On possibilistic dependencies

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Abstract

In this paper we will introduce the expected value of functions of possibility distributions throughout their joint possibility distributions. We will show that the expected value operator is linear and the covariance of weighted sums of possibility distributions can be computed in the same way as in probablity theory.

Keywords: Random variable; Joint probability distribution; Possibility distribution, Joint possibility distribution; Expected value; Covariance; Variance

1 Functions of random variables

In probability theory, the dependency between two random variables can be characterized through their joint probability density function. Namely, if X and Y are two random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then the density function, $f_{X,Y}(x,y)$, of their joint random variable (X,Y), should satisfy the following properties

$$\int_{\mathbb{R}} f_{X,Y}(x,t)dt = f_X(x), \quad \int_{\mathbb{R}} f_{X,Y}(t,y)dt = f_Y(y), \tag{1}$$

for all $x, y \in \mathbb{R}$. Furthermore, $f_X(x)$ and $f_Y(y)$ are called the the marginal probability density functions of random variable (X,Y). X and Y are said to be independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

holds for all x, y. The expected value of random variable X is defined as

$$E(X) = \int_{\mathbb{R}} x f_X(x)dx,$$

and if g is a function of X then the expected value of $g(X)$ can be computed as

$$E(g(X)) = \int_{\mathbb{R}} g(x)f_X(x)dx.$$

Furthermore, if h is a function of X and Y then the expected value of $h(X,Y)$ can be computed as

$$E(h(X,Y)) = \int_{\mathbb{R}^2} h(x,y)f_{X,Y}(x,y)dxdy.$$

Especially,

$$\begin{aligned} E(X+Y) = \int_{\mathbb{R}^2} (x+y)f_{X,Y}(x,y)dxdy = \int_{\mathbb{R}^2} x f_{X,Y}(x,y)dxdy \\ + \int_{\mathbb{R}^2} y f_{X,Y}(x,y)dxdy = \int_{\mathbb{R}} x\Big(\int_{\mathbb{R}} f_{X,Y}(x,y)dy\Big)dx \\ + \int_{\mathbb{R}} y\Big(\int_{\mathbb{R}} f_{X,Y}(x,y)dx\Big)dy = \int_{\mathbb{R}} x f_X(x)dx \\ + \int_{\mathbb{R}} y f_Y(y)dy = E(X) + E(Y), \end{aligned}$$

that is, the the expected value of X and Y can be determined according to their individual density functions (that are the marginal probability functions of random variable (X,Y)). The key issue here is that the joint probability distribution vanishes (even if X and Y are not independent), because of the principle of 'falling integrals' (1).

Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the probability that X takes its value from $[a, b]$ is computed by

$$\mathrm{P}(X \in [a, b]) = \int_a^b f_X(x)dx.$$

The covariance between two random variables X and Y is defined as

$$\begin{aligned}\mathrm{Cov}(X, Y) &= E\big((X - E(X))(Y - E(Y))\big) = E(XY) - E(X)E(Y) \\ &= \int_{\mathbb{R}^2} xy f_{X,Y}(x, y)dxdy - \int_{\mathbb{R}} x f_X(x)dx \int_{\mathbb{R}} y f_Y(y)dy,\end{aligned}$$

and if X and Y are independent then $\mathrm{Cov}(X, Y) = 0$, since $E(XY) = E(X)E(Y)$. The covariance operator is a symmetrical bilinear operator and it is easy to see that $\mathrm{Cov}(\lambda, X) = 0$ for any $\lambda \in \mathbb{R}$.
The variance of random variable X is defined as the covariance between X and itself, that is

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \int_{\mathbb{R}} x^2 f_X(x)dx - \left(\int_{\mathbb{R}} x f_X(x)dx\right)^2.$$

For any random variables X and Y and real numbers $\lambda, \mu \in \mathbb{R}$ the following relationship holds

$$\mathrm{Var}(\lambda X + \mu Y) = \lambda^2 \mathrm{Var}(X) + \mu^2 \mathrm{Var}(Y) + 2\lambda\mu \mathrm{Cov}(X, Y).$$

2 Joint possibility distributions

In this section we shall introduce possibilistic dependencies between possibility distributions. A fuzzy set A in $\mathbb{R}$ is said to be a fuzzy number if it is normal, fuzzy convex and has an upper semi-continuous membership function of bounded support. The family of all fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy set A in $\mathbb{R}^m$ is defined by $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = \mathrm{cl}\{x \in \mathbb{R}^m : A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$. If $A \in \mathcal{F}$ is a fuzzy number then $[A]^\gamma$ is a convex and compact subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$. Fuzzy numbers can be considered as possibility distributions. Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the possibility that $A \in \mathcal{F}$ takes its value from $[a, b]$ is defined by [5]

$$\mathrm{Pos}(A \in [a, b]) = \max_{x \in [a,b]} A(x).$$

Definition 2.1. *A fuzzy set B in $\mathbb{R}^m$ is said to be a joint possibility distribution of fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \ldots, m$, if it satisfies the relationship*

$$\max_{x_j \in \mathbb{R},\, j \neq i} B(x_1, \ldots, x_m) = A_i(x_i),\ \forall x_i \in \mathbb{R}, i = 1, \ldots, m.$$

Furthermore, A_i is called the i-th marginal possibility distribution of B, and the projection of B on the i-th axis is A_i for $i = 1, \ldots, m$.

We emphasise here that the joint possibility distribution always uniquely defines its marginal distributions (the shadow of B on the i-th axis is exactly A_i), but not vice versa. Let B denote a joint possibility distribution of $A_1, A_2 \in \mathcal{F}$. Then B should satisfy the relationships

$$\max_y B(x_1, y) = A_1(x_1), \quad \max_y B(y, x_2) = A_2(x_2), \ \forall x_1, x_2 \in \mathbb{R}.$$

The following theorem shows some important properties of joint possibility distributions.

Theorem 2.1. *Let $A_i \in \mathcal{F}$, $i = 1, \dots, m$, and let B be their joint possibility distribution. Then,*

$$B(x_1, \dots, x_m) \leq \min\{A_1(x_1), \dots, A_m(x_m)\} \text{ and } [B]^\gamma \subseteq [A_1]^\gamma \times \cdots \times [A_m]^\gamma,$$

hold for all $x_1, \dots, x_m \in \mathbb{R}$ and $\gamma \in [0,1]$.

If $m = 2$ then Theorem 2.1 states that any γ-level set of $[B]^\gamma$ should be contained by the rectangle determined by the Cartesian product the γ-level sets of marginal distributions $[A_1]^\gamma \times [A_2]^\gamma$, and it should reach each side of that rectangle.
In the following the biggest (in the sense of subsethood of fuzzy sets) joint possibility distribution will play a special role among joint possibility distributions: it defines the concept of independence of fuzzy numbers.

Definition 2.2. *Fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, m$, are said to be independent if their joint possibility distribution, B, is given by*

$$B(x_1, \dots, x_m) = \min\{A_1(x_1), \dots, A_m(x_m)\},$$

or, equivalently,

$$[B]^\gamma = [A_1]^\gamma \times \cdots \times [A_m]^\gamma,$$

for all $x_1, \dots, x_m \in \mathbb{R}$ and $\gamma \in [0,1]$.

It can easily be seen that constants are always independent from any fuzzy number. In particular, they are independent from each other.

Remark 2.1. *Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'.*

3 Expected value of functions of possibility distributions

Definition 3.1. *Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$. If $[A]^\gamma$ is non-degenerate, i.e. $a_1(\gamma) \neq a_2(\gamma)$ then the central value of $[A]^\gamma$ is defined by*

$$\text{center}([A]^\gamma) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx.$$

The central value of $[A]^\gamma$ is computed by

$$\text{center}([A]^\gamma) = \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x dx = \frac{a_1(\gamma) + a_2(\gamma)}{2},$$

which remains valid in the limit case $a_2(\gamma) - a_1(\gamma) = 0$ for some γ.
That is, the central value of a γ level set $[a_1(\gamma), a_2(\gamma)]$ is simple the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$.

Definition 3.2. *[2] A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition*

$$\int_0^1 f(\gamma) d\gamma = 1.$$

Different weighting functions can give different (case-dependent) importances to γ-levels sets of fuzzy numbers.

Definition 3.3. *The expected value of A with respect to a weighting function f is defined as*

$$E_f(A) = \int_0^1 \text{center}([A]^\gamma) f(\gamma) d\gamma = \int_0^1 \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx f(\gamma) d\gamma,$$

That is,

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma.$$

It should be noted that if $f(\gamma) = 2\gamma$, $\gamma \in [0,1]$ then

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 [a_1(\gamma) + a_2(\gamma)]\, \gamma d\gamma = \bar{M}(A).$$

which is the possibilistic mean value of A introduced by Fullér and Carlsson in [3].
And if $f(\gamma) = 1$, $\gamma \in [0,1]$ then we get

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma.$$

which is the generative expectation of fuzzy numbers introduced by Chanas and M. Nowakowski in ([1], page 47).

Definition 3.4. *Let $A_1, \ldots, A_m \in \mathcal{F}$ be fuzzy numbers, and let $g\colon \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Then $g(A_1, \ldots, A_m)$ is defined by the extension principle [4] as follows*

$$g(A_1, \ldots, A_m)(y) = \sup_{g(x_1, \ldots, x_m) = y} \min\{A_1(x_1), \ldots, A_m(x_m)\}.$$

Definition 3.5. *The central value of the* γ*-level set of* $g(A_1, \dots, A_m)$ *is defined by*

$$\operatorname{center}([g(A_1, \dots, A_m)]^\gamma) = \frac{1}{\int_{[B]^\gamma} dx} \int_{[B]^\gamma} g(x) dx,$$

where B *is the joint distribution of* $A_1, \dots, A_m$ *and* $g(x) = g(x_1, \dots, x_m)$*.*

Definitions 3.4 and 3.5 are crucial for our theory. We always use the 'min' operator in the definition of the extension principle, but we define the central value of $g(A_1, \dots, A_m)$ throughout their joint possibility distribution, which is usually given implicitly.
Now we define the expected value of $g(A_1, \dots, A_m)$ as follows.

Definition 3.6. *The expected value of* $g(A_1, \dots, A_m)$ *with respect to a weighting function* f *is defined by*

$$\begin{aligned} E_f(g(A_1, \dots, A_m)) &= \int_0^1 \operatorname{center}([g(A_1, \dots, A_m)]^\gamma) f(\gamma) d\gamma \\ &= \int_0^1 \frac{1}{\int_{[B]^\gamma} dx} \int_{[B]^\gamma} g(x) dx f(\gamma) d\gamma. \end{aligned}$$

If g is single-variable function then Definitons 3.5 and 3.6 read

Definition 3.7. *Let* $g\colon \mathbb{R} \to \mathbb{R}$ *and let* $A \in \mathcal{F}$ *be a fuzzy number. Then the central value of the* γ*-level set of* $g(A)$ *is defined as*

$$\operatorname{center}([g(A)]^\gamma) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} g(x) dx,$$

and the expected value of $g(A)$ *with respect to a weighting function* f *is defined as*

$$E_f(g(A)) = \int_0^1 \operatorname{center}([g(A)]^\gamma) f(\gamma) d\gamma = \int_0^1 \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} g(x) dx f(\gamma) d\gamma,$$

where $g(A)$ *is defined by the sup–min extension principle.*

If $g(x) = x$ then $\operatorname{center}([g(A)]^\gamma) = \operatorname{center}([A]^\gamma)$ and $E_f(g(A)) = E_f(A)$. In particular, if $A = a$ is a real number then we have $\operatorname{center}([g(A)]^\gamma) = E_f(g(A)) = g(a)$, for all weighting function f.
In the following we prove that the central value operator is linear.

Theorem 3.1. *Let* A *and* B *be fuzzy numbers. Then*

$$\operatorname{center}([A + B]^\gamma) = \operatorname{center}([A]^\gamma) + \operatorname{center}([B]^\gamma),$$

holds for all $\gamma \in [0, 1]$*, where all central values are defined by the joint possibility distribution of* A *and* B*. If* A *and* B *are independent then the central values of* $[A]^\gamma$ *and* $[B]^\gamma$ *can be computed using the individual distributions* A *and* B*, respectively. Moreover, if* A *and* B *are independent we have*

$$\operatorname{center}([AB]^\gamma) = \operatorname{center}([A]^\gamma) \cdot \operatorname{center}([B]^\gamma), \forall \gamma \in [0, 1].$$

As a special case of this theorem we get that if $A \in \mathcal{F}$ and $\lambda \in \mathbb{R}$ then $\text{center}([\lambda A]^\gamma) = \lambda \times \text{center}([A]^\gamma)$ holds for all $\gamma \in [0,1]$.
Now we are in the position to state the theorem about the linearity of the expected value operator.

Theorem 3.2. *The expected value operator is linear, that is*

$$E_f(A+B) = E_f(A) + E_f(B) \tag{2}$$

and

$$E_f(\lambda A) = \lambda E_f(A)$$

hold for all fuzzy numbers A and B and real number λ. Moreover, if A and B are independent then on the right-hand side of (2) the expected value operator is defined by the marginal distributions A and B, respectively.

4 A measure of possibilistic dependency

In this Section we shall introduce a measure of possibilistic dependency through joint possibility distributions.

Definition 4.1. *Let $A, B \in \mathcal{F}$ be fuzzy numbers. Then we will introduce the following (dependency) relation between their γ-level sets*

$$R_\gamma(A,B) = \text{center}([(A - \text{center}([A]^\gamma))(B - \text{center}([B]^\gamma))]^\gamma),\ \gamma \in [0,1].$$

Definition 4.2. *The covariance of A and B with respect to weighting function f is defined by*

$$\text{Cov}_f(A,B) = \int_0^1 R_\gamma(A,B) f(\gamma) d\gamma. \tag{3}$$

That is,

$$\text{Cov}_f(A,B) = \int_0^1 \text{center}([(A - \text{center}([A]^\gamma))(B - \text{center}([B]^\gamma))]^\gamma) f(\gamma) d\gamma.$$

It can easily be seen that, $R_\gamma(A,B)$ (as a function of their joint possibility distribution C) can be written in the form

$$R_\gamma(A,B) = \frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} xydxdy - \frac{1}{\int_{[C]^\gamma} dx} \int_{[C]^\gamma} xdx \times \frac{1}{\int_{[C]^\gamma} dy} \int_{[C]^\gamma} ydy. \tag{4}$$

Furthermore, the covariance of A and B can be computed as

$$\text{Cov}_f(A,B) = \int_0^1 \left[\text{center}([AB]^\gamma) - \text{center}([A]^\gamma) \cdot \text{center}([B]^\gamma)\right] f(\gamma) d\gamma.$$

So, if A and B are independent then

$$R_\gamma(A,B) = \text{center}([AB]^\gamma) - \text{center}([A]^\gamma) \cdot \text{center}([B]^\gamma) = 0$$

for all $\gamma \in [0,1]$, which directly implies the following theorem:

Theorem 4.1. *If $A, B \in \mathcal{F}$ are independent then* $\text{Cov}_f(A,B) = 0$.

5 Variance of possibility distributions

We introduce now the variance of fuzzy numbers using a similar reasoning as in probablity theory.

Definition 5.1. *Let A be a fuzzy number. Then, the self-relation between its γ-level set is defined as*

$$R_\gamma(A, A) = \text{center}([(A - \text{center}([A]^\gamma))(A - \text{center}([A]^\gamma))]^\gamma), \tag{5}$$

for all $\gamma \in [0, 1]$.

Definition 5.2. *The variance of A with respect to weighting function f is defined as*

$$\text{Var}_f(A) = \int_0^1 R_\gamma(A, A) f(\gamma) d\gamma.$$

That is, $\text{Var}_f(A) = \text{Cov}_f(A, A)$, where, of course, the covariance defined through the one-dimensional possibility distribution A.

It can easily be seen that,

$$R_\gamma(A, A) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x^2 dx - \left(\frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx \right)^2. \tag{6}$$

That is,

$$\begin{aligned} R_\gamma(A, A) &= \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x^2 dx - \left(\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x dx \right)^2 \\ &= \frac{(a_2(\gamma) - a_1(\gamma))^2}{12}. \end{aligned}$$

Summarizing these findings we get

$$\text{Var}_f(A) = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

We can see that $R_\gamma(A, A) \geq 0$ holds for all $\gamma \in [0, 1]$, therefore we can state the following theorem.

Theorem 5.1. *If $A \in \mathcal{F}$ then $\text{Var}_f(A) \geq 0$. Moreover, if f is strictly increasing then $\text{Var}_f(A) = 0$ implies that A is constant.*

Remark 5.1. *Let σ_A be the possibilistic variance of A introduced in [3] as*

$$\sigma_A = \frac{1}{2} \int_0^1 [a_2(\gamma) - a_1(\gamma)]^2 \gamma d\gamma.$$

If $f(\gamma) = 2\gamma$, then we find

$$\text{Var}_f(A) = \frac{1}{6} \int_0^1 [a_2(\gamma) - a_1(\gamma)]^2 \gamma d\gamma = \frac{\sigma_A}{3}.$$

Considering expressions (4) and (6) and using the linearity of central and expected value we have the following theorems.

Theorem 5.2. *The relation operator is a symmetrical bilinear operator, that is, the equations* $R_\gamma(A,B) = R_\gamma(B,A)$*,* $R_\gamma(\lambda A,B) = \lambda R_\gamma(A,B)$ *and* $R_\gamma(A+B,C) = R_\gamma(A,C) + R_\gamma(B,C)$ *hold for all fuzzy numbers* A*,* B*,* C *and constant* $\lambda \in \mathbb{R}$*.*

Theorem 5.3. *The self-relation operator satifies the equations* $R_\gamma(A+B,A+B) = R_\gamma(A,A) + R_\gamma(B,B) + 2R_\gamma(A,B)$ *and* $R_\gamma(\lambda A,\lambda A) = \lambda^2 R_\gamma(A,A)$ *for all fuzzy numbers* A*,* B *and constant* $\lambda \in \mathbb{R}$*.*

Now we are in the position to derive the following theorems about the covariance (and variance) of the weighted sum of possibility distributions. The following two theorems show that the 'principle of central values' leads us to the same relationships in possibilitic environment as in probabilitic one. It is why we claim that the principle of 'central values' should play an important role in defining possibilistic dependencies.

Theorem 5.4. *Let* A*,* B *and* C *be fuzzy numbers, and let* $\lambda, \mu \in \mathbb{R}$*. Then*

$$\mathrm{Cov}_f(\lambda A + \mu B, C) = \lambda \mathrm{Cov}_f(A,C) + \mu \mathrm{Cov}_f(B,C),$$

where all terms in this equation are defined through joint possibility distributions.

Theorem 5.5. *Let* A *and* B *be fuzzy numbers, and let* $\lambda, \mu \in \mathbb{R}$*. Then*

$$\mathrm{Var}_f(\lambda A + \mu B) = \lambda^2 \mathrm{Var}_f(A) + \mu^2 \mathrm{Var}_f(B) + 2\lambda\mu \mathrm{Cov}_f(A,B).$$

and if A *and* B *are independent then* $\mathrm{Var}(A+B) = \mathrm{Var}(A) + \mathrm{Var}(B)$*. That is* $\mathrm{Var}(A+B)$ *can be computed using the marginal possibility distributions* A *and* B*.*

6 Illustrations

Let us consider fuzzy numbers $A, B \in \mathcal{F}$ with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$. We have learned that the 'maximal' joint possibility distribution, which can be defined by A and B is distinguished from the other ones, and in this case A and B are independent from each other (see Figure 1).
Now, let A and B be fuzzy numbers, and let us consider their joint possibility distribution C, defined by (see Figure 2)

$$[C]^\gamma = \{t(a_1(\gamma), b_1(\gamma)) + (1-t)(a_2(\gamma), b_2(\gamma)) : t \in [0,1]\}$$

First, we compute the covariance of A and B with respect to C. Let γ be arbitrarily fixed, and let $a_1 = a_1(\gamma), a_2 = a_2(\gamma), b_1 = b_1(\gamma), b_2 = b_2(\gamma)$. Then the γ-level set of C can be calculated as $\{c(y) | b_1 \le y \le b_2\}$ where

$$c(y) = \frac{b_2 - y}{b_2 - b_1} a_1 + \frac{y - b_1}{b_2 - b_1} a_2 = \frac{a_2 - a_1}{b_2 - b_1} y + \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1}.$$

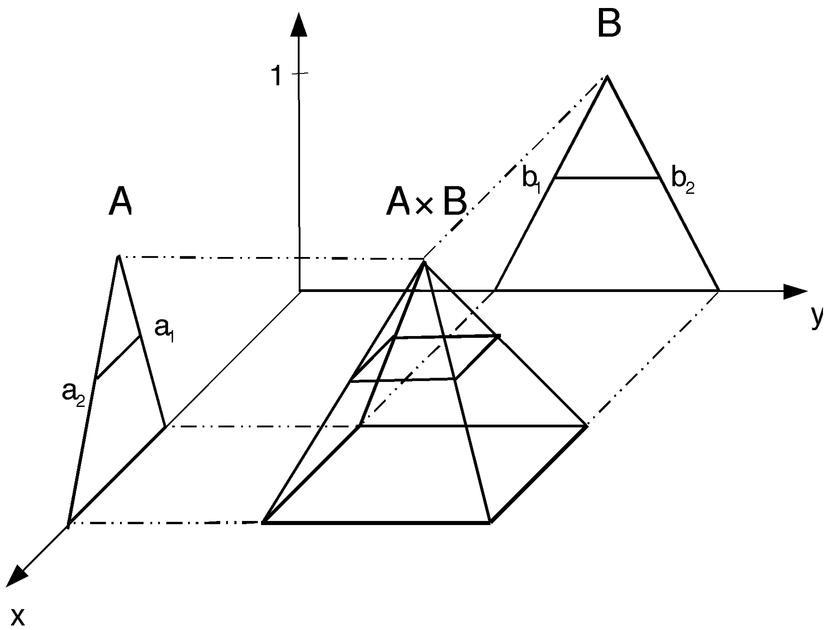

Figure 1: Independent possibility distributions.

Since all γ-level sets of C are degenerated, i.e. their integrals vanish, the following formal calculations can be done

$$\int_{[C]^\gamma} dxdy = \int_{b_1}^{b_2} [x]_{c(y)}^{c(y)} dy$$

and

$$\int_{[C]^\gamma} xydxdy = \int_{b_1}^{b_2} y \left[\frac{x^2}{2}\right]_{c(y)}^{c(y)} dy,$$

which turns into

$$\begin{aligned}
\text{center}([AB]^\gamma) &= \frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} xydxdy \\
&= \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} yc(y)dy \\
&= \frac{(a_2 - a_1)(b_2 - b_1)}{3} + \frac{a_1 b_2 + a_2 b_1}{2}.
\end{aligned}$$

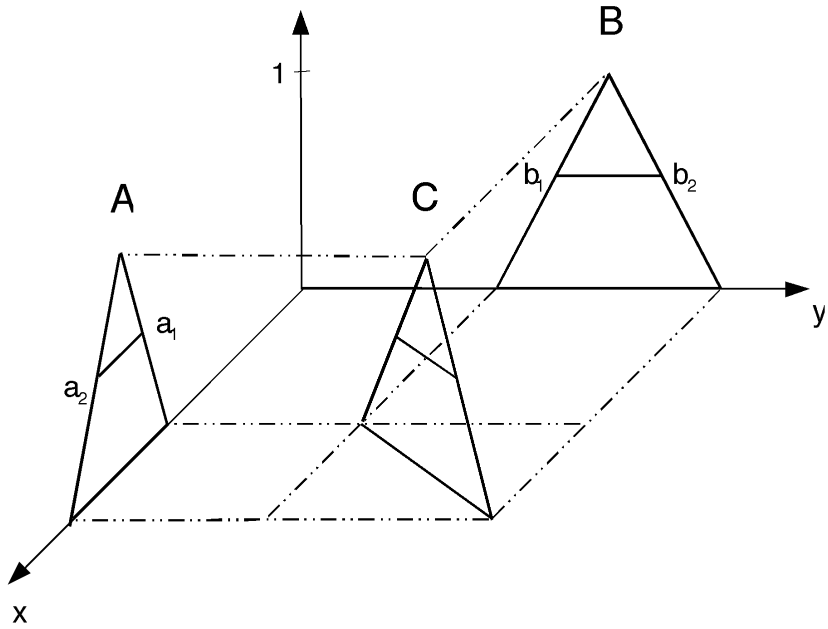

Figure 2: The joint possibility distribution C.

Hence, we have

$$\begin{aligned} R_\gamma(A,B) &= \text{center}([AB]^\gamma) - \text{center}([A]^\gamma)\text{center}([B]^\gamma) \\ &= \frac{(a_2-a_1)(b_2-b_1)}{3} + \frac{a_1b_2+a_2b_1}{2} - \frac{(a_1+a_2)(b_1+b_2)}{4} \\ &= \frac{(a_2-a_1)(b_2-b_1)}{12}, \end{aligned} \tag{7}$$

and, finally, the covariance of A and B with respect to their joint possibility distribution C is

$$\text{Cov}_f(A,B) = \frac{1}{12}\int_0^1 [a_2(\gamma)-a_1(\gamma)][b_2(\gamma)-b_1(\gamma)]f(\gamma)d\gamma.$$

Remark 6.1. *Let* $\text{Cov}(A,B)$ *denote the possibilistic covariance of* A *and* B *introduced in [3] as*

$$\text{Cov}(A,B) = \frac{1}{2}\int_0^1 [a_2(\gamma)-a_1(\gamma)][b_2(\gamma)-b_1(\gamma)]\gamma d\gamma.$$

If $f(\gamma)=2\gamma$*, then we find*

$$\text{Cov}_f(A,B) = \frac{1}{6}\int_0^1 [a_2(\gamma)-a_1(\gamma)][b_2(\gamma)-b_1(\gamma)]\gamma d\gamma,$$

and we have

$$\text{Cov}_f(A,B) = \frac{\text{Cov}(A,B)}{3}.$$

Now, let us consider a joint distribution D given by (see Figure 3)

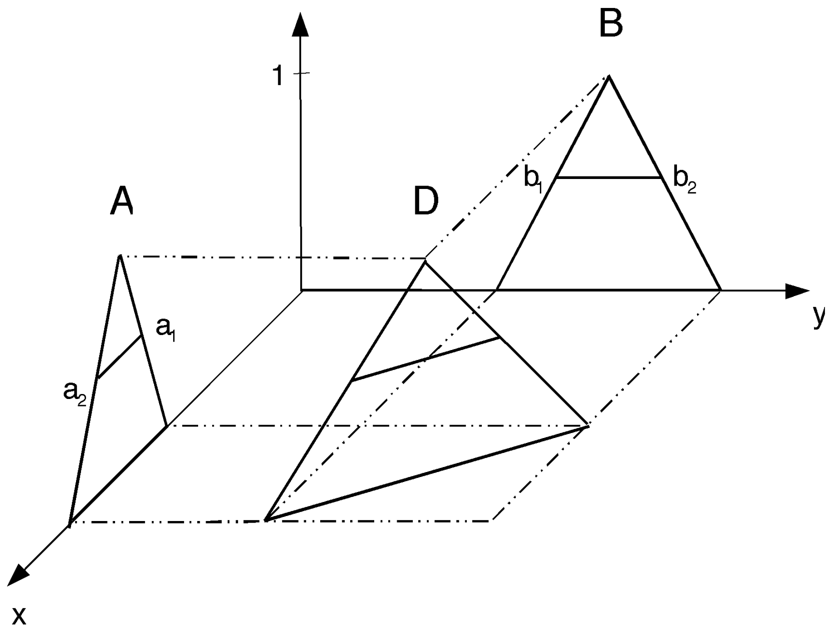

Figure 3: The joint possibility distribution D.

$$[D]^\gamma = \{t(a_1(\gamma), b_2(\gamma)) + (1-t)(a_2(\gamma), b_1(\gamma)) : t \in [0,1]\}.$$

After similar calculations we get

$$\text{center}([AB]^\gamma) = -\frac{(a_2 - a_1)(b_2 - b_1)}{3} + \frac{a_1 b_1 + a_2 b_2}{2},$$

which implies that

$$R_\gamma(A,B) = \text{center}([AB]^\gamma) - \text{center}([A]^\gamma)\text{center}([B]^\gamma) = -\frac{(a_2 - a_1)(b_2 - b_1)}{12},$$

and we find

$$\text{Cov}_f(A,B) = -\frac{1}{12}\int_0^1 [a_2(\gamma) - a_1(\gamma)][b_2(\gamma) - b_1(\gamma)] f(\gamma) d\gamma.$$

If $f(\gamma) = 2\gamma$, then we find

$$\text{Cov}_f(A,B) = -\frac{\text{Cov}(A,B)}{3}.$$

This example clearly shows that the covariance between two possibility distributions can be negative, zero, or positive depending on the definition of their joint possibility distribution.

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