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Abstract

We consider varieties of recognizable subsets of many-sorted finitely generated free algebras over a given variety, varieties of congruences of such algebras, and varieties of finite many-sorted algebras. A variety theorem that establishes bijections between the classes of these three types of varieties is proved. For this, appropriate notions of many-sorted syntactic congruences and algebras are needed. Also an alternative type of varieties is considered where each subset consists of elements of just one sort.

Keywords: Recognizable Sets, Variety Theorem, Many-sorted Algebras, Syntactic Algebras, Varieties of Finite Algebras, Tree Languages

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1 Introduction

S. Eilenberg's [7] famous Variety Theorem establishes a bijective correspondence between varieties of regular languages ($*$–varieties) and varieties of finite monoids or, alternatively, between varieties of regular languages without the empty word ($+$–varieties) and varieties of finite semigroups. The theorem provides a general framework for the classification of regular languages and it describes the families of regular languages that can be characterized by syntactic monoids or by syntactic semigroups.

The Variety Theorem has been extended or adapted to other kinds of regular sets in several ways. A useful addition was the correspondence between varieties of regular languages and varieties of congruences of free monoids or free semigroups introduced by Thérien [20]. Another notable extension of the basic theory is Pin's [15] theory of positive varieties. In [17] Steinby proposes a theory of varieties of recognizable subsets of free algebras that encompasses both Eilenberg's theory and a theory of varieties of regular tree languages as special cases. The idea of recognizable subsets of arbitrary algebras goes back to Mezei and Wright [13]. The similar generalization developed in the more extensive study [1] by Almeida includes also varieties of congruences. Varieties of congruences appear also in the theory of varieties of tree languages presented in [18] and in the theory of generalized varieties of tree languages of [19]. The theories in [1, 17, 18, 19] are all based on syntactic algebras. As one more extension along these lines, we should mention Ésik's theory [9] where the place of varieties of finite algebras is taken by varieties of finitary theories.

It appears that Maibaum [11] was the first one to consider many-sorted tree languages. Many-sorted trees are used also by Engelfriet and Schmidt [8] in their study of the equational semantics of context-free tree languages. Recognizable subsets of general many-sorted algebras were studied by Courcelle [4, 5].

In this paper we join the above two lines of research by developing a theory of varieties of recognizable subsets of free many-sorted algebras. Thus we actually generalize the theories of [17, 18] and [1] to the many-sorted case. It should be mentioned that, although not considered here, Wilke's [21] tree algebras gave an important impetus to this work; they are 3-sorted algebras used for characterizing families of (binary) tree languages. These algebras we study in [16].

In Section 2 we present some basic definitions and our notation for many-sorted algebras. Also some more specialized notions relevant to our work are introduced. The references [10] and [12] may be consulted for general treatments of the theory of many-sorted algebras. In Section 3 recognizable subsets of many-sorted algebras are considered. There are actually two types of these, recognizable sorted subsets and the 'pure' recognizable subsets considered in [5, 8, 11] in which all elements are of some given sort. We mainly consider the former type but we will show how the theory applies also to other kind of sets.

Syntactic congruences and syntactic algebras of subsets of many-sorted algebras are introduced in Section 4, and it is shown that they enjoy all the same general properties as their counterparts for monoids [7, 14] or term algebras, or one-sorted algebras in general [1, 17, 18].

In Section 5 we define our varieties of recognizable sets and varieties of congruences. For this a finite set of sorts S and variety $\mathbf{V}$ of some finite S-sorted type Ω are fixed. A variety of recognizable $\mathbf{V}$-sets consists then of recognizable subsets of the finitely generated free algebras over $\mathbf{V}$. Similarly, a variety of $\mathbf{V}$-congruences consists of congruences of finite index on these algebras. Finally, $\mathbf{V}$-variety of finite algebras is defined as a variety of finite algebras contained in $\mathbf{V}$. In Section 6 we define six mappings that transform varieties of recognizable $\mathbf{V}$-sets, varieties of $\mathbf{V}$-congruences and $\mathbf{V}$-varieties of finite algebras to each other. Then we prove our Variety Theorem that essentially says that these six mappings form three pairs of mutually inverse isomorphisms between the complete lattices of the three kinds of varieties considered. The proof is very similar to the one presented in [18], but there are naturally some technical differences and for the reader's convenience a rather detailed proof is presented.

In Section 7 we define varieties of pure recognizable $\mathbf{V}$-sets in which each recognizable set is a subset of the set of elements of some given sort of a finitely generated free algebra over $\mathbf{V}$. By establishing a natural correspondence between the two types of varieties of recognizable $\mathbf{V}$-sets, a Variety Theorem is derived also for varieties of pure recognizable $\mathbf{V}$-sets.

2 Many-sorted algebras

In what follows, S is always a non-empty set of *sorts*. We will consider various families of objects indexed by S. Such families are said to be S-*sorted*, or just *sorted*. The sort of an object is usually shown as a subscript or in parentheses (to avoid multiple subscripts). An S-*sorted set* $A = \langle A_s \rangle_{s \in S}$ is an S-indexed family of sets; for each $s \in S$, A_s is the set of *elements of sort* s in A, and we write it also as $A(s)$. The basic set-theoretic notions are defined for S-sorted sets in the natural sortwise manner. In particular, for any S-sorted sets $A = \langle A_s \rangle_{s \in S}$ and $B = \langle B_s \rangle_{s \in S}$, $A \subseteq B$ means that $A_s \subseteq B_s$ for every $s \in S$, $A \cup B = \langle A_s \cup B_s \rangle_{s \in S}$, $A \cap B = \langle A_s \cap B_s \rangle_{s \in S}$ and $A - B = \langle A_s - B_s \rangle_{s \in S}$, and general sorted unions and intersections are defined similarly. The notation $\emptyset$ is used also for the S-sorted empty set $\langle \emptyset \rangle_{s \in S}$.

We shall also consider subsets of one given sort of sorted sets. With any subset $T \subseteq A_u$ of some sort $u \in S$ of an S-sorted set $A = \langle A_s \rangle_{s \in S}$ we associate the sorted subset $\langle T \rangle \subseteq A$ such that $\langle T \rangle_u = L$ and $\langle T \rangle_s = \emptyset$ for every $s \in S \setminus \{u\}$.

A *sorted relation* $\theta = \langle \theta_s \rangle_{s \in S}$ on $A = \langle A_s \rangle_{s \in S}$ is an S-sorted family of relations such that for each $s \in S$, θ_s is a relation on A_s. Such a θ could also be viewed as a relation on the disjoint union of the sets A_s ($s \in S$) such that any two θ-related elements are always of the same sort. A *sorted equivalence* on $A = \langle A_s \rangle_{s \in S}$ is a sorted relation $\theta = \langle \theta_s \rangle_{s \in S}$ where θ_s is an equivalence relation on A_s for each $s \in S$. Let $\mathrm{Eq}_S(A)$ denote the set of all sorted equivalences on A. If $\theta = \langle \theta_s \rangle_{s \in S} \in \mathrm{Eq}_S(A)$, then the corresponding *quotient set* is the S-sorted set $A/\theta = \langle A_s/\theta_s \rangle_{s \in S}$, where $A_s/\theta_s = \{a/\theta_s \mid a \in A_s\}$ ($s \in S$). Of course, $\mathrm{Eq}_S(A)$ forms with respect to the sorted inclusion relation a complete lattice in which least upper bounds and greatest lower bounds are formed sortwise as for usual equivalences. The least element is the *sorted diagonal relation* $\Delta_A = \langle \Delta_{A(s)} \rangle_{s \in S}$ and the greatest element is the *sorted universal relation* $\nabla_A = \langle \nabla_{A(s)} \rangle_{s \in S}$, where $\Delta_{A(s)} = \{(a, a) \mid a \in A(s)\}$ and $\nabla_{A(s)} = A(s) \times A(s)$ for each $s \in S$.

A *sorted mapping* $\varphi \colon A \to B$ from an S-sorted set $A = \langle A_s \rangle_{s \in S}$ to an S-sorted set $B = \langle B_s \rangle_{s \in S}$ is an S-sorted family $\varphi = \langle \varphi_s \rangle_{s \in S}$ of mappings $\varphi_s \colon A_s \to B_s$ ($s \in S$). The *kernel* of φ is the sorted equivalence $\ker \varphi = \langle \ker \varphi_s \rangle_{s \in S}$ on A. For any sorted subset $H = \langle H_s \rangle_{s \in S}$ of A, $H\varphi$ denotes the sorted subset $\langle H_s \varphi_s \rangle_{s \in S}$ of B. Similarly, if $H = \langle H_s \rangle_{s \in S}$ is a sorted subset of B, then $H\varphi^{-1}$ denotes the sorted subset $\langle H_s \varphi_s^{-1} \rangle_{s \in S}$ of A. The *composition* of two S-sorted mappings $\varphi \colon A \to B$ and $\psi \colon B \to C$, where $C = \langle C_s \rangle_{s \in S}$ is also an S-sorted set, is defined as the sorted mapping $\varphi\psi \colon A \to C$ such that $(\varphi\psi)_s = \varphi_s \psi_s$ for each $s \in S$. Here the mappings were composed from left to right, as we shall do especially with homomorphisms. Hence, $\varphi_s \psi_s \colon a \mapsto (a\varphi_s)\psi_s$ for all $s \in S$ and $a \in A_s$.

Treating S as an alphabet, S^* denotes the set of finite strings over S, including the empty string e, and S^+ is the set of non-empty strings over S. An *S-sorted signature* Ω is a set of operation symbols each of which has been assigned a *type* that is an element of $S^* \times S$. For any $(w, s) \in S^* \times S$, let $\Omega_{w,s}$ be the set of symbols of type (w, s), and Ω may be given by specifying the non-empty sets $\Omega_{w,s}$. If $f \in \Omega_{w,s}$, then w is the *domain type* of f, and s is its *sort*. In particular, every element of $\Omega_{e,s}$, for the empty word e, is a *constant symbol* of sort s. The fact that $f \in \Omega_{w,s}$ is expressed also by writing $f \colon w \to s$. For a finite S, a finite S-sorted signature is called an *S-sorted ranked alphabet*. Later S is assumed to be finite and Ω is always an S-sorted ranked alphabet. However, the following basic definitions and facts do not depend on this assumption.

An *Ω-algebra* $\mathcal{A} = (A, \Omega)$ consists of an S-sorted set $A = \langle A_s \rangle_{s \in S}$, where $A_s \neq \emptyset$ for every $s \in S$, equipped with constants and operations as follows:

(1) for each constant symbol $c \in \Omega_{e,s}$ of sort $s \in S$, an element $c^{\mathcal{A}} \in A_s$ of sort s is specified;

(2) for any function symbol $f \in \Omega_{w,s}$ with $w \in S^+$ and $s \in S$, there is an operation $f^{\mathcal{A}} \colon A^w \to A_s$ of *type* (w, s), *domain type* w and *sort* s. Here $A^w = A_{s(1)} \times \cdots \times A_{s(m)}$ for $w = s(1) \ldots s(m)$.

Such an algebra $\mathcal{A}$ is said to be S*-sorted*. For each $s \in S$, A_s is the set of elements of $\mathcal{A}$ of *sort* s. The algebra $\mathcal{A}$ is *trivial* if every A_s ($s \in S$) is a singleton set. We may write $\mathcal{A} = (\langle A_s \rangle_{s\in S}, \Omega)$ to emphasize the fact that $\mathcal{A}$ is S-sorted. However, when we speak about the Ω-algebras $\mathcal{A} = (A, \Omega)$, $\mathcal{B} = (B, \Omega)$ and $\mathcal{C} = (C, \Omega)$, it will usually be assumed that $A = \langle A_s \rangle_{s\in S}$, $B = \langle B_s \rangle_{s\in S}$ and $C = \langle C_s \rangle_{s\in S}$.

An Ω-algebra $\mathcal{B} = (B, \Omega)$ such that $B \subseteq A$ is a *subalgebra* of $\mathcal{A} = (A, \Omega)$, and this we may express by writing $\mathcal{B} \subseteq \mathcal{A}$, if

(1) $c^{\mathcal{B}} = c^{\mathcal{A}}$ whenever $c \in \Omega_{e,s}$ for some $s \in S$, and
(2) $f^{\mathcal{B}} = f^{\mathcal{A}}|_{B^w}$ for any $f \in \Omega_{w,s}$ with $w \in S^+$ and $s \in S$.

If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then $B = \langle B_s \rangle_{s\in S}$ is a *closed subset* of $\mathcal{A}$, that is,

(1) $c^{\mathcal{A}} \in B_s$ whenever $c \in \Omega_{e,s}$ and $s \in S$, and
(2) $f^{\mathcal{A}}(b_1, \ldots, b_m) \in B_s$ whenever $m > 0$, $f\colon s(1)\ldots s(m) \to s$ and $b_1 \in B_{s(1)}, \ldots, b_m \in B_{s(m)}$.

On the other hand, any closed subset B with $B_s \neq \emptyset$ for every $s \in S$, is the carrier set of a unique subalgebra of $\mathcal{A}$. Hence, subalgebras coincide with the closed subsets whose all components are non-empty. The set of all closed subsets of $\mathcal{A}$ is denoted by $\mathrm{Sub}(\mathcal{A})$, and let $\mathrm{Sub}^+(\mathcal{A})$ denote the set of closed subsets with non-empty components.

Since the intersection $\bigcap \mathcal{S}$ of any set $\mathcal{S} \subseteq \mathrm{Sub}(\mathcal{A})$ of closed subsets of an Ω-algebra $\mathcal{A}$ is also closed, any subset $H = \langle H_s \rangle_{s\in S}$ of A is contained in a unique minimal closed subset $[H] = \bigcap\{B \mid H \subseteq B,\, B \in \mathrm{Sub}(\mathcal{A})\}$, the *closed subset generated* by H. If $H_s \cup \Omega_{e,s} \neq \emptyset$ for every $s \in S$, then $[H]$ is a subalgebra of $\mathcal{A}$, but note that this may be the case even otherwise. If $[H] \in \mathrm{Sub}^+(\mathcal{A})$, then $[H]$ is called the *subalgebra generated* by H.

A sorted equivalence $\theta = \langle \theta_s \rangle_{s\in S}$ on A is a *congruence* on $\mathcal{A} = (A, \Omega)$ if

$$a_1\, \theta_{s(1)}\, b_1, \ldots, a_m\, \theta_{s(m)}\, b_m \;\Rightarrow\; f^{\mathcal{A}}(a_1, \ldots, a_m)\, \theta_s\, f^{\mathcal{A}}(b_1, \ldots, b_m),$$

whenever $f\colon s(1)\ldots s(m) \to s$ and $a_1, b_1 \in A_{s(1)}, \ldots, a_m, b_m \in A_{s(m)}$. The corresponding *quotient algebra* $\mathcal{A}/\theta = (A/\theta, \Omega)$ is defined by setting

(1) $c^{\mathcal{A}/\theta} = c^{\mathcal{A}}/\theta_s$ for any $c \in \Omega_{e,s}$, and
(2) $f^{\mathcal{A}/\theta}(a_1/\theta_{s(1)}, \ldots, a_m/\theta_{s(m)}) = f^{\mathcal{A}}(a_1, \ldots, a_m)/\theta_s$ for $f\colon s(1)\ldots s(m) \to s$ and $a_1 \in A_{s(1)}, \ldots, a_m \in A_{s(m)}$.

Since θ is a congruence, the operations $f^{\mathcal{A}/\theta}$ are well-defined.

A sorted mapping $\varphi\colon A \to B$ is a *homomorphism* from $\mathcal{A} = (A, \Omega)$ to $\mathcal{B} = (B, \Omega)$, and we express this by writing $\varphi\colon \mathcal{A} \to \mathcal{B}$, if

(1) $c^{\mathcal{A}}\varphi_s = c^{\mathcal{B}}$ whenever $c \in \Omega_{e,s}$ for some $s \in S$, and

(2) $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi_s = f^{\mathcal{B}}(a_1\varphi_{s(1)}, \ldots, a_m\varphi_{s(m)})$ for any $f\colon s(1)\ldots s(m) \to s$ and $a_1 \in A_{s(1)}, \ldots, a_m \in A_{s(m)}$.

A homomorphism φ is a *monomorphism*, an *epimorphism* or an *isomorphism*, if every φ_s ($s \in S$) is injective, surjective or bijective, respectively. If there exists an epimorphism $\mathcal{A} \to \mathcal{B}$, then $\mathcal{B}$ is an *image* of $\mathcal{A}$, and we write $\mathcal{B} \leftarrow \mathcal{A}$. If there is an isomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$, the algebras are *isomorphic*, $\mathcal{A} \cong \mathcal{B}$ in symbols. An Ω-algebra $\mathcal{A}$ *divides* an Ω-algebra $\mathcal{B}$, and we write $\mathcal{A} \preceq \mathcal{B}$, if $\mathcal{A}$ is an image of a subalgebra of $\mathcal{B}$.

We shall sometimes use the observation that $\mathcal{A} \preceq \mathcal{B}$ iff there is an Ω-algebra $\mathcal{C}$ for which there exist a monomorphism $\varphi\colon \mathcal{C} \to \mathcal{B}$ and an epimorphism $\psi\colon \mathcal{C} \to \mathcal{A}$.

The *natural map* corresponding to a sorted equivalence $\theta = \langle \theta_s \rangle_{s \in S}$ on a sorted set A, is the sorted map $\theta^\natural\colon A \to A/\theta$, where $\theta^\natural_s\colon A_s \to A_s/\theta_s$, $a \mapsto a/\theta_s$, for each $s \in S$. It is easy to verify that if θ is a congruence on an Ω-algebra $\mathcal{A}$, then $\theta^\natural$ is an epimorphism from $\mathcal{A}$ onto $\mathcal{A}/\theta$. Moreover, the Homomorphism Theorem (cf. [2], for example) extends in a straightforward manner to many-sorted algebras as follows (cf. [12], for example).

Proposition 2.1 *If* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *is a homomorphism of* Ω*-algebras, then* $\ker \varphi$ *is a congruence on* $\mathcal{A}$ *and* $\psi\colon \mathcal{A}/\ker\varphi \to \mathcal{B}$, $a/\ker\varphi_s \mapsto a\varphi_s$, *is a monomorphism such that* $(\ker\varphi)^\natural\psi = \varphi$*. If* φ *is an epimorphism, then* ψ *is an isomorphism.*

Next we introduce the many-sorted version of a notion that has proved very useful for dealing with congruences.

Let $\mathcal{A} = (A, \Omega)$ be an Ω-algebra. For any pair $s, s' \in S$ of sorts, an *elementary* s, s'*-translation* is any mapping $A_s \to A_{s'}$ of the form

$$\alpha(\xi_s) = f^{\mathcal{A}}(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m),$$

where $m \geq 1$, $f\colon s(1)\ldots s(m) \to s'$, $1 \leq j \leq m$, $s(j) = s$, and $a_i \in A_{s(i)}$ for every $i \neq j$. Here and later, ξ_s is a variable of sort s that does not appear in the other alphabets considered.

Let $\mathrm{ETr}(\mathcal{A}, s, s')$ denote the set of all elementary s, s'-translations of $\mathcal{A}$. The $S \times S$-sorted set $\mathrm{Tr}(\mathcal{A}) = \langle \mathrm{Tr}(\mathcal{A}, s, s') \rangle_{s,s' \in S}$ of all *translations* of $\mathcal{A}$ is now defined inductively by the following clauses:

(1) $\mathrm{ETr}(\mathcal{A}, s, s') \subseteq \mathrm{Tr}(\mathcal{A}, s, s')$ for all $s, s' \in S$,

(2) for each $s \in S$, the identity map $1_{A(s)}\colon A(s) \to A(s)$ is in $\mathrm{Tr}(\mathcal{A}, s, s)$, and

(3) if $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ and $\beta(\xi_{s'}) \in \mathrm{Tr}(\mathcal{A}, s', s'')$, for some $s, s', s'' \in S$, then $\beta(\alpha(\xi_s)) \in \mathrm{Tr}(\mathcal{A}, s, s'')$.

For any $s, s' \in S$, the elements of $\mathrm{Tr}(\mathcal{A}, s, s')$ are the s, s'*-translations* of $\mathcal{A}$.

The following lemma is an immediate generalization of the corresponding fact about one-sorted algebras (see e.g. [2, 3, 6]).

Lemma 2.2 *Let $\mathcal{A} = (A, \Omega)$ be an Ω-algebra. Every congruence $\theta = \langle \theta_s \rangle_{s \in S}$ on $\mathcal{A}$ is invariant with respect to all translations of $\mathcal{A}$, that is to say, $a\, \theta_s\, b$ implies $\alpha(a)\, \theta_{s'}\, \alpha(b)$ for all $s, s' \in S$, $a, b \in A_s$ and $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$. On the other hand, a sorted equivalence θ on A is a congruence if it is invariant with respect to every elementary translation of $\mathcal{A}$.*

The following generalization of a lemma from the one-sorted case [17, 18] is frequently needed.

Lemma 2.3 *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of Ω-algebras from $\mathcal{A} = (A, \Omega)$ to $\mathcal{B} = (B, \Omega)$. For any $s, s' \in S$ and every $\alpha(\xi_s)$ in $\mathrm{Tr}(\mathcal{A}, s, s')$, there exists a translation $\alpha_\varphi(\xi_s) \in \mathrm{Tr}(\mathcal{B}, s, s')$ such that*

$$\alpha(a)\varphi_{s'} = \alpha_\varphi(a\varphi_s)$$

for every $a \in A_s$. If φ is an epimorphism, then for all $s, s' \in S$ and every $\beta(\xi_s)$ in $\mathrm{Tr}(\mathcal{B}, s, s')$ there exists an $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ such that $\beta = \alpha_\varphi$.

Proof. Because the claim clearly holds for the identity translations and all other non-elementary translations are products of elementary translations, it suffices to note that for any elementary s, s'-translation

$$\alpha(\xi_s) = f^{\mathcal{A}}(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m)$$

with $f\colon s(1) \ldots s(m) \to s'$ and $s(j) = s$, we may choose

$$\alpha_\varphi(\xi_s) = f^{\mathcal{B}}(a_1\varphi_{s(1)}, \ldots a_{j-1}\varphi_{s(j-1)}, \xi_s, a_{j+1}\varphi_{s(j+1)} \ldots, a_m\varphi_{s(m)}).$$

If φ is surjective, then every elementary translation of $\mathcal{B}$ is obtained this way, which also holds for all translations of $\mathcal{B}$. $\square$

Translations and their inverses of an Ω-algebra $\mathcal{A} = (A, \Omega)$ are applied to subsets of a given sort and to sorted subsets as follows. Let $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ for some $s, s' \in S$. For any $u \in S$ and $T \subseteq A_u$, let

- $\alpha(T) = \{\alpha(a) \mid a \in T\}$ ($\subseteq A_{s'}$) if $u = s$, and $\alpha(T) = \emptyset$ if $u \neq s$;
- $\alpha^{-1}(T) = \{a \in A_s \mid \alpha(a) \in T\}$ if $u = s'$, and $\alpha^{-1}(T) = \emptyset$ if $u \neq s'$.

Furthermore, for any sorted subset $L = \langle L_s \rangle_{s \in S}$ of A, we set

- $\alpha(L) = \langle K_u \rangle_{u \in S}$, where $K_{s'} = \alpha(L_s)$, and $K_u = \emptyset$ for each $u \neq s'$, and
- $\alpha^{-1}(L) = \langle K_u \rangle_{u \in S}$, where $K_s = \alpha^{-1}(L_{s'})$, and $K_u = \emptyset$ for each $u \neq s$.

The *direct product* of two Ω-algebras $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ is the Ω-algebra $\mathcal{A} \times \mathcal{B} = (A \times B, \Omega)$, where

(1) $A \times B = \langle A_s \times B_s \rangle_{s \in S}$,
(2) $c^{\mathcal{A} \times \mathcal{B}} = (c^{\mathcal{A}}, c^{\mathcal{B}})$ for any $s \in S$ and $c \in \Omega_{e,s}$, and

(3) $f^{\mathcal{A}\times\mathcal{B}}((a_1,b_1),\ldots,(a_m,b_m)) = (f^{\mathcal{A}}(a_1,\ldots,a_m), f^{\mathcal{B}}(b_1,\ldots,b_m))$ for any $a_1 \in A_{s(1)}, b_1 \in B_{s(1)},\ldots,a_m \in A_{s(m)}, b_m \in B_{s(m)}$ and any function symbol $f\colon s(1)\ldots s(m) \to s$.

The direct product $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ of any finite family $\mathcal{A}_1,\ldots,\mathcal{A}_n$, or the direct product $\prod_{i\in I}\mathcal{A}_i$ of a general family $\mathcal{A}_i$ $(i \in I)$ of Ω-algebras, are defined correspondingly.

If $\varphi\colon A \to B$ is a sorted mapping from an S-sorted set $A = \langle A_s\rangle_{s\in S}$ to an S-sorted set $B = \langle B_s\rangle_{s\in S}$ and $\theta = \langle\theta_s\rangle_{s\in S}$ is a sorted equivalence on B, then $\varphi\circ\theta\circ\varphi^{-1}$ is the sorted equivalence on A defined by the condition

$$a_1\,(\varphi\circ\theta\circ\varphi^{-1})_s\,a_2 \;\Leftrightarrow\; a_1\varphi_s\,\theta_s\,a_2\varphi_s \qquad (s \in S,\ a_1,a_2 \in A_s).$$

In the following lemma we note a few basic facts about quotient algebras.

Lemma 2.4 *Let $\mathcal{A} = (A,\Omega)$ and $\mathcal{B} = (B,\Omega)$ be Ω-algebras, $\theta,\theta' \in \mathrm{Con}(\mathcal{A})$, $\rho \in \mathrm{Con}(\mathcal{B})$, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism.*

(1) *If $\theta \subseteq \theta'$, then $\mathcal{A}/\theta' \leftarrow \mathcal{A}/\theta$.*
(2) *$\mathcal{A}/\theta\cap\theta' \subseteq \mathcal{A}/\theta \times \mathcal{A}/\theta'$.*
(3) *The relation $\varphi\circ\rho\circ\varphi^{-1}$ is a congruence on $\mathcal{A}$, and $\mathcal{A}/\varphi\circ\rho\circ\varphi^{-1} \preceq \mathcal{B}/\rho$. If φ is an epimorphism, then $\mathcal{A}/\varphi\circ\rho\circ\varphi^{-1} \cong \mathcal{B}/\rho$*

Proof. Statements (1) and (2) are direct generalizations of well-known facts. In the many-sorted case they follow, for example, from Theorem 3.4.20 and Lemma 4.1.5 of [12].

Let us prove (3). If $a\,(\varphi\circ\rho\circ\varphi^{-1})_s\,b$, for some $s \in S$ and $a,b \in A_s$, then $a\varphi_s\,\rho_s\,b\varphi_s$. By Lemma 2.3, for any $s' \in S$ and every $\alpha \in \mathrm{Tr}(\mathcal{A},s,s')$, there is an $\alpha_\varphi \in \mathrm{Tr}(\mathcal{B},s,s')$ such that $\alpha_\varphi(d\varphi_s) = \alpha(d)\varphi_{s'}$ for every $d \in A_s$. Since $\alpha_\varphi(a\varphi_s)\,\rho_{s'}\,\alpha_\varphi(b\varphi_s)$ by Lemma 2.2, we also have $\alpha(a)\varphi_{s'}\,\rho_{s'}\,\alpha(b)\varphi_{s'}$, that is to say, $\alpha(a)\,(\varphi\circ\rho\circ\varphi^{-1})_{s'}\,\alpha(b)$. Hence, $\varphi\circ\rho\circ\varphi^{-1} \in \mathrm{Con}(\mathcal{A})$ by Lemma 2.2. It is now easy to see that $\psi\colon \mathcal{A}/(\varphi\circ\rho\circ\varphi^{-1}) \to \mathcal{B}/\rho$ is a monomorphism if we define

$$\psi_s\colon A_s/(\varphi\circ\rho\circ\varphi^{-1})_s \to B_s/\rho_s, \quad a/(\varphi\circ\rho\circ\varphi^{-1})_s \mapsto a\varphi_s/\rho_s,$$

for each $s \in S$. Finally, we note that if φ is surjective, then so is ψ. □

The class operators S, H, P and $\mathrm{P_f}$ are defined exactly as in the one-sorted case: for any class $\mathbf{K}$ of Ω-algebras and any Ω-algebra $\mathcal{A}$,

(1) $\mathcal{A} \in \mathrm{S}(\mathbf{K})$ iff $\mathcal{A}$ is isomorphic to a subalgebra of a member of $\mathbf{K}$,
(2) $\mathcal{A} \in \mathrm{H}(\mathbf{K})$ iff $\mathcal{A}$ is an image of some member of $\mathbf{K}$,
(3) $\mathcal{A} \in \mathrm{P}(\mathbf{K})$ iff $\mathcal{A}$ is isomorphic to the direct product of a family of algebras in $\mathbf{K}$, and
(4) $\mathcal{A} \in \mathrm{P_f}(\mathbf{K})$ iff $\mathcal{A}$ is isomorphic to the direct product of a finite family of algebras in $\mathbf{K}$.

A class $\mathbf{K}$ of Ω-algebras is a *variety* if $\mathrm{S}(\mathbf{K}), \mathrm{H}(\mathbf{K}), \mathrm{P}(\mathbf{K}) \subseteq \mathbf{K}$. Birkhoff's well-known theorem by which a class of algebras is definable by equations iff it is a variety, holds also for many-sorted algebras (cf. Section 5 of [12]).

A class $\mathbf{K}$ of finite Ω-algebras is called a *variety of finite Ω-algebras*, an Ω-VFA for short, if it is closed under subalgebras, homomorphic images, and finite direct products, i.e., if $\mathrm{S}(\mathbf{K}), \mathrm{H}(\mathbf{K}), \mathrm{P_f}(\mathbf{K}) \subseteq \mathbf{K}$. It is easy to show that a class $\mathbf{K}$ of finite Ω-algebras is an Ω-VFA iff $\mathcal{A} \in \mathbf{K}$ whenever $\mathcal{A} \preceq \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ for some $n \geq 0$ and $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathbf{K}$. When we deal with varieties of finite Ω-algebras, both S and Ω are assumed to be finite.
Let $X = \langle X_s \rangle_{s \in S}$ be an S-sorted alphabet disjoint from Ω. The S-sorted set $T_\Omega(X) = \langle T_\Omega(X,s) \rangle_{s \in S}$ of *Ω-terms with variables in X* is defined inductively:

(1) $\Omega_{e,s} \cup X_s \subseteq T_\Omega(X,s)$ for every $s \in S$, and
(2) $f(t_1, \ldots, t_m) \in T_\Omega(X,s)$ for any function symbol $f\colon s_1 \ldots s_m \to s$ and terms $t_1 \in T_\Omega(X, s_1)$, …, $t_m \in T_\Omega(X, s_m)$.

The alphabet X is said to be *full* for Ω if $T_\Omega(X,s) \neq \emptyset$ for every sort $s \in S$. Note that a given $T_\Omega(X,s)$ may be non-empty even when $X_s = \Omega_{e,s} = \emptyset$.
If $X = \langle X_s \rangle_{s \in S}$ is full for Ω, then the *ΩX-term algebra* $\mathcal{T}_\Omega(X) = (T_\Omega(X), \Omega)$ is defined in the natural way:

(1) $c^{\mathcal{T}_\Omega(X)} = c$ for any $s \in S$ and $c \in \Omega_{e,s}$, and
(2) $f^{\mathcal{T}_\Omega(X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ whenever $m > 0$, $f\colon s_1 \ldots s_m \to s$ and $t_1 \in T_\Omega(X, s_1)$, …, $t_m \in T_\Omega(X, s_m)$.

Of course, $\mathcal{T}_\Omega(X)$ is *freely generated* by X over the class of all Ω-algebras, that is to say, for any Ω-algebra $\mathcal{A} = (A, \Omega)$, any sorted mapping $\alpha : X \to A$ has a unique extension to a homomorphism $\alpha^{\mathcal{A}}\colon \mathcal{T}_\Omega(X) \to \mathcal{A}$.
More generally, if $\mathbf{V}$ is a class of Ω-algebras, an Ω-algebra $\mathcal{F} = (\langle F_s \rangle_{s \in S}, \Omega)$ is *generated freely* over $\mathbf{V}$ by a sorted subset $G \subseteq F$, if $\mathcal{F} \in \mathbf{V}$, $\mathcal{F}$ is generated by G, and for any $\mathcal{A} = (A, \Omega)$ in $\mathbf{V}$, any sorted mapping $\varphi_0\colon G \to A$ can be extended to a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{A}$. If such an $\mathcal{F}$ exists, it is determined uniquely up to isomorphism by $\mathbf{V}$ and G, and we denote it $\mathcal{F}_{\mathbf{V}}(G) = (\mathrm{F}_{\mathbf{V}}(G), \Omega)$ with $\mathrm{F}_{\mathbf{V}}(G) = \langle \mathrm{F}_{\mathbf{V}}(G, s) \rangle_{s \in S}$.

Let Ω be an S-sorted ranked alphabet and let X be an S-sorted alphabet disjoint from Ω. For each $s \in S$, let ξ_s be again a special symbol of sort s. The $S \times S$-sorted set $C_\Omega(X) = \langle C_\Omega(X, s, s') \rangle_{s,s' \in S}$ of *ΩX-contexts* is defined inductively by the conditions

(1) $\xi_s \in C_\Omega(X, s, s)$ for each $s \in S$, and
(2) $f(t_1, \ldots, t_{j-1}, p, t_{j+1} \ldots, t_m) \in C_\Omega(X, s, s')$ whenever $s, s', s_1, \ldots, s_m \in S$, $m \geq 1$, $f\colon s_1 \ldots s_m \to s'$, $1 \leq j \leq m$, $p \in C_\Omega(X, s, s_j)$, and $t_i \in T_\Omega(X, s_i)$ for all $i \neq j$.

The *product* $p \cdot q = q(p)$ of two ΩX-contexts $p \in C_\Omega(X, s, s')$ and $q \in C_\Omega(X, s', s'')$ ($s, s', s'' \in S$) is the ΩX-context obtained from q when $\xi_{s'}$ is replaced with p.

Let $\mathcal{A} = (A, \Omega)$ be any Ω-algebra. Every translation of $\mathcal{A}$ is *represented* in a natural way by an ΩA-context of a matching type:

(1) an elementary translation $\alpha(\xi_s) = f^{\mathcal{A}}(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m)$ is represented by the ΩA-context $f(a_1, \ldots a_{j-1}, \xi_s, a_{j+1} \ldots, a_m)$,

(2) the identity map $1_{A(s)}\colon A(s) \to A(s)$ is represented by ξ_s, and

(3) if $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$ and $\beta(\xi_{s'}) \in \mathrm{Tr}(\mathcal{A}, s', s'')$ are represented by the ΩA-contexts $p(\xi_s) \in C_\Omega(A, s, s')$ and $q(\xi_{s'}) \in C_\Omega(A, s', s'')$, respectively, then $\beta(\alpha(\xi_s))$ is represented by $q(p(\xi_s)) \in C_\Omega(A, s, s'')$.

That a translation $\alpha(\xi_s)$ is represented by a context $p(\xi_s)$ means that α is the polynomial function (cf. [2], for example) defined by p in $\mathcal{A}$, when p is interpreted as a polynomial symbol with ξ_s as the only variable.

3 Recognizable subsets

An equivalence θ on a set A *saturates* a subset L of A if L is the union of some θ-classes, and that θ is said to be of *finite index* if it has a finite number of equivalence classes. Mezei and Wright [13] call a subset L of an algebra $\mathcal{A}$ *recognizable* if it is saturated by a congruence of finite index on $\mathcal{A}$. Clearly, L is recognizable iff there exist a finite algebra $\mathcal{B}$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a subset H of $\mathcal{B}$ such that $L = H\varphi^{-1}$. We use this condition, where $\mathcal{B}$ may be viewed as a 'recognizer' of L, for defining recognizability in many-sorted algebras. There are two natural types of recognizable subsets of a sorted algebra: the *recognizable sorted subsets* and the *recognizable subsets of a given sort.*
In what follows, S is always a finite set of sorts and Ω is an S-sorted ranked alphabet. An S-sorted set $A = \langle A_s \rangle_{s \in S}$ is said to be *finite* if every A_s ($s \in S$) is finite, and an Ω-algebra $\mathcal{A} = (A, \Omega)$ is *finite* if $A = \langle A_s \rangle_{s \in S}$ is finite.

Definition 3.1 A sorted subset $L \subseteq A$ of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is *recognizable* if there exist a finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a sorted subset H of B such that $L = H\varphi^{-1}$. Then we say also that $\mathcal{B}$ *recognizes* L. Let $\mathrm{Rec}(\mathcal{A})$ denote the set of all recognizable subsets of $\mathcal{A}$.

For any $s \in S$, a subset T of A_s is said to be *recognizable* in $\mathcal{A}$ if if there exist a finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a subset H of B_s such that $T = H\varphi_s^{-1}$. Let $\mathrm{Rec}(\mathcal{A}, s)$ denote the set of all such subsets of A_s. We call such sets also *pure recognizable sets.*

The recognizable tree languages of sort $s \in S$ considered by Maibaum [11] are the pure recognizable subsets of the term algebra $\mathcal{T}_\Omega(\emptyset)$ of sort s, i.e., the elements of $\mathrm{Rec}(\mathcal{T}_\Omega(\emptyset), s)$. Courcelle [4, 5] extends this notion to any S-sorted algebra $\mathcal{A} = (\langle A_s \rangle_{s \in S}, \Omega)$, without assuming the finiteness of S or Ω, by calling a subset $T \subseteq A_s$ recognizable if there exist a locally finite

Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a subset H of B_s such that $T = H\varphi_s^{-1}$; an algebra $\mathcal{B} = (\langle B_s\rangle_{s\in S}, \Omega)$ is *locally finite* if every B_s is finite ($s \in S$). Since we assume that S is finite, this 'locally finite' means here just 'finite', and hence our pure recognizable subsets are exactly Courcelle's recognizable subsets.

Although we are primarily concerned with sorted recognizable sets, we will also note how the theory can be adapted to pure recognizable sets.

A sorted equivalence $\theta = \langle\theta_s\rangle_{s\in S}$ on an S-sorted set $A = \langle A_s\rangle_{s\in S}$ is said to *saturate* a sorted subset $L = \langle L_s\rangle_{s\in S}$ of A if every L_s is the union of some θ_s-classes ($s \in S$), and θ is of *finite index* if every θ_s ($s \in S$) is of finite index. The following lemma is an obvious generalization of the fact noted above.

Lemma 3.2 *A sorted subset of an Ω-algebra $\mathcal{A}$ is recognizable iff it is saturated by a congruence on $\mathcal{A}$ of finite index. Similarly, a subset $L \subseteq A_u$ of some sort $u \in S$ is recognizable iff it is saturated by θ_u for some congruence $\theta = \langle\theta_s\rangle_{s\in S}$ on $\mathcal{A}$ of finite index.*

Next we present a few closure properties that are well-known for recognizable subsets of one-sorted algebras.

Proposition 3.3 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be any Ω-algebras.*

(1) $\emptyset, A \in \mathrm{Rec}(\mathcal{A})$.
(2) *If $K, L \in \mathrm{Rec}(\mathcal{A})$, then $K \cup L, K \cap L, K - L \in \mathrm{Rec}(\mathcal{A})$.*
(3) *If $L \in \mathrm{Rec}(\mathcal{A})$ and $\alpha \in \mathrm{Tr}(\mathcal{A}, s, s')$ for some $s, s' \in S$, then $\alpha^{-1}(L) \in \mathrm{Rec}(\mathcal{A})$.*
(4) *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism and $L \in \mathrm{Rec}(\mathcal{B})$, then $L\varphi^{-1} \in \mathrm{Rec}(\mathcal{A})$.*

Proof. Assertion (1) is trivial, and (2) can be proved as usual by defining the direct product of any two finite algebras recognizing K and L, respectively.

For (3), we recall first that $\alpha^{-1}(L)_s = \alpha^{-1}(L_{s'})$ and $\alpha^{-1}(L)_{s''} = \emptyset$ for every $s'' \neq s$. Assume now that $L = H\varphi^{-1}$, where $\varphi\colon \mathcal{A} \to \mathcal{C}$ is a homomorphism to a finite algebra Ω-algebra $\mathcal{C} = (C, \Omega)$, and $H \subseteq C$. By Lemma 2.3 there is a translation $\alpha_\varphi \in \mathrm{Tr}(\mathcal{C}, s, s')$ such that $\alpha(a)\varphi_{s'} = \alpha_\varphi(a\varphi_s)$ for every $a \in L_s$. Now it is easy to see that $\alpha^{-1}(L) = G\varphi^{-1}$ for the sorted subset G of C defined in such a way that $G_s = \alpha_\varphi^{-1}(H_{s'})$ and $G_{s''} = \emptyset$ for every $s'' \neq s$.

To prove (4), assume that $L = H\psi^{-1}$, where $\psi\colon \mathcal{B} \to \mathcal{C}$ is a homomorphism to a finite algebra Ω-algebra $\mathcal{C} = (C, \Omega)$ and $H \subseteq C$. Then $L\varphi^{-1} = H(\varphi\psi)^{-1} \in \mathrm{Rec}(\mathcal{A})$ as claimed. $\square$

Let us clarify here the relationship between the two notions of recognizable subsets, recognizable sorted subsets and pure recognizable subsets.

The following fact can be derived directly from Definition 3.1.

Lemma 3.4 *Let $\mathcal{A} = (A, \Omega)$ be an S-sorted algebra. For any $s \in S$ and $T \subseteq A_s$, $T \in \mathrm{Rec}(\mathcal{A}, s)$ iff $\langle T \rangle \in \mathrm{Rec}(\mathcal{A})$.*

The forward direction of the following proposition is again a direct consequence of Definition 3.1, and the converse part follows from Lemma 3.4 and Proposition 3.3(2).

Proposition 3.5 *A sorted subset $L = \langle L_s \rangle_{s \in S}$ of an S-sorted algebra $\mathcal{A} = (A, \Omega)$ is recognizable iff $L_s \in \mathrm{Rec}(\mathcal{A}, s)$ for every $s \in S$.*

4 Syntactic congruences and algebras

We shall now present a theory of syntactic congruences and syntactic algebras for S-sorted algebras similar to those known for semigroups, monoids (cf. [7, 14, 15]) or general one-sorted algebras (cf. [1, 17, 18]).

Definition 4.1 The *syntactic congruence* $\approx^L = \langle \approx^L_s \rangle_{s \in S}$ of a sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is defined by

$$a \approx^L_s b \Leftrightarrow (\forall s' \in S)(\forall \alpha \in \mathrm{Tr}(\mathcal{A}, s, s'))(\alpha(a) \in L_{s'} \leftrightarrow \alpha(b) \in L_{s'})$$

for every $s \in S$ and $a, b \in A_s$.

The following basic property of syntactic congruences can be verified exactly as in the one-sorted case.

Lemma 4.2 *The syntactic congruence $\approx^L$ of any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is the greatest congruence on $\mathcal{A}$ that saturates L.*

Of course, we have also the following Nerode-Myhill type theorem.

Proposition 4.3 *For any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$, the following are equivalent:*

(1) $L \in \mathrm{Rec}(\mathcal{A})$;
(2) *L is saturated by a congruence on $\mathcal{A}$ of finite index;*
(3) *$\approx^L$ is of finite index.*

Proof. If there exist a finite Ω-algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a sorted subset H of B such that $L = H\varphi^{-1}$, then $\ker \varphi$ is a congruence on $\mathcal{A}$ of finite index saturating L. On the other hand, if L is saturated by a congruence $\theta \in \mathrm{Con}(\mathcal{A})$ of finite index, then L is recognized by the finite Ω-algebra $\mathcal{A}/\theta$. Hence, (1) and (2) are equivalent. Conditions (2) and (3) are equivalent by Lemma 4.2. $\square$

Also the following facts can be proved similarly as their counterparts in the one-sorted theory. In the proposition, K and L are always sorted subsets.

Proposition 4.4 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be Ω-algebras.*

(1) $\approx^{A-L} = \approx^{L}$, *for every $L \subseteq A$.*
(2) $\approx^{K} \cap \approx^{L} \subseteq \approx^{K\cap L}$, *for every $K, L \subseteq A$.*
(3) $\approx^{L} \subseteq \approx^{\alpha^{-1}(L)}$, *for every $L \subseteq A$ and any translation $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$.*
(4) *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $\varphi\circ \approx^{L} \circ\varphi^{-1} \subseteq \approx^{L\varphi^{-1}}$ for every $L \subseteq B$. If φ is an epimorphism, then $\varphi\circ \approx^{L} \circ\varphi^{-1} = \approx^{L\varphi^{-1}}$.*

For any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$, let $A/L = \langle A_s/L\rangle_{s\in S}$, where $A_s/L = A_s/\approx^{L}_{s}$ for each sort $s \in S$. Moreover, for any $s \in S$ and any $a \in A_s$, let a/L be a shorthand for $a/\approx^{L}_{s}$.

Definition 4.5 The *syntactic algebra* $\mathcal{A}/L = (A/L, \Omega)$ of a sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is the quotient algebra $\mathcal{A}/\approx^{L}$, and the corresponding canonical homomorphism $\varphi^{L} = \langle\varphi^{L}_{s}\rangle_{s\in S}$, where for each $s \in S$,

$$\varphi^{L}_{s}\colon A_s \to A_s/L,\ a \mapsto a/L, \quad (a \in A_s),$$

is called the *syntactic homomorphism* of L.

It is clear that any sorted subset L of an Ω-algebra $\mathcal{A} = (A, \Omega)$ is recognized by its syntactic algebra. Indeed, $L = L\varphi^{L}(\varphi^{L})^{-1}$ for the syntactic homomorphism $\varphi^{L}\colon \mathcal{A} \to \mathcal{A}/L$. It follows from Lemma 4.2 that $\mathcal{A}/L$ is in the following sense the least algebra recognizing L.

Lemma 4.6 *A sorted subset L of an Ω-algebra $\mathcal{A}$ is recognizable iff the syntactic algebra $\mathcal{A}/L$ is finite. An Ω-algebra $\mathcal{B}$ recognizes L iff $\mathcal{A}/L \preceq \mathcal{B}$.*

Proposition 4.7 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be any Ω-algebras.*

(1) $\mathcal{A}/(A - L) = \mathcal{A}/L$, *for any $L \subseteq A$.*
(2) $\mathcal{A}/K \cap L \preceq \mathcal{A}/K \times \mathcal{A}/L$, *for any $K, L \subseteq A$.*
(3) $\mathcal{A}/\alpha^{-1}(L) \preceq \mathcal{A}/L$, *for any $L \subseteq A$, $s, s' \in S$ and $\alpha(\xi_s) \in \mathrm{Tr}(\mathcal{A}, s, s')$.*
(4) $\mathcal{A}/L\varphi^{-1} \preceq \mathcal{B}/L$, *for any homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ and any $L \subseteq B$. Moreover, if φ is an epimorphism, then $\mathcal{A}/L\varphi^{-1} \cong \mathcal{B}/L$.*

Proof. Assertions (1) and (3) follow immediately by the corresponding parts of Proposition 4.4 and Lemma 2.4. For (2) it suffices to note that

$$\mathcal{A}/K \cap L \leftarrow \mathcal{A}/(\approx^{K} \cap \approx^{L}) \subseteq \mathcal{A}/K \times \mathcal{A}/L$$

by Proposition 4.4(2) and Lemma 2.4.

To prove (4), let us first assume that φ is an epimorphism and show that

$$\psi_s\colon A_s/L\varphi^{-1} \to B_s/L,\ a/L\varphi^{-1} \mapsto a\varphi_s/L, \quad (s \in S,\ a \in A_s)$$

defines an isomorphism $\psi = \langle \psi_s \rangle_{s\in S}$ between $\mathcal{A}/L\varphi^{-1}$ and $\mathcal{B}/L$. First we verify that ψ is well-defined and injective: for each $s \in S$ and any $a, a' \in A_s$,

$$\begin{array}{rcl}
(a/L)\psi_s = (a'/L)\psi_s & \Leftrightarrow & a\varphi_s \approx_s^L a'\varphi_s \\
& \Leftrightarrow & (\forall s')(\forall \beta)[\beta(a\varphi_s) \in L_{s'} \leftrightarrow \beta(a'\varphi_s) \in L_{s'}] \\
& \Leftrightarrow & (\forall s')(\forall \alpha)[\alpha_\varphi(a\varphi_s) \in L_{s'} \leftrightarrow \alpha_\varphi(a'\varphi_s) \in L_{s'}] \\
& \Leftrightarrow & (\forall s')(\forall \alpha)[\alpha(a)\varphi_{s'} \in L_{s'} \leftrightarrow \alpha(a')\varphi_{s'} \in L_{s'}] \\
& \Leftrightarrow & (\forall s')(\forall \alpha)[\alpha(a) \in L_{s'}\varphi_{s'}^{-1} \leftrightarrow \alpha(a') \in L_{s'}\varphi_{s'}^{-1}] \\
& \Leftrightarrow & a/L\varphi^{-1} = a'/L\varphi^{-1},
\end{array}$$

where s' ranges over S, α over $\mathrm{Tr}(\mathcal{A}, s, s')$ and β over $\mathrm{Tr}(\mathcal{B}, s, s')$.
Consider now a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ that is not necessarily onto, and let $\mathcal{C} = (\langle A_s\varphi_s\varphi_s^L \rangle_{s\in S}, \Omega)$ be the subalgebra of $\mathcal{B}/L$ obtained as the image of $\mathcal{B}$ under the homomorphism $\varphi\varphi^L : \mathcal{A} \to \mathcal{B}/L$. Then $\eta\colon \mathcal{A} \to \mathcal{C}, a \mapsto a\varphi\varphi^L$, is an epimorphism, and hence $\mathcal{A}/L\varphi^{-1}\eta\eta^{-1} \cong \mathcal{C}/L\varphi^{-1}\eta$. However, this implies $\mathcal{A}/L\varphi^{-1} \preceq \mathcal{B}/L$ since $L\varphi^{-1}\eta\eta^{-1} = L\varphi^{-1}$ and $\mathcal{C}$ is a subalgebra of $\mathcal{B}/L$. $\square$

Lemma 4.8 *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism of Ω-algebras and $L \subseteq B$, then for every $s \in S$,*

$$\varphi_s\circ \approx_s^L \circ\varphi_s^{-1} \ \subseteq\ \bigcap\{\approx_s^{\beta^{-1}(L)\varphi^{-1}} \mid \beta \in \mathrm{Tr}(\mathcal{B}, s, s'),\ s' \in S\},$$

and if φ is an epimorphism, equality holds.

Proof. Let ρ denote the intersection appearing in the claimed equality. Parts (3) and (4) of 4.4 yield for every $\beta \in \mathrm{Tr}(\mathcal{B}, s, s')$,

$$\varphi_s\circ \approx_s^L \circ\varphi_s^{-1} \ \subseteq\ \varphi_s\circ \approx_s^{\beta^{-1}(L)} \circ\varphi_s^{-1} \ \subseteq \approx_s^{\beta^{-1}(L)\varphi^{-1}} .$$

Hence $\varphi_s\circ \approx_s^L \circ\varphi_s^{-1} \subseteq \rho$. Assume now that φ is surjective. The converse inclusion is then obtained by the following chain of implications, where $a, a' \in A_s$, s' and s'' range over S, β and γ are translations of $\mathcal{B}$, and $(\forall\beta)_{s,s'}$ is a shorthand for $(\forall\beta \in \mathrm{Tr}(\mathcal{B}, s, s'))$ etc.:

$$\begin{array}{rcl}
a\,\rho\,a' & \Rightarrow & (\forall s')(\forall\beta)_{s,s'}[a \approx_s^{\beta^{-1}(L)\varphi^{-1}} a'] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}[a\varphi_s \approx_s^{\beta^{-1}(L)} a'\varphi_s] \\
& \Rightarrow & (\forall s', s'')(\forall\beta)_{s,s'}(\forall\gamma)_{s,s''}[\gamma(a\varphi_s) \in \beta^{-1}(L)_{s''} \leftrightarrow \gamma(a'\varphi_s) \in \beta^{-1}(L)_{s''}] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}(\forall\gamma)_{s,s}[\gamma(a\varphi_s) \in \beta^{-1}(L_{s'}) \leftrightarrow \gamma(a'\varphi_s) \in \beta^{-1}(L_{s'})] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}(\forall\gamma)_{s,s}[\beta(\gamma(a\varphi_s)) \in L_{s'} \leftrightarrow \beta(\gamma(a'\varphi_s)) \in L_{s'}] \\
& \Rightarrow & (\forall s')(\forall\beta)_{s,s'}[\beta(a\varphi_s) \in L_{s'} \leftrightarrow \beta(a'\varphi_s) \in L_{s'}] \\
& \Rightarrow & a\varphi_s \approx_s^L a'\varphi_s \\
& \Rightarrow & a\,\varphi_s\circ \approx_s^L \circ\varphi_s^{-1}\, a'.
\end{array}$$

Here we used also the fact that $\beta^{-1}(L)_{s''} = \emptyset$ for every $s'' \neq s$. $\square$

Let us now present the natural generalizations of some basic facts known for monoids [7, 14] and algebras in general in the one-sorted case [17, 18].

Lemma 4.9 *Let $L = \langle L_s \rangle_{s \in S}$ be a sorted subset of an Ω-algebra $\mathcal{A} = (A, \Omega)$. For any $s \in S$ and $a \in A_s$,*

$$a/L \;=\; \bigcap\{\alpha^{-1}(L_{s'}) \mid \alpha(a_s) \in L_{s'}\} \setminus \bigcup\{\alpha^{-1}(L_{s'}) \mid \alpha(a_s) \notin L_{s'}\},$$

where s' ranges over S and α over $\mathrm{Tr}(\mathcal{A}, s, s')$.

Lemma 4.10 *Any congruence θ on an algebra $\mathcal{A} = (A, \Omega)$ is the intersection of some syntactic congruences. In particular, $\theta = \bigcap\{\approx^{\langle a/\theta\rangle} \mid s \in S, a \in A_s\}$.*

Let us call an Ω-algebra $\mathcal{A}$ *syntactic*, if $\mathcal{A} \cong \mathcal{B}/L$ for some Ω-algebra $\mathcal{B}$ and some sorted subset L of $\mathcal{B}$. A sorted subset D of an Ω-algebra $\mathcal{A}$ is *disjunctive* if $\approx^D \,=\, \Delta_A$.

Proposition 4.11 *An Ω-algebra $\mathcal{A}$ is syntactic iff it has a disjunctive subset.*

Subdirect products of Ω-algebras are defined (cf. [12], Section 4.1, or [10], p. 159) exactly as for one-sorted algebras, and by generalizing in an obvious way a well-known theorem of Birkhoff (cf. [2], for example), we may say that an Ω-algebra $\mathcal{A} = (A, \Omega)$ is *subdirectly irreducible* if the intersection of all non-trivial congruences on $\mathcal{A}$ is the diagonal relation Δ_A. By applying Lemma 4.10 to the diagonal relation we get the following result.

Corollary 4.12 *Every subdirectly irreducible Ω-algebra is syntactic.*

Since it is clear that also varieties of many-sorted algebras are generated by their subdirectly irreducible members, Corollary 4.12 implies the following important fact. However, let us note that the result follows also directly from Lemma 4.10: $\mathcal{A} \subseteq \prod\{\mathcal{A}/\approx^{\{a\}} \mid a \in A\}$ for any finite $\mathcal{A} = (A, \Omega)$ since $\Delta_A = \bigcap\{\approx^{\{a\}} \mid a \in A\}$.

Lemma 4.13 *Every Ω-VFA is generated by syntactic algebras. Hence, if $\mathbf{K}$ is an Ω-VFA and $\mathcal{A}$ any finite Ω-algebra, then $\mathcal{A} \in \mathbf{K}$ iff $\mathcal{A} \preceq \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ for some $n \geq 0$ and some syntactic algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathbf{K}$.*

5 Varieties of recognizable V-sets and varieties of V-congruences

Let S and Ω be again a finite set of sorts and an S-sorted ranked alphabet, respectively. We shall consider varieties of recognizable subsets of finitely generated free algebras over a given variety $\mathbf{V}$ of Ω-algebras. If $\mathbf{V}$ is the class of all Ω-algebras, we are actually dealing with varieties of many-sorted tree languages. In what follows, we call finite S-sorted alphabets full for Ω simply *full alphabets*, and $X = \langle X_s \rangle_{s \in S}$ and $Y = \langle Y_s \rangle_{s \in S}$ are always such full alphabets.

The free algebra $\mathcal{F}_{\mathbf{V}}(X) = (\mathrm{F}_{\mathbf{V}}(X), \Omega)$ exists for every full alphabet X, and we call the recognizable subsets of $\mathcal{F}_{\mathbf{V}}(X)$ *recognizable* $\mathbf{V}$*-sets*. The syntactic algebra $\mathcal{F}_{\mathbf{V}}(X)/L$ of a sorted subset L of $\mathcal{F}_{\mathbf{V}}(X)$ is denoted simply $\mathrm{SA}(L)$. It is clear that $\mathrm{SA}(L) \in \mathbf{V}$.

We shall also need the following fact that can proved similarly as its one-sorted counterpart [7, 17, 18].

Lemma 5.1 *Let $\mathcal{A}$ is a finite algebra in $\mathbf{V}$ and let X be a full alphabet such that for some generating set $G = \langle G_s \rangle_{s\in S}$ of $\mathcal{A}$, $|G_s| \leq |X_s|$ for every $s \in S$. Then $\mathcal{A}$ is syntactic iff $\mathcal{A} \cong \mathrm{SA}(L)$ for some $L \in \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X))$.*

A *family of recognizable* $\mathbf{V}$*-sets* is a mapping $\mathcal{R}$ that assigns to each full alphabet X a set $\mathcal{R}(X) \subseteq \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X))$ of recognizable $\mathbf{V}$-sets. We write then $\mathcal{R} = \{\mathcal{R}(X)\}_X$ with the understanding that X ranges over all full alphabets. The inclusion relation and the basic set-operations are defined for families of recognizable $\mathbf{V}$-sets by the natural componentwise conditions. For example, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are any families of recognizable $\mathbf{V}$-sets, then $\mathcal{R}_1 \subseteq \mathcal{R}_2$ iff $\mathcal{R}_1(X) \subseteq \mathcal{R}_2(X)$ for every X.

The for any X and $L \subseteq \mathrm{F}_{\mathbf{V}}(X)$, let $\overline{L}$ denote the complement $\mathrm{F}_{\mathbf{V}}(X) - L$.

Definition 5.2 A family of recognizable $\mathbf{V}$-sets $\mathcal{R} = \{\mathcal{R}(X)\}_X$ is a *variety of recognizable* $\mathbf{V}$*-sets*, a $\mathbf{V}$-VRS for short, if for all full alphabets X and Y,

(1) $\mathcal{R}(X) \neq \emptyset$,
(2) $K, L \in \mathcal{R}(X)$ implies $K \cap L, \overline{L} \in \mathcal{R}(X)$,
(3) if $L \in \mathcal{R}(X)$, then $\alpha^{-1}(L) \in \mathcal{R}(X)$ for every $\alpha \in \mathrm{Tr}(\mathcal{F}_{\mathbf{V}}(X))$, and
(4) if $L \in \mathcal{R}(Y)$, then $L\varphi^{-1} \in \mathcal{R}(X)$ for every $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$.

Let $\mathrm{VRS}(\mathbf{V})$ denote the class of all varieties of recognizable $\mathbf{V}$-sets.

It is clear that the intersection of any family of varieties of recognizable $\mathbf{V}$-sets is again a $\mathbf{V}$-VRS, and hence $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ is a complete (in fact, algebraic) lattice.

If $L = \langle L_s \rangle_{s\in S}$ is a sorted subset of any algebra $\mathcal{A} = (A, \Omega)$ and $s \in S$ is any sort, then $\langle L_s \rangle = 1_{A(s)}^{-1}(L)$. Applied to the algebras $\mathcal{F}_{\mathbf{V}}(X)$, this observation yields the following fact.

Lemma 5.3 *Let $\mathcal{R} = \{\mathcal{R}(X)\}_X$ be a $\mathbf{V}$-VRS. If $L = \langle L_s \rangle_{s\in S} \in \mathcal{R}(X)$ for some X, then $\langle L_s \rangle \in \mathcal{R}(X)$ for every $s \in S$.*

From Lemma 5.3 and Lemma 4.9 we get directly the following fact.

Lemma 5.4 *If $\mathcal{R} = \{\mathcal{R}(X)\}_X$ is a $\mathbf{V}$-VRS and $L \in \mathcal{R}(X)$ for some X, then $\langle a/L \rangle \in \mathcal{R}(X)$ for any $s \in S$ and any $a \in F_V(X)_s$.*

For any full alphabet X, let $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ denote the set of all congruences on $\mathcal{F}_{\mathbf{V}}(X)$ of finite index. Such congruences are called $\mathbf{V}$-*congruences*. A *family of* $\mathbf{V}$-*congruences* is a map Γ that assigns to each X a set $\Gamma(X) \subseteq \mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. We represent such a family in the form $\Gamma = \{\Gamma(X)\}_X$.

Definition 5.5 A family of $\mathbf{V}$-congruences $\Gamma = \{\Gamma(X)\}_X$ is a *variety of* $\mathbf{V}$-*congruences*, a $\mathbf{V}$-VFC for short, if for all X and Y,

(1) $\Gamma(X) \neq \emptyset$,
(2) if $\theta, \theta' \in \Gamma(X)$, then $\theta \cap \theta' \in \Gamma(X)$,
(3) if $\theta \in \Gamma(X)$ and $\theta \subseteq \theta'$, then $\theta' \in \Gamma(X)$, and
(4) if $\theta \in \Gamma(Y)$, then $\varphi \circ \theta \circ \varphi^{-1} \in \Gamma(X)$ for any homomorphism $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$.

Let $\mathrm{VFC}(\mathbf{V})$ denote the class of all varieties of $\mathbf{V}$-congruences.

In other words, a variety of $\mathbf{V}$-congruences is a family of filters of the lattices $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ closed under inverse homomorphisms. It is again easy to see that $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ is an algebraic lattice.

6 The Variety Theorem

Let S, Ω and $\mathbf{V}$ be as in the previous section. By a *variety of finite* $\mathbf{V}$-*algebras*, a $\mathbf{V}$-VFA for short, we mean a variety of finite Ω-algebras contained in $\mathbf{V}$. Let $\mathrm{VFA}(\mathbf{V})$ be the class of all $\mathbf{V}$-VFAs. We shall prove a Variety Theorem that establishes a triple of bijective correspondences between all varieties of recognizable $\mathbf{V}$-sets, all varieties of finite $\mathbf{V}$-algebras, and all varieties of $\mathbf{V}$-congruences. The proof is similar to those of various other Variety Theorems, and in particular to the one of [18]. However, for the convenience of the reader we present a rather detailed proof.

Let us now introduce the six mappings that will yield the Variety Theorem in the form of three pairs of mutually inverse isomorphisms between the three complete lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$, $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$.

Definition 6.1 For any $\mathbf{V}$-VFA $\mathbf{K}$, any $\mathbf{V}$-VRS $\mathcal{R}$, and any $\mathbf{V}$-VFC Γ, let

(1) $\mathbf{K}^r$ be the family of recognizable $\mathbf{V}$-sets such that for each X,

$$\mathbf{K}^r(X) = \{L \subseteq \mathrm{F}_{\mathbf{V}}(X) \mid \mathrm{SA}(L) \in \mathbf{K}\},$$

(2) $\mathbf{K}^c$ be the family of $\mathbf{V}$-congruences such that for each X,

$$\mathbf{K}^c(X) = \{\theta \in \mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X)) \mid \mathcal{F}_{\mathbf{V}}(X)/\theta \in \mathbf{K}\},$$

(3) $\mathcal{R}^a$ be the $\mathbf{V}$-VFA generated by the syntactic algebras $\mathrm{SA}(L)$ with $L \in \mathcal{R}(X)$ for some X,

(4) $\mathcal{R}^c$ be the family of $\mathbf{V}$-congruences such that for each X, $\mathcal{R}^c(X)$ is the filter in $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ generated by the syntactic congruences $\approx^L$ of all sets $L \in \mathcal{R}(X)$,
(5) Γ^a be the $\mathbf{V}$-VFA generated by all algebras $\mathcal{F}_{\mathbf{V}}(X)/\theta$ such that $\theta \in \mathcal{R}(X)$ for some X, and let
(6) Γ^r be the family of recognizable $\mathbf{V}$-sets such that for each X,

$$\Gamma^r(X) \;=\; \{L \subseteq \mathrm{F}_{\mathbf{V}}(X) \mid \approx^L \in \Gamma(X)\}.$$

Lemma 6.2 *For any* $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$, $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$ *and* $\Gamma \in \mathrm{VFC}(\mathbf{V})$,

(1) $\mathcal{R}^a, \Gamma^a \in \mathrm{VFA}(\mathbf{V})$,
(2) $\mathbf{K}^r, \Gamma^r \in \mathrm{VRS}(\mathbf{V})$, *and*
(3) $\mathbf{K}^c, \mathcal{R}^c \in \mathrm{VFC}(\mathbf{V})$.

Moreover, the mappings $\mathbf{K} \mapsto \mathbf{K}^r$, $\mathbf{K} \mapsto \mathbf{K}^c$, $\mathcal{R} \mapsto \mathcal{R}^a$, $\mathcal{R} \mapsto \mathcal{R}^a$, $\Gamma \mapsto \Gamma^a$ *and* $\Gamma \mapsto \Gamma^r$ *are all inclusion-preserving.*

Proof. By definition, $\mathcal{R}^a$ and Γ^a are $\mathbf{V}$-VFAs. That $\mathbf{K}^r$ and Γ^r are $\mathbf{V}$-VRSs, follows from Propositions 4.7 and 4.4. Finally, Lemmas 2.4 and 4.8, and Proposition 4.4 imply that $\mathbf{K}^c$ and $\mathcal{R}^c$ are in $\mathrm{VFC}(\mathbf{V})$. $\square$

We shall show that the six mappings introduced above form three pairs of mutually inverse isomorphisms between the complete lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$, $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$. Since we already know that all of the maps are isotone, it suffices to show that they are pairwise inverses of each other.

Proposition 6.3 *The lattices* $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ *are isomorphic as*

(1) $\mathbf{K}^{ra} = \mathbf{K}$ *for every* $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$, *and*
(2) $\mathcal{R}^{ar} = \mathcal{R}$ *for every* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$.

Proof. It suffices to prove (1) and (2).

Since $\mathbf{K}^{ra}$ is generated by syntactic algebras belonging to $\mathbf{K}$, the inclusion $\mathbf{K}^{ra} \subseteq \mathbf{K}$ is obvious. For the converse inclusion, let us consider any syntactic algebra $\mathcal{A} \in \mathbf{K}$. By Lemma 5.1 there exists an X such that $\mathcal{A} \cong \mathrm{SA}(L)$ for some $L \in \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X))$. Then $L \in \mathbf{K}^r(X)$ and hence $\mathcal{A} \in \mathbf{K}^{ra}$. This implies $\mathbf{K} \subseteq \mathbf{K}^{ra}$ because, by Lemma 4.13, $\mathbf{K}$ is generated by syntactic algebras.

The inclusion $\mathcal{R} \subseteq \mathcal{R}^{ar}$ is obvious: if $L \in \mathcal{R}(X)$ for any X, then $\mathrm{SA}(L) \in \mathcal{R}^a$ and hence $L \in \mathcal{R}^{ar}(X)$. Assume then that $L \in \mathcal{R}^{ar}(X)$ for some X. Then $\mathrm{SA}(L) \in \mathcal{R}^a$ implies that $\mathrm{SA}(L) \preceq \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ for some $k \geq 1$, some full alphabets $X_i = \langle X_i(s) \rangle_{s \in S}$ and sets $L_i \in \mathcal{R}(X_i)$ $(i = 1, \ldots, k)$. For each $i = 1, \ldots, k$, let φ_i denote the syntactic homomorphisms $\varphi^{L_i} \colon \mathcal{F}_{\mathbf{V}}(X_i) \to \mathrm{SA}(T_i)$. Then there is a homomorphism

$$\eta \colon \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$$

such that for every $i = 1, \dots, k$, $\eta\pi_i = \varphi_i\tau_i$, where

$$\pi_i\colon \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k) \longrightarrow \mathrm{SA}(L_i),$$

and

$$\tau_i\colon F_{\mathbf{V}}(X_1) \times \cdots \times F_{\mathbf{V}}(X_k) \longrightarrow F_{\mathbf{V}}(X_i)$$

are the respective projection functions. By Lemma 4.6 there exist a homomorphism $\varphi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ and a subset H of $\mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ such that $L = H\varphi^{-1}$. Since η is an epimorphism, there is a homomorphism $\psi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ such that $\psi\eta = \varphi$. Because H is finite, $L = \bigcup_{u\in H} u\varphi^{-1}$ is the union of finitely many sets $u\varphi^{-1}$ with $u = (u_1, \dots, u_k) \in \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$. For each such $u \in H$,

$$u\varphi^{-1} \;=\; \bigcap\{u_i(\varphi\pi_i)^{-1} \mid 1 \le i \le k\} \;=\; \bigcap\{u_i\varphi_i^{-1}(\psi\tau_i)^{-1} \mid 1 \le i \le k\}.$$

By Lemma 5.4, $u_i\varphi_i^{-1} \in \mathcal{R}(X_i)$ for each $i = 1, \dots, k$, and thus $L \in \mathcal{R}(X)$. $\square$

Lemma 6.4 *For any* $\mathbf{V}$-VFC Γ *and any finite algebra* $\mathcal{A} \in \mathbf{V}$, $\mathcal{A} \in \Gamma^{\mathrm{a}}$ *iff there exist a finite set* X *and an epimorphism* $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{A}$ *such that* $\ker\varphi \in \Gamma(X)$.

Proof. If $\mathcal{A} \in \Gamma^a$, then $\mathcal{A} \preceq \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$ for some $k \ge 1$, some full alphabets $X_1, \cdots, X_k$ and congruences $\theta_1 \in \Gamma(X_1), \cdots, \theta_k \in \Gamma(X_k)$. This means that for some algebra $\mathcal{B}$ there exist an epimorphism $\eta\colon \mathcal{B} \to \mathcal{A}$ and a monomorphism $\varphi\colon \mathcal{B} \to \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$. The algebras $\mathcal{F}_{\mathbf{V}}(X_i)/\theta_i$ are finite members of $\mathbf{V}$ and hence there is for some X an epimorphism $\psi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{B}$. The condition $(a_1, \cdots, a_k)\chi = (a_1/\theta_1, \cdots, a_k/\theta_k)$ defines an epimorphism

$$\chi\colon \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k.$$

For each $i = 1, \dots, k$, let $\pi_i\colon F_{\mathbf{V}}(X_1) \times \cdots \times F_{\mathbf{V}}(X_k) \to F_{\mathbf{V}}(X_i)$ be the i^{th} projection, and let $\omega\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ be the homomorphism such that $\omega\chi = \psi\varphi$. Then $\psi\eta\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{A}$ is an epimorphism, and

$$\ker\psi\eta \supseteq \ker\psi\varphi = \ker\omega\chi = \bigcap\{\omega\pi_i \circ \theta_i \circ (\omega\pi_i)^{-1} \mid 1 \le i \le k\}$$

shows that $\ker\psi\eta \in \Gamma(X)$.

The converse implication is immediately clear by the definition of Γ^a. $\square$

Proposition 6.5 *The lattices* $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ *are isomorphic as*

(1) $\mathbf{K}^{ca} = \mathbf{K}$ *for every* $\mathbf{V}$-VFA $\mathbf{K}$, *and*

(2) $\Gamma^{ac} = \Gamma$ *for every* $\mathbf{V}$-VFCΓ.

Proof. By Lemma 6.4, $\mathcal{A} \in \mathbf{K}^{ca}$ iff for some X there exists an epimorphism $\varphi : \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{A}$ such that $\ker\varphi \in \mathbf{K}^c$. By Proposition 2.1 this is equivalent to $\mathcal{F}_{\mathbf{V}}(X)/\ker\varphi \cong \mathcal{A}$, which is the case exactly when $\mathcal{A} \in \mathbf{K}$. Thus (1) follows.

To prove (2), we consider any X and $\theta \in \mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. If $\theta \in \Gamma^{ac}(X)$, then by Lemma 6.4, there exist a Y and an epimorphism $\psi\colon \mathcal{F}_{\mathbf{V}}(Y) \to \mathcal{F}_{\mathbf{V}}(X)/\theta$ such that $\ker\psi \in \Gamma(Y)$. Since ψ is surjective, there is for any $s \in S$ and every $x \in X_s$ an element $t_s^x \in \mathrm{F}_{\mathbf{V}}(Y)_s$ such that $t_s^x\psi_s = x/\theta_s$. If $\varphi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$ is the homomorphism such that $x\varphi = t_s^x$ for all $s \in S$ and $x \in X_s$, then $\varphi\psi = \theta^\natural$, where $\theta^\natural\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X)/\theta$ is the canonical epimorphism. Hence $\theta = \ker\varphi\psi = \varphi \circ (\ker\psi) \circ \varphi^{-1} \in \Gamma(X)$. The converse inclusion is obvious: if $\theta \in \Gamma(X)$, then $\mathcal{F}_{\mathbf{V}}(X)/\theta \in \Gamma^a$ implies $\theta \in \Gamma^{ac}$. $\square$

Propositions 6.3 and 6.5 already show that the lattices $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ are isomorphic, but the following composition laws imply also that the mappings $\mathcal{R} \mapsto \mathcal{R}^c$ and $\Gamma \mapsto \Gamma^r$ form a pair of mutually inverse isomorphisms between them.

Proposition 6.6 *For any* $\mathbf{V}$-VFA $\mathbf{K}$, $\mathbf{V}$-VRS $\mathcal{R}$, *and* $\mathbf{V}$-VFC Γ,

(1) $\mathbf{K}^{cr} = \mathbf{K}^r$,
(2) $\mathcal{R}^{ac} = \mathcal{R}^c$, *and*
(3) $\Gamma^{ra} = \Gamma^a$.

Proof. For (1) it suffices to note that

$$L \in \mathbf{K}^r(X) \;\Leftrightarrow\; \mathrm{SA}(L) \in \mathbf{K} \;\Leftrightarrow\; \approx^L \in \mathbf{K}^c(X) \;\Leftrightarrow\; L \in \mathbf{K}^{cr}(X),$$

for any X and $L \subseteq \mathrm{F}_{\mathbf{V}}(X)$.

To prove (2), let us consider any X and $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. If $\theta \in \mathcal{R}^c(X)$, then $\approx^{L_1} \cap \cdots \cap \approx^{L_k} \subseteq \theta$ for some $k \geq 1$ and $L_1, \cdots, L_k \in \mathcal{R}(X)$. This implies that $\mathcal{F}_{\mathbf{V}}(X)/\theta \in \mathcal{R}^a$ since $\mathcal{F}_{\mathbf{V}}(X)/\theta \preceq \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$, and therefore $\theta \in \mathcal{R}^{ac}$.

If $\theta \in \mathcal{R}^{ac}(X)$, then $\mathcal{F}_{\mathbf{V}}(X)/\theta \preceq \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$ for some full alphabets $X_1, \cdots, X_k$ and sorted sets $L_1 \in \mathcal{R}(X_1), \cdots, L_k \in \mathcal{R}(X_k)$ $(k \geq 1)$. Hence, there is an Ω-algebra $\mathcal{B}$ such that there exist an epimorphism $\psi\colon \mathcal{B} \to \mathcal{F}_{\mathbf{V}}(X)$ and a monomorphism $\eta\colon \mathcal{B} \to \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$. We may also assume that there is an epimorphism $\varphi\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{B}$ such that $\varphi\psi = \theta^\natural$ (if not, we replace $\mathcal{B}$ with a suitable subalgebra). For each $i = 1, \ldots k$, let π_i be the i^{th} projection from $\mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ onto $\mathcal{F}_{\mathbf{V}}(X_i)$, and let

$$\pi\colon \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathrm{SA}(L_1) \times \cdots \times \mathrm{SA}(L_k)$$

be the homomorphism such that $(t_1, \ldots, t_k) \mapsto (t_1/L_1, \ldots, t_k/L_k)$ for all $s \in S$ and $t_1 \in \mathcal{F}_{\mathbf{V}}(X_1)_s, \ldots t_k \in \mathcal{F}_{\mathbf{V}}(X_k)_s$. Since π clearly is surjective, we may define a homomorphism $\gamma\colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ for which $\gamma\pi = \varphi\eta$. Then

$$\theta \;=\; \ker\varphi\psi \;\supseteq\; \ker\varphi\eta \;=\; \ker\gamma\pi \;=\; \bigcap\{\gamma\pi_i \circ \approx^{L_i} \circ (\gamma\pi_i)^{-1} \mid 1 \le i \le k\},$$

and hence $\theta \in \mathcal{R}^c(X)$.

To prove (3), consider any finite algebra $\mathcal{A} = (A, \Omega)$. Now, $\mathcal{A}$ belongs to Γ^a iff $\mathcal{A} \preceq \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$, for some full alphabets $X_1, \cdots, X_k$ and and some $\theta_1 \in \Gamma(X_1), \cdots, \theta_k \in \Gamma(X_k)$ $(k \ge 1)$. Since any $\Gamma(X)$ is generated by syntactic congruences by Lemma 4.10, we can assume that each θ_i is the syntactic congruence of some $L_i \subseteq \mathrm{F}_{\mathbf{V}}(X_i)$, and then $L_i \in \Gamma^r(X_i)$, and so $\mathcal{A} \in \Gamma^a$ iff $\mathcal{A} \in \Gamma^{ra}$. $\square$

Proposition 6.7 *The lattices* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ *are isomorphic as*

(1) $\mathcal{R}^{cr} = \mathcal{R}$ *for every* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$*, and*
(2) $\Gamma^{rc} = \Gamma$ *for every* $\Gamma \in \mathrm{VFC}(\mathbf{V})$.

Proof. By using the previous three propositions we can see that $\mathcal{R}^{cr} = \mathcal{R}^{acr} = \mathcal{R}^{ar} = \mathcal{R}$ for every $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$. Similarly, $\Gamma^{rc} = \Gamma^{rac} = \Gamma^{ac} = \Gamma$ for every $\Gamma \in \mathrm{VFC}(\mathbf{V})$. $\square$

Let us note that Proposition 6.7 could be obtained also directly in a similar way as the analogous facts are proved in [1]. For example, $\mathcal{R}^{cr} = \mathcal{R}$ can be seen as follows.

The inclusion $\mathcal{R} \subseteq \mathcal{R}^{cr}$ follows directly from the definitions of the two operators. On the other hand, if $L \in \mathcal{R}^{cr}(X)$, then $\approx^{L_1} \cap \ldots \cap \approx^{L_k} \subseteq \approx^L$ for some $L_1, \ldots, L_k \in \mathcal{R}(X)$. This means that each $\approx^L$-class, and hence also L, is a Boolean combination of $\approx^{L_i}$-classes $(1 \le i \le k)$, and since each such class is in $\mathcal{R}(X)$ by Lemma 5.4, this implies $L \in \mathcal{R}(X)$.

We may sum up the results of this section as follows.

Theorem 6.8 (Variety Theorem) *The mappings*

$\mathrm{VFA}(\mathbf{V}) \to \mathrm{VRS}(\mathbf{V}),\ \mathbf{K} \mapsto \mathbf{K}^r,\quad \mathrm{VRS}(\mathbf{V}) \to \mathrm{VFA}(\mathbf{V}),\ \mathcal{R} \mapsto \mathcal{R}^a,$
$\mathrm{VFA}(\mathbf{V}) \to \mathrm{VFC}(\mathbf{V}),\ \mathbf{K} \mapsto \mathbf{K}^c,\quad \mathrm{VFC}(\mathbf{V}) \to \mathrm{VFA}(\mathbf{V}),\ \Gamma \mapsto \Gamma^a,$ *and*
$\mathrm{VRS}(\mathbf{V}) \to \mathrm{VFC}(\mathbf{V}),\ \mathcal{R} \mapsto \mathcal{R}^c,\quad \mathrm{VFC}(\mathbf{V}) \to \mathrm{VRS}(\mathbf{V}),\ \Gamma \mapsto \Gamma^r,$

form three pairs of isomorphisms that are inverses of each other between the lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$*,* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$*, and* $(\mathrm{VFC}(\mathbf{V}), \subseteq)$*. Moreover,* $\mathbf{K}^{cr} = \mathbf{K}^r$*,* $\mathbf{K}^{rc} = \mathbf{K}^c$*,* $\mathcal{R}^{ca} = \mathcal{R}^a$*,* $\mathcal{R}^{ac} = \mathcal{R}^c$*,* $\Gamma^{ra} = \Gamma^a$*, and* $\Gamma^{ar} = \Gamma^r$*, for any* $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$*,* $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$*, and* $\Gamma \in \mathrm{VFC}(\mathbf{V})$.

Proof. That the given mappings form isomorphisms of the claimed kind follows from Propositions 6.3, 6.5, and 6.7. Moreover, Proposition 6.6 contains half of the composition laws, and together with Propositions 6.3, 6.5 and 6.7 it implies also the rest of them. For example, by Propositions 6.6 and 6.3, we get $\mathbf{K}^{rc} = (\mathbf{K}^r)^{ac} = (\mathbf{K}^{ra})^c = \mathbf{K}^c$ for any $\mathbf{K} \in \mathrm{VFA}(\mathbf{V})$. $\square$

7 Varieties of pure recognizable sets

In this section we shall show how the above variety theory can be translated into a theory of varieties of pure recognizable sets.

Let $\mathbf{V}$ be again a given variety of Ω-algebras. For any full alphabet X and any sort $s \in S$, the members of $\mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X), s)$ are called *pure recognizable* $\mathbf{V}X$*-sets of sort* s, or simply *pure recognizable* $\mathbf{V}$*-sets*. A *family of pure recognizable* $\mathbf{V}$*-sets* is a mapping $\mathcal{P}$ that assigns to each X and each s a set $\mathcal{P}(X, s) \subseteq \mathrm{Rec}(\mathcal{F}_{\mathbf{V}}(X), s)$ of pure recognizable $\mathbf{V}X$-sets of sort s, and we write it as $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X,s}$.

Definition 7.1 A *variety of pure recognizable* $\mathbf{V}$*-sets*, a $\mathbf{V}$-VRS for short, is a family of pure recognizable $\mathbf{V}$-sets $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X,s}$ such that for all full alphabets X and Y and all sorts $s, s' \in S$,

(1) $\mathcal{P}(X, s) \neq \emptyset$,
(2) $T, U \in \mathcal{P}(X, s)$ implies $T \cap U, \overline{T} \in \mathcal{P}(X, s)$,
(3) if $T \in \mathcal{P}(X, s')$ and $\alpha \in \mathrm{Tr}(\mathcal{F}_{\mathbf{V}}(X), s, s')$, then $\alpha^{-1}(T) \in \mathcal{P}(X, s)$, and
(4) if $T \in \mathcal{P}(Y, s)$ and $\varphi \colon \mathcal{F}_{\mathbf{V}}(X) \to \mathcal{F}_{\mathbf{V}}(Y)$ is any homomorphism, then also $T\varphi^{-1} \in \mathcal{P}(X, s)$.

Let $\mathrm{VPRS}(\mathbf{V})$ denote the class of all varieties of pure recognizable $\mathbf{V}$-sets.

Of course, $(\mathrm{VPRS}(\mathbf{V}), \subseteq)$ is a complete lattice. We shall now show that there is a natural correspondence between varieties of pure recognizable $\mathbf{V}$-sets and varieties of recognizable $\mathbf{V}$-sets.

Definition 7.2 With any family $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X,s}$ of pure recognizable $\mathbf{V}$-sets we associate the family of recognizable $\mathbf{V}$-sets $\mathcal{P}^r = \{\mathcal{P}^r(X)\}_X$ such that

$$\mathcal{P}^r(X) \;=\; \{L \subseteq \mathrm{F}_{\mathbf{V}}(X) \mid (\forall s \in S) L_s \in \mathcal{P}(X, s)\}$$

for each X. With any family $\mathcal{R} = \{\mathcal{R}(X)\}_X$ of recognizable $\mathbf{V}$-sets we associate the family $\mathcal{R}^p = \{\mathcal{R}^p(X)\}_{X,s}$ of pure recognizable $\mathbf{V}$-sets such that

$$\mathcal{R}^p(X, s) \;=\; \{L_s \mid L \in \mathcal{R}(X)\}$$

for any X and $s \in S$.

Let us first note a few basic facts about these mappings. The notation $\langle T \rangle$ was introduced before Lemma 3.4.

Lemma 7.3 *Let* $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X,s}$ *be a* $\mathbf{V}$*-VPRS and* $\mathcal{R} = \{\mathcal{R}(X)\}_X$ *be a* $\mathbf{V}$*-VRS. For any* X, $s \in S$ *and* $T \subseteq \mathrm{F}_{\mathbf{V}}(X)$,

(1) $T \in \mathcal{P}(X, s)$ *iff* $\langle T \rangle \in \mathcal{P}^r(X)$, *and*
(2) $T \in \mathcal{R}^p(X, s)$ *iff* $\langle T \rangle \in \mathcal{R}(X)$.

Proof. If $T \in \mathcal{P}(X,s)$, then $\langle T\rangle \in \mathcal{P}^r(X)$ since $\langle T\rangle_s = T \in \mathcal{P}(X,s)$ and $T_u = \emptyset \in \mathcal{P}(X,u)$ for every $u \in S, u \neq s$. On the other hand, if $\langle T\rangle \in \mathcal{P}^r(X)$, then $T = \langle T\rangle_s \in \mathcal{P}(X,s)$, and hence (1) holds.

To prove (2), assume first that $T \in \mathcal{R}^p(X,s)$. Then $T = L_s$ for some $L \in \mathcal{R}(X)$. If 1_s denotes the identity translation $\mathrm{F}_{\mathbf{V}}(X) \to \mathrm{F}_{\mathbf{V}}(X)$, then $\langle T\rangle = 1_s^{-1}(L) \in \mathcal{R}(X)$. On the other hand, if $\langle T\rangle \in \mathcal{R}(X)$, then $T = \langle T\rangle_s \in \mathcal{R}^p(X,s)$ by the definition of $\mathcal{R}^p$. □

Lemma 7.4 *The mappings $\mathcal{P} \mapsto \mathcal{P}^r$ and $\mathcal{R} \mapsto \mathcal{R}^p$ are inclusion-preserving. Moreover,*

(1) *if $\mathcal{P} \in \mathrm{VPRS}(\mathbf{V})$, then $\mathcal{P}^r \in \mathrm{VRS}(\mathbf{V})$, and*
(2) *if $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$, then $\mathcal{R}^p \in \mathrm{VPRS}(\mathbf{V})$.*

Proof. The first claim is completely obvious. Now, let $\mathcal{P} \in \mathrm{VPRS}(\mathbf{V})$. That $\mathcal{P}^r$ satisfies the conditions of Definition 5.2 follows easily from the assumption that $\mathcal{P}$ satisfies the corresponding conditions of Definition 7.1. For example,

$$\begin{aligned} K, L \in \mathcal{P}^r(X) \quad &\Rightarrow \quad (\forall s \in S)\ K_s, L_s \in \mathcal{P}(X,s) \\ &\Rightarrow \quad (\forall s \in S)\ K_s \cap L_s \in \mathcal{P}(X,s) \\ &\Rightarrow \quad (\forall s \in S)\ (K \cap L)_s \in \mathcal{P}(X,s) \\ &\Rightarrow \quad K \cap L \in \mathcal{P}^r(X), \end{aligned}$$

for any X and $K, L \subseteq \mathrm{F}_{\mathbf{V}}(X)$. Similarly, if $L \in \mathcal{P}^r(X)$ and α is a translation in $\mathrm{Tr}(\mathcal{F}_{\mathbf{V}}(X), s, s')$ for some X and $s, s' \in S$, then $L_{s'} \in \mathcal{P}(X,s')$ implies that $\alpha^{-1}(L)_s = \alpha^{-1}(L_{s'}) \in \mathcal{P}(X,s)$, and hence $\alpha^{-1}(L) \in \mathcal{P}^r(X)$ as $\alpha^{-1}(L)_u = \emptyset \in \mathcal{P}(X,u)$ for every $u \in S, u \neq s$. Assertion (2) has a similar proof. □

Proposition 7.5 *The lattices* $(\mathrm{VPRS}(\mathbf{V}), \subseteq)$ *and* $(\mathrm{VRS}(\mathbf{V}), \subseteq)$ *are isomorphic because*

(1) *$\mathcal{P}^{rp} = \mathcal{P}$ for every $\mathcal{P} \in \mathrm{VPRS}(\mathbf{V})$, and*
(2) *$\mathcal{R}^{pr} = \mathcal{R}$ for every $\mathcal{R} \in \mathrm{VRS}(\mathbf{V})$.*

Proof. In view of Lemma 7.4 it suffices to prove (1) and (2), and these claims follow directly from Definition 7.2. For example, let $\mathcal{P} \in \mathrm{VPRS}(\mathbf{V})$ and consider any X and $s \in S$. If $T \in \mathcal{P}(X,s)$, then $\langle T\rangle \in \mathcal{P}^r(X)$ by Lemma 7.3, and hence $T = \langle T\rangle_s \in \mathcal{P}^{rp}$. Conversely: if $T \in \mathcal{P}^{rp}$, then there is an $L \in \mathcal{P}^r(X)$ such that $T = L_s$. But $L \in \mathcal{P}^r(X)$ means that $L_u \in \mathcal{P}(X,u)$ for every $u \in S$, and therefore, in particular, $T = L_s \in \mathcal{P}(X,s)$. Assertion (2) can be verified similarly. □

Proposition 7.5 already implies that $(\mathrm{VPRS}(\mathbf{V}), \subseteq)$ is isomorphic also to the lattices $(\mathrm{VFA}(\mathbf{V}), \subseteq)$ and $(\mathrm{VFC}(\mathbf{V}), \subseteq)$ via $(\mathrm{VRS}(\mathbf{V}), \subseteq)$, but we shall also exhibit direct isomorphisms. However, let us first consider generally syntactic congruences and algebras of subsets of a given sort.

Courcelle [5] defines the *syntactic congruence* of a subset $T \subseteq A_u$ of sort $u \in S$ of an Ω-algebra $\mathcal{A} = (A, \Omega)$ as the sorted equivalence $\sim^T = \langle \sim_s^T \rangle_{s \in S}$ on A such that for each $s \in S$ and any $a, b \in A$,

$$a \sim_s^T b \quad \Leftrightarrow \quad (\forall \alpha \in \mathrm{Tr}(\mathcal{A}, s, u))(\alpha(a) \in T \leftrightarrow \alpha(b) \in T).$$

It is easy to see that $\sim^T$ is the greatest congruence θ on $\mathcal{A}$ such that θ_u saturates T. Let us call such congruences *pure syntactic congruences.*

The following lemma is quite obvious.

Lemma 7.6 *Let $\mathcal{A} = (A, \Omega)$ be an Ω-algebra. Then $\sim^T = \approx^{\langle T \rangle}$ for any subset $T \subseteq A_u$ of any given sort $u \in S$. On the other hand,*

$$\approx^L = \bigcap \{ \sim^{L_s} \mid s \in S \},$$

for any sorted subset $L = \langle L_s \rangle_{s \in S}$ of $\mathcal{A}$.

Hence, any pure syntactic congruence is a syntactic congruence in our sense, while every syntactic congruence is the intersection of finitely many pure syntactic congruences.

The *syntactic algebra* of a subset $T \subseteq A_s$ of any sort $s \in S$ of an Ω-algebra $\mathcal{A} = (A, \Omega)$, is defined in [5] as the quotient algebra $\mathcal{A}/\sim^T$. Let us call an algebra *pure syntactic* if it is isomorphic to such a syntactic algebra.

Proposition 7.7 *Any pure syntactic Ω-algebra is syntactic, and any syntactic Ω-algebra is a subdirect product of a finite family of pure syntactic Ω-algebras. Furthermore, every subdirectly irreducible Ω-algebra is pure syntactic.*

Proof. The first two assertions are immediate consequences of Lemma 7.6. If $\mathcal{A} = (A, \Omega)$ is subdirectly irreducible, then $\bigcap \{ \sim^{\{a\}} \mid a \in A_s, s \in S \} = \Delta_A$ implies that $\sim^{\{a\}} = \Delta_A$ for at least some $s \in S$ and $a \in A_s$, and hence $\mathcal{A} \cong \mathcal{A}/\sim^{\{a\}}$ is pure syntactic. $\square$

Corollary 7.8 *Every* $\mathbf{V}$*-VFA is generated by pure syntactic algebras.*

Let us return to pure recognizable $\mathbf{V}$-sets. The syntactic algebra $\mathcal{F}_{\mathbf{V}}(X)/\sim^T$ of a subset $T \subseteq \mathrm{F}_{\mathbf{V}}(X)_s$ of sort some $s \in S$ of $\mathcal{F}_{\mathbf{V}}(X)$ is denoted simply $\mathrm{PSA}(T)$. Note that $\mathrm{PSA}(T) = \mathrm{SA}(\langle T \rangle)$ by the first assertion of Lemma 7.6.

For any $\mathbf{V}$-VPRS $\mathcal{P}$, let $\mathcal{P}^a$ be the $\mathbf{V}$-VFA generated by the pure syntactic algebras $\mathrm{PSA}(T)$, where $T \in \mathcal{P}(X, s)$ for some X and $s \in S$.

Lemma 7.9 (1) $\mathcal{P}^a = \mathcal{P}^{ra}$ *for any* $\mathbf{V}$*-VPRS* $\mathcal{P}$*, and*

(2) $\mathcal{R}^a = \mathcal{R}^{pa}$ *for any* $\mathbf{V}$*-VRS* $\mathcal{R}$.

Proof. To prove (1) it suffices to show that the syntactic algebras generating $\mathcal{P}^a$ are in $\mathcal{P}^{ra}$, and conversely. For any X, $s \in S$ and $T \in \mathcal{P}(X, s)$, we have $\mathrm{PSA}(T) = \mathrm{SA}(\langle T \rangle) \in \mathcal{P}^{ra}$ since $\langle T \rangle \in \mathcal{P}^r$. Conversely, if $L \in \mathcal{P}^r(X)$ for some X, then $\mathrm{SA}(L)$ is by Proposition 7.7 a subdirect product of the pure syntactic algebras $\mathrm{PSA}(L_s)$ ($s \in S$). Because $L_s \in \mathcal{P}(X, s)$, and therefore $\mathrm{PSA}(L_s) \in \mathcal{P}^a$, for every $s \in S$, this means that $\mathrm{SA}(L) \in \mathcal{P}^a$. Assertion (2) has an equally straightforward proof. □

Now it is clear that $\mathcal{P} \mapsto \mathcal{P}^a$ defines an isomorphism from $(\mathrm{VPRS}(\mathbf{V}), \subseteq)$ to $(\mathrm{VFA}(\mathbf{V}), \subseteq)$. In fact, it is the composition of the two isomorphisms $\mathcal{P} \mapsto \mathcal{P}^r$ and $\mathcal{R} \mapsto \mathcal{R}^a$. This converse can be defined explicitly as follows: for any $\mathbf{V}$-VFA $\mathbf{K}$, let $\mathbf{K}^p$ be the family of pure recognizable $\mathbf{V}$-sets such that for any X, $s \in S$ and $T \subseteq \mathrm{F}_{\mathbf{V}}(X)_s$, $T \in \mathbf{K}^p(X, s)$ iff $\mathrm{PSA}(T) \in \mathbf{K}$.

Corresponding to Lemma 7.9 the following facts hold.

Lemma 7.10 *For any* $\mathbf{V}$*-VFA* $\mathbf{K}$*,* (1) $\mathbf{K}^p = \mathbf{K}^{rp}$*, and* (2) $\mathbf{K}^r = \mathbf{K}^{pr}$.

Proof. To prove (1) we note that for any X, $s \in S$ and $T \subseteq \mathrm{F}_{\mathbf{V}}(X)_s$,

$$T \in \mathbf{K}^p(X, s) \;\Leftrightarrow\; \mathrm{PSA}(T) \in \mathbf{K} \;\Leftrightarrow\; \mathrm{SA}(\langle T \rangle) \in \mathbf{K} \;\Leftrightarrow\; \langle T \rangle \in \mathbf{K}^r \;\Leftrightarrow T \in \mathbf{K}^{rp}.$$

Now (2) follows since $\mathbf{K}^{pr} = \mathbf{K}^{rpr} = \mathbf{K}^r$ by Proposition 7.5. □

Let us now consider the connections between pure recognizable $\mathbf{V}$-sets and $\mathbf{V}$-congruences. Proposition 7.5 and the Variety Theorem yield the isomorphisms $\mathrm{VPRS}(\mathbf{V}) \to \mathrm{VFC}(\mathbf{V}), \mathcal{P} \mapsto \mathcal{P}^{rc}$ and $\mathrm{VFC}(\mathbf{V}) \to \mathrm{VPRS}(\mathbf{V}), \Gamma \mapsto \Gamma^{cp}$, via $\mathrm{VRS}(\mathbf{V})$, but we can also define them directly as follows.

For any $\mathbf{V}$-VPRS $\mathcal{P}$, let $\mathcal{P}^c$ be the family of $\mathbf{V}$-congruences such that for each X, $\mathcal{P}^c(X)$ is the filter of $\mathrm{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ generated by the pure syntactic congruences $\sim^T$, where $T \in \mathcal{P}(X, s)$ for some $s \in S$. Conversely, for any $\mathbf{V}$-VFC Γ, let Γ^p be the family of pure recognizable $\mathbf{V}$-sets such that for any X and $s \in S$,

$$\Gamma^p(X, s) \;=\; \{T \subseteq \mathrm{F}_{\mathbf{V}}(X)_s \mid \sim^T \in \Gamma(X)\}.$$

Lemma 7.11 (1) $\mathcal{P}^c = \mathcal{P}^{rc}$ *for any* $\mathbf{V}$*-VPRS* $\mathcal{P}$*, and*
(2) $\mathcal{R}^c = \mathcal{R}^{pc}$ *for any* $\mathbf{V}$*-VRS* $\mathcal{R}$.

Proof. To prove (1), we show that for any X, the generators of $\mathcal{P}^c(X)$ are in $\mathcal{P}^{rc}(X)$, and the generators of $\mathcal{P}^{rc}(X)$ are in $\mathcal{P}^c(X)$.

For any $s \in S$ and $T \in \mathcal{P}(X, s)$, $\langle T \rangle \in \mathcal{P}^r(X)$ and $\sim^T = \approx^{\langle T \rangle} \in \mathcal{P}^{rc}$. On the other hand, if $L \in \mathcal{P}^r(X)$, then $L_s \in \mathcal{P}(X, s)$ for each $s \in S$, and hence $\approx^L \in \mathcal{P}^c(X)$ by Lemma 7.6. Assertion (2) can be verified similarly. □

Lemma 7.12 *For any* $\mathbf{V}$*-VFC* Γ*, (1)* $\Gamma^p = \Gamma^{rp}$*, and (2)* $\Gamma^r = \Gamma^{pr}$*.*

Proof. To prove (1), it suffices to note that for any X, $s \in S$ and $T \subseteq \mathrm{F}_{\mathbf{V}}(X)_s$, $T \in \Gamma^p(X,s) \;\Leftrightarrow\; \sim^T \in \Gamma(X) \;\Leftrightarrow\; \approx^{\langle T\rangle} \in \Gamma(X) \;\Leftrightarrow\; \langle T\rangle \in \Gamma^r(X) \;\Leftrightarrow\; T \in \Gamma^{rp}(X)$, where Lemma 7.3 is used in the last step.

Assertion (2) follows from (1) and Proposition 7.5: $\Gamma^{pr} = \Gamma^{rpr} = \Gamma^r$. $\square$

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