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A Unifying Study Between Modal-like Operators, Topologies, and Fuzzy Sets

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Abstract

The paper introduces both lattice-theoretical and topological approaches on studying connections between Fuzzy Set Theory and Rough Set Theory or, more precisely, L-sets and modal-like operators. Various results for certain type of L-sets are presented, and it is shown that modal-like operators can be determined by means of L-sets. Moreover, it is shown that a certain subcategory of a category of variable-basis L-sets is isomorphic to the category of Alexandroff topological spaces as well as the category of quasi-ordered sets.

Keywords: Modal-like operators, quasi-orders, modifiers, rough sets, fuzzy sets, topologies, category theory

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1 Introduction

In this paper, we study some relationships between Rough Set Theory and Fuzzy Set Theory, and the paper in its current form is written in a self-contained manner, if possible.

Järvinen has studied Rough Set Theory, especially from lattice-theoretical point of view in [14, 15], and Kortelainen has earlier studied Fuzzy Set Theory, particularly L-sets, from a topological viewpoint (see e.g. [20]). Notice that the term Fuzzy Sets is generic including the L-sets, for example. The authors have also studied a concept of definability in Rough Set Theory from a topological point of view [16]. There are well-known connections between quasi-orders and Alexandroff topologies, and naturally modal-like operators may be determined by means of quasi-orders. These notions play a key role in this paper.

The paper is organized as follows: In Section 2, we recall and develop some notions and notation considering lattices, topological spaces, L-sets, and certain modal-like operators. Especially, Alexandroff topological spaces are introduced. Section 3 presents connections between L-sets, quasi-ordered sets, and Alexandroff topological spaces. We show, for example, that L-sets may be used to determine modal-like operators, and that each quasi-order R on U induces a certain type of L-set, called an L_R-set on U. Especially, the L_R-sets are determined in such a way that they carry the same ordering information as L-sets. In Section 4, some notions and notation concerning category theory are presented, and we complete the paper by interpreting the study using categorical concepts. For example, we determine a category with L_R-sets as its objects, and show that this category is isomorphic to the category of Alexandroff topological spaces.

2 Preliminaries

This subsection is reserved for some basic notions and notation. For a set U, let $\wp(U)$ denote the *powerset* of U, that is, the family of all subsets of U. Given a family $\mathcal{H} \subseteq \wp(U)$ of sets, the *union* of $\mathcal{H}$, $\bigcup \mathcal{H}$, is defined by $a \in \bigcup \mathcal{H}$ if and only if $a \in X$ for some $X \in \mathcal{H}$. The *intersection* $\bigcap \mathcal{H}$ of $\mathcal{H}$ is defined by $a \in \bigcap \mathcal{H}$ if and only if $a \in X$ for all $X \in \mathcal{H}$. For any $X \subseteq U$, we denote by X^c the *complement* $U - X$ of the set X.

2.1 Lattices

The most notions presented in this subsection can be found, for example, in the books by Birkhoff [7], and Davey and Priestley [8]. A binary relation $\leq$ on a set P is called an *order*, if it is reflexive, antisymmetric, and transitive. An *ordered set* is a pair $(P, \leq)$, with P being a set and $\leq$ an order on P. Usually we denote an ordered set $(P, \leq)$ simply by P. The greatest element of P, if it exists, is called the *top element* of P and denoted by $\top$. Similarly, the least element of P, if such an element exists, is called the *bottom element* and it is denoted by $\bot$. If P has top and bottom elements, it is *bounded*.

Let P be an ordered set and $S \subseteq P$. We denote by $\bigvee S$ and $\bigwedge S$ the supremum and the infimum of S respectively. Furthermore, we write $a \vee b$ in place of $\bigvee\{a, b\}$ and $a \wedge b$ in place of $\bigwedge\{a, b\}$.

An ordered set L is a *lattice*, if for any two elements x and y in L, $x \vee y$ and $x \wedge y$ always exist. The operations $\vee$ and $\wedge$ are also called as *join* and *meet*, respectively. A lattice L is a *complete lattice* if $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq L$. Note that in a complete lattice L, the existence of $\bigvee L$ and $\bigwedge L$ guarantee the top element $\top$ and the bottom element $\bot$, respectively. It is known (see e.g. [8]) that P is a complete lattice if $\bigwedge S$ exists for all $S \subseteq P$. Clearly, $\bigvee S = \bigwedge \{x \in P \mid x \geq a \text{ for all } a \in S\}$.

For two complete lattices L and M, a mapping $f \colon L \to M$ is *join-preserving* if for all $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$. Similarly, f is *completely join-preserving* if for all $S \subseteq L$, $f(\bigvee S) = \bigvee f(S)$, where $f(S) = \{f(a) \mid a \in S\}$. The notions of *meet-preserving* and *completely meet-preserving* mappings are defined analogously. It is easy to see that if a mapping $f \colon L \to M$ is join- or meet-preserving, then f is *order-preserving*, that is, $a \leq b$ implies $f(a) \leq f(b)$ for all $a, b \in L$.

2.2 Topological Spaces

Here we briefly introduce topological spaces. A detailed discussion on topological spaces can be found, for example, in [18].

A *topological space* $(U, \mathcal{T})$ consists of a set U and a family $\mathcal{T} \subseteq \wp(U)$ such that

(T1) $\emptyset \in \mathcal{T}$ and $U \in \mathcal{T}$,

(T2) a finite intersection of members of $\mathcal{T}$ is in $\mathcal{T}$,

(T3) an arbitrary union of members of $\mathcal{T}$ is in $\mathcal{T}$.

The family $\mathcal{T}$ is called a *topology* on U and the members of $\mathcal{T}$ are called *open sets*. The complement of an open set is called a *closed set*. The family of closed sets is denoted by

$$\mathcal{L}_{\mathcal{T}} = \{X^c \mid X \in \mathcal{T}\}.$$

The *Kuratowski closure axioms* allow us to define a topology on U by means of an operator $C \colon \wp(U) \to \wp(U)$:

(K1) $X \subseteq C(X)$ for all $X \subseteq U$,

(K2) $C(C(X)) = C(X)$ for all $X \subseteq U$,

(K3) $C(X \cup Y) = C(X) \cup C(Y)$ for all $X, Y \subseteq U$,

(K4) $C(\emptyset) = \emptyset$.

The *closure* $C_{\mathcal{T}}(X)$ of a set $X \subseteq U$ in the topology $\mathcal{T}$ is defined to be the smallest closed set containing X. Obviously,

$$C_{\mathcal{T}}(X) = \bigcap \{Y \in \mathcal{L}_{\mathcal{T}} \mid X \subseteq Y\}.$$

It is well-known that the operator $C_{\mathcal{T}} \colon \wp(U) \to \wp(U), X \mapsto C_{\mathcal{T}}(X)$, satisfies the Kuratowski closure axioms. In addition, if an operator $C \colon \wp(U) \to \wp(U)$ satisfies (K1)–(K4), then the family $\{C(X)^c \mid X \subseteq U\}$ is a topology on U.

If $X \subseteq Y$ and $Y \in \mathcal{T}$, then Y is called a *neighbourhood of* X. Further, any neighbourhood of the singleton set $\{x\}$ is called a *neighbourhood of the point* x.

A topology $\mathcal{T}$ on U is called an *Alexandroff topology* if the intersection of every family of open sets is also open. It is obvious that each $X \subseteq U$ has the smallest neighbourhood in an Alexandroff topology [4]. Note that Alexandroff topologies and *principal topologies* (see e.g. [9, 26]) are the same concepts. It is now clear that Alexandroff topologies can be defined in different ways, for example, a topology $\mathcal{T}$ is called an Alexandroff topology if every point has the smallest neighbourhood (see e.g. [5, 26]), or a topology $\mathcal{T}$ is called an Alexandroff topology if it forms a *complete ring of sets* (see [6]). Interesting enough, Birkhoff called Alexandroff topologies also as *completely distributive topologies* in [6]. Indeed, it is well known that any topology can be considered as a complete lattice (see e.g. [7]), and clearly any Alexandroff topology $\mathcal{T}$ forms a complete lattice $(\mathcal{T}, \subseteq)$ such that $\bigvee \mathcal{H} = \bigcup \mathcal{H}$ and $\bigwedge \mathcal{H} = \bigcap \mathcal{H}$ for all $\mathcal{H} \subseteq \mathcal{T}$. This means that any Alexandroff topology forms a completely distributive lattice (see e.g. [8]).

A family of sets $\mathcal{B} \subseteq \mathcal{T}$ is called a *base for a topology* $\mathcal{T}$ if each member of $\mathcal{T}$ is the union of some members of $\mathcal{B}$. Moreover, it is known that a family of sets $\mathcal{S}$ is a base for some topology on $U = \bigcup \mathcal{S}$ if and only if for all $X, Y \in \mathcal{S}$ and for each point $x \in X \cap Y$, there is a $Z \in \mathcal{S}$ such that $x \in Z$ and $Z \subseteq X \cap Y$ [18].

Let $(U, \mathcal{T})$ be an Alexandroff topological space. We denote for any $X \subseteq U$ by $n(X) = \bigcap \{Y \in \mathcal{T} \mid X \subseteq Y\}$ the smallest neighbourhood of X. Similarly, let $n(x)$ denote the smallest neighbourhood of the point x. Now, consider any $X \in \mathcal{T}$. Then $x \in n(x)$ implies $X \subseteq \bigcup \{n(x) \mid x \in X\}$. For all $x \in X$, $n(x) \subseteq n(X) = X$, which gives $\bigcup \{n(x) \mid x \in X\} \subseteq X$. Thus, $\{n(x) \mid x \in U\}$ is a base for $\mathcal{T}$. Next we show that $\{n(x) \mid x \in U\}$ is the smallest base.

Let $\mathcal{B}$ be a base for $\mathcal{T}$, and suppose that $n(x) \notin \mathcal{B}$ for some x. Because $\mathcal{B}$ is a base, $\bigcup \mathcal{S} = n(x)$ for some $\mathcal{S} \subseteq \mathcal{B}$. Since $x \in n(x)$, there exists an $X \in \mathcal{S}$ such that $x \in X$. Now $X \subseteq n(x)$, but because $n(x)$ is the smallest open set containing x, we have that $X = n(x)$, that is, $n(x) \in \mathcal{B}$, a contradiction!

2.3 L-Sets

The fuzzy sets were defined originally by Zadeh in [29] as mappings from a non-empty set U into the unit interval $[0, 1]$. Goguen generalized fuzzy sets to L-sets in [11] such that an *L-set φ on U* is a mapping $\varphi : U \to L$. Usually L is at least a complete lattice.

It is well-known that the family of all L-sets on U may be ordered with the *pointwise order*:

$$\varphi \leq \psi \iff (\forall x \in U)\ \varphi(x) \leq \psi(x).$$

Because L is a complete lattice, also the family of all L-sets on U is a complete lattice in which arbitrary joins and arbitrary meets are defined pointwise. Note that if $\varphi(x) \in \{\bot, \top\}$ for all $x \in U$, then φ is the characteristic function for some conventional subset of U. Further, the pointwise ordered set of all $\{\bot, \top\}$-sets on U can be identified with $(\wp(U), \subseteq)$.

The *α-level set* of $\varphi\colon U \to L$ is defined for all $\alpha \in L$,

$$\varphi_\alpha = \{x \in U \mid \varphi(x) \geq \alpha\}.$$

It is clear that $\alpha \geq \beta$ implies $\varphi_\alpha \subseteq \varphi_\beta$ for all $\alpha, \beta \in L$.

Note that any collection of subsets of U may be considered as a knowledge about U (see e.g. [25]). Thus, the collection of α-level sets of φ is knowledge about U, and for each $\alpha \in L$, φ_α can be viewed as a certain piece of knowledge. It is now clear that L-sets can be employed to represent knowledge and this interpretation for L-sets is studied more detailed in Sections 3 and 4.

In the following, we present an example demonstrating a connection between L-sets and Alexandroff topologies.

Example 2.1. Let U be a non-empty set, L a complete lattice, and $\varphi\colon U \to L$. Then, the subfamily of α-level sets $\{\varphi_{\varphi(x)} \mid x \in U\}$ is a base for some topology. Indeed, now $U = \bigcup\{\varphi_{\varphi(x)} \mid x \in U\}$, because $x \in \varphi_{\varphi(x)}$ for all $x \in U$. If $x \in \varphi_{\varphi(y)} \cap \varphi_{\varphi(z)}$, then $\varphi(x) \geq \varphi(y)$ and $\varphi(x) \geq \varphi(z)$, which gives $\varphi_{\varphi(x)} \subseteq \varphi_{\varphi(y)}$ and $\varphi_{\varphi(x)} \subseteq \varphi_{\varphi(z)}$, that is, $\varphi_{\varphi(x)} \subseteq \varphi_{\varphi(y)} \cap \varphi_{\varphi(z)}$. Hence, $\{\varphi_{\varphi(x)} \mid x \in U\}$ is a base for some topology on U, say $\mathcal{T}_\varphi$.

Now every point $x \in U$ has the smallest neighbourhood $\varphi_{\varphi(x)}$, because $x \in \varphi_{\varphi(x)} \in \mathcal{T}_\varphi$, and $x \in X \in \mathcal{T}_\varphi$ implies $X = \bigcup\{\varphi_{\varphi(y)} \mid y \in S\}$ for some $S \subseteq U$. Thus, $x \in \varphi_{\varphi(y)}$ for some $y \in S$, which gives $\varphi(x) \geq \varphi(y)$ and $\varphi_{\varphi(x)} \subseteq \varphi_{\varphi(y)} \subseteq X$. Therefore, $(U, \mathcal{T}_\varphi)$ is an Alexandroff topological space and, in fact, $\{\varphi_{\varphi(x)} \mid x \in U\}$ is the smallest base for $\mathcal{T}_\varphi$.

Let us consider the family of α-level sets $\{\varphi_\alpha \mid \alpha \in L\}$. Now for each $\alpha \in L$, φ_α is an open set, because $x \in \varphi_\alpha$ implies $\varphi(x) \geq \alpha$ and $\varphi_{\varphi(x)} \subseteq \varphi_\alpha$. Hence, $\bigcup\{\varphi_{\varphi(x)} \mid x \in \varphi_\alpha\} \subseteq \varphi_\alpha$. On the other hand, $x \in \varphi_{\varphi(x)}$ implies $\varphi_\alpha \subseteq \bigcup\{\varphi_{\varphi(x)} \mid x \in \varphi_\alpha\}$. Thus, $\varphi_\alpha = \bigcup\{\varphi_{\varphi(x)} \mid x \in \varphi_\alpha\}$. Because $\{\varphi_{\varphi(x)} \mid x \in U\} \subseteq \{\varphi_\alpha \mid \alpha \in L\} \subseteq \mathcal{T}_\varphi$, also $\{\varphi_\alpha \mid \alpha \in L\}$ is a base for $\mathcal{T}_\varphi$. It is also easy to observe that for all $S \subseteq L$,

$$\bigcap_{\alpha \in S} \varphi_\alpha = \varphi_\beta,$$

where $\beta = \bigvee S$ (see [13, 23]). This means that $\{\varphi_\alpha \subseteq U \mid \alpha \in L\}$ is a base for $\mathcal{T}_\varphi$ which is actually a complete lattice with respect to $\subseteq$, while the join-operation may not be the union of sets (see Section 2.1). Thus, the base $\{\varphi_\alpha \subseteq U \mid \alpha \in L\}$ can be considered as a completion of the smallest base $\{\varphi_{\varphi(x)} \mid x \in U\}$ which is not necessarily a lattice.

2.4 Modal-Like Operators

By modal-like operators we mean operators which are syntactically similar to modal operators. In this subsection, we consider modal-like operators with two different interpretations: rough approximation operators and compositional modifiers. Both these interpretations for modal operators are binary relation based set-operations [17].

The Rough Set Theory introduced by Pawlak [24] deals with situations in which knowledge about objects of a certain universe of discourse U is limited by a binary relation, originally an equivalence relation. The idea is to define two operators ${}^{\blacktriangle}\colon \wp(U) \to \wp(U)$ and ${}^{\blacktriangledown}\colon \wp(U) \to \wp(U)$ that map any set $X \subseteq U$ to sets consisting of elements that possible and certainly belong in X, respectively.

Let R be an arbitrary binary relation on U, and let us denote for all $x \in U$,

$$R(x) = \{y \in U \mid x\,R\,y\}.$$

The *upper approximation* of X is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\} \tag{2.1}$$

and the *lower approximation* of $X \subseteq U$ is

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}. \tag{2.2}$$

The operators ${}^{\blacktriangle}\colon \wp(U) \to \wp(U)$ and ${}^{\blacktriangledown}\colon \wp(U) \to \wp(U)$ are called *rough approximation operators*. Also other type of definitions for rough approximation operators, similar to (2.1) and (2.2), can be found in the literature (see e.g. [9] for a recent survey). It is well-known that the operators ${}^{\blacktriangle}$ and ${}^{\blacktriangledown}$ are *dual*, that is, for all $X \subseteq U$,

$$X^{c\blacktriangle} = X^{\blacktriangledown c} \text{ and } X^{c\blacktriangledown} = X^{\blacktriangle c}.$$

Let us denote for any $\mathcal{H} \subseteq \wp(U)$, $\mathcal{H}^{\blacktriangle} = \{X^{\blacktriangle} \mid X \in \mathcal{H}\}$ and $\mathcal{H}^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \in \mathcal{H}\}$. The operator ${}^{\blacktriangle}\colon \wp(U) \to \wp(U)$ is completely union-preserving and the operator ${}^{\blacktriangledown}\colon \wp(U) \to \wp(U)$ is completely intersection-preserving in the sense of Section 2.1, that is, $\bigcup \mathcal{H}^{\blacktriangle} = (\bigcup \mathcal{H})^{\blacktriangle}$ and $\bigcap \mathcal{H}^{\blacktriangledown} = (\bigcap \mathcal{H})^{\blacktriangledown}$ for all $\mathcal{H} \subseteq \wp(U)$; further, $\emptyset^{\blacktriangle} = \emptyset$ and $U^{\blacktriangledown} = U$ (see [17]). Obviously, $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are complete lattices such that $(\wp(U)^{\blacktriangledown}, \subseteq) \cong (\wp(U)^{\blacktriangle}, \supseteq)$; the order-isomorphism is $X^{\blacktriangle} \mapsto X^{c\blacktriangledown}$. Note that if R is symmetric, then $(\wp(U)^{\blacktriangledown}, \subseteq) \cong (\wp(U)^{\blacktriangle}, \subseteq)$ (see e.g. [15]).

A relation R is said to be *serial* if for all $x \in U$ there exists $y \in U$ such that $x\,R\,y$. Obviously, reflexivity implies seriality. The following correspondences can be easily found by applying the Ackermann Lemma [1].

$$X^{\blacktriangledown} \subseteq X^{\blacktriangle} \iff R \text{ is serial.} \tag{2.3}$$

$$X^{\blacktriangledown} \subseteq X \iff X \subseteq X^{\blacktriangle} \iff R \text{ is reflexive.} \tag{2.4}$$

$$X^{\blacktriangledown\blacktriangle} \subseteq X \iff X \subseteq X^{\blacktriangle\blacktriangledown} \iff R \text{ is symmetric.} \tag{2.5}$$

$$X^{\blacktriangledown} \subseteq X^{\blacktriangledown\blacktriangledown} \iff X^{\blacktriangle\blacktriangle} \subseteq X^{\blacktriangle} \iff R \text{ is transitive.} \tag{2.6}$$

It is also known that if R is an equivalence (reflexive, symmetric, and transitive relation), then $X^{\blacktriangle\blacktriangledown} = X^{\blacktriangle}$ and $X^{\blacktriangledown\blacktriangle} = X^{\blacktriangledown}$ for any $X \subseteq U$; this gives $\wp(U)^{\blacktriangledown} = \wp(U)^{\blacktriangle}$.

Modifier operators, modifiers for short, interpreted as modal-like operators are studied by Mattila in [22]. Keeping this interpretation in mind, Kortelainen has defined compositional modifiers (see e.g. [19, 20]), and corresponding modified sets may be written as

$$X^{\triangle} = \{x \in U \mid R^{-1}(x) \cap X \neq \emptyset\}, \tag{2.7}$$

where $R^{-1} = \{(x, y) \mid y\, R\, x\}$ is the *inverse* of R, and

$$X^{\triangledown} = \{x \in U \mid R^{-1}(x) \subseteq X\}. \tag{2.8}$$

If R is reflexive, then $X^{\triangle}$ is called the *R-weakened set of $X \subseteq U$* and $X^{\triangledown}$ is called the *R-substantiated set of $X \subseteq U$*. The operators ${}^{\triangle} \colon \wp(U) \to \wp(U)$ and ${}^{\triangledown} \colon \wp(U) \to \wp(U)$ may be interpreted as linguistic hedges "more or less" and "very", respectively. Note that if R is reflexive, symmetric, or transitive, then R^{-1} is also reflexive, symmetric, or transitive, respectively. This implies that the operators ${}^{\triangle}$ and ${}^{\triangledown}$ have the same properties as ${}^{\blacktriangle}$ and ${}^{\blacktriangledown}$ presented above. However, condition (2.3) is now of the form

$$X^{\triangledown} \subseteq X^{\triangle} \iff R^{-1} \text{ is serial.}$$

To complete this subsection, we present some interesting syntactical connections between the rough approximation operators and the compositional modifiers.

For two ordered sets P and Q, a pair $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ of mappings ${}^{\blacktriangleright} \colon P \to Q$ and ${}^{\blacktriangleleft} \colon Q \to P$ is a *Galois connection between P and Q* if for all $p \in P$ and $q \in Q$,

$$p^{\blacktriangleright} \leq q \iff p \leq q^{\blacktriangleleft}.$$

It is well-known that the pairs $({}^{\triangle}, {}^{\blacktriangledown})$ and $({}^{\blacktriangle}, {}^{\triangledown})$ form Galois connections on $\wp(U)$. This implies, for example, that the mappings $X \mapsto X^{\triangle\blacktriangledown}$ and $X \mapsto X^{\blacktriangle\triangledown}$ are closure operators, and $X \mapsto X^{\triangledown\blacktriangle}$ and $X \mapsto X^{\blacktriangledown\triangle}$ are interior operators (cf. [9]). Furthermore,

$$X^{\blacktriangle} = X^{\blacktriangle\triangledown\blacktriangle} \quad \text{and} \quad X^{\triangledown} = X^{\triangledown\blacktriangle\triangledown}$$

for all $X \subseteq U$.

Because the pair $({}^{\triangle}, {}^{\blacktriangledown})$ is a Galois connection on $\wp(U)$, the map $X^{\triangle} \mapsto X^{\triangle\blacktriangledown}$ is an order-isomorphism between $(\wp(U)^{\triangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$. This implies

$$(\wp(U)^{\blacktriangledown}, \subseteq) \cong (\wp(U)^{\blacktriangle}, \supseteq) \cong (\wp(U)^{\triangle}, \subseteq) \cong (\wp(U)^{\triangledown}, \supseteq),$$

and, if R is symmetric, then

$$(\wp(U)^{\blacktriangledown}, \subseteq) = (\wp(U)^{\triangledown}, \subseteq) \cong (\wp(U)^{\blacktriangle}, \subseteq) = (\wp(U)^{\triangle}, \subseteq).$$

Note also that if R is an equivalence, then

$$\wp(U)^{\blacktriangledown} = \wp(U)^{\triangledown} = \wp(U)^{\blacktriangle} = \wp(U)^{\triangle}.$$

For any $x \in U$, $\{x\}^{\blacktriangle} = R^{-1}(x)$ and $\{x\}^{\triangle} = R(x)$. Hence, for all $X \subseteq U$ and for any arbitrary binary relation R on U,

$$X^{\triangle} = \bigcup_{x \in X} R(x) \quad \text{and} \quad X^{\blacktriangle} = \bigcup_{x \in X} R^{-1}(x).$$

Let P be a lattice with a smallest element $\bot$. Mappings ${}^{\triangleright} \colon P \to P$ and ${}^{\blacktriangleright} \colon P \to P$ are *conjugate* (cf. [17]), and $({}^{\triangleright}, {}^{\blacktriangleright})$ forms a conjugate pair, if for any $p, q \in P$,

$$p^{\triangleright} \wedge q = \bot \iff p \wedge q^{\blacktriangleright} = \bot.$$

It was proved in [17] that $({}^{\triangle}, {}^{\blacktriangle})$ is a conjugate pair of operators on $\wp(U)$.

3 L-Sets and Quasi-Orders

In [19] it was proved that the compositional modifier ${}^{\triangle} \colon \wp(U) \to \wp(U)$, determined by a quasi-order R (reflexive and transitive relation) satisfies the Kuratowski closure axioms.[1] Hence, the family $\{X^{\triangle c} \mid X \subseteq U\} = \{X^{c\triangledown} \mid X \subseteq U\} = \{X^{\triangledown} \mid X \subseteq U\}$ is a topology on U. Let us denote

$$\mathcal{T}_R = \{X^{\triangledown} \mid X \subseteq U\}.$$

Similarly, the operator ${}^{\blacktriangle} \colon \wp(U) \to \wp(U)$ satisfies the Kuratowski closure axioms, and the family $\{X^{\blacktriangledown} \mid X \subseteq U\}$ is also a topology on U. Further, the operators ${}^{\triangledown} \colon \wp(U) \to \wp(U)$ and ${}^{\blacktriangledown} \colon \wp(U) \to \wp(U)$ are interior operators.

It is known that

$$X^{\blacktriangle} = X^{\blacktriangle\triangledown} \quad \text{and} \quad X^{\triangledown} = X^{\triangledown\blacktriangle}, \tag{3.1}$$

as discussed in [16, 20]. This gives that $\{X^{\triangledown} \mid X \subseteq U\} = \{X^{\blacktriangle} \mid X \subseteq U\}$ and $\{X^{\blacktriangledown} \mid X \subseteq U\} = \{X^{\triangle} \mid X \subseteq U\}$. Hence, for instance, any $X \subseteq U$ has the upper approximation $X^{\blacktriangle}$ as its smallest neighbourhood in $\mathcal{T}_R$. This means that $\mathcal{T}_R$ is an Alexandroff topology, because each point x has the smallest neighbourhood $\{x\}^{\blacktriangle}$. Note that now

$$(\wp(U)^{\blacktriangledown}, \subseteq) = (\wp(U)^{\triangle}, \subseteq) \cong (\wp(U)^{\blacktriangle}, \supseteq) = (\wp(U)^{\triangledown}, \supseteq).$$

We have shown that each quasi-order induces an Alexandroff topology. Analogously, every Alexandroff topology $\mathcal{T}$ determines a quasi-order R defined by $x\,R\,y$ if and only if y belongs to every $\mathcal{T}$-open set including x. In fact, it is known that there exists an anti-isomorphism between the ordered sets of all quasi-orders and Alexandroff topologies on U [6, 26].

[1] Essentially, this result is already in [6] and [17].

Let $\varphi\colon U \to L$ be an L-set. A binary relation $\lesssim$ on U is defined by setting for all $x, y \in U$,

$$x \lesssim y \iff \varphi(x) \geq \varphi(y), \tag{3.2}$$

(see e.g. [20]). It is easy to see that $\lesssim$ is a quasi-order. Thus, the family

$$\mathcal{T}_{\lesssim} = \{X^{\triangledown} \mid X \subseteq U\} = \{X^{\blacktriangle} \mid X \subseteq U\} \tag{3.3}$$

is an Alexandroff topology on U, where $^{\blacktriangle}$ and $^{\triangledown}$ are determined by $\lesssim$. This topology has obviously the family $\{\{x\}^{\blacktriangle} \mid x \in U\}$ as its base (cf. [20]), since for all $X \in \mathcal{T}_{\lesssim}$, we have $X = X^{\blacktriangle} = \bigcup \{\{x\}^{\blacktriangle} \mid x \in X\}$. We point out that we write sometimes $\lesssim_{\varphi}$ instead of $\lesssim$, defined in formula (3.2), to avoid confusion.

Because $\varphi_{\varphi(x)} = \{y \in U \mid \varphi(y) \geq \varphi(x)\} = \{y \in U \mid y \lesssim x\} = \{x\}^{\blacktriangle}$, we have that the Alexandroff topology $\mathcal{T}_{\varphi}$ created in Example 2.1 and the Alexandroff topology $\mathcal{T}_{\lesssim}$ in formula (3.3), coincide. We may now write,

$$\mathcal{T}_{\varphi} = \{X^{\triangledown} \mid X \subseteq U\}.$$

We have now noted that in $\mathcal{T}_{\varphi}$, both $\{x\}^{\blacktriangle}$ and $\varphi_{\varphi(x)}$ are the smallest neighbourhoods of the point x. Hence, the equal families $\{\varphi_{\varphi(x)} \mid x \in U\}$ and $\{\{x\}^{\blacktriangle} \mid x \in U\}$ are the smallest bases for $\mathcal{T}_{\varphi}$.

Šešelja and Tepavčević have presented representations and completions of ordered structures using Fuzzy Sets in [27, 28]. The following proposition is worth of presenting by means of quasi-orders also in the current paper, although the proposition is obvious by the preceding discussion and the work in [27].

Proposition 3.1. *For any quasi-order R on U, there exists a completely distributive lattice L_R and an L_R-set $\rho\colon U \to L_R$ such that R is equal to $\lesssim$. Moreover, for all $x \in U$, $\rho(x) = \rho_{\rho(x)}$.*

Proof. Let us denote $(\mathcal{T}_R, \supseteq)$ by $(L_R, \leq)$. Obviously, L_R is a completely distributive lattice. Let us determine an L_R-set $\rho\colon U \to L_R$ by setting

$$\rho(x) = \{x\}^{\blacktriangle} \quad (\in \mathcal{T}_R)$$

Next we show that R is equal to $\lesssim$. If $x\,R\,y$, then $x \in \{y\}^{\blacktriangle}$ and $\{x\}^{\blacktriangle} \subseteq \{y\}^{\blacktriangle}$, that is, $\rho(y) \leq \rho(x)$ in L_R, and so $x \lesssim y$. On the other hand, $x \lesssim y$ implies $\rho(y) \leq \rho(x)$ in L_R, and $x \in \{x\}^{\blacktriangle} \subseteq \{y\}^{\blacktriangle}$, that is, $x\,R\,y$.

Further, $\rho_{\rho(x)} = \{x\}^{\blacktriangle} = \rho(x)$ for all $x \in U$. □

As we have mentioned, for any set U, the correspondence between Alexandroff topologies on U and quasi-orders on U is bijective. We also know that for any complete lattice L, each L-set $\varphi\colon U \to L$ induces a quasi-order $\lesssim_{\varphi}$ and an Alexandroff topology $\mathcal{T}_{\varphi}$. On the other hand, Proposition 3.1 shows that each quasi-order R (and hence every Alexandroff topology) determines a completely distributive lattice L_R and an L_R-set $\rho\colon U \to L_R, x \mapsto \{x\}^{\blacktriangle}$, such that $\lesssim_{\rho}$ is equal to R.

Note that since each L-set φ determines a completely distributive lattice $(\mathcal{T}_\varphi, \supseteq)$, we have a machinery for each complete lattice to attach a completely distributive lattice. Further, for any L-set $\varphi\colon U \to L$, only the images $\varphi(x)$ are important in the following sense: They determine the quasi-order $\lesssim$ and the family $\{\varphi_{\varphi(x)} \mid x \in U\}$ which is the smallest base for the topology $\mathcal{T}_\varphi$. Therefore, the mapping $x \mapsto \varphi_{\varphi(x)}$ can be identified as a *canonical representation* of φ suggested in [27].

Example 3.2. Let $U = \{1, 2, 3\}$ and let L be a complete (non-distributive) lattice depicted in Figure 1.

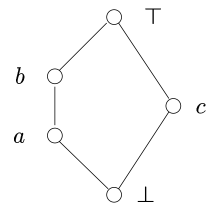

Figure 1: The lattice L.

Let us define an L-set $\varphi\colon U \to L$ by $\varphi(1) = a$, $\varphi(2) = b$, and $\varphi(3) = c$. For all $x \in U$, $\{x\}^\blacktriangle = \{y \mid y \lesssim x\} = \{y \mid \varphi(x) \leq \varphi(y)\}$. Hence, $\{1\}^\blacktriangle = \{1, 2\}$, $\{2\}^\blacktriangle = \{2\}$, and $\{3\}^\blacktriangle = \{3\}$. The Alexandroff topology $\mathcal{T}_\varphi$ is illustrated in Figure 2; its smallest base $\{\varphi_{\varphi(x)} \mid x \in U\}$ is marked with filled circles.

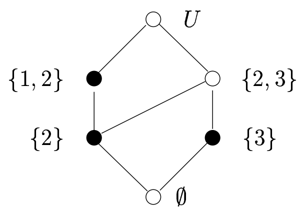

Figure 2: The topology $\mathcal{T}_\varphi$.

For equivalence relations, we may write the following proposition as a special case of Proposition 3.1.

Proposition 3.3. *For any equivalence E on U, there exists a complete atomic Boolean lattice L_E and an L_E-set $\varepsilon\colon U \to L_E$ such that E is equal to $\lesssim$. Moreover, for all $x \in U$, $\varepsilon(x) = \varepsilon_{\varepsilon(x)}$.*

Proof. It is obvious that $(\mathcal{T}_E, \subseteq) \cong (\mathcal{T}_E, \supseteq) = (L_E, \leq)$ is a complete Boolean lattice, since $X \in \mathcal{T}_E$ means that $X = X^\blacktriangle$ and so $X^c = X^{\blacktriangle c} = X^{c\blacktriangledown} = X^{c\blacktriangledown\blacktriangle} \in \mathcal{T}_E$.

The set of atoms of $(\mathcal{T}_E, \subseteq)$ is $\{\{x\}^\blacktriangle \mid x \in U\}$, since for all $X \in \mathcal{T}_E$, $\emptyset \subset X \subseteq \{x\}^\blacktriangle$ implies that for any $y \in X$, $y \in \{x\}^\blacktriangle$. Thus, $x \in \{y\}^\blacktriangle$ for all

$y \in X$ and $x \in \bigcup_{y \in X} \{y\}^{\blacktriangle} = X^{\blacktriangle} = X$ and $\{x\}^{\blacktriangle} \subseteq X^{\blacktriangle} = X$. Thus, L_E is also atomic, because $(\mathcal{T}_E, \subseteq)$ is atomic. □

It is now clear that if R and S are quasi-orders on U such that $R \subseteq S$, then $\mathcal{T}_S \subseteq \mathcal{T}_R$ and $L_S \subseteq L_R$. Let ρ and σ be the corresponding L_R- and L_S-sets on U in the sense of Proposition 3.1, and $R \subseteq S$. Obviously, it is possible to identify σ with some L_R-set on U, because images of σ are members of L_R. Interpreting σ as an L_R-set on U and comparing L_R-sets by the pointwise order induced by L_R, the following lemma can be presented.

Lemma 3.4. *Let R and S be quasi-orders on U. Then, $R \subseteq S$ if and only if $\sigma \leq \rho$ with respect to the pointwise order induced by L_R.*

Proof. $R \subseteq S$ implies $\rho(x) \subseteq \sigma(x)$ for all $x \in U$, because $\rho(x)$ and $\sigma(x)$ are the smallest neighbourhoods of the point x in the Alexandroff topologies $\mathcal{T}_R$ and $\mathcal{T}_S$, respectively, and $\mathcal{T}_S \subseteq \mathcal{T}_R$. This means $\sigma(x) \leq \rho(x)$ for all $x \in U$ in L_R.

On the other hand, if $\sigma \leq \rho$ with respect to the pointwise order induced by L_R, then $R^{-1}(x) = \rho(x) \subseteq \sigma(x) = S^{-1}(x)$ for all $x \in U$, that is, $R \subseteq S$. □

We can also compare L-sets on U by applying the notion of "fineness" studied earlier in [20]. We may define a quasi-order $\preceq$ on the set of all L-sets on U by setting

$$\varphi \preceq \psi \iff \lesssim_\varphi \text{ is included in } \lesssim_\psi. \tag{3.4}$$

The following corollary is now obvious by Lemma 3.4.

Corollary 3.5. Let φ and ψ be L-sets on U, R be equal to $\lesssim_\varphi$, and S be equal to $\lesssim_\psi$. Then, $\varphi \preceq \psi$ if and only if $\sigma \leq \rho$ with respect to the pointwise order induced by L_R.

We end this section by presenting a result from [20] that shows how modal-like operators can be determined by means of an L-set φ on U directly without applying the corresponding quasi-order $\lesssim$.

Lemma 3.6. *Let $\varphi\colon U \to L$ be an L-set on U, and assume that $^{\triangledown}$ and $^{\blacktriangle}$ are determined by $\lesssim$. Then, for all $X \subseteq U$,*

$$X^{\triangledown} = \bigcup \left\{ \varphi_{\varphi(x)} \mid \varphi_{\varphi(x)} \subseteq X \right\},$$
$$X^{\blacktriangle} = \bigcup \left\{ \varphi_{\varphi(x)} \mid x \in X \right\}.$$

Note that now $(\mathcal{T}_\varphi, \subseteq)$ is a completely distributive lattice and $\left\{ \varphi_{\varphi(x)} \mid x \in U \right\}$ is the smallest join-dense set of $(\mathcal{T}_\varphi, \subseteq)$. This hints that we may generalize modal-like operators by considering completely distributive lattices and the smallest join-dense sets (cf. [15]).

4 Some Categorical Considerations

This section is reserved for categorical considerations. In Section 4.1 we present some categorical notions and notation. If the reader is familiar with Category Theory, then Section 4.1 may be omitted. In Section 4.2, we interpret the study, presented especially in Section 3, by means of Category Theory. In fact, we enhance some results, given in [20], on connections between topological spaces, quasi-ordered sets and L-sets.

4.1 Basic Notions and Notation

The authors think that it is useful to recall the most notions used in Section 4.2. The notions and results presented in this subsection can be found in or deduced from [2]. Moreover, much of the written language and many notations are borrowed from [2].

At first, *classes* are collections of objects. *Large classes*, called also as *proper classes*, are distinguished from *small classes*, called *sets*. We refer [2] for more detailed discussion on the underlying set theory.

A *category* is a quadruple $\mathtt{A} = (Ob, \hom, id, \circ)$ such that

- Ob is the class of $\mathtt{A}$-*objects*, and we also write $Ob(\mathtt{A})$ when emphasizing the category $\mathtt{A}$.

- For any pair of $\mathtt{A}$-objects (A, B) we can form a set $\hom(A, B)$, and a member f of $\hom(A, B)$ is called an $\mathtt{A}$-*morphism* from A to B. When emphasizing the category $\mathtt{A}$, we write also $\hom_{\mathtt{A}}(A, B)$. Morphisms are usually denoted by

  $$A \xrightarrow{f} B.$$

  For each $\mathtt{A}$-object A, a morphism $A \xrightarrow{id} A$ is the $\mathtt{A}$-identity on A. When emphasizing the object A, we write also id_A.

- The *composite of the* $\mathtt{A}$-*morphisms* $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ is also an $\mathtt{A}$-morphism denoted by

  $$A \xrightarrow{g \circ f} C.$$

  The composition of morphisms is associative, and for $\mathtt{A}$-morphisms $A \xrightarrow{f} B$ we have $id_B \circ f = f$ and $f \circ id_A = f$.

- The sets $\hom_{\mathtt{A}}(A, B)$ are pairwise disjoint.

Some familiar categories are needed in this paper: The category `Set`, all sets as its objects and all mappings between sets as its morphisms. The category `Prost`, all quasi-ordered (preordered)[2] sets as its objects and all order-preserving mappings as its morphisms. The category `CLat`, all complete

[2]Some authors call quasi-orders as preorders (see e.g. [2]).

lattices as its objects and all join-preserving and meet-preserving mappings as its morphisms.

A category $\mathtt{A}$ is a *subcategory* of a category $\mathtt{B}$ if $Ob(\mathtt{A}) \subseteq Ob(\mathtt{B})$ and for all $A, A' \in Ob(\mathtt{A})$, $\hom_{\mathtt{A}}(A, A') \subseteq \hom_{\mathtt{B}}(A, A')$. Moreover, $\mathtt{A}$ inherits identity morphisms from $\mathtt{B}$, and the composities in $\mathtt{A}$ are restrictions from the composities in $\mathtt{B}$. A subcategory $\mathtt{A}$ of a category $\mathtt{B}$ is *full* if for all $A, A' \in Ob(\mathtt{A})$, $\hom_{\mathtt{A}}(A, A') = \hom_{\mathtt{B}}(A, A')$.

An $\mathtt{A}$-morphism $A \xrightarrow{f} B$ is called an *isomorphism* if there exists an $\mathtt{A}$-morphism $B \xrightarrow{g} A$ with $g \circ f = id_A$ and $f \circ g = id_B$. In this case A is said to be isomorphic to B. Notice that g is also called an *inverse* of f, denoted by $g = f^{-1}$. Clearly, f^{-1} is an isomorphism if f is an isomorphism. Moreover, the composite of isomorphisms is also an isomorphism. A full subcategory $\mathtt{A}$ of a category $\mathtt{B}$ is called *isomorphism-dense* if each $\mathtt{B}$-object is isomorphic to some $\mathtt{A}$-object, and in this special case $\mathtt{A}$ and $\mathtt{B}$ are *equivalent* categories.

Consider $\mathtt{A}$ is a subcategory of $\mathtt{B}$ and B is a $\mathtt{B}$-object. An $\mathtt{A}$-*reflection* for B is a morphism $B \xrightarrow{g} A$ from B to an $\mathtt{A}$-object A if the following holds: for any morphism $B \xrightarrow{f} A'$ from B to an $\mathtt{A}$-object A', there exists a unique $\mathtt{A}$-morphism $A \xrightarrow{f'} A'$ such that the triangle in Figure 3 commutes, that is, $f = f' \circ g$. If each $\mathtt{B}$-object has an $\mathtt{A}$-reflection then $\mathtt{A}$ is called a *reflective subcategory* of $\mathtt{B}$.

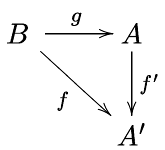

Figure 3: A reflective subcategory.

A (covariant) *functor* F from $\mathtt{A}$ to $\mathtt{B}$, denoted by $F\colon \mathtt{A} \to \mathtt{B}$, is a mapping that assigns to each $A \in Ob(\mathtt{A})$ a $\mathtt{B}$-object $F(A)$, and to each $\mathtt{A}$-morphism $A \xrightarrow{f} A'$ a $\mathtt{B}$-morphism

$$F(A) \xrightarrow{F(f)} F(A').$$

Functors should preserve composities and identity morphisms, that is, $F(f \circ g) = F(f) \circ F(g)$ when $f \circ g$ is defined, and for all $A \in Ob(\mathtt{A})$, $F(id_A) = id_{F(A)}$. For any category $\mathtt{A}$ there exists the identity functor $id_{\mathtt{A}}$ such that $id_{\mathtt{A}}(A \xrightarrow{f} A') = A \xrightarrow{f} A'$. Moreover, the composition of functors is defined canonically.

A functor $F\colon \mathtt{A} \to \mathtt{B}$ is *faithful* if for any two $\mathtt{A}$-objects A and A', the restriction $F\colon \hom_{\mathtt{A}}(A, A') \to \hom_{\mathtt{B}}(F(A), F(A'))$ is injective. A functor $F\colon \mathtt{A} \to \mathtt{B}$ is an *isomorphism* if there exists a functor $G\colon \mathtt{B} \to \mathtt{A}$ such that $G \circ F = id_{\mathtt{A}}$ and $F \circ G = id_{\mathtt{B}}$. In this case G is denoted by F^{-1}.

A *concrete category over* $\mathtt{B}$ is a pair $(\mathtt{A}, F)$, where $\mathtt{A}$ and $\mathtt{B}$ are categories and $F : \mathtt{A} \to \mathtt{B}$ is a faithful functor. Concrete categories over $\mathtt{Set}$ are called *constructs*, and it should not be confusing to write simply $\mathtt{A}$ meaning the construct $(\mathtt{A}, F)$, when F is known. Indeed, if A is an $\mathtt{A}$-object and $(\mathtt{A}, F)$ is a construct, then we usually demand that $F(A) = U$ is the underlying set for A and $F(f)$ is the underlying mapping for morphisms, which should be clear from the context. In the sequel, we give notions only for constructs while originally in [2] they may be defined in a more general setting.

Let $(\mathtt{A}, F)$ and $(\mathtt{B}, G)$ be constructs. A functor $C\colon (\mathtt{A}, F) \to (\mathtt{B}, G)$ is a *concrete isomorphism* if C is an isomorphism and $F = G \circ C$. Notice that if C is a concrete isomorphism then C^{-1} is a concrete isomorphism. Moreover, the composite of concrete isomorphisms is a concrete isomorphism.

Let $(\mathtt{A}, F)$ be a construct, A and A' be $\mathtt{A}$-objects such that $F(A) = U$. An $\mathtt{A}$-morphism $A \xrightarrow{f} A'$ is called *identity carried* if $F(f) = id_U$. Notice that F is injective on $\hom_{\mathtt{A}}(A, A')$, so it should not be confusing to say that $id_U\colon F(A) \to F(A')$ is an $\mathtt{A}$-morphism, when we mean actually the identity carried $\mathtt{A}$-morphism $A \xrightarrow{f} A'$.

Let $(\mathtt{B}, G)$ be a construct and $(\mathtt{A}, F)$ a subconstruct of $(\mathtt{B}, G)$. This means that $\mathtt{A}$ is a subcategory of $\mathtt{B}$, and $E : \mathtt{A} \hookrightarrow \mathtt{B}$ is the naturally associated *inclusion* functor, thus, E is injective on morphisms in a natural way, such that $F = G \circ E$. The subconstruct $(\mathtt{A}, G \circ E)$ is a *reflective modification* of $(\mathtt{B}, G)$ if each $\mathtt{B}$-object B has an identity carried $\mathtt{A}$-reflection.

Let $(\mathtt{A}, F)$ be a construct. The $\mathtt{A}$-*fibre* of U is a quasi-ordered class of all $\mathtt{A}$-objects A with $F(A) = U$, and the quasi-order $\leq$ on the $\mathtt{A}$-fibre of U is defined by

$$A \leq A' \iff id_U\colon F(A) \to F(A') \text{ is an } \mathtt{A}\text{-morphism.}$$

A construct $(\mathtt{A}, F)$ is *fibre-small* if all of its fibres are small classes, that is, sets. Moreover, a construct $(\mathtt{A}, F)$ is *fibre-complete* if all of its fibres are complete lattices.

4.2 Categorical Notes

In this subsection we study connections between L-sets, Alexandroff topological spaces, quasi-ordered sets, and L_R-sets, where $(L_R, \leq)$ is a completely distributive lattice introduced in Section 3.

On categorical point of view, we may consider a category for L-sets, denoted by L-$\mathtt{Set}$, such that its objects are all pairs (U, φ), where φ is an L-set on U and L is a fixed completely distributive lattice. Morphisms $(U, \varphi) \xrightarrow{f} (V, \psi)$ are all mappings $f\colon U \to V$ for which $\varphi \leq \psi \circ f$, and composition of morphisms is the composition of mappings (see [10, 12]).

It may be useful to consider also a category of *variable basis* L-sets. Using our notations, we loosely follow Eklund in [10] such that the category $\mathtt{SetCdl}$ has triples (L, U, φ) as its objects with L a completely distributive lattice and φ an L-set on U. Morphisms of $\mathtt{SetCdl}$ are $(L, U, \varphi) \xrightarrow{(g,f)} (M, V, \psi)$, where $g\colon L \leftarrow M$ is a completely meet- and completely join-preserving mapping, and $f\colon U \to V$ is such that for all $\alpha \in M$, $x \in U$,

$$g(\alpha) \leq \varphi(x) \implies \alpha \leq (\psi \circ f)(x). \tag{4.1}$$

The composition of morphisms is defined canonically. For more detailed discussion on categories for variable basis L-sets we refer [10], also.

Now, if in formula (4.1) we demand

$$g \circ (\psi \circ f) = \varphi, \tag{4.2}$$

then we have for all $x, y \in U$,

$$\varphi(y) \leq \varphi(x) \implies (\psi \circ f)(y) \leq (\psi \circ f)(x). \tag{4.3}$$

The insight of the work [10] will give us a suitable generalization for the notion "fineness" discussed in Section 3 and, for example, in [20]. The following definition is then given.

Definition 4.1. Let U, V be non-empty sets, L and M complete lattices, φ an L-set on U, and ψ an M-set on V. A mapping $f\colon U \to V$ is called *membership order-preserving*, if for all $x, y \in U$,

$$\varphi(y) \leq \varphi(x) \implies (\psi \circ f)(y) \leq (\psi \circ f)(x). \tag{4.4}$$

Moreover, if $f = id_U$ then φ is called *finer* than ψ, or equivalently ψ is called *coarser* than φ, and we denote this by $\varphi \preceq \psi$. If we need to emphasize the basis lattices and the underlying sets, we may also write $(L, U, \varphi) \preceq (M, U, \psi)$.

Notice that if $L = M$ in Definition 4.1 and L^U denotes the family of all L-sets on U, then we can determine $(L^U, \preceq)$, which is a quasi-ordered set. Thus, Definition 4.1 gives formula (3.4) as a special case.

Clearly, the mapping f in formula (4.3) is membership order-preserving. We also point out that there may be such membership order-preserving mappings, which do not satisfy equation (4.2) for any g. Indeed, the next example demonstrates this situation.

Example 4.2. Let $U = V = \{1, 2, 3\}$. Further, let the complete lattices L and M, and the L-set φ on U and the M-set ψ on V be depicted in Figure 4.

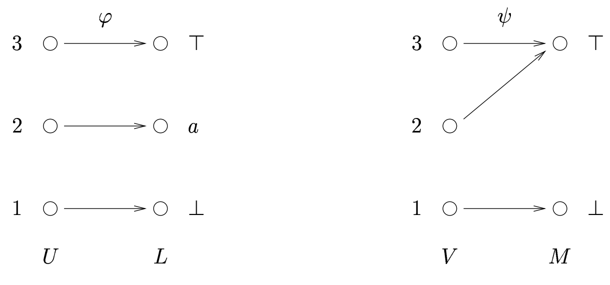

Figure 4: The L-set φ on U and the M-set ψ on V.

Let us define for all $x \in U$, $f(x) = x$. It is clear that for all $x, y \in U$,

$$\varphi(y) \leq \varphi(x) \implies (\psi \circ f)(y) \leq (\psi \circ f)(x),$$

thus, f is membership order-preserving. Moreover, because $f = id_U$ we have $\varphi \preceq \psi$, but there do not exist any $g\colon L \leftarrow M$ such that equation (4.2) would be satisfied.

Example 4.2 motivates us to define another category for variable basis L-sets.

Definition 4.3. The category CLSet has triples (L, U, φ) as its objects, where L is a complete lattice and φ is an L-set on U. The morphisms $(L, U, \varphi) \xrightarrow{f} (M, V, \psi)$ are all membership order-preserving mappings $f\colon U \to V$. The composition of morphisms is the composition of mappings.

In the sequel, it is reasonable to consider $(\mathtt{CLSet}, F)$ as a construct such that for all $(L, U, \varphi) \in Ob(\mathtt{CLSet})$, $F((L, U, \varphi)) = U$ and for each CLSet-morphism we have naturally $F(f)$ as the underlying mapping. Indeed, we have earlier pointed out that L-sets may be used to represent knowledge, also. In this sense U is understood to be endowed by an L-set φ. An illustrative example is presented for instance in [21].

Recall that if R is a quasi-order on U, then the corresponding L_R-set ρ on U may be now denoted by the triple (L_R, U, ρ), where for all $x \in U$, $\rho(x) = \{x\}^{\blacktriangle}$. Notice that L_R and ρ are unique in the sense of the discussion in Section 3. We can now consider the category RSet, which is a full subcategory of CLSet with objects of the form (L_R, U, ρ). It is worth noticing again that for any L_R there exists only one RSet-object, namely (L_R, U, ρ).

At first, we show that the fineness relation has a special categorical interpretation. The following is given:

Proposition 4.4. *All* CLSet*-fibres are endowed by the fineness relation* $\preceq$.

Proof. Let (L, U, φ) and (M, U, ψ) be objects of CLSet, and let us define the CLSet-fibre of U by assigning

$$(L, U, \varphi) \leq (M, U, \psi) \iff id_U\colon F((L, U, \varphi)) \to F((M, U, \psi)), \quad id_U \text{ is membership order-preserving.}$$

By Definition 4.1 we can replace $\leq$ by $\preceq$. □

Notice that each CLSet-fibre is a proper class. Indeed, let $U = \{1\}$, and for any $L \in Ob(\mathtt{CLat})$ let us determine a special L-set, say τ_L on U, such that $\tau_L(1) = \top$. Clearly, the class of objects $(L, \{1\}, \tau_L)$ form a proper class, because $Ob(\mathtt{CLat})$ is a proper class. It is also obvious that the CLSet-fibres are usually (only) quasi-ordered classes. Thus, the category CLSet is not fibre-complete.

In the following, we present some relationships between the categories CLSet and RSet.

Proposition 4.5. *The categories* RSet *and* CLSet *are equivalent categories.*

Proof. At first, we point out that RSet is a full subcategory of CLSet. It is clear, also by the study in [27], that each CLSet-object (L, U, φ) is isomorphic to an RSet-object (L_R, U, ρ) by choosing R being equal to $\lesssim_\varphi$ and choosing the identity carried morphism $(L, U, \varphi) \xrightarrow{g} (L_R, U, \rho)$. Clearly g is an isomorphism by Definition 4.1. Hence, RSet is isomorphism-dense, which means that RSet and CLSet are equivalent categories. □

Now, the following proposition is immediate by the proof of Proposition 4.5, while we still give here a detailed proof.

Proposition 4.6. *The category* RSet *is a reflective subcategory of* CLSet*.*

Proof. Notice that RSet is a full subcategory of CLSet. Let (L, U, φ) be an arbitrary CLSet-object, and let us choose an RSet-object (L_R, U, ρ), where R is equal to $\lesssim_\varphi$. We also choose the identity carried morphism $(L, U, \varphi) \xrightarrow{g} (L_R, U, \rho)$. Now, for any morphism $(L, U, \varphi) \xrightarrow{f} (L_S, V, \sigma)$, where (L_S, V, σ) is an RSet-object, we have a unique morphism $(L_R, U, \rho) \xrightarrow{f'} (L_S, V, \sigma)$ such that the triangle in Figure 5 commutes. In fact, f and f' must be equal underlying mappings. Therefore, g is an RSet-reflection for (L, U, φ). □

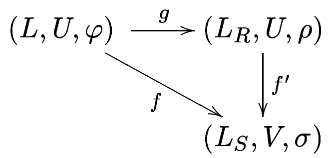

Figure 5: The category RSet is a reflective subcategory of CLSet.

Because the morphism

$$(L, U, \varphi) \xrightarrow{g} (L_R, U, \rho), \tag{4.5}$$

in the proof of Proposition 4.6 is identity carried, the category RSet is a reflective modification of CLSet.

Recall that the category CLSet is not fibre-small nor fibre-complete. Moreover, it is known that the category of all Alexandroff topological spaces with all continuous mappings (see also [3, 5]), denoted in the current paper by Alex, and Prost are isomorphic categories (see e.g. [3]). It is interesting to find out that RSet is a fibre-small and fibre-complete construct, which is isomorphic to Alex, for example. Indeed, we have the following:

Proposition 4.7. *The categories* Alex *and* RSet *are isomorphic categories. Furthermore,* RSet *is fibre-small and fibre-complete.*

Proof. Since Alex and Prost are isomorphic, there exists an isomorphism, say F, between Alex and Prost. At first, we show that there is a concrete isomorphism between Prost and RSet.

Let us define a functor $C\colon \texttt{Prost} \to \texttt{RSet}$ such that

$$C\Big((U,R) \xrightarrow{f} (V,S)\Big) = (L_R, U, \rho) \xrightarrow{g} (L_S, V, \sigma),$$

where f and g are equal underlying mappings. Clearly, C is bijective on objects, because R is equal to $\lesssim_\rho$ and S is equal to $\lesssim_\sigma$. Moreover, f is order-preserving if and only if f is membership order-preserving. Indeed, for all $x, y \in U$,

$$x\,R\,y \implies f(x)\,S\,f(y)$$

if and only if for all $x, y \in U$,

$$\rho(y) \leq \rho(x) \implies (\sigma \circ f)(y) \leq (\sigma \circ f)(x),$$

by Definition 4.1 and the discussion in Section 3. Obviously, the functor C is a concrete isomorphism, and the triangle in Figure 6 commutes. Hence,

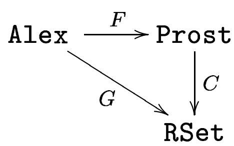

Figure 6: Isomorphisms between categories.

$G = C \circ F$ is an isomorphism. Moreover, it is known that `Prost` is fibre-small and fibre-complete. Actually, we can now completely substitute each `Prost`-object by an `RSet`-object keeping the same underlying mappings (cf. [2]). Hence, the order for the `Prost`-fibres can be substituted by the relation $\preceq$ for the `RSet`-fibres. Each `RSet`-fibre must now be a set and a complete lattice. $\square$

We end this discussion by the following: the category `RSet` might be a natural choice for modelling (only) order based fuzziness, because `RSet` and `CLSet` are equivalent categories, but `RSet` is also fibre-complete (see Remark 5.34 in [2], also).

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