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Positivity of Second Order Linear Recurrent Sequences

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Abstract

We prove that the following problem, which is called the Positivity Problem in the literature, is algorithmically solvable: Given a sequence $(u_n)_{n=0}^{\infty}$ of integers satisfying a linear second order recurrence relation, determine whether or not u_n is non-negative for all n. This problem has connection to other fields of mathematics through the theory of matrices.

Keywords: integer sequence; second order; homogeneous linear recurrence relation; positivity

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1. Introduction. In general, a (homogeneous) *linear recurrence relation* over integers has the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}, \tag{1}$$

where $a_1, a_2, \ldots, a_k$ are fixed integers for $n \geq k$. Such a recurrence equation defines in a unique fashion an integer sequences $(u_n)_{n=0}^{\infty}$ when the first k initial terms $u_0, u_1, \ldots, u_{k-1}$ are given. A sequence $(u_n)_{n=0}^{\infty}$ is said to be *recurrent* if it is defined by a recurrence equation. The integer k in the relation (1) is called the *order* of the recurrence and also of the defined recurrent sequence.

In the following we are mainly interested in second order recurrence sequences. We shall write (1) in the form

$$u_n = a u_{n-1} + b u_{n-2}, \tag{2}$$

where $a, b \in \mathbb{Z}$. Fibonacci numbers form probably the most well known recurrent sequence of second order. The recurrence relation for the Fibonacci numbers is given by $u_n = u_{n-1} + u_{n-2}$. The initial terms are $u_0 = u_1 = 1$. Note that, in general, we shall not assume that the coefficients in (2) are positive.

We shall consider the following problem of recurrent sequences.

Positivity Problem. Let a recurrence relation (1) be given together with the initial terms u_i for $i = 0, 1, \ldots, k-1$. Is the defined sequence $(u_n)_{n=0}^{\infty}$ *positive,* i.e., does it hold that $u_n \geq 0$ for all n?

We shall show that the positivity problem is algorithmically solvable for the second order linear recurrence equations. Our solution methods are elementary and they use only the basic properties of complex numbers.

As an example, consider the linear recurrent sequence $(u_n)_{n=0}^{\infty}$, where

$$u_n = 9u_{n-1} - 21u_{n-2},$$

$u_0 = 1$ and $u_1 = 14$. The first negative member of this sequence is $u_{17} = -344532183345$. Before that the maximum value is $u_{15} = 17954992251$.

The Positivity Problem is well known also in the theory of integer matrices, and through matrix theory it has connections to many other fields of mathematics (cf. [8, 9]). Indeed, it is straightforward to show that a sequence $(u_n)_{n=1}^{\infty}$ defined for a $k \times k$ integer matrix M a sequence $(u_n)_{n=1}^{\infty}$ defined by

$$u_n = \mathbf{v} M^n \mathbf{u}^T$$

where $\mathbf{v} = (1, 0, \ldots, 0) \in \mathbb{Z}^k$ and $\mathbf{u} = (0, \ldots, 0, 1) \in \mathbb{Z}^k$, is a linear recurrent sequence of order k. Therefore if the Positivity Problem for second order recurrence equations can be algorithmically solved also the problem asking whether or not, for a 2×2 integer matrix M all powers M^n have a non-negative right upper corner element $(M^n)_{1,2}$, can be solved. Let us mention that the above connection between matrices and linear recurrent sequences works also to the other direction, but then the dimension of the matrix may grow. Indeed, for any linear recurrent sequence $(u_n)_{n=0}^{\infty}$ of order k, there exists a matrix $M \in \mathbb{Z}^{k+1 \times k+1}$, such that $u_n = (M^n)_{1,k+1}$ for all $n \geq 1$.

Before going into more details of this problem we mention other related questions on the elements of recurrent sequences. In the *Skolem's Problem* it is required to determine whether or not a given sequence defined by (1) has a zero term, $u_n = 0$ for some $n \geq 0$. It is known, and highly nontrivial to prove, that the Skolem's Problem is algorithmically solvable for sequences of order 5 or less; see [5]. It is also known that if Positivity Problem is algorithmically solvable in general (i.e., for all orders $k \geq 1$), then also the Skolem's Problem is solvable in the general case (see [7], p. 129). Indeed, this follows by the fact that if $(u_n)_{n=0}^{\infty}$ is linear recurrent sequence, then so is $(u_n^2 - 1)_{n=0}^{\infty}$ (for the details of the construction of $(u_n^2 - 1)_{n=0}^{\infty}$ see also [3]). On the other hand, it is algorithmically solvable for recurrence equations of any order whether or not they produce infinitely many zeroes; see [1].

Let us also mention another old problem which remains open for 2×2 integer matrices. In the *Mortality Problem* a finite set $M_1, M_2, \ldots, M_k$ of 2×2 integer matrices is given, and one is to determine whether or not $M_{i_1} M_{i_2} \ldots M_{i_n} = 0$ for some product of these matrices. It is not known whether this problem is algorithmically solvable; see [10]. If we increase the size of the matrices by one to 3×3 integer matrices then the problem becomes unsolvable; see e.g. [4].

2. Preliminaries. For a complex number α we use the representation $\alpha = |\alpha| e^{i\theta}$, where $\theta \in [0, 2\pi)$ is the *phase* of α.

We begin with some basic facts about roots of unity. An rth *root of unity*, for $r \in \mathbb{N}$, is a complex number ζ satisfying $\zeta^r = 1$. The smallest positive r such that $\zeta^r = 1$ is called the *order* of ζ and it is denoted by $r = \mathrm{ord}(\zeta)$. Evidently $|\zeta| = 1$ for the roots of unity, and the number of rth roots of unity is exactly r. An rth root of unity is called *primitive* if $\zeta^k \neq 1$ for each $1 \leq k < r$, i.e., if its order is r. All rth roots of unity are obtained as powers of a primitive root of unity. It is always possible to choose $\zeta_r = e^{\frac{2\pi i}{r}}$ as a primitive rth root of unity. For any primitive rth root of unity ζ_r, it is easy to see that ζ_r^k is primitive if and only if $\gcd(r, k) = 1$. The rth *cyclotomic polynomial* is defined as

$$\phi_r(x) = \prod_{\substack{k=0 \\ \gcd(k,r)=1}}^{r-1} (x - \zeta_r^k).$$

Clearly ϕ_r has all the primitive rth roots of unity as zeros. It can be shown also that $\phi_r(x) \in \mathbb{Z}[x]$, and that $\phi_r(x)$ is an irreducible polynomial. It follows that $\deg(\phi_r) = \varphi(r)$, where $\varphi(r)$ is the *Euler's function*. Furthermore,

$$x^r - 1 = \prod_{s|r} \phi_s(x),$$

since every rth root of unity is a primitive sth root of unity for exactly one positive divisor s of r. Indeed, every rth root of unity has a minimal polynomial $\phi_s(x)$ for some s, and the degree of the minimal polynomial is $\varphi(s)$.

For a given real number x, we use notation $\lfloor x \rfloor = \max\{n \in \mathbb{N} \mid n \leq x\}$. Inequalities $x - 1 < \lfloor x \rfloor \leq x$ are immediate. Moreover, we denote by $\{x\} = x - \lfloor x \rfloor$ its fractional part. Clearly, $0 \leq \{x\} < 1$ for all real numbers x.

We use the following well known result in our proof. In fact, Weyl's Criterion (cf. [2]) implies easily that the fractional parts of any irrational number are equidistributed in $[0, 1]$.

Lemma 1. *For any irrational number α, the set $M = \{\{n\alpha\} \mid n \in \mathbb{N}\}$ is dense in the interval $[0, 1]$.*

Proof. Since α is irrational, fractional parts $\{n\alpha\}$ are distinct for distinct n, and consequently the set M is infinite.

Given any k, divide the interval $[0, 1]$ into k subintervals $[0, \frac{1}{k}]$, $[\frac{1}{k}, \frac{2}{k}]$, ..., $[\frac{k-1}{k}, 1]$, and consider the $k + 1$ first fractional parts $\{0 \cdot \alpha\}$, $\{1 \cdot \alpha\}$ ..., $\{k \cdot \alpha\}$. By the pigeon hole principle, there exists a subinterval including two of them, say $\{k_1\alpha\}$ and $\{k_2\alpha\}$, where $0 \leq k_1 < k_2 \leq k$. Hence $|\{k_1\alpha\} - \{k_2\alpha\}| < \frac{1}{k}$ and

$$(k_2 - k_1)\alpha = \lfloor k_2\alpha \rfloor - \lfloor k_1\alpha \rfloor + \{k_2\alpha\} - \{k_1\alpha\}.$$

If $\{k_1\alpha\} < \{k_2\alpha\}$, then $\{(k_2 - k_1)\alpha\} = \{k_2\alpha\} - \{k_1\alpha\} < \frac{1}{k}$. It follows straightforwardly that each interval $[\frac{1}{k}, \frac{2}{k}]$, ..., $[\frac{k-1}{k}, 1]$ contains a fractional part of a multiple of $(k_2 - k_1)\alpha$.

On the other hand, if $\{k_1\alpha\} > \{k_2\alpha\}$, then

$$(k_2 - k_1)\alpha = \lfloor k_2\alpha \rfloor - \lfloor k_1\alpha \rfloor - 1 + 1 + \{k_2\alpha\} - \{k_1\alpha\}$$

has fractional part $1 + \{k_2\alpha\} - \{k_1\alpha\} \in (1 - \frac{1}{k}, 1)$. Since $\{n(1-\alpha)\} = 1 - \{n\alpha\}$, each interval $[0, \frac{1}{k}]$ $[\frac{1}{k}, \frac{2}{k}]$, ..., $[\frac{k-1}{k}, 1]$ contains some $\{n\alpha\}$. The claim follows, since k is arbitrary. $\square$

3. Solution of linear recurrent sequence. Assume that $(u_n)_{n=0}^{\infty}$ is a recurrent sequence defined by the linear recurrence relation

$$u_n = au_{n-1} + bu_{n-2}. \tag{3}$$

Then the polynomial

$$p(x) = x^2 - ax - b \tag{4}$$

is called the *characteristic polynomial* of $(u_n)_{n=0}^{\infty}$. The roots of $p(x)$ are called the *characteristic roots* of $(u_n)_{n=0}^{\infty}$.

The following theorem is based on the well known result in combinatorics stating the form of the solutions of a linear recurrence relations for any order; cf. [6], Thm. 3.1.1, p. 23. We state it only for the second order case, where the roots are simple.

Theorem 1. *Let $u_n = au_{n-1}+bu_{n-2}$ be a linear recurrent sequence with $a, b \neq 0$, and let $p(x) = x^2 - ax - b$ be its characteristic polynomial. Let also $D = a^2 + 4b$ be the discriminant of $p(x)$.*

1. *If $D > 0$, then $u_n = A\lambda_1^n + B\lambda_2^n$, where $\lambda_1 \neq \lambda_2$ are real roots of $p(x)$, $A = \frac{u_1-u_0\lambda_2}{\sqrt{D}}$, and $B = \frac{u_0\lambda_1-u_1}{\sqrt{D}}$.*

2. *If $D = 0$, then $u_n = (A + Bn)\lambda^n$, where $\lambda = \frac{a}{2}$ is a double root of $p(x)$, $A = u_0$, and $B = \frac{2u_1-u_0a}{a}$*

3. *If $D < 0$, then $u_n = A\lambda^n + \overline{A\lambda}^n$, where λ and $\overline{\lambda}$ are the complex roots of $p(x)$, and A is as in Case 1.*

Proof. To determine A and B in Case 1, we note that the pair of equations

$$\begin{cases} u_0 = A + B \\ u_1 = A\lambda_1 + B\lambda_2 \end{cases}$$

can be solved by Cramer's rule, since the determinant $-\sqrt{D}$ of the system is nonzero. The solutions are as in the statement. It is then a simple task to show by induction on n that the claimed solution indeed satisfies the recurrence. The same conclusion holds for Case 3, where in addition we notice that $B = \overline{A}$, since in this case $\sqrt{D}$ is a pure imaginary number.

In Case 2 we have $b = -\frac{a^2}{4}$ and the pair of equations becomes

$$\begin{cases} u_0 = A \\ u_1 = A\lambda + B\lambda \end{cases}$$

Now that $a \neq 0$, the determinant, now λ, is again nonzero and Cramer's rule applies. The solutions of the system are as stated, and it is again easy to verify that the recurrence relation is satisfied. $\square$

The expression of u_n in the three cases of Theorem 1 is usually called the *solution* of the $(u_n)_{n=0}^{\infty}$. Note that the roots are nonzero, since $b \neq 0$.

4. Solving the Positivity. We begin with the easiest cases not covered by Theorem 1, and which, actually, refer to Positivity Problem of a first order recurrent equations. Indeed, the Positivity Problem can be solved for first order recurrent sequences in a trivial way, since then $u_n = a^n u_0$ for any $n \geq 0$, and $(u_n)_{n=0}^{\infty}$ is positive if and only if $a \geq 0$ and $u_0 \geq 0$.

Lemma 2. *If $a = 0$ or $b = 0$ in* (3)*, then the Positivity Problem can be solved.*

Proof. Assume first that $b = 0$. Then the sequence consists of u_0 and $u_n = a^{n-1}u_1$ for $n \geq 1$.

If $a = 0$, then $u_n = bu_{n-2}$, and we can divide $(u_n)_{n=0}^{\infty}$ into two sequences, namely

$$u_n = \begin{cases} b^m u_0, & \text{if } n = 2m, \\ b^m u_1, & \text{if } n = 2m+1. \end{cases}$$

In both cases solvability follows from the case of first order recurrence equations. □

We then study the cases of Theorem 1 separately. In Case 1 the asymptotic behaviour of the sequence $(u_n)_{n=0}^{\infty}$ is essentially determined by the root which has maximum absolute value.

Lemma 3. *Assume that, for all $n \geq 0$, $u_n = A\lambda_1^n + B\lambda_2^n$, where $\lambda_1 \neq \lambda_2$ are real and nonzero, and $A, B \in \mathbb{R}$. Then the Positivity Problem of $(u_n)_{n=0}^{\infty}$ can be solved.*

Proof. Note first that $\lambda_1 \neq -\lambda_2$, since the equality would imply $a = \lambda_1 - \lambda_2 = 0$. This implies also that $|\lambda_1| \neq |\lambda_2|$. Assume that $|\lambda_1| > |\lambda_2|$. Now

$$u_n = \lambda_1^n (A + B(\frac{\lambda_2}{\lambda_1})^n),$$

where $|\frac{\lambda_2}{\lambda_1}| < 1$. Note that if $A = B = 0$, then the positivity of the sequence is trivial. Therefore, assume that this is not the case. We have two cases according to the sign of λ_1.

If $\lambda_1 < 0$, then $u_n \geq 0$ if and only if the sign of

$$A + B(\frac{\lambda_2}{\lambda_1})^n \tag{5}$$

is equal to the sign of λ_1^n or zero. Necessarily $\lambda_2 > 0$ and $B \neq 0$. Since $(\frac{\lambda_2}{\lambda_1})^n$ tends to zero as n grows, there can be only one zero in (5), and the sign of (5) can change only if $A = 0$. Since $\frac{\lambda_2}{\lambda_1} < 0$, necessarily $B > 0$. Therefore, in this case, the sequence $(u_n)_{n=0}^{\infty}$ is positive if and only if $\lambda_2 > 0$, $A = 0$ and $B > 0$.

If $\lambda_1 > 0$, then the sequence $(u_n)_{n=0}^{\infty}$ is positive, if

$$A + B(\frac{\lambda_2}{\lambda_1})^n \geq 0 \tag{6}$$

for all $n \geq 0$. If $B = 0$, then $(u_n)_{n=0}^{\infty}$ is positive when $A \geq 0$; otherwise (6) is equivalent to

$$(\frac{\lambda_2}{\lambda_1})^n \geq \frac{-A}{B} \tag{7}$$

for all $n \geq 0$. We have two cases according to whether $\lambda_2 > 0$ or not. First, if $\lambda_2 > 0$, then (7) is satisfied if and only if $\frac{-A}{B} \leq 0$, since $(\frac{\lambda_2}{\lambda_1})^n$ tends to zero. In the second case we have $\lambda_2 < 0$, again $(\frac{\lambda_2}{\lambda_1})^n$ tends to zero, but now the sign changes. Now it is easy to see that $(u_n)_{n=0}^{\infty}$ is positive, i.e., (7) holds for all $n \geq 0$ if and only if (7) holds for $n = 1$. Indeed, in this case $(\frac{\lambda_2}{\lambda_1})^n \geq (\frac{\lambda_2}{\lambda_1})$ for all $n \geq 0$.

Clearly, each of the above cases provide a method for determining positivity of the sequence $(u_n)_{n=0}^{\infty}$. □

Next we consider the Positivity Problem for Case 2 of Theorem 1.

Lemma 4. *Assume that* $u_n = (A + Bn)\lambda^n$ *for all* $n \geq 0$. *Then the Positivity Problem of the sequence* $(u_n)_{n=0}^{\infty}$ *can be solved.*

Proof. Note first that the sign of $A + Bn$ can change only once when n grows. Therefore, if $\lambda < 0$, then $(u_n)_{n=0}^{\infty}$ cannot be positive. On the other hand, if $\lambda > 0$, then $(u_n)_{n=0}^{\infty}$ is positive if and only if $A + Bn \geq 0$ for all $n \geq 0$, which is equivalent to the condition $n \geq \frac{-A}{B}$, and thus to the condition $\frac{-A}{B} \leq 0$, whenever $B \neq 0$. If $B = 0$, then the positivity is equivalent to $A \geq 0$. $\square$

What is left is the case of two complex roots, i.e., Case 3. Assume that

$$u_n = A\lambda^n + \overline{A\lambda}^n,$$

for all $n \geq 0$, and let $\lambda = |\lambda|e^{i\theta}$ and $A = |A|e^{i\phi}$ for some $\theta, \phi \in [0, 2\pi)$. We can now write

$$\begin{aligned} u_n &= |A|\, e^{i\phi}(|\lambda|\, e^{i\theta})^n + |A|\, e^{-i\phi}(|\lambda|\, e^{-i\theta})^n \\ &= |A|\, |\lambda|^n \left(e^{i(\phi+n\theta)} + e^{-i(\phi+n\theta)}\right) \\ &= 2\, |A|\, |\lambda|^n \cos(\phi + n\theta), \end{aligned} \tag{8}$$

where θ and ϕ are the phases of λ and A as above. It is clear that we need to study the sign of $\cos(\phi + n\theta)$ in order to solve the Positivity Problem.

Lemma 5. *If* $e^{i\theta}$ *is not a root of unity, then the set* $\{\cos(\phi + n\theta) \mid n \in \mathbb{N}\}$ *is dense in* $[-1, 1]$. *In particular, if* $e^{i\theta}$ *in* (8) *is not a root of unity, then* $(u_n)_{n=0}^{\infty}$ *is not positive.*

Proof. Since $e^{i\theta}$ is not a root of unity, we have $\theta = 2\pi\theta_1$, where θ_1 is irrational. It follows that the numbers $n\theta = 2\pi n\theta_1 \pmod{2\pi}$ are dense in $[0, 2\pi]$, which implies that their translations $\phi + 2\pi n\theta_1 \pmod{2\pi}$ are dense in $[\phi, \phi + 2\pi]$. Now since the cosine function is a continuous bijection from $[\phi, \phi + 2\pi)$ onto $[-1, 1]$, it maps a dense set into a dense set. $\square$

On the other hand, for in the case where $e^{i\theta}$ is a root of unity we have a positive answer.

Lemma 6. *Let* $e^{i\theta}$ *in* (8) *be a root of unity. Then the Positivity Problem of* $(u_n)_{n=0}^{\infty}$ *can be solved.*

Proof. Since $e^{i\theta}$ is a root of unity, we have $\theta = 2\pi\frac{r}{s}$ with $r \in \mathbb{Z}$ and $s = \mathrm{ord}(e^{i\theta})$, and then $\cos(\phi + (n + s)\theta) = \cos(\phi + n\theta + 2r\pi) = \cos(\phi + n\theta)$. This is to say that the sequence $u_n / |\lambda|^n$ has period s, and the positivity of the sequence members can be determined by computing the first s terms of the sequence. $\square$

In the present case, Case 3, we still need to describe how to check whether $e^{i\theta}$ is a root of unity or not, and how to find its order. It is obvious by the definition that $e^{i\theta}$ is a root of unity if and only if $e^{2i\theta}$ is so. Indeed, either $\operatorname{ord}(e^{i\theta}) = \operatorname{ord}(e^{2i\theta})$ or $\operatorname{ord}(e^{i\theta}) = 2\operatorname{ord}(e^{2i\theta})$.

We will describe a method how to determine if $e^{2i\theta}$ is a root of unity. Notice that since now $D = a^2 + 4b < 0$, necessarily $b < 0$. Moreover, because $x^2 - ax - b = (x - \lambda)(x - \overline{\lambda})$, it is easy to see that $a = \lambda + \overline{\lambda}$, $|\lambda|^2 = -b$. By squaring the first relation we learn that $a^2 = |\lambda|^2 (e^{2i\theta} + 2 + e^{-2i\theta})$, which together with the second equality and by multiplying the equation by $e^{2i\theta}$ gives

$$be^{4i\theta} + (a^2 + 2b)e^{2i\theta} + b = 0,$$

which shows that $p(x) = x^2 + \frac{a^2+2b}{b}x + 1$ has $e^{2i\theta}$ as a root. In fact, $p(x)$ is the minimal polynomial of $e^{2i\theta}$, since otherwise $e^{2i\theta}$ would be of degree 1 (and hence real), which would imply that also λ^2 is real. But this would mean that either $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$. The first case is impossible, since we assumed that we have Case 3 of Theorem 1, and the second case would imply that $a = 0$, which is against the assumption of Theorem 1.

Lemma 7. *$e^{2i\theta}$ is a root of unity if and only if one of the following conditions holds:*

1. *$a^2 = -b$, in which case, $e^{2i\theta}$ has order 3.*

2. *$a^2 = -2b$, in which case, $e^{2i\theta}$ has order 4.*

3. *$a^2 = -3b$, in which case, $e^{2i\theta}$ has order 6.*

Proof. Assume that $e^{2i\theta}$ is a root of unity of order s. Since $e^{2i\theta}$ is an algebraic number of degree 2, we have that $\varphi(s) = 2$. Clearly, $\varphi(n) \geq \pi(n-1)$, where $\pi(i)$ is the number of primes p such that $p \leq i$. Now $\pi(n-1) > 2$ whenever $n > 6$. Therefore, the only candidates for s are 3, 4, and 6, for which $\varphi(3) = \varphi(4) = \varphi(6) = 2$.

The minimal polynomial of $e^{2i\theta}$ gives

$$(e^{2i\theta})^2 = -\frac{a^2 + 2b}{b}e^{2i\theta} - 1, \tag{9}$$

which, when multiplied by $e^{2i\theta}$, shows that

$$(e^{2i\theta})^3 = \left(\left(\frac{a^2 + 2b}{b}\right)^2 - 1\right)e^{2i\theta} + \frac{a^2 + 2b}{b} \tag{10}$$

Since $e^{2i\theta} \notin \mathbb{R}$, this expression equals to 1 if and only if $\frac{a^2+2b}{b} = 1$, which is equivalent to $a^2 = -b$.

The order of $e^{2i\theta}$ is 4 if and only if $(e^{2i\theta})^2 = -1$. By (9) this happens if and only if $\frac{a^2+2b}{b} = 0$, which is equivalent to $a^2 = -2b$.

The order of $e^{2i\theta}$ is 6 if and only if $(e^{2i\theta})^3 = -1$. By (10) this happens if and only if $\frac{a^2+2b}{b} = -1$, which is equivalent to $a^2 = -3b$. □

Note that it follows from Lemma 7 that if $e^{i\theta}$ is a root of unity, then it has order 3, 4, 6, 8 or 12.

Corollary 1. *If* $u_n = A\lambda^n + \overline{A\lambda}^n$, *for all* $n \geq 0$, *then the Positivity Problem of* $(u_n)_{n=0}^{\infty}$ *can be solved.*

Proof. According to Lemma 7, it can be checked whether $e^{2i\theta}$, and also $e^{i\theta}$, is a root of unity, and then the order can be found by checking the possible orders. After that, the Positivity Problem is solved using the proof of Lemma 6. □

From Lemmas 2, 3, 4 and Corollary 1 follows

Theorem 2. *The Positivity Problem is algorithmically solvable for the second order linear recurrent sequences.*

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