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Abstract

Pin's variety theorem for positive varieties of string languages and varieties of finite ordered semigroups is proved for trees, i.e., a bijective correspondence between positive varieties of tree languages and varieties of finite ordered algebras is established. This, in turn, is extended to generalized varieties of finite ordered algebras, which corresponds to Steinby's generalized variety theorem. Also, families of tree languages and classes of ordered algebras that are definable by ordered (syntactic or translation) monoids are characterized.

Keywords: Tree languages, Tree automata, Variety theorem, Ordered algebra, Ordered monoids

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1 Introduction

The story of variety theory begins with Eilenberg's celebrated variety theorem [5] which was motivated by characterizations of several families of string languages by syntactic monoids or semigroups (see [5, 12]), above all by Schützenberger's [20] theorem connecting star-free languages and aperiodic monoids. A fascinating feature of this variety theorem is the existence of lots of *instances* of it. Indeed most of interesting classes of algebraic structures form varieties, and similarly, most of interesting families of tree or string languages in the literature turn out to be varieties of some kind.

Eilenberg's theorem has since been extended in various directions. One of these extensions, which is generalized in this paper for trees, is Pin's positive variety theorem [13] which established a bijective correspondence between positive varieties of string languages and varieties of ordered semigroups. Another extension is Thérien's [24] which includes also varieties of congruences on free monoids.

Concerning trees, which are studied on the level of universal algebra, Steinby's variety theorem [21] for varieties of tree languages and varieties of finite algebras was the first one of this kind. The correspondence to varieties of congruences, and some other generalizations, were added later by Almeida [1] and Steinby [22, 23]. Another variety theorem for trees is Ésik's [6] correspondence between families of tree languages and varieties of theories (see also [7]).

As Ésik [6] notes any variety theorem connects families of tree languages to a class of some structures via their 'syntactic structures'. One of these syntactic structures is the syntactic semigroup/monoid of a tree language introduced by Thomas [25] and further studied by Salomaa [16]. A different formalism, based on the essentially same concept, was brought up by Nivat and Podelski [11, 15]. Very recently a variety theorem for syntactic semigroups/monoids was proved by Salehi [17]. The newest syntactic structure for binary trees is the 'syntactic tree algebra' introduced by Wilke [26] for which a variety theorem is proved by Salehi and Steinby [18].

In Section 2, we review the basic notions of ordered algebras, ideals and quotient ordered algebras. Ordered algebras play an important role in the field, as Bloom and Wright [4] put it "Ever since Scott popularized their use in [19], ordered algebras have been used in many places in theoretical computer science".

In Section 3, positive varieties of tree languages are introduced and a variety theorem for these varieties and varieties of finite ordered algebras is proved. Informally speaking, a positive variety is a family of recognizable languages which satisfies the definition of a variety except for being closed under complements. Several families of (tree or string) languages are known to be closed under all the variety operations, including intersections and unions, but not under complementation. Pin's positive variety theorem [13] provides a characterization for these families via their syntactic ordered semigroups, see also [8, 14].

In Section 4, positive variety theorem from Section 3 is extended to generalized varieties. Generalized varieties were introduced by Steinby [23] where generalization refers to omitting the condition of having a fixed ranked alphabet; indeed a generalized variety of tree languages or of finite algebras contains tree languages or algebras over any ranked alphabet. This is used for proving a variety theorem for trees and ordered monoids in Section 5.

In Section 5, the results of [17] are extended to ordered monoids. Roughly speaking, a triple correspondence between generalized varieties of finite ordered algebras, generalized positive varieties of tree languages and varieties of finite ordered monoids is presented. This suggests the thesis that once the condition of being closed under complements is removed from the definition of variety, the resulted family (called positive variety) corresponds to a class of ordered syntactic structures of the variety; see also the positive variety theorem by Ésik in [6] Section 12.

Throughout the paper some examples are presented for illustrating the theories and their applicabilities.

2 Ordered Algebras

In this section after reviewing the terminology of ordered sets and ordered algebras we define the notions of ideals, quotient ordered algebras and syntactic ordered algebras, cf. [3].

2.1 Basic Notions

Let A be a set. The diagonal relation on A is denoted by Δ_A, that is $\Delta_A = \{(a,a) \mid a \in A\}$. For a binary relation $\delta \subseteq A \times A$, the *reverse* of δ is the relation $\delta^{-1} = \{(b,a) \mid (a,b) \in \delta\}$, and if σ is also a binary relation on A, the composition of δ and σ is

$$\delta \circ \sigma = \{(a,c) \mid (a,b) \in \delta \,\&\, (b,c) \in \sigma \text{ for some } b \in A\}.$$

Let δ be a binary relation on A. The relation δ is

- *reflexive*, if it contains the diagonal relation, i.e., $\Delta_A \subseteq \delta$;
- *anti-symmetric*, if the intersection of it with its reverse is contained in the diagonal relation, i.e., $\delta \cap \delta^{-1} \subseteq \Delta_A$;
- *symmetric*, if it equals to its reverse, i.e., $\delta = \delta^{-1}$; and
- *transitive*, if it contains its composition with itself, i.e., $\delta \circ \delta \subseteq \delta$.

A binary relation on A is called

- a *quasi-order* on A, if it is reflexive and transitive;
- an *order* on A, if it is reflexive, anti-symmetric and transitive; and
- an *equivalence* on A, if it is reflexive, symmetric and transitive.

For an equivalence relation θ on A, the equivalence θ-class of an $a \in A$ is $a/\theta = \{b \in A \mid a\theta b\}$ and the quotient set A/θ is $\{a/\theta \mid a \in A\}$.

It is easy to see that for a quasi-order $\preccurlyeq$ on A, the relation $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ is an equivalence relation on A, called the *equivalence relation of* $\preccurlyeq$, and the relation $\leqslant$ defined on the quotient set A/θ by $a/\theta \leqslant b/\theta \iff a \preccurlyeq b$ for

$a, b \in A$, is a well-defined order on A/θ. This order $\leqslant$ on A/θ is called the *order induced by the quasi-order* $\preccurlyeq$ on A.
A finite set of function symbols is called a *ranked alphabet*. If Σ is a ranked alphabet, the set of m-ary function symbols of Σ is denoted by Σ_m $(m \geq 0)$. In particular, Σ_0 is the set of constant symbols of Σ. For a ranked alphabet Σ, a Σ-*algebra* is a structure $\mathcal{A} = (A, \Sigma)$ where A is a set, and the operations of Σ are interpreted in A, that is to say, any $c \in \Sigma_0$ is interpreted by an element $c^{\mathcal{A}} \in A$, and any $f \in \Sigma_m$ $(m > 0)$ is interpreted by an m-ary function $f^{\mathcal{A}} : A^m \to A$.

Let Σ be a ranked alphabet. An ordered Σ-algebra is a structure $\mathcal{A} = (A, \Sigma, \leqslant)$ where the structure (A, Σ) is an algebra and $\leqslant$ is an order on A which is *compatible with* Σ, that is to say, for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m, b_1, \cdots, b_m \in A$,

$$\text{if } a_1 \leqslant b_1, \cdots, a_m \leqslant b_m, \text{ then } f^{\mathcal{A}}(a_1, \cdots, a_m) \leqslant f^{\mathcal{A}}(b_1, \cdots, b_m).$$

An equivalence relation θ on A is Σ-*congruence*, if for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m, b_1, \cdots, b_m \in A$,

$$\text{if } a_1 \,\theta\, b_1, \cdots, a_m \,\theta\, b_m, \text{ then } f^{\mathcal{A}}(a_1, \cdots, a_m) \,\theta\, f^{\mathcal{A}}(b_1, \cdots, b_m).$$

We note that any algebra (A, Σ) in the classical sense is an ordered algebra (A, Σ, Δ_A) in which the order relation is equality.
Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras.

- The structure $\mathcal{B}$ is an *order subalgebra* of $\mathcal{A}$, in notation $\mathcal{B} \subseteq \mathcal{A}$, if (B, Σ) is a subalgebra of (A, Σ) and $\leqslant'$ is the restriction of $\leqslant$ on B.
- A mapping $\varphi : A \to B$ is an *order morphism*, if it is a Σ-morphism, that is to say $c^{\mathcal{A}}\varphi = c^{\mathcal{B}}$ and $f^{\mathcal{A}}(a_1, \cdots, a_m)\varphi = f^{\mathcal{B}}(a_1\varphi, \cdots, a_m\varphi)$ for any $c \in \Sigma_0, f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m \in A$, and preserves the orders, i.e., for any $a, b \in A$, if $a \leqslant b$ then $a\varphi \leqslant' b\varphi$. In that case we write $\varphi : \mathcal{A} \to \mathcal{B}$. The order morphism φ is an *order epimorphism*, if it is surjective, and then $\mathcal{B}$ is an *order epimorphic image of* $\mathcal{A}$, in notation $\mathcal{B} \leftarrow \mathcal{A}$. If $\mathcal{B}$ is an order epimorphic image of an order subalgebra of $\mathcal{A}$, then we say that $\mathcal{B}$ *divides* $\mathcal{A}$ and we write $\mathcal{B} \prec \mathcal{A}$. If φ is injective, then it is an *order monomorphism*. When φ is bijective and its reverse is also an order morphism, then it is an *order isomorphism*. We write $\mathcal{A} \cong \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic.
- The direct product of $\mathcal{A}$ and $\mathcal{B}$ is the structure $(A \times B, \Sigma, \leqslant \times \leqslant')$ where $(A \times B, \Sigma)$ is the product of the algebras (A, Σ) and (B, Σ), and the relation $\leqslant \times \leqslant'$ is defined on $A \times B$ by $(a, b) \leqslant \times \leqslant' (c, d) \iff a \leqslant b \;\&\; c \leqslant' d$ for $(a, b), (c, d) \in A \times B$. It is easy to see that the structure $(A \times B, \Sigma, \leqslant \times \leqslant')$ is an order algebra which is denoted by $\mathcal{A} \times \mathcal{B}$.

A *variety of finite ordered algebras*, a VFOA for short, is a class of finite ordered algebras closed under order subalgebras, order epimorphic images, and direct products.

2.2 Ideals and Quotient Ordered Algebras

Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra.

Definition 2.1 A *quasi-order on* $\mathcal{A}$ is a quasi-order $\preccurlyeq$ on A that contains $\leqslant$, i.e., $\preccurlyeq \supseteq \leqslant$, and is compatible with Σ, i.e., for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m, b_1, \cdots, b_m \in A$, if $a_1 \preccurlyeq b_1, \cdots, a_m \preccurlyeq b_m$, then

$$f^{\mathcal{A}}(a_1, \cdots, a_m) \preccurlyeq f^{\mathcal{A}}(b_1, \cdots, b_m).$$

Let $\preccurlyeq$ be a quasi-order on $\mathcal{A}$. The relation $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ is a congruence on (A, Σ). So, the quotient structure $(A/\theta, \Sigma)$ is a Σ-algebra. Moreover, the relation $\leq$ defined on A/θ by $a/\theta \leq b/\theta \iff a \preccurlyeq b$ for $a, b \in A$, is a well-defined order compatible with Σ. Hence the structure $(A/\theta, \Sigma, \leq)$ is an ordered algebra. It can be noticed that quasi-orders on ordered algebras play the role of congruences on ordinary algebras.

Definition 2.2 For a quasi-order $\preccurlyeq$ on $\mathcal{A}$, the *quotient* of $\mathcal{A}$ under $\preccurlyeq$ is the structure $\mathcal{A}/\preccurlyeq = (A/\theta, \Sigma, \leq)$ where $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ is the Σ-congruence induced by $\preccurlyeq$ and $\leq$ is the order induced by $\preccurlyeq$.

Lemma 2.3 For ordered algebras $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$, and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$, if $\preccurlyeq$ is a quasi-order on $\mathcal{B}$, then the relation $\varphi \circ \preccurlyeq \circ \varphi^{-1}$ satisfies $a\ \varphi \circ \preccurlyeq \circ \varphi^{-1}\ c \iff a\varphi \preccurlyeq c\varphi$ for all $a, c \in A$, and is a quasi-order on $\mathcal{A}$.

Moreover, if θ is the congruence on $\mathcal{B}$ induced by $\preccurlyeq$, then the congruence on $\mathcal{A}$ induced by $\varphi \circ \preccurlyeq \circ \varphi^{-1}$ is $\varphi \circ \theta \circ \varphi^{-1}$.

Proof. The first claim is obvious. For the second we note that
$$\varphi \circ \preccurlyeq \circ \varphi^{-1} \cap (\varphi \circ \preccurlyeq \circ \varphi^{-1})^{-1} = \varphi \circ (\preccurlyeq \cap \preccurlyeq^{-1}) \circ \varphi^{-1} = \varphi \circ \theta \circ \varphi^{-1}. \qquad \square$$

Proposition 2.4 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras, $\preccurlyeq$ be a quasi-order on B, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism. Then

(1) the image of $\mathcal{A}$, $\mathcal{A}\varphi = (A\varphi, \Sigma, \leqslant'')$ where $\leqslant''$ is the restriction of $\leqslant'$ on $A\varphi$, is an order subalgebra of $\mathcal{B}$,

(2) $\mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \cong \mathcal{A}\varphi/\preccurlyeq'$ where $\preccurlyeq'$ is the restriction of $\preccurlyeq$ on $A\varphi$, and

(3) moreover, if φ is an order epimorphism, then $\mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \cong \mathcal{B}/\preccurlyeq$.

Proof. The statement (1) is straightforward and (3) follows from (2). For proving (2) we note that the mapping $\psi : \mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \to \mathcal{A}\varphi/\preccurlyeq'$ defined by $(a/\varphi \circ \theta \circ \varphi^{-1})\psi = a\varphi/\theta$ for $a \in A$, where $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$, is an order isomorphism, cf. Lemma 2.3. $\square$

The particular case of the Proposition 2.4 when $\preccurlyeq = \leqslant'$ is of interest: then $\theta = \Delta_B$ and $\varphi \circ \theta \circ \varphi^{-1} = \ker\varphi$, hence we get the first homomorphism theorem for ordered algebras, namely $\mathcal{A}/\varphi \circ \leqslant' \circ \varphi^{-1} \cong \mathcal{A}\varphi$. Similar results for semigroups can be found in [9, 10].

Proposition 2.5 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra, and $\preccurlyeq, \preccurlyeq'$ be two quasi-orders on $\mathcal{A}$.

(1) If $\preccurlyeq \subseteq \preccurlyeq'$, then $\mathcal{A}/\preccurlyeq' \leftarrow \mathcal{A}/\preccurlyeq$.

(2) The relation $\preccurlyeq \cap \preccurlyeq'$ is a quasi-order on $\mathcal{A}$ and the following holds:

$$\mathcal{A}/\preccurlyeq \cap \preccurlyeq' \subseteq \mathcal{A}/\preccurlyeq \times \mathcal{A}/\preccurlyeq' .$$

The proof is straightforward.

Let us recall the definition of translations of an algebra (see e.g. [21, 22, 23]). For an algebra $\mathcal{A} = (A, \Sigma)$, an m-ary function symbol $f \in \Sigma_m$ $(m > 0)$ and elements $a_1, \cdots, a_m \in A$, the term $f^{\mathcal{A}}(a_1, \cdots, \xi, \cdots, a_m)$ where the new symbol ξ sits in the i-th position (for some $i \leq m$) determines a unary function $A \to A$ defined by $a \mapsto f^{\mathcal{A}}(a_1, \cdots, a, \cdots, a_m)$ which is an *elementary translation* of $\mathcal{A}$. The set of translations of $\mathcal{A}$ denoted by $\mathrm{Tr}(\mathcal{A})$ is the smallest set that contains the identity function and elementary translations, and is closed under the compositions of unary functions. The composition of translations p and q is denoted by $q \cdot p$, that is $(q \cdot p)(a) = p(q(a))$ for all $a \in A$. We note that the set $\mathrm{Tr}(\mathcal{A})$ equipped with the composition operation is a monoid, called the *translation monoid* of $\mathcal{A}$.

Definition 2.6 For an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$, a subset $I \subseteq A$ is an *ideal* of $\mathcal{A}$, in notation $I \trianglelefteq A$, if for every $a, b \in A$, $a \leqslant b \in I$ implies $a \in I$. For any $a \in A$, $(a] = \{b \in A \mid b \leqslant a\}$ is the ideal of $\mathcal{A}$ *generated* by a.
The *syntactic quasi-order* of an ideal I, denoted by $\preccurlyeq_I$, is defined by

$$a \preccurlyeq_I b \equiv \big(\forall p \in \mathrm{Tr}(\mathcal{A})\big)\big(p(b) \in I \Rightarrow p(a) \in I\big)$$

for $a, b \in A$. Note that $\preccurlyeq_I$ is a quasi-order on $\mathcal{A}$. The *syntactic ordered algebra* of I is the quotient ordered algebra $\mathrm{SOA}(I) = \mathcal{A}/\preccurlyeq_I$, also denoted by $\mathcal{A}/I$ (cf. [13]).

We note that the equivalence relation of $\preccurlyeq_I$ for any ideal I is the syntactic congruence of I in the classical sense:

$$a(\preccurlyeq_I \cap \preccurlyeq_I^{-1})b \iff (\forall p \in \mathrm{Tr}(\mathcal{A}))(p(a) \in I \Leftrightarrow p(b) \in I),$$

which is denoted by θ_I, that is $\theta_I = \preccurlyeq_I \cap \preccurlyeq_I^{-1}$ (see e.g. [21, 22]).

Trivially, any subset $I \subseteq A$ of the ordered algebra $\mathcal{A} = (A, \Sigma, \Delta_A)$ is an ideal of $\mathcal{A}$. The following is essentailly Lemma 3.2 of [22].

Proposition 2.7 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism. The mapping φ induces a monoid morphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_\varphi$ such that $p(a)\varphi = p_\varphi(a\varphi)$ for all $a \in A$. Moreover, if φ is an order epimorphism then the induced map is a monoid epimorphism.

Proof. For any elementary translation $p = f^{\mathcal{A}}(a_1, \cdots, \xi, \cdots, a_m)$ of $\mathcal{A}$ where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m \in A$, the unary function p_φ on B defined by $b \mapsto f^{\mathcal{B}}(a_1\varphi, \cdots, b, \cdots, a_m\varphi)$ is an elementary translation of $\mathcal{B}$, and if φ is surjective then every elementary translation of $\mathcal{B}$ is in this form. The mapping $p \mapsto p_\varphi$ can be extended to all translations by setting $(1_A)_\varphi = 1_B$ and $(p \cdot q)_\varphi = p_\varphi \cdot q_\varphi$. The identity $p_\varphi(a\varphi) = p(a)\varphi$ obviously holds for all $a \in A$ and $p \in \mathrm{Tr}(\mathcal{A})$. $\square$

For a subset $D \subseteq A$ and a translation $p \in \mathrm{Tr}(\mathcal{A})$, the inverse translation of D under p is $p^{-1}(D) = \{a \in A \mid p(a) \in D\}$, and for an order morphism $\varphi : \mathcal{B} \to \mathcal{A}$, the inverse image of D under φ is $D\varphi^{-1} = \{b \in B \mid b\varphi \in D\}$. Positive Boolean operations are intersection and union of sets, while Boolean operations also include the complement operation.

Lemma 2.8 The collection of all ideals of any ordered algebra is closed under positive Boolean operations, inverse translations and inverse order morphisms. That is to say, for ordered algebras $\mathcal{A}$ and $\mathcal{B}$, ideals $I, J \trianglelefteq \mathcal{A}$, $K \trianglelefteq \mathcal{B}$, and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$, the sets $I \cap J, I \cup J, p^{-1}(I)$ and $K\varphi^{-1}$ are ideals of $\mathcal{A}$.

The proof is straightforward (cf. [13]). We note that the complement of an ideal may not be an ideal.

Proposition 2.9 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be ordered algebras, $I, J \trianglelefteq \mathcal{A}$, $K \trianglelefteq \mathcal{B}$ be ideals, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism. Then the following inclusions hold:

(1) $\preccurlyeq_{I\cap J}, \preccurlyeq_{I\cup J} \supseteq \preccurlyeq_I \cap \preccurlyeq_J$.

(2) $\preccurlyeq_{p^{-1}(I)} \supseteq \preccurlyeq_I$.

(3) $\preccurlyeq_{K\varphi^{-1}} \supseteq \varphi\circ \preccurlyeq_K \circ\varphi^{-1}$, and if φ is an order epimorphism then the equality holds: $\preccurlyeq_{K\varphi^{-1}} = \varphi\circ \preccurlyeq_K \circ\varphi^{-1}$.

Proof. The statements (1) and (2) are obvious. We prove (3): suppose $(a, b) \in \varphi\circ \preccurlyeq_K \circ\varphi^{-1}$ for some $a, b \in A$. Then $a\varphi \preccurlyeq_K b\varphi$. So, for any $p \in \mathrm{Tr}(\mathcal{A})$,

$$\begin{array}{rcl} p(b) \in K\varphi^{-1} & \Rightarrow & p(b)\varphi \in K \\ & \Rightarrow & p_\varphi(b\varphi) \in K \\ & \Rightarrow & p_\varphi(a\varphi) \in K \\ & \Rightarrow & p(a)\varphi \in K \\ & \Rightarrow & p(a) \in K\varphi^{-1}. \end{array}$$

Therefore $a \preccurlyeq_{K\varphi^{-1}} b$, and hence $\varphi\circ \preccurlyeq_K \circ\varphi^{-1} \subseteq \preccurlyeq_{K\varphi^{-1}}$. In the case when φ is surjective we note that by Proposition 2.7 every translation $q \in \mathrm{Tr}(\mathcal{B})$ is in the form p_φ for some $p \in \mathrm{Tr}(\mathcal{A})$. Thus in this case $\preccurlyeq_{K\varphi^{-1}} \subseteq \varphi\circ \preccurlyeq_K \circ\varphi^{-1}$ holds and so does the equality $\preccurlyeq_{K\varphi^{-1}} = \varphi\circ \preccurlyeq_K \circ\varphi^{-1}$. □

Combining Propositions 2.9, 2.5 and 2.4 we get the following.

Corollary 2.10 For ordered algebras $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$, ideals $I, J \trianglelefteq \mathcal{A}$ and $K \trianglelefteq \mathcal{B}$, and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$,
(1) $\mathrm{SOA}(I \cap J), \mathrm{SOA}(I \cup J) \prec \mathrm{SOA}(I) \times \mathrm{SOA}(J)$.
(2) $\mathrm{SOA}(p^{-1}(I)) \leftarrow \mathrm{SOA}(I)$.
(3) $\mathrm{SOA}(K\varphi^{-1}) \prec \mathrm{SOA}(K)$ and, moreover, if φ is an order epimorphism then $\mathrm{SOA}(K\varphi^{-1}) \cong \mathrm{SOA}(K)$.

2.3 Examples

We introduce some classes of ordered algebras and prove some of their elementary properties which will be used later.

For an algebra $\mathcal{A} = (A, \Sigma)$, the set of *non-trivial* translations $\mathrm{TrS}(\mathcal{A})$ of $\mathcal{A}$ consists of the elementary translations $f^{\mathcal{A}}(a_1, \cdots, \xi, \cdots, a_m)$ for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m \in A$, and their compositions. We note that $\mathrm{TrS}(\mathcal{A})$ does not automatically include the identity translation 1_A. The set $\mathrm{TrS}(\mathcal{A})$ is a semigroup with the composition operation which is the *translation semigroup* of $\mathcal{A}$.

2.3.1 Ordered nilpotent algebras

Definition 2.11 An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *ordered n-nilpotent* $(n \in \mathbb{N})$, when $p_1 \cdots p_n(a) \leqslant b$ holds for all $a, b \in A$ and non-trivial translations $p_1, \cdots, p_n \in \mathrm{TrS}(\mathcal{A})$.

An ordered algebra is *ordered nilpotent* if it is ordered n-nilpotent for some $n \in \mathbb{N}$.

The class of all ordered nilpotent Σ-algebras is denoted by $\mathbf{Nil}(\Sigma)$.
An element $a_0 \in A$ is a *trap* of $\mathcal{A}$, if $p(a_0) = a_0$ holds for any $p \in \mathrm{Tr}(\mathcal{A})$.

Lemma 2.12 Every order n-nilpotent algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ has a unique trap which is the least element of the algebra.

Proof. For every $p_1, \cdots, p_n, q_1, \cdots, q_n \in \mathrm{TrS}(\mathcal{A})$ and $a, b \in A$ we have

$$p_1 \cdots p_n(a) \leqslant q_1 \cdots q_n(b) \leqslant p_1 \cdots p_n(a).$$

Thus $p_1 \cdots p_n(a) = q_1 \cdots q_n(b)$ and let a_0 be this element. Clearly $p(a_0) = a_0$ and $a_0 \leqslant a$ for every $p \in \mathrm{TrS}(\mathcal{A})$ and every $a \in A$. So, a_0 is the unique trap of $\mathcal{A}$ which is the least element. □

Proposition 2.13 The class $\mathbf{Nil}(\Sigma)$ of all ordered nilpotent Σ-algebras is a VFOA (variety of finite ordered algebras).

Proof. It can be easily seen that the class of ordered n-nilpotent algebras is closed under order subalgebras and direct products. To see that it is closed under order epimorphic images, let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras such that $\mathcal{A}$ is an ordered n-nilpotent algebra and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an order epimorphism. Let $b, d \in B$ be two elements and $q_1, \cdots, q_n \in \mathrm{TrS}(\mathcal{B})$ be non-trivial translations. There are $a, c \in A$ such that $b = a\varphi$ and $d = c\varphi$, and by Proposition 2.7, there are $p_1, \cdots, p_n \in \mathrm{TrS}(\mathcal{A})$ such that $(p_j)_\varphi = q_j$ for all $j \leq n$. From $p_1 \cdots p_n(a) \leqslant c$, the inequality $p_1 \cdots p_n(a)\varphi \leqslant' c\varphi$ follows and this implies $(p_1)_\varphi \cdots (p_n)_\varphi(a\varphi) \leqslant' c\varphi$, thus $q_1 \cdots q_n(b) \leqslant' d$ holds. Hence, $\mathcal{B}$ is an ordered n-nilpotent algebra.

Finally, the claim follows from the fact that an ordered n-nilpotent algebra is an ordered $(n+1)$-nilpotent algebra as well. □

Lemma 2.14 If $\mathcal{A} = (A, \Sigma, \leqslant)$ is an order n-nilpotent algebra, then the translation semigroup $\mathrm{TrS}(\mathcal{A})$ of $\mathcal{A}$ is a nilpotent semigroup.

Proof. For every $p_1, \cdots, p_n, q_1, \cdots, q_n \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$ we have

$$p_1 \cdots p_n(a) \leqslant q_1 \cdots q_n(a) \leqslant p_1 \cdots p_n(a).$$

Thus $p_1 \cdots p_n = q_1 \cdots q_n$, so $p_1 \cdots p_n \in \mathrm{TrS}(\mathcal{A})$ is the zero element of $\mathrm{TrS}(\mathcal{A})$ and the product of every n elements of this semigroup is zero. □

2.3.2 Semilattice algebras and symbolic ordered algebras

Sequence of elements of a set D are denoted in the bold face, for example $\mathbf{d}$ is a (possibly empty) sequence $\langle d_1, \cdots, d_m \rangle$ where $d_1, \cdots, d_m \in D$. For simplicity we write $\mathbf{d} \in D$ to mean that the components of the sequence $\mathbf{d}$ belong to D. In that case for a function symbol $f \in \Sigma_{m+1}$, $f(d, \mathbf{d})$ stands for $f(d, d_1, \cdots, d_m)$.

Definition 2.15 An algebra $\mathcal{A} = (A, \Sigma)$ is a *semilattice* algebra, if it satisfies the following two identities for every $f, g \in \Sigma$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, a \in A$:

$$f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{b}) = f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b});$$

$$f^{\mathcal{A}}(\mathbf{a}, g^{\mathcal{A}}(\mathbf{c}, a, \mathbf{d}), \mathbf{b}) = g^{\mathcal{A}}(\mathbf{c}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{d}).$$

A monoid $(M, \cdot)$ is a *semilattice* monoid, if it is commutative and idempotent, i.e., for every $a, b \in M$, $a \cdot a = a$ and $a \cdot b = b \cdot a$ hold.

Lemma 2.16 An algebra is semilattice if and only if its translation monoid is semilattice.

Fix a semilattice algebra $\mathcal{A} = (A, \Sigma)$ where $\mathrm{Tr}(\mathcal{A})$ is its translation monoid.

Lemma 2.17 For an element $a \in A$ and translations $p, q \in \mathrm{Tr}(\mathcal{A})$, if $p(q(a)) = a$ then $p(a) = q(a) = a$.

Proof. Suppose $p, q \in \mathrm{Tr}(\mathcal{A})$. Since $q \cdot q = q$, $p \cdot p = p$ and $q \cdot p = p \cdot q$, we have $q(a) = q(p(q(a))) = q(q(p(a))) = q(p(a)) = p(q(a)) = a$, and similarly $p(a) = p(p(q(a))) = p(q(a)) = a$. □

Corollary 2.18 For $a, b \in A$ and $p, q \in \mathrm{Tr}(\mathcal{A})$, if $p(a) = b$ and $a = q(b)$ then $a = b$.

Lemma 2.19 For $\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b \in A$ and $f \in \Sigma$,

(s1) $f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})$.

Proof.

$$\begin{aligned}
&f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = \\
&f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}), \mathbf{b}, b, \mathbf{c}) = \\
&f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, b, \mathbf{c}), \mathbf{c}) = \\
&f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}), \mathbf{c}) = \\
&f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}), \mathbf{b}, b, \mathbf{c}) = \\
&p(f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})).
\end{aligned}$$

where $p = f^{\mathcal{A}}(\mathbf{a}, \xi, \mathbf{b}, b, \mathbf{c})$. By the same argument and swapping a and b it can be proved that $f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}) = q(f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}))$ for some $q \in \mathrm{Tr}(\mathcal{A})$. Thus, from Corollary 2.18, it follows that $f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})$. □

Lemma 2.20 For $\mathbf{a}, a, b \in A$ and $f \in \Sigma$,

(s2) $f^{\mathcal{A}}(a, a, b, \mathbf{a}) = f^{\mathcal{A}}(a, b, b, \mathbf{a})$.

Proof. The equality of the third and fourth lines of the following is implied by (s1) Lemma 2.19:

$$\begin{aligned}
&f^{\mathcal{A}}(a, a, b, \mathbf{a}) = \\
&f^{\mathcal{A}}(f^{\mathcal{A}}(a, a, b, \mathbf{a}), a, b, \mathbf{a}) = \\
&f^{\mathcal{A}}(a, a, f^{\mathcal{A}}(b, a, b, \mathbf{a}), \mathbf{a}) = \\
&f^{\mathcal{A}}(f^{\mathcal{A}}(a, b, b, \mathbf{a}), a, a, \mathbf{a}) = \\
&p(f^{\mathcal{A}}(a, b, b, \mathbf{a}))
\end{aligned}$$

where $p = f^{\mathcal{A}}(\xi, a, a, \mathbf{a})$. By the same argument and swapping a and b it can be proved that $f^{\mathcal{A}}(a, b, b, \mathbf{a}) = q(f^{\mathcal{A}}(a, a, b, \mathbf{a}))$ for some $q \in \mathrm{Tr}(\mathcal{A})$. Hence, $f^{\mathcal{A}}(a, a, b, \mathbf{a}) = f^{\mathcal{A}}(a, b, b, \mathbf{a})$ by Corollary 2.18. □

Lemma 2.21 For $\mathbf{a}, \mathbf{b}, a, b \in A$ and $f, g \in \Sigma$,
(s3) $f^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{a}), b, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(b, \mathbf{a}), a, \mathbf{b})$.

Proof. The second equality follows from (s1) Lemma 2.19:

$$
\begin{aligned}
&f^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{a}), b, \mathbf{b}) = \\
&g^{\mathcal{A}}(f^{\mathcal{A}}(a, b, \mathbf{b}), \mathbf{a}) = \\
&g^{\mathcal{A}}(f^{\mathcal{A}}(b, a, \mathbf{b}), \mathbf{a}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(b, \mathbf{a}), a, \mathbf{b}).
\end{aligned}
$$

□

Lemma 2.22 For $a_1, a_2, \cdots, a_m \in A$ and $f \in \Sigma_m$,
(s4) $f^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \cdots, a_1), a_2, \cdots, a_m) = f^{\mathcal{A}}(a_1, a_2, \cdots, a_m)$.

Proof. The third equality is implied by (s2) Lemma 2.20:

$$
\begin{aligned}
&f^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \cdots, a_1), a_2, \cdots, a_m) = \\
&f^{\mathcal{A}}(a_1, \cdots, a_1, f^{\mathcal{A}}(a_1, a_2, \cdots, a_m)) = \\
&f^{\mathcal{A}}(a_1, f^{\mathcal{A}}(a_1, \cdots, a_1, a_1, a_2), a_3, \cdots, a_m) = \\
&f^{\mathcal{A}}(a_1, f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, a_2), a_3, \cdots, a_m) = \\
&f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)).
\end{aligned}
$$

Now, we show for any $j < m$,

$$
\begin{aligned}
&f^{\mathcal{A}}(a_1, \cdots, a_1, a_1, a_2, \ldots, a_j, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)) = \\
&f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, a_{j+1}, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)),
\end{aligned}
$$

as follows, where the second equality follows from (s1), (s2) and Lemma 2.20,

$$
\begin{aligned}
&f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)) = \\
&f^{\mathcal{A}}(a_1, a_2, \cdots, a_j, f^{\mathcal{A}}(a_1, \cdots, a_1, a_1, a_2, \ldots, a_j, a_{j+1}), a_{j+2}, \cdots, a_m) = \\
&f^{\mathcal{A}}(a_1, a_2, \cdots, a_j, f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, a_{j+1}, a_{j+1}), a_{j+2}, \cdots, a_m) = \\
&f^{\mathcal{A}}(a_1, \cdots, a_1, a_2, \ldots, a_j, a_{j+1}, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)).
\end{aligned}
$$

By repeating this argument $m - 1$ times, we get

$$
\begin{aligned}
&f^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \cdots, a_1), a_2, \cdots, a_m) = \\
&f^{\mathcal{A}}(a_1, \cdots, a_{m-1}, f^{\mathcal{A}}(a_1, a_2, a_3, \cdots, a_m)) = \\
&f^{\mathcal{A}}(a_1, a_2, \cdots, a_m).
\end{aligned}
$$

□

Lemma 2.23 For $a, b, \mathbf{a}, \mathbf{b} \in A$ and $f \in \Sigma$,
(s5) $f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), b, \mathbf{b})$.

Proof. We distinguish two cases:

(1) The sequence $\mathbf{a}$ is empty: By using identities (s4), (s3), (s1), (s3), (s3) and (s4) consecutively, we get:

$$\begin{aligned}
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b),a,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,g^{\mathcal{A}}(b,b)),a,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(b,g^{\mathcal{A}}(a,b)),a,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a,b),b),a,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a,b),a),b,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a,a),b),b,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b),b,\mathbf{b}).
\end{aligned}$$

(2) The sequence $\mathbf{a}$ is not empty: Write $\mathbf{a} = (c, \mathbf{c})$ and use identities (s3), (s1), (s2) and (s3) consecutively:

$$\begin{aligned}
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,\mathbf{a}),a,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,c,\mathbf{c}),a,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,a,\mathbf{c}),c,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,a,b,\mathbf{c}),c,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,b,\mathbf{c}),c,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,c,\mathbf{c}),b,\mathbf{b}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,\mathbf{a}),b,\mathbf{b}).
\end{aligned}$$

$\square$

Lemma 2.24 For $f \in \Sigma_m$, $g \in \Sigma_n$ where $m \le n$ $(n \ge 2)$ and $a, b, \mathbf{a}, \mathbf{b}, \mathbf{c} \in A$, the following identity, where the sequence $\overline{b}$ consists of $n - m$ times b, holds:

(s6) $f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,\mathbf{a}),\mathbf{b}),\mathbf{c}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a,\mathbf{b},\overline{b}),b,\mathbf{a}),\mathbf{c})$.

Proof. Use identities (s1), (s3) and (s4) alternatively:

$$\begin{aligned}
&f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a,b,\mathbf{a}),\mathbf{b}),\mathbf{c}) = \\
&f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a,g^{\mathcal{A}}(b,\cdots,b),\mathbf{a}),\mathbf{b}),\mathbf{c}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(b,\cdots,b),\mathbf{b}),a,\mathbf{a}),\mathbf{c}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(b,\cdots,b),\overline{b},\mathbf{b}),a,\mathbf{a}),\mathbf{c}) = \\
&g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(b,\cdots,b),\overline{b},\mathbf{b}),\mathbf{c}),a,\mathbf{a}) = \\
&g^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(f^{\mathcal{A}}(b,\cdots,b),\mathbf{c}),\overline{b},\mathbf{b}),a,\mathbf{a}) = \\
&g^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(b,\mathbf{c}),\overline{b},\mathbf{b}),a,\mathbf{c}) = \\
&g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(b,\overline{b},\mathbf{b}),\mathbf{c}),a,\mathbf{a}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(b,\overline{b},\mathbf{b}),a,\mathbf{a}),\mathbf{c}) = \\
&f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a,\mathbf{b},\overline{b}),b,\mathbf{a}),\mathbf{c}).
\end{aligned}$$

$\square$

We note that the identity corresponding to (s6) for $m = n = 1$ also holds, i.e., $f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a))) = f^{\mathcal{A}}(g^{\mathcal{A}}(a)) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a)))$.

Definition 2.25 An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *symbolic*, if it is a semilattice algebra and $f^{\mathcal{A}}(a_1, \cdots, a_m) \leqslant a_j$ holds for every $a_1, \cdots, a_m \in A$, $f \in \Sigma_m$ $(m > 0)$ and $j \leq m$.

An ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$ is *symbolic*, if it is a semilattice monoid and the unit is the greatest element of the monoid, i.e., $m \lesssim 1_M$ holds for all $m \in M$ and the unit element $1_M \in M$.

The class of all semilattice Σ-algebras is denoted by $\mathbf{SL}(\Sigma)$ and $\mathbf{Sym}(\Sigma)$ denotes the class of all symbolic ordered Σ-algebras.

The proofs of the followings are easy and thus we omit them.

Lemma 2.26 An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is symbolic if and only if it is a semilattice algebra such that $p(a) \leqslant a$ holds for all $a \in A$ and $p \in \mathrm{Tr}(\mathcal{A})$.

Proposition 2.27 The class $\mathbf{SL}(\Sigma)$ is a variety of finite algebras and the class $\mathbf{Sym}(\Sigma)$ is a VFOA.

3 Positive Variety Theorem

Recall that a ranked alphabet is a finite set of function symbols, and if Σ is a ranked alphabet, the set of m-ary function symbols of Σ is denoted by Σ_m (for every $m \geq 0$); in particular, Σ_0 is the set of constant symbols of Σ. For a ranked alphabet Σ and a *leaf alphabet* X, the set of ΣX*-trees* $\mathrm{T}(\Sigma, X)$ is the smallest set satisfying

(1) $\Sigma_0 \cup X \subseteq \mathrm{T}(\Sigma, X)$, and for any $m > 0$

(2) $f(t_1, \cdots, t_m) \in \mathrm{T}(\Sigma, X)$ for all $f \in \Sigma_m$, $t_1, \cdots, t_m \in \mathrm{T}(\Sigma, X)$.

Any subset of $\mathrm{T}(\Sigma, X)$ is a *tree language*.
The ΣX*-term algebra* $\mathcal{T}(\Sigma, X) = (\mathrm{T}(\Sigma, X), \Sigma)$ is defined by setting
(1) $c^{\mathcal{T}(\Sigma,X)} = c$ for each $c \in \Sigma_0$, and
(2) $f^{\mathcal{T}(\Sigma,X)}(t_1, \cdots, t_m) = f(t_1, \cdots, t_m)$ for all $m > 0$, $f \in \Sigma_m$, and $t_1, \cdots, t_m \in \mathrm{T}(\Sigma, X)$.
Let ξ be a (special) symbol which does not appear in any ranked alphabet or leaf alphabet considered here. The set of ΣX*-contexts*, denoted by $\mathrm{C}(\Sigma, X)$, consists of the $\Sigma(X \cup \{\xi\})$-trees in which ξ appears exactly once. For $P, Q \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$ the context $Q \cdot P$, the composite of P and Q, results from P by replacing the special leaf ξ with Q, and the term $t \cdot P$ results from P by replacing ξ with t. Note that $\mathrm{C}(\Sigma, X)$ is a monoid with the composition operation, and that $t \cdot (Q \cdot P) = (t \cdot Q) \cdot P$ holds for all $P, Q \in \mathrm{C}(\Sigma, X)$, $t \in \mathrm{T}(\Sigma, X)$. There is a bijective correspondence between $\mathrm{C}(\Sigma, X)$ and the translations of the term algebra $\mathrm{Tr}(\mathcal{T}(\Sigma, X))$ in a natural way: an elementary context $P = f(t_1, \cdots, \xi, \cdots, t_m)$ corresponds with $P^{\mathcal{T}(\Sigma,X)} = f^{\mathcal{T}(\Sigma,X)}(t_1, \cdots, \xi, \cdots, t_m)$, and the composition $P \cdot Q$ of the contexts P and Q corresponds with the composition $P^{\mathcal{T}(\Sigma,X)} \cdot Q^{\mathcal{T}(\Sigma,X)}$ of translations.

Definition 3.1 For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ the *syntactic quasi-order* $\preccurlyeq_T$ of T is defined by the following for $t, s \in \mathrm{T}(\Sigma, X)$

$$t \preccurlyeq_T s \iff (\forall P \in \mathrm{C}(\Sigma, X))\big(s \cdot P \in T \Rightarrow t \cdot P \in T\big).$$

We note that the equivalence relation $\theta_T = \preccurlyeq_T \cap \preccurlyeq_T^{-1}$ of $\preccurlyeq_T$ is the *syntactic congruence* of T:

$$t\,\theta_T\,s \iff (\forall P \in \mathrm{C}(\Sigma, X))\big(t \cdot P \in T \Leftrightarrow s \cdot P \in T\big).$$

The *syntactic ordered algebra* of T is the structure

$$\mathrm{SOA}(T) = (\mathrm{T}(\Sigma, X)/\theta_T, \Sigma, \leqslant_T),$$

where $\leqslant_T$ is the order induced by $\preccurlyeq_T$:

$$t/\theta_T \leqslant_T s/\theta_T \Leftrightarrow t \preccurlyeq_T s \text{ for } t, s \in \mathrm{T}(\Sigma, X).$$

The *syntactic morphism* of T is the mapping $\varphi_T : \mathcal{T}(\Sigma, X) \to \mathrm{SOA}(T)$ defined by $t\varphi_T = t/\theta_T$ for $t \in \mathrm{T}(\Sigma, X)$.

It can be easily seen that not every ordered algebra is the syntactic ordered algebra of a tree language. However, these syntactic ordered algebras can be characterized as follows (cf. [22] Proposition 3.6).

Proposition 3.2 A finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is order isomorphic to the syntactic ordered algebra of a tree language if and only if there exists an ideal $I \trianglelefteq \mathcal{A}$ such that $\preccurlyeq_I = \leqslant$.

Proof. First, suppose $\mathcal{A} \cong \mathrm{SOA}(T)$ for some tree language T. Then the subset $I = T/\theta_T = \{t/\theta_T \mid t \in T\}$ is an ideal of $\mathrm{SOA}(T)$ and $\preccurlyeq_I = \leqslant_T$ holds. Next, suppose for $I \trianglelefteq \mathcal{A}$, $\preccurlyeq_I = \leqslant$ holds. Let the Σ-morphism $\varphi : \mathcal{T}(\Sigma, A) \to \mathcal{A}$ be resulted by extending the identity mapping $1_A : A \to A$. Since φ is an epimorphism then $\preccurlyeq_{I\varphi^{-1}} = \varphi \circ \preccurlyeq_I \circ \varphi^{-1}$ by Proposition 2.9(3). Hence, Proposition 2.4 implies that $\mathcal{T}(\Sigma, A)/\preccurlyeq_{I\varphi^{-1}} \cong \mathcal{A}/\preccurlyeq_I$, and since $\preccurlyeq_I = \leqslant$, then $\mathrm{SOA}(I\varphi^{-1}) \cong \mathcal{A}$. □

3.1 Recognizability by ordered algebras

Let Σ be a ranked alphabet, X be a leaf alphabet, and $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *recognized* by $\mathcal{A}$, if there exist an ideal $I \trianglelefteq \mathcal{A}$ and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $T = I\varphi^{-1}$.

In fact every homomorphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ is uniquely determined by a mapping $\alpha : X \to A$ which is an *initial assignment* for $\mathcal{A}$. It can be extended to the homomorphism $\alpha^{\mathcal{A}} : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ inductively by $c\alpha^{\mathcal{A}} = c^{\mathcal{A}}$ and $f(t_1, \cdots, t_m)\alpha^{\mathcal{A}} = f^{\mathcal{A}}(t_1\alpha^{\mathcal{A}}, \cdots, t_m\alpha^{\mathcal{A}})$ for all $c \in \Sigma_0$, $f \in \Sigma_m$ $(m > 0)$ and $t_1, \cdots, t_m \in \mathrm{T}(\Sigma, X)$. In that case we say that T is *recognized* by $(\mathcal{A}, \alpha, I)$ or, in other words, $T = \{t \in \mathrm{T}(\Sigma, X) \mid t\alpha^{\mathcal{A}} \in I\}$.

Proposition 3.3 For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ and an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$, $\mathrm{SOA}(T) \prec \mathcal{A}$ if and only if T is recognized by $\mathcal{A}$.

Proof. Suppose $T = I\varphi^{-1}$ for a morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and an ideal $I \trianglelefteq \mathcal{A}$. Let the ordered Σ-algebra $\mathcal{B}$ be the image of φ, and define the mapping $\psi : \mathcal{B} \to \mathrm{SOA}(T)$ by $(t\varphi)\psi = t/\theta_T$ for $t \in \mathrm{T}(\Sigma, X)$.

We show that $t\varphi \leqslant s\varphi$ implies $t \preccurlyeq_T s$ for all $t, s \in \mathrm{T}(\Sigma, X)$. This also proves that ψ is well-defined. Suppose $t\varphi \leqslant s\varphi$, then $t\varphi \preccurlyeq_I s\varphi$ since $\leqslant \subseteq \preccurlyeq_I$. Now, for any translation $p \in \mathrm{Tr}(\mathcal{A})$,

$$\begin{aligned} p(s) \in T \quad &\Rightarrow \quad p(s)\varphi \in I \\ &\Rightarrow \quad p_\varphi(s\varphi) \in I \\ &\Rightarrow \quad p_\varphi(t\varphi) \in I \\ &\Rightarrow \quad p(t)\varphi \in I \\ &\Rightarrow \quad p(t) \in T. \end{aligned}$$

That is $t \preccurlyeq_T s$. It can also be seen that ψ is a Σ-morphism. Thus ψ is an order epimorphism, hence $\mathrm{SOA}(T) \leftarrow \mathcal{B} \subseteq \mathcal{A}$.

Now suppose for an ordered algebra $\mathcal{B}$, $\mathrm{SOA}(T) \leftarrow \mathcal{B} \subseteq \mathcal{A}$, and let $\psi : \mathcal{B} \to \mathrm{SOA}(T)$ be an order epimorphism. A Σ-morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ can be defined by choosing $x\varphi$ to be an element of $\mathcal{B}$ such that $(x\varphi)\psi = x/\theta_T$ for every $x \in X \cup \Sigma_0$. By induction on t it can be shown that $t\varphi\psi = t/\theta_T$ holds for every $t \in \mathrm{T}(\Sigma, X)$. The set $\{t/\theta_T \in \mathrm{SOA}(T) \mid t \in T\}\psi^{-1}$ is an ideal of $\mathcal{B}$. If I is the ideal of $\mathcal{A}$ generated by this set, then $T = I\varphi^{-1}$. □

From Proposition 3.3 it follows that the syntactic ordered algebra of a tree language is the least ordered algebra which recognizes the tree language.

Let us recall that for a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, a context $P \in \mathrm{C}(\Sigma, X)$, and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, the inverse translation of T under P is $P^{-1}(T) = \{t \in \mathrm{T}(\Sigma, X) \mid t \cdot P \in T\}$, and the inverse morphism of T under φ is $T\varphi^{-1} = \{t \in \mathrm{T}(\Sigma, Y) \mid t\varphi \in T\}$ (cf. [22]).

The following is an immediate consequence of Corollary 2.10.

Corollary 3.4 For tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$, a context $P \in \mathrm{C}(\Sigma, X)$, and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$,

(1) $\mathrm{SOA}(T \cap T'), \mathrm{SOA}(T \cup T') \prec \mathrm{SOA}(T) \times \mathrm{SOA}(T')$.

(2) $\mathrm{SOA}(P^{-1}(T)) \leftarrow \mathrm{SOA}(T)$.

(3) $\mathrm{SOA}(T\varphi^{-1}) \prec \mathrm{SOA}(T)$ and, moreover, when φ is surjective then $\mathrm{SOA}(T\varphi^{-1}) \cong \mathrm{SOA}(T)$.

From the clause (2) of the above corollary it follows that for a recognizable tree language T the set $\{P^{-1}(T) \mid P \in \mathrm{C}(\Sigma, X)\}$ is finite, since the ordered algebra $\mathrm{SOA}(T)$ is finite.

3.2 Positive Variety Theorem

Let Σ be a fixed ranked alphabet.

Recall that a class of finite ordered Σ-algebras is a variety (of finite ordered algebras) if it is closed under order subalgebras, order epimorphic images, and finite direct products.

Definition 3.5 An *indexed family* of recognizable tree languages is a family $\mathscr{V} = \{\mathscr{V}(X)\}$ in which $\mathscr{V}(X)$ consists of a collection of recognizable ΣX-tree languages for any leaf alphabet X. An indexed family is a *positive variety of tree languages*, abbreviated by PVTL, if it is closed under positive Boolean operations (intersections and unions), inverse translations, and inverse morphisms.

Definition 3.6 For a variety of finite ordered algebras $\mathscr{K}$, let the indexed family $\mathscr{K}^{\mathtt{t}} = \{\mathscr{K}^{\mathtt{t}}(X)\}$ be the family of tree languages whose syntactic ordered algebras are in $\mathscr{K}$, that is

$$\mathscr{K}^{\mathtt{t}}(X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOA}(T) \in \mathscr{K}\}.$$

For a positive variety of tree languages $\mathscr{V}$, let $\mathscr{V}^{\mathtt{a}}$ be the variety of finite ordered algebras generated by syntactic ordered algebras of tree languages in $\mathscr{V}$, that is $\mathscr{V}^{\mathtt{a}}$ is the VFOA generated by the class

$$\{\mathrm{SOA}(T) \mid T \in \mathscr{V}(X) \text{ for a leaf alphabet } X\}.$$

By Corollary 3.4, for a variety of finite ordered algebras $\mathscr{K}$, the family $\mathscr{K}^{\mathtt{t}}$ is a positive variety of tree languages.

Lemma 3.7 (1) The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are monotone, i.e., if $\mathscr{K} \subseteq \mathscr{L}$ and $\mathscr{V} \subseteq \mathscr{W}$, then $\mathscr{K}^{\mathtt{t}} \subseteq \mathscr{L}^{\mathtt{t}}$ and $\mathscr{V}^{\mathtt{a}} \subseteq \mathscr{W}^{\mathtt{a}}$.

(2) $\mathscr{V} \subseteq \mathscr{V}^{\mathtt{at}}$, and $\mathscr{K}^{\mathtt{ta}} \subseteq \mathscr{K}$.

Proof. The statement (1) and the inclusion $\mathscr{V} \subseteq \mathscr{V}^{\mathtt{at}}$ are obvious. For the inclusion $\mathscr{K}^{\mathtt{ta}} \subseteq \mathscr{K}$ we note that if $\mathcal{A} \in \mathscr{K}^{\mathtt{ta}}$, then for some $T_1, \cdots, T_n$ in $\mathscr{K}^{\mathtt{t}}$, $\mathcal{A} \prec \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_n)$ holds. Since T_j's are in $\mathscr{K}^{\mathtt{t}}$, then by definition, $\mathrm{SOA}(T_j) \in \mathscr{K}$ for every j, hence $\mathcal{A} \in \mathscr{K}$. $\square$

The following was proved for classical algebras in [16].

Lemma 3.8 For any finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ there are tree languages $T_1, \cdots, T_m$ recognizable by $\mathcal{A}$ such that

$$\mathcal{A} \subseteq \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_m).$$

Proof. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra, and suppose the epimorphism $\psi : \mathcal{T}(\Sigma, A) \to \mathcal{A}$ is obtained by extending the identity mapping $\Delta_A : A \to A$. Recall that for any $a \in A$, $(a] = \{b \in A \mid b \leqslant a\}$ is the ideal of $\mathcal{A}$ generated by a. By Corollary 2.10(3), $\mathrm{SOA}((a]\psi^{-1}) \cong \mathcal{A}/(a]$ for every $a \in A$. We show $\mathcal{A} \subseteq \prod_{a\in A} \mathcal{A}/(a]$. This will finish the proof since $(a]\psi^{-1}$ is recognizable by $\mathcal{A}$. Define the mapping $\phi : \mathcal{A} \to \prod_{a\in A} \mathcal{A}/(a]$ by $u\phi = \big(u/\theta_{(a]}\big)_{a\in A}$ for $u \in A$. Clearly ϕ is an order morphism. All we have to show is that ϕ is injective. Suppose $u\phi = v\phi$ for $u, v \in A$. Then $u/\theta_{(a]} = v/\theta_{(a]}$ for every $a \in A$. In particular $u/\theta_{(u]} = v/\theta_{(u]}$ and $u/\theta_{(v]} = v/\theta_{(v]}$, which imply $v \in (u]$ and $u \in (v]$, respectively. So, $u \leqslant v$ and $v \leqslant u$, thus $u = v$. □

Corollary 3.9 (1) Every VFOA is generated by syntactic ordered algebras of some tree languages.

(2) For any PVTL $\mathscr{V}$ and any finite ordered algebra $\mathcal{A}$, if every tree language recognizable by $\mathcal{A}$ belongs to $\mathscr{V}$, then $\mathcal{A} \in \mathscr{V}^{\mathtt{a}}$.

Lemma 3.10 For every variety of finite ordered algebras $\mathscr{K}$, $\mathscr{K} \subseteq \mathscr{K}^{\mathtt{ta}}$.

Proof. By Corollary 3.9(1), it is enough to show that ordered syntactic algebras of tree languages that belong to $\mathscr{K}$ are in $\mathscr{K}^{\mathtt{ta}}$. Suppose for a tree language T, $\mathrm{SOA}(T) \in \mathscr{K}$. Then T is in $\mathscr{K}^{\mathtt{t}}$ by definition, so $\mathrm{SOA}(T) \in \mathscr{K}^{\mathtt{ta}}$ which finishes the proof. □

The essential part of the positive variety theorem is the following.

Lemma 3.11 For every positive variety of tree languages $\mathscr{V}$, $\mathscr{V}^{\mathtt{at}} \subseteq \mathscr{V}$.

Proof. Suppose $T \in \mathscr{V}^{\mathtt{at}}(X)$. Then there are leaf alphabets $X_1, \cdots, X_n$ and tree languages $T_1 \in \mathscr{V}(X_1), \cdots, T_n \in \mathscr{V}(X_n)$ such that $\mathrm{SOA}(T)$ divides the product $\mathcal{A} = \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_n)$. So, by Proposition 3.3, T is recognized by $\mathcal{A}$, and so there is an order morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and an ideal $I \trianglelefteq \mathcal{A}$ such that $T = I\varphi^{-1}$. Let $\mathrm{SOA}(T_j) = \mathcal{A}_j = (A_j, \Sigma, \leqslant_j)$ for each $j \leq n$. Recall that $(\mathbf{a}]$ is the ideal of $\mathcal{A}$ generated by $\mathbf{a} \in \prod_i A_i$, and similarly $(a_j]$ is the ideal of $\mathcal{A}_j$ generated by $a_j \in A_j$. For any $\mathbf{a} = (a_1, \cdots, a_n) \in \prod_i A_i$ we have $\big(\mathbf{a}\big] = (a_1] \times \cdots \times (a_n]$. Let $\varphi_j : \mathcal{T}(\Sigma, X) \to \mathcal{A}_j$ be the composition of φ with the j-th projection mapping $\prod_i A_i \to A_j$. Then

$$T = I\varphi^{-1} = \bigcup_{\mathbf{a}\in I} (\mathbf{a}]\varphi^{-1} = \bigcup_{(a_1,\cdots,a_n)\in I} \bigcap_{j\leq n} (a_j]\varphi_j^{-1}.$$

We aim at showing $T \in \mathscr{V}(X)$. It is enough to show $(a_j]\varphi_j^{-1} \in \mathscr{V}(X)$ for every $j \leq n$. Fix a $j \leq n$. Let $\varphi_{T_j} : \mathcal{T}(\Sigma, X_j) \to \mathcal{A}_j$ be the syntactic morphism of T_j. One can construct a Σ-morphism $\chi_j : \mathcal{T}(\Sigma, X) \to \mathcal{T}(\Sigma, X_j)$ such that $\chi_j\varphi_{T_j} = \varphi_j$. Then $(a_j]\varphi_j^{-1} = (a_j]\varphi_{T_j}^{-1}\chi_j^{-1}$, and since $\mathscr{V}$ is closed under inverse morphisms, for showing $(a_j]\varphi_j^{-1} \in \mathscr{V}(X)$ it is enough to show $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(X_j)$.

There exists a $t \in \mathrm{T}(\Sigma, X_j)$ such that $a_j = t\varphi_{T_j}$. We show

$$(a_j]\varphi_{T_j}^{-1} = \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma, X_j), P(t) \in T_j\}.$$

For any $s \in \mathrm{T}(\Sigma, X_j)$,

$$\begin{array}{rcl} s \in (a_j]\varphi_{T_j}^{-1} & \Leftrightarrow & s\varphi_{T_j} \leqslant_j a_j = t\varphi_{T_j} \\ & \Leftrightarrow & s \preccurlyeq_{T_j} t \\ & \Leftrightarrow & (\forall P \in \mathrm{C}(\Sigma, X_j))(P(t) \in T_j \Rightarrow P(s) \in T_j) \\ & \Leftrightarrow & (\forall P \in \mathrm{C}(\Sigma, X_j))(P(t) \in T_j \Rightarrow s \in P^{-1}(T_j)) \\ & \Leftrightarrow & s \in \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma, X_j), P(t) \in T_j\}. \end{array}$$

By $T_j \in \mathscr{V}(X_j)$ and the fact that $\mathscr{V}$ is closed under inverse translations and positive Boolean operations, it follows that $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(X_j)$. So, $(a_j]\varphi_j^{-1}$ belongs to $\mathscr{V}(X)$ for all j, thus $T \in \mathscr{V}(X)$. $\square$

Summing up, we showed the following.

Proposition 3.12 (Positive Variety Theorem) The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are mutually inverse lattice isomorphisms between the class of all varieties of finite ordered algebras and the class of all positive varieties of recognizable tree languages, i.e., $\mathscr{V}^{\mathtt{at}} = \mathscr{V}$ and $\mathscr{K}^{\mathtt{ta}} = \mathscr{K}$.

3.3 Examples

Here, we introduce some families of tree languages and provide some instances for Positive Variety Theorem (Proposition 3.12).

3.3.1 Cofinite tree languages

Definition 3.13 A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *cofinite* if its complement $\mathrm{T}(\Sigma, X) \setminus T$ is finite.
The family of cofinite ΣX-tree languages is denoted by $\mathrm{Cof}(\Sigma, X)$, and $\mathrm{Cof}_\Sigma = \{\mathrm{Cof}(\Sigma, X)\}$ is the family of cofinite tree languages for all leaf alphabets X.

Proposition 3.14 A language $T \subseteq \mathrm{T}(\Sigma, X)$ is cofinite if and only if it can be recognized by a finite ordered nilpotent algebra.

Proof. Suppose $T \subseteq \mathrm{T}(\Sigma, X)$ is cofinite. There exists an $n \in \mathbb{N}$ such that $P_1 \cdots P_n(t) \in T$ holds for all $P_1, \cdots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t \in \mathrm{T}(\Sigma, X)$. Therefore, $P_1 \cdots P_n(t) \preccurlyeq_T s$ holds for all $P_1, \cdots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and all $t, s \in \mathrm{T}(\Sigma, X)$. This immediately implies that the syntactic algebra $\mathrm{SOA}(T)$

of T satisfies $p_1 \cdots\ p_n(a) \leqslant_T b$ for all $p_1, \cdots, p_n \in \mathrm{TrS}(\mathrm{SOA}(T))$ and all $a, b \in \mathrm{SOA}(T)$. Thus, $\mathrm{SOA}(T)$ is an ordered n-nilpotent algebra.

Conversely, suppose that a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is recognized by an ordered n-nilpotent algebra $\mathcal{A} = (A, \Sigma, \leqslant)$. Let $\varphi : \mathrm{T}(\Sigma, X) \to \mathcal{A}$ be an order morphism and $I \trianglelefteq A$ be an ideal such that $T = I\varphi^{-1}$. The mapping $\varphi_* : \mathrm{C}(\Sigma, X) \setminus \{\xi\} \to \mathrm{TrS}(\mathcal{A})$ obtained from setting

$$f(t_1, \cdots, \xi, \cdots, t_m)\varphi_* = f^{\mathcal{A}}(t_1\varphi, \cdots, \xi, \cdots, t_m\varphi)$$

for all $f \in \Sigma_m$ $(m > 0)$ and $t_1, \cdots, t_m \in \mathrm{T}(\Sigma, X)$, and $(P \cdot Q)\varphi_* = P\varphi_* \cdot Q\varphi_*$, is a semigroup morphism which satisfies $P\varphi_*(t\varphi) = P(t)\varphi$ for all $t \in \mathrm{T}(\Sigma, X)$, $P \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$. Since $\mathcal{A}$ is an ordered n-nilpotent algebra, then $p_1 \cdots p_n(a) \in I$ holds for all $p_1, \cdots, p_n \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$. In particular, $P_1\varphi_* \cdots P_n\varphi_*(t\varphi) \in I$ holds for all $P_1, \cdots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t \in \mathrm{T}(\Sigma, X)$. The statement $P_1\varphi_* \cdots P_n\varphi_*(t\varphi) \in I$ is equivalent to $P_1 \cdots P_n(t)\varphi \in I$ and $P_1 \cdots P_n(t) \in I\varphi^{-1} = T$. Hence, T is cofinite. $\square$

Corollary 3.15 The family Cof_Σ is a PVTL and $\mathrm{Cof}_\Sigma = \mathbf{Nil}(\Sigma)^{\mathfrak{t}}$.

Proof. This follows immediately from Propositions 3.14, 2.13 and 3.12; though it can be verified directly that the family of cofinite tree languages is closed under finite unions and intersection, inverse translations and inverse morphisms. $\square$

3.3.2 Semilattice and symbolic tree languages

We can assume that the leaf alphabets X are always disjoint from the ranked alphabet Σ.

Definition 3.16 For a tree $t \in \mathrm{T}(\Sigma, X)$, the contents $\mathrm{c}(t)$ of t is the set of symbols from $\Sigma \cup X$ which appear in t. It can be defined inductively as:

(1) $\mathrm{c}(x) = \{x\}$ for $x \in \Sigma_0 \cup X$;
(2) $\mathrm{c}(f(t_1, \cdots, t_m)) = \{f\} \cup \mathrm{c}(t_1) \cup \cdots \cup \mathrm{c}(t_m)$ for $t_1, \cdots, t_m \in \mathrm{T}(\Sigma, X)$ and $f \in \Sigma_m$.

For a subset $Z \subseteq \Sigma \cup X$, the tree language $T(Z)$ consists of trees in which all symbols of Z appear, i.e.,

$$T(Z) = \{t \in \mathrm{T}(\Sigma, X) \mid Z \subseteq \mathrm{c}(t)\}.$$

A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *symbolic*, if it is a union of tree languages of the form $T(Z)$ for some subsets $Z \subseteq \Sigma \cup X$.

The family of symbolic ΣX-tree languages is denoted by $\mathrm{Sym}(\Sigma, X)$, and $\mathrm{Sym}_\Sigma = \{\mathrm{Sym}(\Sigma, X)\}$ is the family of symbolic tree languages for all leaf alphabets X.

Lemma 3.17 For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ the following properties are equivalent:

(1) T is symbolic;

(2) For all trees $t, t' \in \mathrm{T}(\Sigma, X)$, $\mathrm{c}(t) \subseteq \mathrm{c}(t')$ and $t \in T$ imply $t' \in T$;

(3) $T = \bigcup_{t \in T} T(\mathrm{c}(t))$.

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are straightforward. For the implication (2) $\Rightarrow$ (3), the inclusion $T \subseteq \bigcup_{t \in T} T(\mathrm{c}(t))$ always holds. Suppose $t' \in T(\mathrm{c}(t))$ for some $t \in T$. Then $\mathrm{c}(t) \subseteq \mathrm{c}(t')$, thus $t' \in T$. Hence, the opposite inclusion $\bigcup_{t \in T} T(\mathrm{c}(t)) \subseteq T$ holds too. $\square$

Definition 3.18 A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is a *semilattice* tree language, if for all trees $t, t' \in \mathrm{T}(\Sigma, X)$, $\mathrm{c}(t) = \mathrm{c}(t')$ and $t \in T$ imply $t' \in T$.

The family of semilattice ΣX-tree languages is denoted by $\mathrm{SL}(\Sigma, X)$, and $\mathrm{SL}_\Sigma = \{\mathrm{SL}(\Sigma, X)\}$ is the family of semilattice tree languages for all leaf alphabets X.

The rest of this subsection is devoted to proving the facts that semilattice tree languages are definable by semilattice algebras and symbolic tree languages are definable by symbolic ordered algebras, i.e., $\mathrm{SL}_\Sigma = \mathbf{SL}(\Sigma)^{\mathfrak{t}}$ and $\mathrm{Sym}_\Sigma = \mathbf{Sym}(\Sigma)^{\mathfrak{t}}$.

Fix a ranked alphabet Σ and a leaf alphabet X. The sequences of trees are denoted by bold face fonts, e.g., $\mathbf{t}$ is a sequence like $(t_1, \cdots, t_m)$ for $t_1, \cdots, t_m \in \mathrm{T}(\Sigma, X)$.

Let σ be a Σ-congruence on $\mathcal{T}(\Sigma, X)$ such that $\mathcal{T}(\Sigma, X)/\sigma$ is a semilattice algebra, i.e., it satisfies the following relations for all function symbols $f, g \in \Sigma$ and trees $\mathbf{t}, \mathbf{r}, \mathbf{u}, \mathbf{v}, t \in \mathrm{T}(\Sigma, X)$:

(d1) $f(\mathbf{t}, f(\mathbf{t}, t, \mathbf{r}), \mathbf{r}) \ \sigma \ f(\mathbf{t}, t, \mathbf{r})$; and

(d2) $f(\mathbf{t}, g(\mathbf{u}, t, \mathbf{v}), \mathbf{r}) \ \sigma \ g(\mathbf{u}, f(\mathbf{t}, t, \mathbf{r}), \mathbf{v})$.

The following lemma is implied by Lemmas 2.19, 2.20, 2.21, 2.22, 2.23 and 2.24.

Lemma 3.19 The following relations hold for any $f \in \Sigma_m$, $g \in \Sigma_n$, and any ΣX-trees $t, s, \mathbf{r}, \mathbf{t}, \mathbf{s}$:

(s1) $f(\mathbf{t}, t, \mathbf{r}, r, \mathbf{u}) \ \sigma \ f(\mathbf{t}, r, \mathbf{r}, t, \mathbf{u})$;

(s2) $f(t, t, r, \mathbf{t}) \ \sigma \ f(t, r, r, \mathbf{t})$;

(s3) $f(g(t, \mathbf{t}), r, \mathbf{r}) \ \sigma \ f(g(r, \mathbf{t}), t, \mathbf{r})$;

(s4) $f(f(t, \cdots, t), \mathbf{t}) \ \sigma \ f(t, \mathbf{t})$;

(s5) $f(g(t,r,\mathbf{t}),t,\mathbf{r})\ \sigma\ f(g(t,r,\mathbf{t}),r,\mathbf{r})$;

(s6) $f(f(g(t,s,\mathbf{t}),\mathbf{r}),\mathbf{u})\ \sigma\ f(g(g(t,\mathbf{r},\overline{r}),r,\mathbf{t}),\mathbf{u})$
where $m \leq n$ and the sequence $\overline{r}$ consists of $n-m$ times r.

The family of Σ-congruences on $\mathcal{T}(\Sigma, X)$ satisfying (d1) and (d2) is closed under intersections and contains the universal relation $\mathrm{T}(\Sigma, X) \times \mathrm{T}(\Sigma, X)$, and so has the smallest element τ.

In the forthcoming we prove that for any trees t_1 and t_2,

$$t_1\ \tau\ t_2 \text{ iff } \mathrm{c}(t_1) = \mathrm{c}(t_2).$$

Suppose the elements of $\Sigma \setminus \Sigma_0$ are linearly ordered in such a way that the function symbols with smaller arity are smaller than the function symbols with greater arity. Similarly suppose the leaves $X \cup \Sigma_0$ are linearly ordered.

Let $\mathrm{c}_\Sigma(t) \subseteq \Sigma \setminus \Sigma_0$ be the set of nodes of a tree $t \in \mathrm{T}(\Sigma, X)$ and $\mathrm{c}_X(t) \subseteq X \cup \Sigma_0$ be its set of leaves.

A tree t is in the *canonical form* if:

(1) either $t \in X \cup \Sigma_0$, or
(2) $t = f(t_1, x_2, \ldots, x_m)$ where
   (a) t_1 is in the canonical form and $x_2, \cdots, x_m \in \Sigma_0 \cup X$,
   (b) f is the smallest in $\mathrm{c}_\Sigma(t)$,
   (c) either $f \notin \mathrm{c}_\Sigma(t_1)$ or $\mathrm{c}_\Sigma(t_1) = \{f\}$,
   (d) if $|\mathrm{c}_X(t)| \geq m-1$, then $x_2, \ldots, x_m$ are the smallest $m-1$ elements in $\mathrm{c}_X(t)$ in that order, and
   (e) otherwise, if $\mathrm{c}_X(t) = \{x_2, \ldots, x_n\}$ with $x_2 \lneqq \ldots \lneqq x_n$, $n < m$, then $x_{n+1} = \ldots = x_m = x_n$ and $\mathrm{c}_X(t_1) = \{x_n\}$.

In other words, a tree is in the canonical form if on each its level only the leftmost node is from $\Sigma \setminus \Sigma_0$, all the others are leaves from $\Sigma_0 \cup X$, nodes grow from root downwards and the leaves grow from left to right and from top to down. As soon as the set of nodes or leaves are exhausted, the last symbol from the set is repeated as long as there are still symbols in the other set to be used.

Let us fix σ to be any congruence on $\mathcal{T}(\Sigma, X)$ satisfying (d1) and (d2). Our aim is to show that every tree t is σ-equivalent to a tree t' in the canonical form where $\mathrm{c}(t) = \mathrm{c}(t')$.

We are proving this by induction on the complexity of t. The claim clearly holds for $t \in \Sigma_0 \cup X$. Suppose $t = f(t_1, t_2, \ldots, t_n)$ and that $t_1, \ldots, t_m$ are in the canonical form. The transformation consists of several phases:

Step 1. Shaping the tree into a leftmost branching tree while arranging the nodes in the increasing order from top to down.

Step 2. Organizing repetitions of nodes so that any repetition of a smaller node is changed by a repetition of the next greater node.

Step 3. Arranging the order of leaves.

Step 4. Organizing repetitions of leaves so that any repetition of a smaller leaf is changed by a repetition of the next greater leaf.

Step 5. Fold the unnecessary part.

Step 1 : Let $g = \min\{\mathrm{root}(t_1), \dots, \mathrm{root}(t_m)\}$. Without loosing generality, by (s1), we can assume that $g = \mathrm{root}(t_1)$. Write $t_1 = g(t_1', x_2, \dots, x_n)$. We distinguish two cases:

First $g \leq f$: Then $n \leq m$ and by (d2) we have

$t = f(g(t_1', x_2, \dots, x_n), t_2, \dots, t_m) \ \sigma \ g(f(t_1', t_2, \dots, t_m), x_2, \dots, x_n)$,

and now we can apply the induction hypotheses to $f(t_1', t_2, \dots, t_m)$.

Second $f < g$: Then $m \leq n$ and by (d2),(s3) we have

$t = f(g(t_1', x_2, \dots, x_n), t_2, \dots, t_m) \ \sigma$

$\sigma \quad f(g(t_1', t_2, \dots, t_m, x_2, \dots, x_{n-m+1}), x_{n-m+2}, \dots, x_n)$,

and then we can continue by induction.

We get a tree of the the desired shape, the order of the nodes is increasing from top to down, but there may be repetitions of a same node following each other.

Step 2 : The clause (s6) of Lemma 3.19 provides a transformation that pushes repetitions, i.e., if $f \leq g$ then the transformation will replace a copy of f by a copy of g.

After these transformations we get a tree σ-equivalent to t, branching only in the leftmost node and with increasing nodes where only the greatest node is possibly repeated. The tree is still not in the canonical form since leaves are not necessarily already arranged.

Step 3 : The sequence of leaves is read starting from left to right and from top downwards. This sequence can be sorted using standard algorithms for sorting sequences which assumes comparing the first symbol with the rest one by one and when a smaller one appears swap them and continue comparing with the rest of the variables. After this the smallest leaf is on the first place. Repeat the same with the second one and rest of the sequence, etc. We note that this swapping is supported by σ, since places of leaves on the same level can be changed by (s1), and if they are on different levels then (s3) is applied.

After this, leaves will be in increasing order, but there are possibly repetitions of those leaves which are not the greatest.

Step 4 : The idea is the same as in Step 2, the repetition of a smaller leaf is replaced by the repetition of the next greater leaf, so that repetitions are pushed trough the sequence and finaly only the greatest leaf may be repeated. In other words, if $x < y$ then the subsequence xxy is replaced by xyy. We distinguish four cases.

First, xxy appears on the same level, i.e., as the components of the same node. This case is solved by applying (s2).

Second, the first x is on one level and x and y are both on the next. This is solved easily by applying first (s5) and so changing the first x into y, and then applying (s3) to swap x and outer y:

$f(g(t, x, y, \mathbf{x}), x, \mathbf{y})\ \sigma\ f(g(t, x, y, \mathbf{x}), y, \mathbf{y})\ \sigma\ f(g(t, y, y, \mathbf{x}), x, \mathbf{y})$.

Third, both x's are on the upper level and y is on the lower. We proceed as

$$\begin{array}{rl} & f(g(t, y, \mathbf{x}), \mathbf{y}, x, x) \\ \sigma & f(g(x, y, \mathbf{x}), \mathbf{y}, x, t) \\ \sigma & f(g(x, y, \mathbf{x}), \mathbf{y}, y, t) \\ \sigma & f(g(t, y, \mathbf{x}), \mathbf{y}, y, x) \\ \sigma & f(g(t, y, \mathbf{x}), \mathbf{y}, x, y). \end{array}$$

Note that t plays an important role here and existence of such a tree follows from the fact that $f < g$ and thus the arity of g is at least 2.

Fourth, all three leaves appear on different levels. The tree is in the form $f(g(h(t, y, \mathbf{z}), x), x)$ where $f, g \in \Sigma_2$. The first x should be changed into y. The transformation is as follows:

$$\begin{array}{rl} & f(g(h(t, y, \mathbf{z}), x), x) \\ \sigma & f(g(h(x, y, \mathbf{z}), t), x) \\ \sigma & f(g(h(x, y, \mathbf{z}), x), t) \\ \sigma & f(g(h(x, y, \mathbf{z}), y), t) \\ \sigma & f(g(h(x, y, \mathbf{z}), t), y) \\ \sigma & f(g(h(t, y, \mathbf{z}), x), y) \\ \sigma & f(g(h(t, y, \mathbf{z}), y), x). \end{array}$$

After this, our tree has almost the canonical form, the only disturbing thing may be too long subtree at the end having only the greatest symbol from $c_\Sigma(t)$ as nodes and the greatest element from $c_X(t)$ as leaves.

Step 5 : Applying (s4) as many times as needed the tree is folded into one without repetitions of the greatest symbol from $c_\Sigma(t)$, or with its repetitions but not with only the greatest element of $c_X(t)$ as leaves on the deepest level.

Clearly, the procedure results a unique tree in the canonical form which is σ-equivalent to a given tree.

For example suppose $h \in \Sigma_3$, $f, g \in \Sigma_2$, $c \in \Sigma_0$, $x \in X$, and the ordering on the nodes and leaves are as $f < g < h$ and $x < c$. Let

$t = h(g(x, f(x,c)), x, g(x,c))$. Then by applying the above steps we get the tree r_j in the j-th step as follows:

$$\begin{array}{rl} t\ \sigma & r_1 = f(g(g(h(x,x,x),c),c),x) \\ \sigma & r_2 = f(g(h(h(x,x,x),x,x),c),x) \\ \sigma & r_3 = f(g(h(h(c,x,x),x,x),x),x) \\ \sigma & r_4 = f(g(h(h(c,c,c),c,c),c),x) \\ \sigma & r_5 = f(g(h(c,c,c),c),x). \end{array}$$

It can be noticed that the canonical form tree corresponding to a given tree t is determined by $c(t)$ and can be constructed from this set. The procedure can roughly be described as follows:

1. put the smallest node in the root of the tree, draw the necessary branches, put the next smallest symbol from $c_\Sigma(t)$ in the left most node, continue doing this as long as $c_\Sigma(t)$ is not exhausted;
2. put the smallest leaf in the first drawn leaf place, choose the next smallest and put in the next place, etc.
3. if there are still empty nodes put the greatest symbol from $c_X(t)$ there;
4. if not all $c_X(t)$ is used, continue building the tree by shifting all symbols from $X \cup \Sigma_0$ on the last level by one place to the right, return the last leaf to $c_X(t)$, put the greatest element of $c_\Sigma(t)$ to the leftmost place, add its arity new branches, fill them with remaining symbols from $c_X(t)$, and repeat this step as many times as needed.

Recall that τ is the smallest congruence satisfying (d1) and (d2).

Lemma 3.20 For any trees t_1 and t_2, $t_1 \tau t_2$ iff $\mathrm{c}(t_1) = \mathrm{c}(t_2)$.

Proof. Define τ' by $t_1 \tau' t_2$ iff $\mathrm{c}(t_1) = \mathrm{c}(t_2)$. Then τ' satisfies (d1) and (d2). Let σ be any congruence satisfying (d1) and (d2). We are proving that $\tau' \subseteq \sigma$. Assume $t_1 \tau' t_2$. There are canonical trees t_1' and t_2' such that $t_1 \sigma t_1'$ and $t_2 \sigma t_2'$. Then $\mathrm{c}(t_1') = \mathrm{c}(t_2')$ and since the canonical tree is uniquely determined by its contents, it follows that $t_1' = t_2'$ which immediately implies that $t_1 \sigma t_2$. Therefore, $\tau = \tau'$. □

For a context $P \in \mathrm{C}(\Sigma, X)$, the set of contents $\mathrm{c}(P)$ of P is the set of symbols from $\Sigma \cup X$ which appear in P. We note that $\mathrm{c}(P(t)) = \mathrm{c}(P) \cup \mathrm{c}(t)$ holds for any context $P \in \mathrm{C}(\Sigma, X)$ and tree $t \in \mathrm{T}(\Sigma, X)$.

Proposition 3.21 (1) A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is semilattice if and only if it is recognizable by a finite semilattice algebra.

(2) A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is symbolic if and only if it is recognizable by a finite symbolic ordered algebra.

Proof. (1) By Lemma 3.20, T is a semilattice tree language if and only if $\tau \subseteq \theta_T$ if and only if the syntactic algebra of T is a semilattice algebra.

(2) Every symbolic tree language is also a semilattice tree language. So, if T is symbolic then the syntactic algebra of T is semilattice. On the other hand, since $\mathrm{c}(t) \subseteq \mathrm{c}(P(t))$ holds for all $t \in \mathrm{T}(\Sigma, X)$ and $P \in \mathrm{C}(\Sigma, X)$, then $P(t) \preccurlyeq_T t$ always holds. This shows that $\mathrm{SOA}(T)$ is a symbolic ordered algebra. Conversely, if $\mathrm{SOA}(T)$ is a symbolic ordered algebra, then $\tau \subseteq \theta_T$ and $P(t) \preccurlyeq_T t$. Suppose for trees t and t', $\mathrm{c}(t) \subseteq \mathrm{c}(t')$ and $t \in T$ hold. Then there exists a context P such that $\mathrm{c}(t') = \mathrm{c}(P(t))$. By Lemma 3.20, $t' \ \tau \ P(t)$, and so $t' \ \theta_T \ P(t)$ holds. On the other hand $P(t) \preccurlyeq_T t$ implies $t' \preccurlyeq_T t$, and this by $t \in T$ implies $t' \in T$. Hence, T is a symbolic tree language by Lemma 3.17. □

Corollary 3.22 The family SL_Σ is a variety of tree languages and $\mathrm{SL}_\Sigma = \mathbf{SL}(\Sigma)^{\mathfrak{t}}$, also the family Sym_Σ is a positive variety of tree languages and $\mathrm{Sym}_\Sigma = \mathbf{Sym}(\Sigma)^{\mathfrak{t}}$.

We note that SL_Σ is closed under complements while Sym_Σ is not. Clearly, varieties of tree languages are special cases of positive varieties, thus theory of positive varieties applies to varieties in general.

4 Generalized Positive Variety Theorem

Generalized varieties of tree languages and generalized varieties of finite algebras were introduced by Steinby [23] who proved a generalized variety theorem for these classes. A variety of finite algebras is a class of finite algebras over a fixed ranked alphabet as the notions of subalgebras, homomorphic images and direct products are defined for algebras over the same ranked alphabet. These notions can be generalized for algebras over different ranked alphabets. A generalized variety of finite algebras is a class of finite algebras over any ranked alphabet that satisfies certain closure properties. Similarly a generalized variety of tree languages is defined.

In this section we generalize our Positive Variety Theorem 3.12 for generalized positive varieties of tree languages and generalized varieties of finite ordered algebras.

The following definition is the ordered version of Definitions 3.1, 3.2, 3.3, 3.14 from [23].

Definition 4.1 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ be ordered algebras.

• The algebra $\mathcal{B}$ is an *order g-subalgebra* of $\mathcal{A}$, in notation $\mathcal{B} \subseteq_g \mathcal{A}$, if $B \subseteq A$, $\Omega_m \subseteq \Sigma_m$ for all $m \geq 0$, $f^{\mathcal{B}}$ is the restriction of $f^{\mathcal{A}}$ to B for every $f \in \Omega_m$, and $\leqslant'$ is the restriction of $\leqslant$ on B.

• An *assignment* is a mapping $\kappa : \Sigma \to \Omega$ such that $\kappa(\Sigma_m) \subseteq \Omega_m$ for all $m \geq 0$. An *order g-morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair (κ, φ) where the mapping $\kappa : \Sigma \to \Omega$ is an assignment and $\varphi : A \to B$ is an order preserving mapping satisfying $f^{\mathcal{A}}(a_1, \cdots, a_m)\varphi = (f\kappa)^{\mathcal{B}}(a_1\varphi, \cdots, a_m\varphi)$ for any $m \geq 0$, $f \in \Sigma_m$, and $a_1, \cdots, a_m \in A$. Note that order preserving means that $a \leqslant b$ implies $a\varphi \leqslant' b\varphi$ for all $a, b \in A$. If both κ and φ are surjective, then (κ, φ) is a *order g-epimorphism*, and in that case we write $\mathcal{B} \leftarrow_g \mathcal{A}$ meaning that $\mathcal{B}$ is an *order g-epimorphic image of* $\mathcal{A}$. When $\mathcal{B}$ is an order g-epimorphic image of an order g-subalgebra of $\mathcal{A}$, we write $\mathcal{B} \prec_g \mathcal{A}$. When both κ and φ are bijective and $(\kappa^{-1}, \varphi^{-1})$ is an order g-morphism, (κ, φ) is an *order g-isomorphism*, and $\mathcal{B} \cong_g \mathcal{A}$ means that $\mathcal{B}$ and $\mathcal{A}$ are order g-isomorphic.

• Let $\Sigma^1, \cdots, \Sigma^n$ and Γ be ranked alphabets. The product $\Sigma^1 \times \cdots \times \Sigma^n$ is a ranked alphabet such that $(\Sigma^1 \times \cdots \times \Sigma^n)_m = \Sigma^1_m \times \cdots \times \Sigma^n_m$ for every $m \geq 0$. For any assignment $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ and any finite number of ordered algebras $\mathcal{A}_1 = (A_1, \Sigma^1, \leqslant_1), \cdots, \mathcal{A}_n = (A_n, \Sigma^n, \leqslant_n)$, the *$\kappa$-product* of $\mathcal{A}_1, \cdots, \mathcal{A}_n$ is the ordered Γ-algebra

$$\kappa(\mathcal{A}_1, \cdots, \mathcal{A}_n) = (A_1 \times \cdots \times A_n, \Gamma, \leqslant_1 \times \cdots \times \leqslant_n)$$

defined by the following:
For $c \in \Gamma_0$, $f \in \Gamma_m$ $(m > 0)$ and $\mathbf{a}_i = (a_{i1}, \cdots, a_{in}) \in A_1 \times \cdots \times A_n$ $(i \leq m)$,

(1) $c^{\kappa(\mathcal{A}_1, \cdots, \mathcal{A}_n)} = (c_1^{\mathcal{A}_1}, \cdots, c_n^{\mathcal{A}_n})$ where $c\kappa = (c_1, \cdots, c_n)$,

(2) $f^{\kappa(\mathcal{A}_1, \cdots, \mathcal{A}_n)}(\mathbf{a}_1, \cdots, \mathbf{a}_m) = (f_1^{\mathcal{A}_1}(a_{11}, \cdots, a_{m1}), \cdots, f_n^{\mathcal{A}_n}(a_{1n}, \cdots, a_{mn}))$ where $f\kappa = (f_1, \cdots, f_n)$, and

(3) $\mathbf{a}_1 \leqslant_1 \times \cdots \times \leqslant_n \mathbf{a}_2 \iff a_{11} \leqslant_1 a_{21} \;\&\; \cdots \;\&\; a_{1n} \leqslant_n a_{2n}$.

Without specifying the assignment κ, such algebras are *g-products*.
A *generalized variety of finite ordered algebras*, a gVFOA for short, is a class $\mathscr{K} = \{\mathscr{K}(\Sigma)\}$ which consists of a class of finite ordered Σ-algebras $\mathscr{K}(\Sigma)$ for any ranked alphabet Σ, and is closed under order g-subalgebras, ordered g-epimorphic images, and g-products.

In the sequel we prove the necessary facts about generalized algebras needed for generalizing our positive variety theorem.

Proposition 4.2 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ be two ordered algebras, $\preccurlyeq$ be a quasi-order on $\mathcal{B}$ and $(\kappa, \varphi) : \mathcal{A} \to \mathcal{B}$ be an order g-morphism. Then

(1) the image of $\mathcal{A}$, $\mathcal{A}(\kappa, \varphi) = (A\varphi, \Sigma\kappa, \leqslant'')$ where $\leqslant''$ is the restriction of $\leqslant'$ on $A\varphi$, is an order g-subalgebra of $\mathcal{B}$,

(2) the relation $\varphi\circ \preccurlyeq \circ\varphi^{-1}$ is a quasi-order on $\mathcal{A}$ and when $\preccurlyeq'$ is the restriction of $\preccurlyeq$ on $A\varphi$, then $\mathcal{A}/\varphi\circ \preccurlyeq \circ\varphi^{-1} \cong_g \mathcal{A}\varphi/ \preccurlyeq'$, and

(3) moreover, if φ is an order g-epimorphism, then $\mathcal{A}/\varphi\circ \preccurlyeq \circ\varphi^{-1} \cong_g \mathcal{B}/ \preccurlyeq$.

The proof is a direct generalization of that of Proposition 2.4.

Proposition 4.3 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ be two ordered algebras, and $(\kappa, \varphi) : \mathcal{A} \to \mathcal{B}$ be an order g-morphism. The mappings (κ, φ) induce a monoid morphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_{(\kappa,\varphi)}$ such that $p(a)\varphi = p_{(\kappa,\varphi)}(a\varphi)$ for all $a \in A$. Moreover, if (κ, φ) is an order g-epimorphism then the induced map is a surjection.

Proof. For any elementary translation $p = f^{\mathcal{A}}(a_1, \cdots, \xi, \cdots, a_m)$ of $\mathcal{A}$ where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m \in A$, the unary function $p_{(\kappa,\varphi)}$ defined by $b \mapsto (f\kappa)^{\mathcal{B}}(a_1\varphi, \cdots, b, \cdots, a_m\varphi)$ is an elementary translation of $\mathcal{B}$, and if κ and φ are surjective then every elementary translation of $\mathcal{B}$ is of this form. $\square$

Lemma 4.4 For ordered algebras $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$, ideal $K \trianglelefteq \mathcal{B}$, and order g-morphism $(\kappa, \varphi) : \mathcal{A} \to \mathcal{B}$, $K\varphi^{-1}$ is an ideal of $\mathcal{A}$ and $\leqslant_{K\varphi^{-1}} \supseteq \varphi\circ \leqslant_K \circ\varphi^{-1}$, also $\mathrm{SOA}(K\varphi^{-1}) \prec_g \mathrm{SOA}(K)$.
Moreover, if (κ, φ) is an order g-epimorphism, then $\leqslant_{K\varphi^{-1}} = \varphi\circ \leqslant_K \circ\varphi^{-1}$ and $\mathrm{SOA}(K\varphi^{-1}) \cong_g \mathrm{SOA}(K)$.

The proof is very similar to that of Proposition 2.9.
Let Σ and Ω be ranked alphabets, X be a leaf alphabet, and $\mathcal{A} = (A, \Omega, \leqslant)$ be an ordered algebra. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *g-recognized* by $\mathcal{A}$, if there exist an ideal $I \trianglelefteq \mathcal{A}$ and an order g-morphism $(\kappa, \varphi) : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $T = I\varphi^{-1}$.

Lemma 4.5 A tree language T is g-recognized by $\mathcal{A}$ if $\mathrm{SOA}(T) \prec_g \mathcal{A}$.

Proof. If $\mathrm{SOA}(T) \leftarrow_g \mathcal{B} \subseteq_g \mathcal{A}$ and $(\iota, \psi) : \mathcal{B} \to \mathrm{SOA}(T)$ is an order g-epimorphism, then there exists an order g-morphism $(\kappa, \varphi) : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $(t\varphi)\psi = t/\theta_T$ and $(f\kappa)\iota = f$ for any $t \in \mathrm{T}(\Sigma, X)$, $f \in \Sigma$. Moreover, if I is the ideal of $\mathcal{A}$ generated by the set $\{t/\theta_T \mid t \in T\}\psi^{-1}$, then $T = I\varphi^{-1}$. $\square$

Contrary to Proposition 3.3, the converse of Lemma 4.5 does not hold, for more details see the definition of reduced syntactic algebra in Section 6 of [23].

By Lemma 4.4, for any g-morphism $(\kappa, \varphi) : \mathcal{T}(\Omega, Y) \to \mathcal{T}(\Sigma, X)$ and tree language $T \subseteq \mathrm{T}(\Sigma, X)$, $\mathrm{SOA}(T\varphi^{-1}) \prec_g \mathrm{SOA}(T)$ holds, and if (κ, φ) is a g-epimorphism then $\mathrm{SOA}(T\varphi^{-1}) \cong_g \mathrm{SOA}(T)$.

Definition 4.6 A *family* of recognizable tree languages is a family $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ where $\mathscr{V}(\Sigma, X)$ consists of a collection of recognizable ΣX-tree languages for any ranked alphabet Σ and leaf alphabet X is a *generalized positive variety of tree languages*, abbreviated by gPVTL, if it is closed under positive Boolean operations (intersections and unions), inverse translations, and inverse g-morphisms.

Definition 4.7 Let $\mathscr{K} = \{\mathscr{K}(\Sigma)\}$ be a gVFOA. Define the family $\mathscr{K}^{\mathsf{t}} = \{\mathscr{K}^{\mathsf{t}}(\Sigma, X)\}$ to be the family of tree languages whose syntactic ordered algebras are in $\mathscr{K}$, that is

$$\mathscr{K}^{\mathsf{t}}(\Sigma, X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOA}(T) \in \mathscr{K}(\Sigma)\}.$$

For a gPVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, let $\mathscr{V}^{\mathsf{a}} = \{\mathscr{V}^{\mathsf{a}}(\Sigma)\}$ be the gVFOA generated by the class $\{\mathrm{SOA}(T) \mid T \in \mathscr{V}(\Sigma, X) \text{ for some } \Sigma, X\}$.

Both of the following lemmas can be proved by arguments similar to their classical counterparts, Lemmas 3.7, 3.10 and Corollary 3.9.

Lemma 4.8 (1) The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathsf{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathsf{a}}$ are monotone, i.e., if $\mathscr{K} \subseteq \mathscr{L}$ and $\mathscr{V} \subseteq \mathscr{W}$, then $\mathscr{K}^{\mathsf{t}} \subseteq \mathscr{L}^{\mathsf{t}}$ and $\mathscr{V}^{\mathsf{a}} \subseteq \mathscr{W}^{\mathsf{a}}$.

(2) $\mathscr{V} \subseteq \mathscr{V}^{\mathsf{at}}$, and $\mathscr{K}^{\mathsf{ta}} \subseteq \mathscr{K}$.

(3) For a gVFOA $\mathscr{K}$, the family $\mathscr{K}^{\mathsf{t}}$ is a gPVTL, and $\mathscr{K} \subseteq \mathscr{K}^{\mathsf{ta}}$ holds.

Lemma 4.9 (1) Every gVFOA is generated by syntactic ordered algebras of some tree languages.

(2) For any gPVTL $\mathscr{V}$ and any finite ordered algebra $\mathcal{A}$, if every tree language recognizable by $\mathcal{A}$ belongs to $\mathscr{V}$, then $\mathcal{A} \in \mathscr{V}^{\mathsf{a}}$.

The essential part of the positive variety theorem can be generalized as follows.

Lemma 4.10 For every gPVTL $\mathscr{V}$, $\mathscr{V}^{\mathsf{at}} \subseteq \mathscr{V}$.

Proof. Suppose $T \in \mathscr{V}^{\mathsf{at}}(\Sigma, X)$. There are ranked alphabets $\Sigma^1, \cdots, \Sigma^n$, leaf alphabets $X_1, \cdots, X_n$ and tree languages

$$T_1 \in \mathscr{V}(\Sigma^1, X_1), \cdots, T_n \in \mathscr{V}(\Sigma^n, X_n)$$

such that $\mathrm{SOA}(T) \prec_g \kappa(\mathrm{SOA}(T_1), \cdots, \mathrm{SOA}(T_n))$ where $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ is an assignment for a ranked alphabet Γ. Let $\mathcal{A}_j = \mathrm{SOA}(T_j)$ for $j \leq n$. By Lemma 4.5, T is g-recognized by $\kappa(\mathcal{A}_1, \cdots, \mathcal{A}_n)$. So, there exist an order g-morphism $(\lambda, \varphi) : \mathcal{T}(\Sigma, X) \to \kappa(\mathcal{A}_1, \cdots, \mathcal{A}_n)$ and an ideal $I \trianglelefteq \kappa(\mathcal{A}_1, \cdots, \mathcal{A}_n)$ such that $T = I\varphi^{-1}$. Let $\varphi_j : \mathrm{T}(\Sigma, X) \to A_j$ be the composition of φ with the j-th projection function $\prod_i A_i \to A_j$, and $\lambda_j : \Sigma \to \Sigma^j$ be the composition of $\lambda\kappa : \Sigma \to \Sigma^1 \times \cdots \times \Sigma^n$ with the j-th projection $\Sigma^1 \times \cdots \times \Sigma^n \to \Sigma^j$. Then $(\lambda_j, \varphi_j) : \mathcal{T}(\Sigma, X) \to \mathcal{A}_j$ is an order g-morphism, and similarly to the proof of Lemma 3.11,

$$T = I\varphi^{-1} = \bigcup_{\mathbf{a} \in I} (\mathbf{a}]\varphi^{-1} = \bigcup_{(a_1, \cdots, a_n) \in I} \bigcap_{j \leq n} (a_j]\varphi_j^{-1}.$$

For showing $T \in \mathscr{V}(\Sigma, X)$ it is enough to show $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ for every $j \leq n$. Fix a $j \leq n$. Let $\varphi_{T_j} : \mathcal{T}(\Sigma^j, X_j) \to \mathcal{A}_j$ be the syntactic morphism of T_j. One can construct a g-morphism $(\lambda_j, \chi_j) : \mathcal{T}(\Sigma, X) \to \mathcal{T}(\Sigma^j, X_j)$ such that $\chi_j \varphi_{T_j} = \varphi_j$. Then $(a_j]\varphi_j^{-1} = (a_j]\varphi_{T_j}^{-1}\chi_j^{-1}$, and since $\mathscr{V}$ is closed under inverse g-morphisms, for showing $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ it is enough to show $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(\Sigma^j, X_j)$. In the proof of Lemma 3.11, it was shown that $(a_j]\varphi_{T_j}^{-1} = \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma^j, X_j), P(t) \in T_j\}$ for some $t \in \mathrm{T}(\Sigma^j, X_j)$. So, by $T_j \in \mathscr{V}(\Sigma^j, X_j)$ and the fact that $\mathscr{V}$ is closed under inverse translations and positive Boolean operations, it follows that $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(\Sigma^j, X_j)$. So, $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ for all j, thus $T \in \mathscr{V}(\Sigma, X)$. □

Proposition 4.11 (Generalized Positive Variety Theorem) The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are mutually inverse lattice isomorphisms between the class of all gVFOA's and the class of gPVTL's, i.e., $\mathscr{V}^{\mathtt{at}} = \mathscr{V}$ and $\mathscr{K}^{\mathtt{ta}} = \mathscr{K}$.

4.1 Examples

The examples of families of recognizable tree languages and classes of finite ordered algebras in the previous sections do not heavily depend on their ranked alphabets. Here we will see that the collection of those varieties for various ranked alphabets form generalized varieties.

4.1.1 Order nilpotent algebras and cofinite tree languages

Let $\mathbf{Nil} = \{\mathbf{Nil}(\Sigma)\}$ be the class of all ordered nilpotent algebras for every ranked alphabet Σ, and $\mathrm{Cof} = \{\mathrm{Cof}(\Sigma, X)\}$ be the family of all cofinite tree languages for all ranked alphabets Σ and leaf alphabets X.

Proposition 4.12 The class $\mathbf{Nil}$ is a gVFOA, the family Cof is a gPVTL, and $\mathrm{Cof} = \mathbf{Nil}^{\mathtt{t}}$.

That Cof is a gPVTL can be verified directly: the family is closed under positive Boolean operations, inverse translations and inverse g-morphisms. Similarly, $\mathbf{Nil}$ can be proved to be a gVFOA. From Proposition 3.14 it follows that for all tree languages $T \subseteq \mathrm{T}(\Sigma, X)$,

$$T \in \mathrm{Cof}(\Sigma, X) \Leftrightarrow \mathrm{SOA}(T) \in \mathbf{Nil}(\Sigma),$$

which implies that $\mathrm{Cof} = \mathbf{Nil}^{\mathtt{t}}$.

4.1.2 Semilattice algebras, semilattice tree languages, symbolic ordered algebras and symbolic tree languages

Let $\mathbf{SL} = \{\mathbf{SL}(\Sigma)\}$ and $\mathbf{Sym} = \{\mathbf{Sym}(\Sigma)\}$ be respectively the classes of all semilattice algebras and symbolic ordered algebras for every ranked alphabet

Σ, and $\mathrm{SL} = \{\mathrm{SL}(\Sigma, X)\}$ and $\mathrm{Sym} = \{\mathrm{Sym}(\Sigma, X)\}$ be respectively the families of all semilattice and symbolic tree languages for all ranked alphabets Σ and leaf alphabets X.

The following instance of generalized positive variety theorem holds.

Proposition 4.13 (1) The class **SL** is a generalized variety of finite algebras and the family SL is a generalized variety of recognizable tree languages, moreover $\mathrm{SL} = \mathbf{SL}^{\mathfrak{t}}$

(2) The class **Sym** is a gVFOA and the family Sym is a gPVTL, moreover $\mathrm{Sym} = \mathbf{Sym}^{\mathfrak{t}}$.

Proposition 4.14 For a semilattice algebra $\mathcal{A} = (A, \Sigma)$, the structure $\mathcal{A}_s = (A, \Sigma, \leqslant)$ where $\leqslant$ is defined by

$$a \leqslant b \iff \text{there is a } p \in \mathrm{Tr}(\mathcal{A}) \text{ such that } a = p(b)$$

for all $a, b \in A$, is a symbolic ordered algebra.

Proof. The relation $\leqslant$ is an order:
- $a \leqslant a$ holds since $1_A(a) = a$ for the identity translation 1_A;
- if $a \leqslant b$ and $b \leqslant a$ then $a = p(b)$ and $b = q(a)$ for some $p, q \in \mathrm{Tr}(\mathcal{A})$, so $a = b$ by Corollary 2.18;
- if $a \leqslant b$ and $b \leqslant c$ then $a = p(b)$ and $b = q(c)$ for some $p, q \in \mathrm{Tr}(\mathcal{A})$ thus $a = p(q(c))$ whence $a \leqslant c$.

The order $\leqslant$ is compatible with Σ since :
- if $a \leqslant b$ then $a = p(b)$ for some $p \in \mathrm{Tr}(\mathcal{A})$, so $q(a) = q(p(b)) = p(q(b))$ for every $q \in \mathrm{Tr}(\mathcal{A})$, thus $q(a) \leqslant q(b)$ for every $q \in \mathrm{Tr}(\mathcal{A})$; and
- it satisfies $p(a) \leqslant a$ since $p(a) = p(a)$.

Hence, $\mathcal{A}_s$ is a symbolic ordered algebra by Lemma 2.26. □

Definition 4.15 For a semilattice algebra (A, Σ), a subset $D \subseteq A$ is *translation closed*, if $d \in D$ then $p(d) \in D$ for all $p \in \mathrm{Tr}(\mathcal{A})$.

Lemma 4.16 A subset $D \subseteq A$ for a semilattice algebra $\mathcal{A} = (A, \Sigma)$ is translation closed if and only if D is an ideal of the symbolic ordered algebra $\mathcal{A}_s$ where $\mathcal{A}_s$ is defined in Proposition 4.14.

Proposition 4.17 Let $T \subseteq \mathrm{T}(\Sigma, X)$ be a tree language.
(1) T is a semilattice tree language if and only if there exist a finite semilattice algebra $\mathcal{A} = (A, \Sigma)$, a morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and a subset $F \subseteq A$ such that $T = F\varphi^{-1}$.
(2) T is a symbolic tree language if and only if there exist a finite semilattice algebra $\mathcal{A} = (A, \Sigma)$, a morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and a translation closed subset $F \subseteq A$ such that $T = F\varphi^{-1}$.

5 Definability by Ordered Monoids

An important class of ordered algebras is the class of ordered monoids. Let us recall that an ordered monoid is a structure $\mathcal{M} = (M, \cdot, \lesssim)$ where $(M, \cdot)$ is a monoid and $\lesssim$ is an order on M compatible with $\cdot$ (called "stable order" in [13]), i.e., for any $a, b, m, m' \in M$, if $a \lesssim b$ then $m \cdot a \cdot m' \lesssim m \cdot b \cdot m'$.

5.1 Ordered Algebras Definable by Ordered Monoids

The translations of ordered algebras can be ordered as follows:

Definition 5.1 For an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$, the *ordered translation monoid* of $\mathcal{A}$ is the structure $\mathrm{OTr}(\mathcal{A}) = (\mathrm{Tr}(\mathcal{A}), \cdot, \lesssim_{\mathcal{A}})$ where $(\mathrm{Tr}(\mathcal{A}), \cdot)$ is the translation monoid of $\mathcal{A}$ and the binary relation $\lesssim_{\mathcal{A}}$ is defined on $\mathrm{Tr}(\mathcal{A})$ by the following for $p, q \in \mathrm{Tr}(\mathcal{A})$,

$$p \lesssim_{\mathcal{A}} q \iff (\forall a \in A)\big(p(a) \leqslant q(a)\big).$$

The relation $\lesssim_{\mathcal{A}}$ is indeed an order on $\mathrm{Tr}(\mathcal{A})$ compatible with the composition of translations: if $p \lesssim_{\mathcal{A}} q$ then $p \cdot r \lesssim_{\mathcal{A}} q \cdot r$ and $r \cdot p \lesssim_{\mathcal{A}} r \cdot q$ for any $p, q, r \in \mathrm{Tr}(\mathcal{A})$.

The following proposition is the ordered version of Lemma 10.7 of [23].

Proposition 5.2 For any finite ordered algebras $\mathcal{A}$ and $\mathcal{B}$,
(1) if $\mathcal{A} \subseteq_g \mathcal{B}$, then $\mathrm{OTr}(\mathcal{A}) \prec \mathrm{OTr}(\mathcal{B})$;
(2) if $\mathcal{A} \leftarrow_g \mathcal{B}$, then $\mathrm{OTr}(\mathcal{A}) \leftarrow \mathrm{OTr}(\mathcal{B})$; and
(3) $\mathrm{OTr}(\kappa(\mathcal{A}, \mathcal{B})) \subseteq \mathrm{OTr}(\mathcal{A}) \times \mathrm{OTr}(\mathcal{B})$ for any g-product $\kappa(\mathcal{A}, \mathcal{B})$.

Proof. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$.

(1) Let $\mathcal{M}$ be the order submonoid of $\mathrm{OTr}(\mathcal{B})$ generated by the elementary translations of the form $f^{\mathcal{B}}(a_1, \cdots, \xi, \cdots, a_m)$ for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m \in A$. The mapping

$$f^{\mathcal{B}}(a_1, \cdots, \xi, \cdots, a_m) \mapsto f^{\mathcal{A}}(a_1, \cdots, \xi, \cdots, a_m)$$

can be uniquely extended to an order monoid epimorphism $\mathcal{M} \to \mathrm{OTr}(\mathcal{A})$. Thus $\mathrm{OTr}(\mathcal{A}) \leftarrow \mathcal{M} \subseteq \mathrm{OTr}(\mathcal{B})$.

(2) Suppose $(\kappa, \varphi) : \mathcal{B} \to \mathcal{A}$ is an order g-epimorphism. By Proposition 4.3, the mapping $\mathrm{OTr}(\mathcal{B}) \to \mathrm{OTr}(\mathcal{A})$, $p \mapsto p_{(\kappa,\varphi)}$ is a monoid epimorphism. We show that it also preserves the translation orders. For any $p, q \in \mathrm{OTr}(\mathcal{B})$,

$$\begin{aligned} p \lesssim_{\mathcal{B}} q \quad &\Rightarrow \quad p(b) \leqslant' q(b) \text{ for all } b \in B \\ &\Rightarrow \quad p(b)\varphi \leqslant q(b)\varphi \text{ for all } b \in B \\ &\Rightarrow \quad p_{(\kappa,\varphi)}(b\varphi) \leqslant q_{(\kappa,\varphi)}(b\varphi) \text{ for all } b \in B \\ &\Rightarrow \quad p_{(\kappa,\varphi)}(a) \leqslant q_{(\kappa,\varphi)}(a) \text{ for all } a \in A \\ &\Rightarrow \quad p_{(\kappa,\varphi)} \lesssim_{\mathcal{A}} q_{(\kappa,\varphi)}. \end{aligned}$$

(3) Let Γ be a ranked alphabet and $\kappa : \Gamma \to \Sigma \times \Omega$ be an assignment. It is easy to verify that the mapping

$$g^{\kappa(\mathcal{A},\mathcal{B})}\big((a_1,b_1),\cdots,\xi,\cdots,(a_m,b_m)\big) \mapsto \Big(f^{\mathcal{A}}(a_1,\cdots,\xi,\cdots,a_m), h^{\mathcal{B}}(b_1,\cdots,\xi,\cdots,b_m)\Big)$$

for $a_1,\cdots,a_m \in A, b_1,\cdots,b_m \in B$ and $g \in \Gamma_m$ $(m > 0)$ where $g\kappa = (f,h)$, can be extended to a monomorphism $\psi : \mathrm{OTr}(\kappa(\mathcal{A},\mathcal{B})) \to \mathrm{OTr}(\mathcal{A}) \times \mathrm{OTr}(\mathcal{B})$ which satisfies $p(a,b) = (p\psi_1(a), p\psi_2(b))$ for all $a \in A$, $b \in B$ and $p \in \mathrm{Tr}(\kappa(\mathcal{A},\mathcal{B}))$, where ψ_1 and ψ_2 are the components of ψ: $p\psi = (p\psi_1, p\psi_2)$. We show that ψ is also order preserving. For $p,q \in \mathrm{Tr}(\kappa(\mathcal{A},\mathcal{B}))$,

$$\begin{array}{lll} p \lesssim_{\kappa(\mathcal{A},\mathcal{B})} q & \Rightarrow & p(a,b) \leqslant \times \leqslant' q(a,b) \text{ for all } a \in A, b \in B \\ & \Rightarrow & p\psi_1(a) \leqslant q\psi_1(a) \;\&\; p\psi_2(b) \leqslant' q\psi_2(b) \text{ for all } a \in A, b \in B \\ & \Rightarrow & p\psi_1 \lesssim_{\mathcal{A}} q\psi_1 \;\&\; p\psi_2 \lesssim_{\mathcal{B}} q\psi_2 \\ & \Rightarrow & (p\psi_1, p\psi_2) \lesssim_{\mathcal{A}} \times \lesssim_{\mathcal{B}} (q\psi_1, q\psi_2) \\ & \Rightarrow & p\psi \lesssim_{\mathcal{A}} \times \lesssim_{\mathcal{B}} q\psi. \end{array}$$

□

Definition 5.3 A *variety of finite ordered monoids*, VFOM in notation, is a class of finite ordered monoids closed under order submonoids, order epimorphic images and finite direct products.

For a VFOM $\mathbf{M}$, $\mathbf{M}^{\mathsf{a}}$ is the class of all finite ordered algebras whose ordered translation monoids are in $\mathbf{M}$, i.e.,

$$\mathbf{M}^{\mathsf{a}} = \{\mathcal{A} \mid \mathcal{A} \text{ is an ordered algebra such that } \mathrm{OTr}(\mathcal{A}) \in \mathbf{M}\}.$$

A class of finite ordered algebras $\mathscr{K}$ is said to be *definable by ordered translation monoids*, if there is a VFOM $\mathbf{M}$ such that $\mathbf{M}^{\mathsf{a}} = \mathscr{K}$.

Corollary 5.4 For any VFOM $\mathbf{M}$, the class $\mathbf{M}^{\mathsf{a}}$ is a gVFOA.

It is well-known that not every gVFOA is definable by syntactic ordered monoids. In this section we give necessary and sufficient conditions for a gVFOA to be of the form $\mathbf{M}^{\mathsf{a}}$ for some VFOA $\mathbf{M}$.

Definition 5.5 For any set D, let $\Lambda_D = \{\overline{d} \mid d \in D\}$ be the unary ranked alphabet consisting of unary function symbols $\overline{d}$ for each $d \in D$.

Let $\mathcal{M} = (M, \cdot, \lesssim)$ be a finite ordered monoid. The unary ordered algebra $\mathcal{M}^{\nu} = (M, \Lambda_M, \lesssim)$ is defined by $\overline{m}^{\mathcal{M}^{\nu}}(a) = a \cdot m$ for all $a, m \in M$.

The structure $\mathcal{M}^{\nu}$ for a finite ordered monoid $\mathcal{M}$ is indeed an ordered algebra, since for any $a, b, m \in M$,

$$a \lesssim b \Rightarrow a \cdot m \lesssim b \cdot m \Rightarrow \overline{m}^{\mathcal{M}^{\nu}}(a) \lesssim \overline{m}^{\mathcal{M}^{\nu}}(b).$$

Proposition 5.6 For a finite ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$,

$$\mathrm{OTr}(\mathcal{M}^\nu) \cong \mathcal{M}.$$

Proof. The elementary translations of $\mathcal{M}^\nu$ are of the form $\overline{m}^{\mathcal{M}^\nu}(\xi)$ where $m \in M$, and clearly $\overline{m}^{\mathcal{M}^\nu}(\xi) \cdot \overline{m'}^{\mathcal{M}^\nu}(\xi) = \overline{m \cdot m'}^{\mathcal{M}^\nu}(\xi)$ for all $m, m' \in M$. For the identity element 1_M of $\mathcal{M}$, the translation $\overline{1_M}^{\mathcal{M}^\nu}(\xi)$ is the identity translation of $\mathcal{M}^\nu$. This means that $\mathrm{Tr}(\mathcal{M}^\nu) = \{\overline{m}^{\mathcal{M}^\nu}(\xi) \mid m \in M\}$. Moreover, $\overline{m}^{\mathcal{M}^\nu}(\xi) \neq \overline{m'}^{\mathcal{M}^\nu}(\xi)$ whenever $m \neq m'$, since $\overline{m}^{\mathcal{M}^\nu}(\xi) = \overline{m'}^{\mathcal{M}^\nu}(\xi)$ implies

$$m = 1_M \cdot m = \overline{m}^{\mathcal{M}^\nu}(1_M) = \overline{m'}^{\mathcal{M}^\nu}(1_M) = 1_M \cdot m' = m'.$$

Hence, the mapping $\mathcal{M} \to \mathrm{OTr}(\mathcal{M}^\nu)$, $m \mapsto \overline{m}^{\mathcal{M}^\nu}(\xi)$ is a monoid isomorphism. We show that it is also an order isomorphism. For any $m, m' \in M$,

$$\begin{aligned} m \lesssim m' \quad &\Leftrightarrow \quad a \cdot m \lesssim a \cdot m' \text{ for all } a \in M \\ &\Leftrightarrow \quad \overline{m}^{\mathcal{M}^\nu}(a) \lesssim \overline{m'}^{\mathcal{M}^\nu}(a) \text{ for all } a \in M \\ &\Leftrightarrow \quad \overline{m}^{\mathcal{M}^\nu}(\xi) \lesssim_{\mathcal{M}^\nu} \overline{m'}^{\mathcal{M}^\nu}(\xi). \end{aligned}$$

□

Proposition 5.7 For all finite ordered monoids $\mathcal{M}$ and $\mathcal{P}$,

(1) if $\mathcal{M} \subseteq \mathcal{P}$, then $\mathcal{M}^\nu \subseteq_g \mathcal{P}^\nu$;

(2) if $\mathcal{M} \leftarrow \mathcal{P}$, then $\mathcal{M}^\nu \leftarrow_g \mathcal{P}^\nu$; and

(3) $(\mathcal{M} \times \mathcal{P})^\nu \cong_g \kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$ for some g-product $\kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$.

Proof. Write $\mathcal{M} = (M, \cdot, \lesssim)$ and $\mathcal{P} = (P, \cdot, \lesssim')$.

The statement (1) is obvious. For (2), we note that if $\varphi : \mathcal{P} \to \mathcal{M}$ is an order monoid epimorphism, then $(\overline{\varphi}, \varphi) : \mathcal{P}^\nu \to \mathcal{M}^\nu$, where $\overline{\varphi} : \Lambda_P \to \Lambda_M$ is defined by $(\overline{m})\overline{\varphi} = \overline{m\varphi}$, is an order g-epimorphism. For proving (3) define the assignment $\kappa : \Lambda_{M \times P} \to \Lambda_M \times \Lambda_P$ by $\overline{(m,p)}\kappa = (\overline{m}, \overline{p})$ for $m \in M, p \in P$, and let $\kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$ be the corresponding g-product of $\mathcal{M}^\nu$ and $\mathcal{P}^\nu$. It is easy to verify that the mappings $(\lambda, \varphi) : (\mathcal{M} \times \mathcal{P})^\nu \to \kappa(\mathcal{M}^\nu, \mathcal{P}^\nu)$ where λ is the identity mapping on $\Lambda_{M \times P}$ and φ is the identity mapping on $M \times P$, is an order g-isomorphism. □

The clause (3) of Proposition 5.7 can be generalized to any finite number of finite ordered monoids $\mathcal{M}_1, \cdots, \mathcal{M}_n$: $(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n)^\nu \cong_g \kappa(\mathcal{M}_1^\nu, \cdots, \mathcal{M}_n^\nu)$ for some g-product $\kappa(\mathcal{M}_1^\nu, \cdots, \mathcal{M}_n^\nu)$.

Definition 5.8 For a finite ordered algebra $\mathcal{A}$, the unary algebra $\mathcal{A}^\rho$ is defined to be $\big(\mathrm{OTr}(\mathcal{A})\big)^\nu$.

An immediate consequence of Proposition 5.7 is the following.

Corollary 5.9 For any finite ordered algebras $\mathcal{A}, \mathcal{A}_1, \cdots, \mathcal{A}_n$,

if $\mathrm{OTr}(\mathcal{A}) \prec \mathrm{OTr}(\mathcal{A}_1) \times \cdots \times \mathrm{OTr}(\mathcal{A}_n)$ then $\mathcal{A}^\rho \prec_g \kappa(\mathcal{A}_1^\rho, \cdots, \mathcal{A}_n^\rho)$ for some g-product $\kappa(\mathcal{A}_1^\rho, \cdots, \mathcal{A}_n^\rho)$.

Our characterization of gVFOA's definable by syntactic ordered monoids is the following.

Proposition 5.10 For a class $\mathscr{K}$ of finite ordered algebras the following conditions are equivalent:

(1) $\mathscr{K}$ is definable by ordered translation monoids;
(2) $\mathscr{K}$ is a gVFOA such that for all finite ordered algebras $\mathcal{A}$ and $\mathcal{B}$, if $\mathrm{OTr}(\mathcal{A}) \cong \mathrm{OTr}(\mathcal{B})$ and $\mathcal{A} \in \mathscr{K}$ then $\mathcal{B} \in \mathscr{K}$;
(3) $\mathscr{K}$ is a gVFOA such that $\mathcal{A} \in \mathscr{K}$ if and only if $\mathcal{A}^\rho \in \mathscr{K}$ for any $\mathcal{A}$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows from Proposition 5.6. For (3) $\Rightarrow$ (1), suppose the gVFOA $\mathscr{K}$ satisfies the equivalence $\mathcal{A} \in \mathscr{K} \Leftrightarrow \mathcal{A}^\rho \in \mathscr{K}$ for any finite ordered algebra $\mathcal{A}$. Let $\mathbf{M}$ be the VFOM generated by $\{\mathrm{OTr}(\mathcal{A}) \mid \mathcal{A} \in \mathscr{K}\}$. We show that $\mathscr{K} = \mathbf{M}^{\mathsf{a}}$. Obviously $\mathscr{K} \subseteq \mathbf{M}^{\mathsf{a}}$. For the opposite inclusion let $\mathcal{B} \in \mathbf{M}^{\mathsf{a}}$. So, $\mathrm{OTr}(\mathcal{B}) \prec \mathrm{OTr}(\mathcal{A}_1) \times \cdots \times \mathrm{OTr}(\mathcal{A}_n)$ for some $\mathcal{A}_1, \cdots, \mathcal{A}_n \in \mathscr{K}$. By Corollary 5.9, $\mathcal{B}^\rho \prec_g \kappa(\mathcal{A}_1^\rho, \cdots, \mathcal{A}_n^\rho)$ for some g-product $\kappa(\mathcal{A}_1^\rho, \cdots, \mathcal{A}_n^\rho)$. Since $\mathcal{A}_1^\rho, \cdots, \mathcal{A}_n^\rho \in \mathscr{K}$ then $\mathcal{B}^\rho \in \mathscr{K}$, hence $\mathcal{B} \in \mathscr{K}$. Thus $\mathbf{M}^{\mathsf{a}} \subseteq \mathscr{K}$. $\square$

Remark 5.11 Proposition 5.7 and the proof of Proposition 5.10 also yield the fact that for any gVFOA $\mathscr{K}$ definable by ordered translation monoids, the class $\{\mathrm{OTr}(\mathcal{A}) \mid \mathcal{A} \in \mathscr{K}\}$ is a variety of finite ordered monoids.

5.1.1 Examples

Ordered nilpotent algebras

Lemma 5.12 If $\mathcal{A} = (A, \Sigma, \leqslant)$ is an ordered n-nilpotent algebra, then the ordered translation semigroup $\mathrm{OTrS}(\mathcal{A}) = (\mathrm{TrS}(\mathcal{A}), \cdot, \lesssim_{\mathcal{A}})$ of $\mathcal{A}$ is a nilpotent semigroup where zero element is the least element.

Proof. It was shown in Lemma 2.14 that $\mathrm{TrS}(\mathcal{A})$ is a nilpotent semigroup, where $p_1 \cdots p_n$ is the zero element for every $p_1, \cdots, p_n \in \mathrm{TrS}(\mathcal{A})$. On the other hand $p_1 \cdots p_n(a) \leqslant q(a)$ holds for all $q \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$. Thus $p_1 \cdots p_n \lesssim_{\mathcal{A}} q$, so zero is the least element of the semigroup $\mathrm{TrS}(\mathcal{A})$. $\square$

The converse of Lemma 5.12 does not hold:

Example 5.13 Let $\Lambda = \Lambda_1 = \{\alpha\}$ and $A = \{a, b\}$, $B = \{a, b, c\}$. Define the ordered Λ-algebras $\mathcal{A} = (A, \Lambda, \leqslant)$ and $\mathcal{B} = (B, \Lambda, \leqslant')$ by $\alpha^{\mathcal{A}}(a) = \alpha^{\mathcal{A}}(b) = b$, $\alpha^{\mathcal{B}}(a) = \alpha^{\mathcal{B}}(b) = b$, $\alpha^{\mathcal{B}}(c) = c$, and $\leqslant = \{(a, a), (b, a), (b, b)\}$, $\leqslant' = \{(a, a), (b, a), (b, b), (c, c)\}$. Then the ordered translation semigroups of $\mathcal{A}$ and $\mathcal{B}$ are the trivial semigroup which consists of a single zero element, while $\mathcal{A}$ is an ordered nilpotent algebra and $\mathcal{B}$ is not.

Hence, **Nil** is not definable by ordered translation monoids or semigroups.

Semilattice algebras

By Lemma 2.16 the class **SL** is definable by semilattice monoids.

Symbolic ordered algebras

Lemma 5.14 An ordered algebra is symbolic if and only if its ordered translation monoid is a symbolic monoid.

Proof. By Lemma 2.26, an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is symbolic if and only if (A, Σ) is a semilattice algebra and $p(a) \leq a$ holds for all $a \in A$ and $p \in \mathrm{Tr}(\mathcal{A})$. The statement "$p(a) \leq a$ for all $a \in A$" is equivalent to $p \lesssim_{\mathcal{A}} 1_A$. Thus, from Lemma 2.16, it follows that $\mathcal{A}$ is symbolic if and only if $\mathrm{OTr}(\mathcal{A})$ is a symbolic ordered monoid. $\square$

So, the class **Sym** is definable by symbolic ordered monoids.

5.2 Tree Languages Definable by Ordered Monoids

Let Σ be a ranked alphabet and X be a leaf alphabet.

Definition 5.15 For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the quasi-order $\precsim_T$ is defined on ΣX-contexts by

$$P \precsim_T Q \iff (\forall R \in \mathrm{C}(\Sigma, X))(\forall t \in \mathrm{T}(\Sigma, X))\big(t \cdot Q \cdot R \in T \Rightarrow t \cdot P \cdot R \in T\big)$$

for $P, Q \in \mathrm{C}(\Sigma, X)$.

We note that the equivalence relation of $\precsim_T$ is the m-congruence of T:

$$P \mu_T Q \iff (\forall R \in \mathrm{C}(\Sigma, X))(\forall t \in \mathrm{T}(\Sigma, X))\big(t \cdot P \cdot R \in T \Leftrightarrow t \cdot Q \cdot R \in T\big).$$

Note that the quotient structure $(\mathrm{C}(\Sigma, X)/\mu_T, \cdot)$ is a monoid where the operation $(\cdot)$ is defined by $P/\mu_T \cdot Q/\mu_T = (P \cdot Q)/\mu_T$ for $P, Q \in \mathrm{C}(\Sigma, X)$. This is called the syntactic monoid of T.
The *syntactic ordered monoid* of T is the structure

$$\mathrm{SOM}(T) = (\mathrm{C}(\Sigma, X)/\theta_T, \cdot, \lesssim_T),$$

where $\lesssim_T$ is the order induced by $\precsim_T$: $P/\mu_T \lesssim_T Q/\mu_T \Leftrightarrow P \precsim_T Q$ for $P, Q \in \mathrm{C}(\Sigma, X)$; cf. [23] or [25]. It is easy to verify that $P \precsim_T Q$ implies $R \cdot P \cdot S \precsim_T R \cdot Q \cdot S$ for any $P, Q, R, S \in \mathrm{C}(\Sigma, X)$. Thus the structure $\mathrm{SOM}(T)$ is indeed an ordered monoid.

It is known that the syntactic monoid of a tree language is the translation monoid of the syntactic algebra of the language ([16, 23]). Here we show the corresponding proposition for ordered translation monoids and syntactic ordered algebras.

Proposition 5.16 For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$,

$$\mathrm{OTr}(\mathrm{SOA}(T)) \cong \mathrm{SOM}(T).$$

Proof. It is easy to see that the mapping

$$f(t_1, \cdots, \xi, \cdots, t_m) \mapsto f^{\mathrm{SOA}(T)}(t_1/\theta_t, \cdots, \xi, \cdots, t_m/\theta_T)$$

can be extended to a monoid epimorphism $\varphi : \mathrm{C}(\Sigma, X) \to \mathrm{OTr}(\mathrm{SOA}(T))$ which satisfies $P\varphi(t/\theta_T) = (t \cdot P)/\theta_T$ for all $t \in \mathrm{T}(\Sigma, X), P \in \mathrm{C}(\Sigma, X)$. We show that for any $P, Q \in \mathrm{C}(\Sigma, X)$, $P \precsim_T Q$ iff $P\varphi \lesssim_{\mathrm{SOA}(T)} Q\varphi$:

$$\begin{array}{rcl} P \precsim_T Q & \Leftrightarrow & (\forall t \in \mathrm{T}(\Sigma, X))\, (\forall R \in \mathrm{C}(\Sigma, X))\, \big(t \cdot Q \cdot R \in T \to t \cdot P \cdot R \in T\big) \\ & \Leftrightarrow & t \cdot P \preccurlyeq_T t \cdot Q \text{ for all } t \in \mathrm{T}(\Sigma, X) \\ & \Leftrightarrow & (t \cdot P)/\theta_T \leqslant_T (t \cdot Q)/\theta_T \text{ for all } t \in \mathrm{T}(\Sigma, X) \\ & \Leftrightarrow & P\varphi(t/\theta_T) \leqslant_T Q\varphi(t/\theta_T) \text{ for all } t \in \mathrm{T}(\Sigma, X) \\ & \Leftrightarrow & P\varphi \lesssim_{\mathrm{SOA}(T)} Q\varphi. \end{array}$$

Thus $\varphi\circ \lesssim_{\mathrm{SOA}(T)} \circ\varphi^{-1} = \precsim_T$, and then from Proposition 2.4 it follows that $\mathrm{SOM}(T) \cong \mathrm{OTr}(\mathrm{SOA}(T))$. □

The following is implied by Corollary 3.4, Lemma 4.4 and Propositions 5.2 and 5.16.

Corollary 5.17 For ranked alphabets Σ and Ω, leaf alphabets X and Y, a ΣX-context $P \in \mathrm{C}(\Sigma, X)$, an order g-morphism $(\kappa, \varphi) : \mathcal{T}(\Omega, Y) \to \mathcal{T}(\Sigma, X)$, and tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$,
(1) $\mathrm{SOM}(T \cap T'), \mathrm{SOM}(T \cup T') \prec \mathrm{SOM}(T) \times \mathrm{SOM}(T')$.
(2) $\mathrm{SOM}(P^{-1}(T)) \leftarrow \mathrm{SOM}(T)$.
(3) $\mathrm{SOM}(T\varphi^{-1}) \prec \mathrm{SOM}(T)$ and, moreover, if (κ, φ) is a g-epimorphism then $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$.

Definition 5.18 For a VFOM $\mathbf{M}$, let $\mathbf{M}^{\mathrm{t}}$ be the family of all recognizable tree languages whose syntactic ordered monoids are in $\mathbf{M}$, that is to say, for any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, $T \in \mathbf{M}^{\mathrm{t}}(\Sigma, X) \Leftrightarrow \mathrm{SOM}(T) \in \mathbf{M}$ holds.

A family of recognizable tree languages $\mathscr{V}$ is said to be *definable by syntactic ordered monoids* if there is a VFOM $\mathbf{M}$ such that $\mathbf{M}^{\mathrm{t}} = \mathscr{V}$.

By Corollary 5.17, the family $\mathbf{M}^{\mathsf{t}}$ for any VFOM $\mathbf{M}$ is a gPVTL. In this subsection we characterize the gPVTL's that are definable by syntactic ordered monoids.

Lemma 5.19 For any VFOM $\mathbf{M}$ the following hold:
(1) $\mathbf{M}^{\mathsf{at}} = \mathbf{M}^{\mathsf{t}}$,
(2) $\mathbf{M}^{\mathsf{ta}} = \mathbf{M}^{\mathsf{a}}$.

Proof. (1) For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$ by Proposition 5.16,
$T \in \mathbf{M}^{\mathsf{at}}(\Sigma, X) \Leftrightarrow \mathrm{SOA}(T) \in \mathbf{M}^{\mathsf{a}} \Leftrightarrow \mathrm{OTr}(\mathrm{SOA}(T)) \in \mathbf{M} \Leftrightarrow \mathrm{SOM}(T) \in \mathbf{M} \Leftrightarrow T \in \mathbf{M}^{\mathsf{t}}(\Sigma, X)$.
(2) By (1) and Lemma 4.8, $(\mathbf{M}^{\mathsf{t}})^{\mathsf{a}} = (\mathbf{M}^{\mathsf{at}})^{\mathsf{a}} = (\mathbf{M}^{\mathsf{a}})^{\mathsf{ta}} = \mathbf{M}^{\mathsf{a}}$. $\square$

Corollary 5.20 (1) A gPVTL $\mathscr{V}$ is definable by syntactic ordered monoids iff $\mathscr{V}^{\mathsf{a}}$ is a gVFOA definable by ordered translation monoids.
(1) A gVFOA $\mathscr{K}$ is definable by ordered translation monoids iff $\mathscr{K}^{\mathsf{t}}$ is a gPVTL definable by syntactic ordered monoids.

Definition 5.21 Let Σ, Ω be ranked alphabets and X, Y be leaf alphabets.
A *tree homomorphism* is a mapping $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ determined by some mappings $\varphi_X : X \to \mathrm{T}(\Omega, Y)$ and $\varphi_m : \Sigma_m \to \mathrm{T}(\Omega, Y \cup \{\xi_1, \cdots, \xi_m\})$ where $\Sigma_m \neq \emptyset$ and the ξ_i's are new variables, inductively as follows
(1) $x\varphi = \varphi_X(x)$ for $x \in X$, $c\varphi = \varphi_0(c)$ for $c \in \Sigma_0$, and
(2) $f(t_1, \cdots, t_n)\varphi = \varphi_n(f)[\xi_1 \leftarrow t_1\varphi, \cdots, \xi_n \leftarrow t_n\varphi]$ in which ξ_i is replaced with $t_i\varphi$ for all $i \leq n$ (cf. [23], page 7).
A tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is *regular* if for every $f \in \Sigma_m$ ($m \geq 1$) each $\xi_1, \cdots, \xi_m$ appears exactly once in $\varphi_m(f)$, cf. [16].

For a regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$, the unique extension $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ to contexts is obtained by setting $\varphi_*(\xi) = \xi$ (cf. [23], Proposition 10.3). We note that the identities $(Q \cdot P)\varphi_* = Q\varphi_* \cdot P\varphi_*$ and $(t \cdot Q \cdot P)\varphi = t\varphi \cdot Q\varphi_* \cdot P\varphi_*$ hold for all $P, Q \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$.

For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ the *syntactic monoid morphism* of T is the mapping $\lambda_T : \mathrm{C}(\Sigma, X) \to \mathrm{SOM}(T)$ defined by $P\lambda_T = P/\mu_T$ for $P \in \mathrm{C}(\Sigma, X)$.

Definition 5.22 A regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is said to be *full with respect to* a tree language $T \subseteq \mathrm{T}(\Omega, Y)$, if the mappings $\varphi\varphi_T : \mathrm{T}(\Sigma, X) \to \mathrm{SOA}(T)$, $t\varphi\varphi_T = t\varphi/\theta_T$ and $\varphi_*\lambda_T : \mathrm{C}(\Sigma, X) \to \mathrm{SOM}(T)$, $P\varphi_*\lambda_T = P\varphi_*/\mu_T$ are surjective.

An equivalent definition is:

Lemma 5.23 A regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is full with respect to $T \subseteq \mathrm{T}(\Omega, Y)$ iff for every $Q \in \mathrm{C}(\Omega, Y)$ and every $s \in \mathrm{T}(\Omega, Y)$, there are $P \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$ such that $Q \, \mu_T \, P\varphi_*$ and $s \, \theta_T \, t\varphi$ hold.

Lemma 5.24 If $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is a regular tree homomorphism and $T \subseteq \mathrm{T}(\Omega, Y)$, then $\mathrm{SOM}(T\varphi^{-1}) \prec \mathrm{SOM}(T)$, and if φ is full with respect to T, then $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$.

Proof. We note that $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid homomorphism. Let $S \subseteq \mathrm{C}(\Omega, Y)$ be the image of φ_*, $\precsim$ be the restriction of $\precsim_T$ to S and μ be the equivalence relation of $\precsim$. Then S/μ is a submonoid of $\mathrm{C}(\Omega, Y)/\mu_T$. We show that $P\varphi_* \precsim Q\varphi_*$ implies $P \precsim_{T\varphi^{-1}} Q$ for all $P, Q \in \mathrm{C}(\Sigma, X)$. Suppose $P\varphi_* \precsim Q\varphi_*$ and take arbitrary $t \in \mathrm{T}(\Sigma, X)$ and $R \in \mathrm{C}(\Sigma, X)$. Then

$$
\begin{aligned}
t \cdot Q \cdot R \in T\varphi^{-1} \quad &\Rightarrow \quad t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T \\
&\Rightarrow \quad t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T \\
&\Rightarrow \quad t \cdot P \cdot R \in T\varphi^{-1},
\end{aligned}
$$

that is $P \precsim_{T\varphi^{-1}} Q$. So the mapping $\psi : S/\mu \to \mathrm{C}(\Sigma, X)/\mu_{T\varphi^{-1}}$ defined by $((P\varphi_*)\mu)\psi = P\mu_{T\varphi^{-1}}$ is well-defined, order preserving and surjective. It is also a monoid morphism, since $((P\varphi_*)\mu \cdot (Q\varphi_*)\mu)\psi = ((P \cdot Q)\varphi_*\mu)\psi = (P \cdot Q)\mu_{T\varphi^{-1}} = P\mu_{T\varphi^{-1}} \cdot Q\mu_{T\varphi^{-1}} = ((P\varphi_*)\mu)\psi \cdot ((Q\varphi_*)\mu)\psi$ for all $P, Q \in \mathrm{C}(\Sigma, X)$. Hence $\mathrm{SOM}(T\varphi^{-1}) \leftarrow S/\precsim\, \subseteq \mathrm{SOM}(T)$, so $\mathrm{SOM}(T\varphi^{-1}) \prec \mathrm{SOM}(T)$.

Now, suppose φ is full with respect to T. We show $P \precsim_{T\varphi^{-1}} Q$ iff $P\varphi_* \precsim_T Q\varphi_*$ for any $P, Q \in \mathrm{C}(\Sigma, X)$. Clearly, $P\varphi_* \precsim_T Q\varphi_*$ implies $P \precsim_{T\varphi^{-1}} Q$ (see above). For the converse, suppose $P \precsim_{T\varphi^{-1}} Q$, and take arbitrary $R' \in \mathrm{C}(\Omega, Y)$ and $t' \in \mathrm{T}(\Omega, Y)$. There are $R \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$ such that $R\varphi_* \, \mu_T \, R'$ and $t\varphi \, \theta_T \, t'$. Hence

$$
\begin{aligned}
t' \cdot Q\varphi_* \cdot R' \in T \quad &\Rightarrow \quad t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T \\
&\Rightarrow \quad (t \cdot Q \cdot R)\varphi \in T \\
&\Rightarrow \quad t \cdot Q \cdot R \in T\varphi^{-1} \\
&\Rightarrow \quad t \cdot P \cdot R \in T\varphi^{-1} \\
&\Rightarrow \quad t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T \\
&\Rightarrow \quad t' \cdot P\varphi_* \cdot R' \in T,
\end{aligned}
$$

which shows that $P\varphi_* \precsim_T Q\varphi_*$. Hence $P \precsim_{T\varphi^{-1}} Q$ iff $P\varphi_* \precsim_T Q\varphi_*$, and since the function $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid homomorphism, then by Proposition 2.4, $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$. □

In the following two lemmas some connections between tree languages recognizable by a finite ordered algebra $\mathcal{A}$ and tree languages recognizable by $\mathcal{A}^\rho$ are presented. Recall that unary ranked alphabet of the algebra $\mathcal{A}^\rho$ is $\{\overline{p} \mid p \in \mathrm{Tr}(\mathcal{A})\}$; for simplicity we denote this alphabet by $\Lambda_\mathcal{A}$.

Suppose $\mathcal{A} = (A, \Sigma)$ is a finite algebra. Every context in $\mathrm{C}(\Sigma, A)$ corresponds to a translation in $\mathrm{Tr}(\mathcal{A})$ in a natural way: With the elementary context $f(a_1, \cdots, \xi, \cdots, a_m)$ we associate the elementary translation $f^\mathcal{A}(a_1, \cdots, \xi, \cdots, a_m)$ where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \cdots, a_m \in A$. This correspondence can be extended to a mapping $-^\mathcal{A} : \mathrm{C}(\Sigma, A) \to \mathrm{Tr}(\mathcal{A})$ which satisfies $\xi^\mathcal{A} = 1_A$ (the identity translation) and $(P \cdot Q)^\mathcal{A} = P^\mathcal{A} \cdot Q^\mathcal{A}$ for all

$P, Q \in \mathrm{C}(\Sigma, A)$. We note that for any translation $p \in \mathrm{Tr}(\mathcal{A})$, there is a $P \in \mathrm{C}(\Sigma, A)$ such that $P^{\mathcal{A}} = p$ and this P may not be unique. In other words, $-^{\mathcal{A}}$ is a non-injective monoid epimorphism.

We also note that the mapping $-^{\mathcal{A}} : \mathrm{C}(\Sigma, A) \setminus \{\xi\} \to \mathrm{TrS}(\mathcal{A})$ is a semigroup epimorphism that assigns non-unit contexts of $\mathrm{C}(\Sigma, A)$ to translations of $\mathcal{A}$.

Lemma 5.25 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra, and X be a leaf alphabet disjoint from A. For any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^{\rho}$ there exists a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ and a tree language $T \subseteq \mathrm{T}(\Sigma, X \cup A)$ such that $L = T\varphi^{-1}$ and T can be recognized by a finite power $\mathcal{A}^n$ where $n = |A|$.

Proof. Let $\alpha : X \to \mathrm{Tr}(\mathcal{A})$ be an initial assignment for $\mathcal{A}^{\rho}$ and $F \subseteq \mathrm{Tr}(\mathcal{A})$ be an ideal of $\mathrm{OTr}(\mathcal{A})$ such that $L = \{t \in \mathrm{T}(\Lambda_{\mathcal{A}}, X) \mid t\alpha^{\mathcal{A}^{\rho}} \in F\}$. Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ by $\varphi_X(x) = x$ for all $x \in X$, and for every $p \in \mathrm{Tr}(\mathcal{A})$ choose a $\varphi_1(\overline{p}) \in \mathrm{C}(\Sigma, A)$ such that $\varphi_1(\overline{p})^{\mathcal{A}} = p$. Obviously φ is a regular tree homomorphism. Suppose that $A = \{a_1, \cdots, a_n\}$. Let F' be the ideal of $\mathcal{A}^n$ generated by $\{(p(a_1), \cdots, p(a_n)) \in A^n \mid p \in F\}$, i.e.,

$$(b_1, \cdots, b_m) \in F' \Leftrightarrow \text{for some } p \in F \ \big(b_j \leqslant p(a_j) \text{ for all } j \leq n\big),$$

and define the initial assignment $\beta : X \cup A \to A^n$ for $\mathcal{A}^n$ by $a\beta = (a, \cdots, a) \in A^n$ for all $a \in A$ and $x\beta = \big((x\alpha)(a_1), \cdots, (x\alpha)(a_n)\big)$ for all $x \in X$.

Let T be the subset of $\mathrm{T}(\Sigma, X \cup A)$ recognized by $(\mathcal{A}^n, \beta, F')$, that is $T = \{t \in \mathrm{T}(\Sigma, X \cup A) \mid t\beta^{\mathcal{A}^n} \in F'\}$. We show that $L = T\varphi^{-1}$. Every tree w in $\mathrm{T}(\Lambda_{\mathcal{A}}, X)$ is of the form

$$w = \overline{p_1}\Big(\overline{p_2}\big(\cdots \overline{p_k}(x) \cdots\big)\Big)$$

for some $p_1, \cdots, p_k \in \mathrm{Tr}(\mathcal{A})$ $(k \geq 0)$ and $x \in X$. For such a tree w,

$$w\alpha^{\mathcal{A}^{\varrho}} = x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1, \text{ and}$$

$$(w\varphi)\beta^{\mathcal{A}^n} = \Big(x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a_1), \cdots, x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a_n)\Big). \text{ So,}$$

$$\begin{aligned} w\varphi \in T \quad &\Leftrightarrow \quad (w\varphi)\beta^{\mathcal{A}^n} \in F' \\ &\Leftrightarrow \quad \text{for some } p \in F,\ x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1(a) \leqslant p(a) \text{ for all } a \in A \\ &\Leftrightarrow \quad \text{for some } p \in F,\ x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1 \lesssim_{\mathcal{A}} p \\ &\Leftrightarrow \quad x\alpha \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1 \in F \\ &\Leftrightarrow \quad w\alpha^{\mathcal{A}^{\varrho}} \in F \\ &\Leftrightarrow \quad w \in L. \end{aligned}$$

□

Lemma 5.26 Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra and X be a leaf alphabet disjoint from $A \cup \Sigma$. For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$ recognized by $\mathcal{A}$ there exists a unary ranked alphabet Λ and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ such that φ is full with respect to T, and for every $z \in X \cup \Sigma_0$, $T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^\rho$.

Proof. Let $\mathcal{B} = (B, \Sigma, \leqslant')$ be the syntactic ordered algebra of T. Then $\mathcal{B} \prec \mathcal{A}$. Suppose $T = \{t \in \mathrm{T}(\Sigma, X) \mid t\beta^{\mathcal{B}} \in F\}$ where $\beta : X \to B$ is an initial assignment for $\mathcal{B}$ and $F \trianglelefteq \mathcal{B}$. Since $\mathcal{B}$ is the least ordered algebra that recognizes T, the algebra $\mathcal{B}$ is generated by $\beta(X)$. The mapping $\beta : X \to B$ can be uniquely extended to a monoid homomorphism $\beta_{\mathrm{c}} : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Sigma, B)$. Since B is generated by $\beta(X)$, the mapping $\beta_{\mathrm{c}}^{\mathcal{B}} : \mathrm{C}(\Sigma, X) \to \mathrm{Tr}(\mathcal{B})$, $\beta_{\mathrm{c}}^{\mathcal{B}}(Q) = \beta_c(Q)^{\mathcal{B}}$ is surjective. Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ by $\varphi_X(x) = x$ for all $x \in X \cup \Sigma_0$, and for every $q \in \mathrm{Tr}(\mathcal{B})$ choose a $\varphi_1(\overline{q}) = Q \in \mathrm{C}(\Sigma, X)$ such that $\beta_{\mathrm{c}}(Q)^{\mathcal{B}} = q$. Note that φ is a regular tree homomorphism. It remains to show that φ is full with respect to T and that for every $z \in X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{B}^\rho$. This will finish the proof since $\mathrm{OTr}(\mathcal{B}) \prec \mathrm{OTr}(\mathcal{A})$ follows from $\mathcal{B} \prec \mathcal{A}$ by Proposition 5.2, and so $\mathcal{B}^\rho \prec \mathcal{A}^\rho$ by Proposition 5.7, which implies that L_z can also be recognized by $\mathcal{A}^\rho$.
First, we show that φ is full with respect to T. Let $Q \in \mathrm{C}(\Sigma, X)$ be a context. For $q = \beta_{\mathrm{c}}(Q)^{\mathcal{B}} \in \mathrm{Tr}(\mathcal{B})$, $\overline{q}(\xi)\varphi_* \, \mu_T \, Q$ holds. By induction on the height of t we show that for any $t \in \mathrm{T}(\Sigma, X)$ there is an $s \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$ such that $t \, \theta_T \, s\varphi$. If $t = x \in X \cup \Sigma_0$, then $s\varphi \, \theta_T \, t$ for $s = t$. If $t = t' \cdot P$ for some $P \in \mathrm{C}(\Sigma, X)$ and $t' \in \mathrm{T}(\Sigma, X)$ such that the height of t' is less than the height of t, then by the induction hypothesis there is an $s' \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$ such that $t' \, \theta_T \, s'\varphi$. Also, for some $p \in \mathrm{Tr}(\mathcal{B})$, $\overline{p}(\xi)\varphi_* \, \mu_T \, P$ holds. Let $s = \overline{p}(s')$. Then $s\varphi = s'\varphi \cdot \overline{p}(\xi)\varphi_* \, \theta_T \; t' \cdot P = t$. Thus, φ is full with respect to T by Lemma 5.23.
Second, we show that L_z can be recognized by $\mathcal{B}^\rho$ for a fixed $z \in X \cup \Sigma_0$. Let 1_B be the identity translation of $\mathcal{B}$. Define the initial assignment $\alpha : \{z\} \to \mathrm{Tr}(\mathcal{B})$ for $\mathcal{B}^\rho$ by $z\alpha = 1_B$, and let $F_z = \{q \in \mathrm{Tr}(\mathcal{B}) \mid q(z\beta^{\mathcal{B}}) \in F\}$. We show that $F_z \trianglelefteq \mathcal{B}^\rho$ and L_z is recognized by $(\mathcal{B}^\rho, \alpha, F_z)$. For $p, q \in \mathrm{Tr}(\mathcal{B})$, if $p \lesssim_{\mathcal{B}} q \in F_z$, then $p(z\beta^{\mathcal{B}}) \leqslant' q(z\beta^{\mathcal{B}}) \in F$, so $p(z\beta^{\mathcal{B}}) \in F$, thus $p \in F_z$. Hence $F_z \trianglelefteq \mathcal{B}^\rho$. Every $w \in \mathrm{T}(\Lambda_{\mathcal{B}}, \{z\})$ can be written in the form

$$w = \overline{q_1}\Big(\overline{q_2}\big(\cdots \overline{q_h}(z) \cdots\big)\Big)$$

for some $q_1, \cdots, q_h \in \mathrm{Tr}(\mathcal{B})$ $(h \geq 0)$. For such a tree w,

$$w\alpha^{\mathcal{B}^\rho} = 1_B \cdot q_h \cdot \ldots \cdot q_2 \cdot q_1, \text{ and } (w\varphi)\beta^{\mathcal{B}} = q_h \cdot \ldots \cdot q_2 \cdot q_1(z\beta^{\mathcal{B}}). \text{ Thus,}$$

$$\begin{aligned} w \in L_z \;\Leftrightarrow\; w\varphi \in T \quad &\Leftrightarrow \quad (w\varphi)\beta^{\mathcal{B}} \in F \\ &\Leftrightarrow \quad q_h \cdot \ldots \cdot q_2 \cdot q_1(z\beta^{\mathcal{B}}) \in F \\ &\Leftrightarrow \quad q_h \cdot \ldots \cdot q_2 \cdot q_1 \in F_z \\ &\Leftrightarrow \quad w\alpha^{\mathcal{B}^\rho} \in F_z. \end{aligned}$$

So, $L_z = \{w \in \mathrm{T}(\Lambda, \{z\}) \mid w\alpha^{\mathcal{B}^\rho} \in F_z\}$. $\square$

Now, we are almost ready to characterize the gPVTL's definable by syntactic ordered monoids. Before that we note a remark.

Remark 5.27 Let Λ be a unary ranked alphabet. For every leaf alphabet X and every subset $Y \subseteq X$, $\mathrm{C}(\Lambda, Y) = \mathrm{C}(\Lambda, X)$, and the quasi-order $\precsim_T$ for a tree language $T \subseteq \mathrm{T}(\Lambda, Y)$ on $\mathrm{C}(\Lambda, Y)$ is the same relation $\precsim_T$ on $\mathrm{C}(\Lambda, X)$ when T is viewed as a subset of $\mathrm{T}(\Lambda, X)$.

So, if a family of tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ is definable by syntactic ordered monoids, then for any unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.

Proposition 5.28 A family of recognizable tree languages $\mathscr{V}$ is definable by syntactic ordered monoids if and only if $\mathscr{V}$ is a gPVTL that satisfies the following properties:
(1) The family $\mathscr{V}$ is closed under inverse regular tree homomorphisms.
(2) For every unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.
(3) For any regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ which is full with respect to a tree language $T \subseteq \mathrm{T}(\Omega, Y)$, if $T\varphi^{-1} \in \mathscr{V}(\Sigma, X)$ then $T \in \mathscr{V}(\Omega, Y)$.

Proof. The fact that for any VFOM $\mathbf{M}$, $\mathbf{M}^{\mathsf{t}}$ is a gPVTL follows from Corollary 5.17, that it satisfies the conditions (1) and (3) follows from Proposition 5.24 and that it satisfies the condition (2) follows from Remark 5.27.

For the converse, suppose a gPVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ satisfies the conditions of the proposition. By Corollary 5.20 it is enough to show that $\mathscr{V}^{\mathsf{a}}$ satisfies the condition of Proposition 5.10.

Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra in $\mathscr{V}^{\mathsf{a}}$. By Lemma 5.25, any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^\rho$ can be written as $L = T\varphi^{-1}$ where $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ is a regular tree homomorphism, and T is a tree language recognized by some power $\mathcal{A}^n$ of $\mathcal{A}$. Then $\mathcal{A}^n \in \mathscr{V}^{\mathsf{a}}$ implies that $T \in \mathscr{V}(\Sigma, X \cup A)$, and hence $L = T\varphi^{-1} \in \mathscr{V}(\Lambda_{\mathcal{A}}, X)$ by (1). This holds for every tree language L recognizable by $\mathcal{A}^\rho$, so by Lemma 4.9, $\mathcal{A}^\rho \in \mathscr{V}^{\mathsf{a}}$.

Now, suppose $\mathcal{A}^\rho \in \mathscr{V}^{\mathsf{a}}$ for a finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$. Let $T \subseteq \mathrm{T}(\Sigma, X)$ be a tree language recognizable by $\mathcal{A}$. By Lemma 5.26, there exists a unary ranked alphabet Λ and a full regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ with respect to T such that for every z in $X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^\rho$. So, $L_z \in \mathscr{V}(\Lambda, \{z\})$, thus $L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$ by (2). Hence, $T\varphi^{-1} = \bigcup_{z \in X \cup \Sigma_0} L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$. Since φ is full with respect to T, then $T \in \mathscr{V}(\Sigma, X)$ by (3). This holds for every tree language T recognizable by $\mathcal{A}$, hence $\mathcal{A} \in \mathscr{V}^{\mathsf{a}}$ by Lemma 4.9. $\square$

5.2.1 Examples

Cofinite tree languages

It can be shown that the gPVTL Cof is closed under inverse regular tree homomorphisms. Since $\mathbf{Nil} = \text{Cof}^{\mathsf{a}}$ is not definable by ordered translation monoids, then Cof is not definable by syntactic ordered monoids. We can show this directly: Let $\Lambda = \Lambda_1 = \{\alpha\}$ be a unary ranked alphabet and $X = \{x, y\}$ be a leaf alphabet. Let $T = \{\alpha(y), \alpha(\alpha(y)), \alpha(\alpha(\alpha(y))), \cdots\}$. Clearly $T \in \text{Cof}(\Lambda, \{y\})$, but $T \notin \text{Cof}(\Lambda, X)$. Hence, Cof does not satisfy the condition (2) of Proposition 5.28.

Semilattice tree languages

The family SL is definable by syntactic monoids, since a tree language is semilattice if and only if its translation monoid is a semilatiice monoid.

Symbolic tree languages

A tree language is symbolic if and only if its ordered translation monoid is a symbolic ordered monoid, thus the family Sym is definable by syntactic ordered monoids.

6 Conclusions

We proved three variety theorems:

(1) a variety theorem connecting families of recognizable tree languages to classes of finite ordered algebras,

(2) generalized form of the above variety theorem, and

(3) a variety theorem connecting families of recognizable tree languages to classes of finite ordered monoids.

We also characterized classes of finite ordered algebras that are definable by ordered monoids.

Three examples were studied along the paper:

(1) the family Cof of cofinite tree languages is a gPVTL, is characterizable by ordered nilpotent algebras but is not definable by ordered monoids or semigroups,

(2) the family SL of semilattice tree languages is a generalized variety of tree languages, is characterizable by semilattice algebras and definable by semilattice monoids, and

(3) the family Sym of symbolic tree languages is a gPVTL, is characterizable by symbolic ordered algebras and definable by symbolic ordered monoids.

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