On constrained OWA aggregations

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Abstract

Yager considered the problem of maximizing an OWA aggregation of a group of variables that are interrelated and constrained by a collection of linear inequalities and he showed how this problem can be modeled as an integer linear programming problem. In this short communication we show a simple algorithm for exact computation of optimal solutions to constrained OWA aggregation problems with a single constraint on the sum of all decision variables.

Keywords: OWA operators, Constrained optimization

1 Constrained OWA aggregations

An Ordered Weighted Averaging (OWA) is a mapping $F\colon \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $w = (w_1, \dots, w_n)^T$ of having the properties $w_1 + \cdots + w_n = 1$, $0 \le w_i \le 1,\ i = 1, \dots, n$, and such that

$$F(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i y_i,$$

where y_j is the jth largest element of the bag $\{x_1, \dots, x_n\}$.

The constrained OWA aggregation problem [1] can be expressed as the following mathematical programming problem

$$\max\ F(x_1, \dots, x_n);\ \text{subject to } \{Ax \le b,\ x \ge 0\},$$

where $F(x_1, \dots, x_n) = w^T y = w_1 y_1 + \cdots + w_n y_n$ and y_j denotes the jth largest element of the bag $\{x_1, \dots, x_n\}$.

In this paper we shall show a simple algorithm for solving the following constrained OWA aggregation problem

$$\max\ w^T y;\ \text{subject to } \{x_1 + \cdots + x_n \le 1,\ x \ge 0\}, \tag{1}$$

which is an n-dimensional extension of the 3-dimensional problem analyzed by Yager in [1].

First using the relations $y_1 \ge y_2 \ge \cdots \ge y_n \ge 0$, we rewrite (1) in the form

$$\max\ w^T y;\ \text{subject to } \hat{G} y \le q, \tag{2}$$

where

$$\hat{G} = \begin{bmatrix} e^T \\ G \end{bmatrix},$$

and $q = (1, 0, 0, \dots, 0)^T \in \mathbb{R}^{n+1}$, $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$, and $G = (g_{ij})$ with $g_{ij} = 1$ if $i = j - 1$, $g_{ij} = -1$ if $i = j$, and $g_{ij} = 0$ otherwise, for $i, j = 1, \dots, n$.

We note here that the condition $y \ge 0$ is implicitly included in problem (2). The dual problem of (2) can be formulated as

$$\min q^T \hat{z};\ \text{subject to } \{\hat{z}^T \hat{G} = w^T,\ \hat{z} \ge 0\}, \tag{3}$$

where $\hat{z} = [t, z_1, \dots, z_n]^T \in \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$ is a real number. It is easy to see that problem (3) can be written as

$$\min t;\ \text{subject to } \{t - z_1 = w_1,\ t - z_2 + z_1 = w_2, \dots, t - z_n + z_{n-1} = w_n\}, \tag{4}$$

where $t \ge 0$ and $z \ge 0$. Summing up the first k conditions of (4) for $k = 1, \dots, n$, we get $kt - z_k = w_1 + \cdots + w_k$, that is,

$$t = \frac{w_1 + \cdots + w_k}{k} + \frac{z_k}{k},\ k = 1, \dots, n. \tag{5}$$

So problem (3) is equivalent to the problem

$$\min t; \text{ subject to } \left\{ t = \frac{w_1 + \cdots + w_k}{k} + \frac{z_k}{k}, k = 1, \ldots, n \right\}, \tag{6}$$

where $z_1, \ldots, z_n \geq 0$. From $z_k/k \geq 0$ it follows that any solution, t^*, to problem (6) should satisfy the inequality,

$$t^* \geq \max_{k=1,2,\ldots,n} \frac{w_1 + \cdots + w_k}{k}.$$

Introducing the notations

$$z_j^* = j \times \max_{k=1,2,\ldots,n} \frac{w_1 + \cdots + w_k}{k} - (w_1 + \cdots + w_j),\ j = 1, \ldots, n.$$

we find that $z_j^* \geq 0$ for $j = 1, \ldots, n$ and, furthermore,

$$\begin{aligned} t^* = \frac{w_1 + \cdots + w_j}{j} + \frac{z_j^*}{j} = \max_{k=1,2,\ldots,n} \frac{w_1 + \cdots + w_k}{k} = \\ \max\left\{ w_1, \frac{w_1 + w_2}{2}, \ldots, \frac{w_1 + \cdots + w_n}{n} \right\},\ j = 1, \ldots, n. \end{aligned} \tag{7}$$

Therefore, $\hat{z}^* = [t^*, z_1^*, \ldots, z_n^*]$ is an optimal solution to problem (3). Let us introduce the notations

$$y^k = (\overbrace{1/k}^{\text{1-st}}, \ldots \overbrace{1/k}^{k\text{-th}}, 0, \ldots, 0)^T \in \mathbb{R}^n,\ k = 1, \ldots, n. \tag{8}$$

It can easily be checked that each y^k satisfy all conditions of problem (2). Using the duality theorem we have

$$\begin{aligned} \max_{k=1,\ldots,n} \frac{w_1 + \cdots + w_k}{k} = \max_{k=1,\ldots,n} w^T y^k \leq \max\{w^T y | \hat{G} y \leq q\} \leq \\ \min\{q^T \hat{z} | \hat{z}^T \hat{G} = w^T, \hat{z} \geq 0\} = \max_{k=1,\ldots,n} \frac{w_1 + \cdots + w_k}{k}, \end{aligned}$$

which means that any optimal solution to problem (2) should belong to the set $\{y^1, \ldots, y^n\}$.

Summary 1.1. *To find an optimal solution to (2) we should proceed as follows: select the maximal element of the set*

$$\max\left\{ w_1, \frac{w_1 + w_2}{2}, \ldots, \frac{w_1 + \cdots + w_n}{n} \right\},$$

and then choose the corresponding element from (8).

2 Illustration

As an example, consider the following 4-dimensional constrained OWA aggregation problem

$$\max\; F(x_1, x_2, x_3, x_4);\ \text{subject to } \{x_1 + x_2 + x_3 + x_4 \le 1, x \ge 0\}. \quad (9)$$

Then the set of all conceivable optimal values is constructed as

$$H = \left\{ w_1, \frac{w_1 + w_2}{2}, \frac{w_1 + w_2 + w_3}{3}, \frac{w_1 + w_2 + w_3 + w_4}{4} \right\}$$

and, the correspending optimal solutions are

1. If $\max H = w_1$ then an optimal solution to problem (9) will be $x_1^* = 1, x_2^* = x_3^* = x_4^* = 0$ with $F(x^*) = w_1$.
2. If $\max H = (w_1 + w_2)/2$ an optimal solution to problem (9) will be $x_1^* = x_2^* = 1/2, x_3^* = x_4^* = 0$ with $F(x^*) = (w_1 + w_2)/2$.
3. If $\max H = (w_1 + w_2 + w_3)/3$ an optimal solution to problem (9) will be $x_1^* = x_2^* = x_3^* = 1/3, x_4^* = 0$ with $F(x^*) = (w_1 + w_2 + w_3)/3$.
4. If $\max H = (w_1 + w_2 + w_3 + w_4)/4$ an optimal solution to problem (9) will be $x_1^* = x_2^* = x_3^* = x_4^* = 1/4$ with $F(x^*) = (w_1 + w_2 + w_3 + w_4)/4$.

Remark 2.1. *From the commutativity of OWA operators it follows that all permutations of the coordinates of an optimal solution are also optimal solutions to constrained OWA aggregation problems.*

References

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