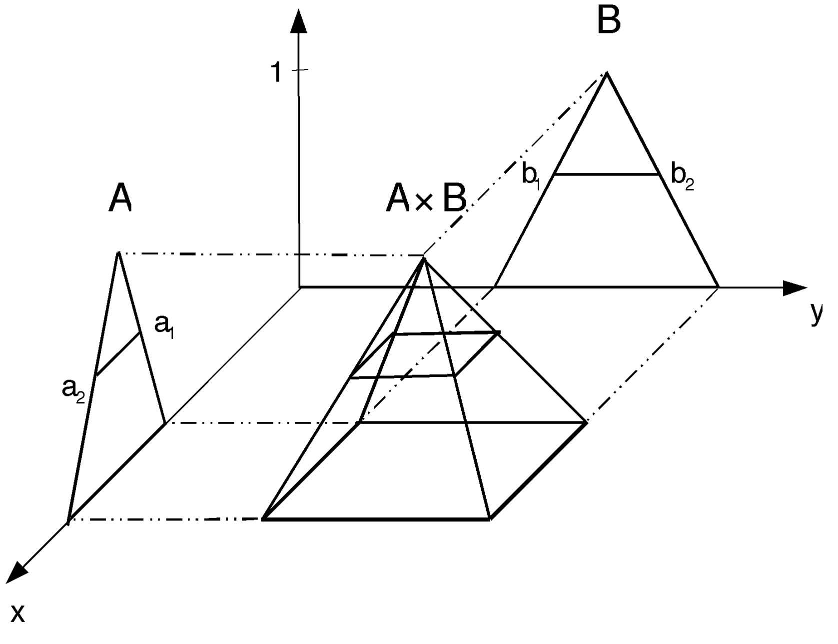

Figure 1: If A and B are independent then $\rho_f(A, B) = 0$.

Consider now the case depicted in Fig. 2. It can be shown [2] that the dependency relation between the γ-level sets of A and B (with respect to their joint possibility distribution C) is

$$\begin{aligned}\mathrm{Rel}_C([A]^\gamma,[B]^\gamma) &= \mathcal{C}_C([AB]^\gamma)-\mathcal{C}_C([A]^\gamma)\mathcal{C}_C([B]^\gamma)\\ &= \frac{(a_2(\gamma)-a_1(\gamma))(b_2(\gamma)-b_1(\gamma))}{12}.\end{aligned}$$

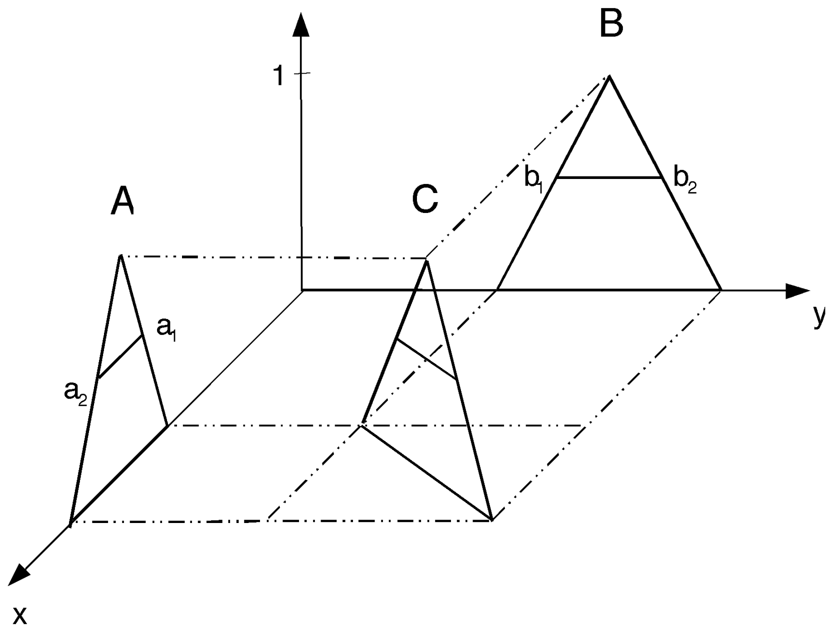

Figure 2: The case of $\rho_f(A,B)=1$.

Furthermore, the variances of A and B are computed by

$$\mathrm{Var}_f(A)=\mathrm{Cov}_f(A,A)=\int_0^1\frac{(a_2(\gamma)-a_1(\gamma))^2}{12}f(\gamma)d\gamma.$$

and,

$$\mathrm{Var}_f(B)=\mathrm{Cov}_f(B,B)=\int_0^1\frac{(b_2(\gamma)-b_1(\gamma))^2}{12}f(\gamma)d\gamma.$$

Therefore, we get

$$\begin{aligned}\rho_f(A,B) &= \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}\\ &= \frac{\int_0^1\frac{(a_2(\gamma)-a_1(\gamma))(b_2(\gamma)-b_1(\gamma))}{12}f(\gamma)d\gamma}{\sqrt{\int_0^1\frac{(a_2(\gamma)-a_1(\gamma))^2}{12}f(\gamma)d\gamma}\sqrt{\int_0^1\frac{(b_2(\gamma)-b_1(\gamma))^2}{12}f(\gamma)d\gamma}}\end{aligned}$$

From the linearity of A and B we have that there exists $w \in \mathbb{R}$, with $w \geq 0$ such that equality

$$a_2(\gamma) - a_1(\gamma) = w(b_2(\gamma) - b_1(\gamma)),$$

holds for any $\gamma \in [0, 1]$. Using this relationship we find that

$$\rho_f(A, B) = 1,$$

holds for any weighting function f.

Consider now the case depicted in Fig. 3. It can be shown [2] that the dependency relation between the γ-level sets of A and B (with respect to their joint possibility distribution D) is

$$\begin{aligned}\mathrm{Rel}_D([A]^\gamma, [B]^\gamma) &= \mathcal{C}_D([AB]^\gamma) - \mathcal{C}_D([A]^\gamma)\mathcal{C}_D([B]^\gamma) \\ &= -\frac{(a_2(\gamma) - a_1(\gamma))(b_2(\gamma) - b_1(\gamma))}{12}.\end{aligned}$$

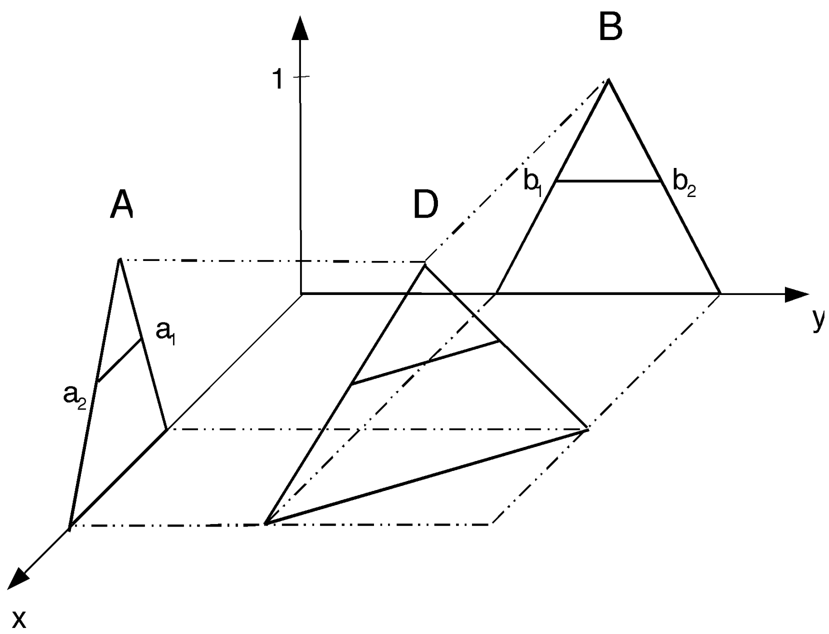

Figure 3: The case of $\rho_f(A, B) = -1$.

Therefore, we get

$$\rho_f(A,B) = \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}$$
$$= -\frac{\int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))(b_2(\gamma)-b_1(\gamma))}{12} f(\gamma)d\gamma}{\sqrt{\int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))^2}{12} f(\gamma)d\gamma}\sqrt{\int_0^1 \frac{(b_2(\gamma)-b_1(\gamma))^2}{12} f(\gamma)d\gamma}}$$

From the linearity of A and B we have that there exists $w \in \mathbb{R}$, with $w \geq 0$ such that the equality

$$a_2(\gamma) - a_1(\gamma) = w(b_2(\gamma) - b_1(\gamma)),$$

holds for any $\gamma \in [0,1]$. Using this relation we find that

$$\rho_f(A,B) = -1$$

holds for any weighting function f.

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