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The square tiling problem is NP-complete for deterministic tile sets

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Abstract

It is shown that the square tiling problem of Garey, Johnson and Papadimitrou is NP-complete even if the given tile set is deterministic by any two sides, i.e. the colors of any two sides uniquely determine a tile within the given tile set.

Keywords: deterministic tiles, NP-completeness, square tiling problem, tiling problem, Wang tiles

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1 Introduction

A *Wang tile* is simply a unit square with colored edges. The edges of a Wang tile are called *north*-, *east*-, *west*- and *south*-edges in a natural way. A *Wang tile set* T is a finite set containing Wang tiles. A *tiling* is a mapping $f : \mathbb{Z}^2 \to T$, which assigns a unique Wang tile for each integer pair of the plane. A tiling f is said to be *valid*, if for every pair $(x, y) \in \mathbb{Z}^2$ the tile $f(x, y) \in T$ matches its neighboring tiles (i.e. the south side of tile $f(x, y)$ has the same color as the north side of tile $f(x, y - 1)$ etc.).

A Wang tile set T is said to be *NW-deterministic*, if within the tile set there does not exist two different tiles with the same colors on the north- and west-sides. In general, a Wang tile set is *XY-deterministic*, if the X- and Y-sides uniquely determine a tile in the given Wang tile set. A Wang tile set is *4-way deterministic*, if it is NE-, NW-, SE- and SW-deterministic.

The *tiling problem* is the following: "Does there exist a valid tiling of the plane for the given tile set?" It was shown by Berger [2], that the tiling problem is undecidable. It was shown by Kari [5], that the tiling problem is undecidable even when restricted to NW-deterministic tile sets. It is not known whether the tiling problem is decidable for 4-way deterministic tile sets. It is known that there does exist a 4-way deterministic tile set, which is aperiodic [6].

The *square tiling problem* is defined as follows [4]: "Given an integer N and a set of Wang tiles T with at least N different colors, does there exist a valid tiling of an $N \times N$-square by these given tiles?"

Theorem 1.1 (Garey, Johnson, Papadimitrou, 1977 [4]). *The square tiling problem is NP-complete.*

In this article it is shown that the square tiling problem given in [4] is NP-complete even if the given tile set is 4-way deterministic, or even when it is deterministic by any two sides. The construction is based on the tile set given by Aggarwal et al. [1], which again is based on the construction of Lagoudakis and LaBean [8].

2 Determinism by adjacent sides

In this section it is shown that the NP-complete 3-satisfiability problem can be reduced to the square tiling problem for 4-way deterministic tile sets.

A *boolean variable* x_i is a variable that contains either value 'true' (denoted by $\top$) or value 'false' (denoted by $\bot$). A *positive literal* is an expression of the form x_i for some boolean variable x_i. A *negative literal* is an expression of the form $\neg x_i$ for some boolean variable x_i. A *clause* is an expression $l_{i_1} \vee l_{i_2} \vee l_{i_3}$ where for every literal l_i either $l_i = x_i$ or $l_i = \neg x_i$ holds. A *valuation* is a mapping

$$f : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$$

assigning a truth value for every boolean variable.

The *3-satisfiability problem* (*3SAT* for short) is the following: "Given a boolean formula

$$\bigwedge_{i=1}^{m} (l_{i,1} \vee l_{i,2} \vee l_{i,3}) \,, \tag{1}$$

where $l_{i,j} \in \{x_k, \neg x_k\}$ for every i and j (and some k depending on i and j), does there exist such a valuation that formula (1) gets value true?"

Theorem 2.1 (Cook, 1971 [3]). *The 3SAT problem is NP-complete.*

2.1 The tile set for 3SAT

In the following, the variables are denoted by x_i and the clauses are denoted by C_i. The construction of the Wang tile set is done as follows:

1. First add the *seed tile* in figure 1, and for every clause C_i add the *clause tile* shown in figure 2. These tiles are simply used as auxiliary tiles. The clauses are represented by the columns in the rectangle to be assembled.

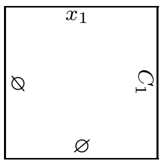

Figure 1: The seed tile.

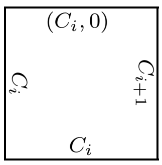

Figure 2: The clause tile for clause C_i.

2. For every variable x_i in the given instance of 3SAT, add the *valuation tiles* in figure 3. A column of these tiles is used to represent all the possible valuations. The tile with the color $(x_i, \top)$ on its west edge represents valuation $x_i = \top$ and the tile with the color $(x_i, \bot)$ on its west edge represents valuation $x_i = \bot$. Each of the variables is represented by a row in the rectangle.

3. For every clause C_i, where $1 \leq i \leq m$, and variable x_j, where $1 \leq j < n$, add only one set of the tiles from figures 4, 5 and 6. For every clause C_i, where $1 \leq i \leq m$, and for the last variable x_n, add only those tiles (from one of the figures 4, 5 and 6), which have north side colors different from $(C_i, 0)$. This restriction is set to force the uppermost row to be tiled if, and only if, all the clauses (i.e. columns) have a literal, which is true in the arbitrary valuation given by the valuation tiles in the first column.

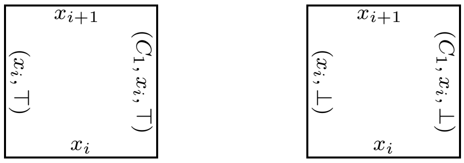

Figure 3: The valuation tiles for variable x_i.

The rectangle is constructed upwards so that the tile corresponding to variable x_j in clause C_i has color $(C_i, k+1)$ on its north side if it has color (C_i, k) (with $k > 0$) on its south side. The north side color is $(C_i, 1)$ if the corresponding literal gets value true in the valuation and the southside color is $(C_i, 0)$. Otherwise the north side color of the tile is $(C_i, 0)$. In other words, the first literal in clause C_i that gets value $\top$ initiates a counter, which is incremented by 1 on every row above.

The west side color $(C_i, x_j, \top)$ represents value $x_j = \top$ for the tile in the row that represents clause C_i. Likewise, the color $(C_i, x_j, \bot)$ represents value $x_j = \bot$.

(a) If the positive literal x_j belongs to clause C_i, add the tiles in figure 4. If the south side color is (C_i, k) where $k \neq 0$, the north color is $(C_i, k+1)$. If the south side color is (C_i, k) and $x_j = \top$ (west side color is $(C_i, x_j, \top)$) the north side color is $(C_i, 1)$. Otherwise the north side color is $(C_i, 0)$.

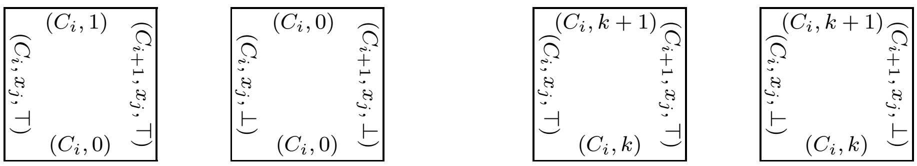

Figure 4: The tiles for positive literals x_j in clause C_i.

(b) If the negative literal $\neg x_j$ belongs to clause C_i, add the tiles in figure 5. If the south side color is (C_i, k) where $k \neq 0$, the north color is $(C_i, k+1)$. If the south side color is (C_i, k) and $x_j = \bot$ the north side color is $(C_i, 1)$. Otherwise the north side color is $(C_i, 0)$.

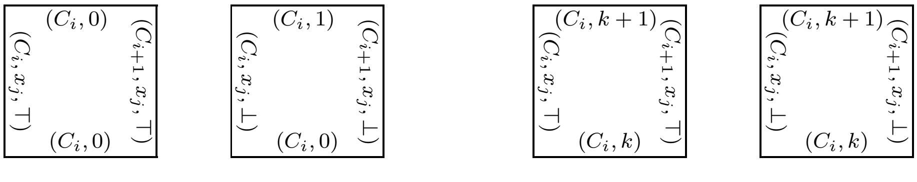

Figure 5: The tiles for negative literals $\neg x_j$ in clause C_i.

(c) If neither the positive literal x_j nor the negative literal $\neg x_j$ belongs to clause C_i, add the tiles in figure 6. These tiles simply move the information on the truth state of the clause upwards. If the south side color is (C_i, k) where $k \neq 0$, the north color is $(C_i, k+1)$. If the south side color is $(C_i, 0)$ then also the north side color is $(C_i, 0)$.

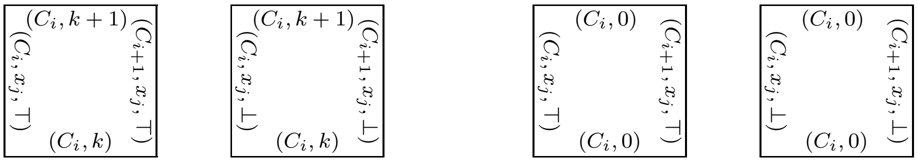

Figure 6: The tiles for literals x_j not in clause C_i.

Theorem 2.2. *The square tiling problem is NP-complete even for instances with a 4-way deterministic tile set.*

Proof. It is quite obvious that the tile set constructed above is 4-way deterministic. All the tiles are determined uniquely by any two adjacent sides. The only case that is not as obvious is the uniqueness within the tiles in figures 4, 5 and 6. However, for every clause C_i and variable x_j only one of these sets is chosen and within each of these sets the tile is uniquely determined.

It is not necessary to show that a square can be tiled with the given tile set. For every instance of 3SAT it is possible to add dummy variables that belong to no clause so that the number of variables and the number of clauses are the same.

It remains to be shown that an instance of 3SAT with m clauses and n variables has a solution if, and only if, an $(m+1) \times (n+1)$-rectangle can be tiled with the tile set corresponding to the instance.

It can be seen from the colors of the east and west sides of the tiles, that only rows of length $m+1$ at most can be constructed. In a row of length $m+1$ there has to be a valuation tile or a seed tile at the left end of the row. On the other hand, $n+1$ such rows are required to construct an $(m+1) \times (n+1)$-rectangle. These rows must match everywhere, so the only way to place one row atop another is to have their leftmost tiles match. This happens only if the leftmost column consists of the seed tile and all the valuation tiles in the correct order. Hence, a correctly tiled rectangle must have a column consisting of the seed tile and the valuation tiles as its leftmost column. Likewise, its lowermost row must consist of the seed tile and all the clause tiles in correct order.

By the construction, it is always possible to construct all the rows (excluding the uppermost row) so that they match their neighboring rows. The uppermost row can be tiled to match the row below if, and only if, the valuation represented by the valuation tiles is a positive solution for the given instance of 3SAT. Therefore, the rectangle can be tiled if, and only if, the given instance of 3SAT has a solution.

The reduction from the given instance of 3SAT to a 4-way deterministic tile set can clearly be done in polynomial time. Hence, the square tiling problem is NP-complete for 4-way deterministic tile sets. □

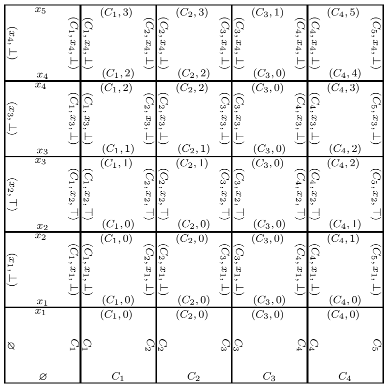

Figure 7: A valid tiling of a rectangle for formula $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee x_4)$ with values $x_1 = \bot$, $x_2 = \top$, $x_3 = \bot$ and $x_4 = \bot$.

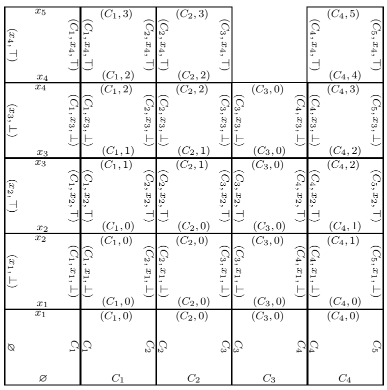

Figure 8: An incomplete tiling of a rectangle for formula $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee x_4)$ with values $x_1 = \bot$, $x_2 = \top$, $x_3 = \bot$ and $x_4 = \top$. There is no matching tile for the negative literal $\neg x_4$ in the third clause.

Figures 7 and 8 present examples of a valid tiling of a rectangle and an incomplete tiling of a rectangle. In figure 7 the valuation (i.e. the leftmost column) is such that the rectangle can be completed. In figure 8 the valuation is such that the column representing the third clause cannot be tiled.

2.2 The tile set for one-in-three 3SAT

A variant of the 3SAT problem is the *one-in-three 3SAT*: "Given a boolean formula of form (1), does there exist such a valuation that the given formula gets value true and every clause has exactly one true literal?"

Theorem 2.3 (Schaefer, 1978 [9]). *The one-in-three 3SAT problem is NP-complete. The problem remains NP-complete even if the given formula does not contain negative literals.*

The reduction from one-in-three 3SAT to the square tiling problem can be done in a similar way as in the case of simple 3SAT. Instead of using the tiles in figures 4, 5 and 6 for tile set construction, the tiles in figures 9 and 10 are used.

For every positive literal x_j in the clause C_i only the tiles in figure 9 are added to the tile set. For the last variable x_n the second tile in figure 9 is left out of the tile set.

By theorem 2.3 it is reasonable to assume that the instance of one-in-three 3SAT contains no negative literals, so no tiles are needed for negative literals.

For every literal x_j not in clause C_i the tiles in figure 10 are added to the tile set. For the last variable x_n the first and the third tile in figure 9 are left out of the tile set.

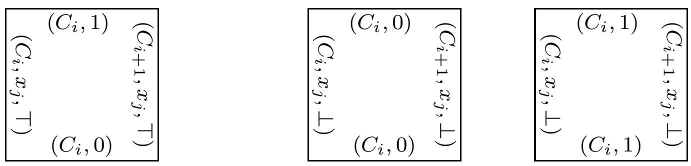

Figure 9: The tiles for literals x_j in clause C_i.

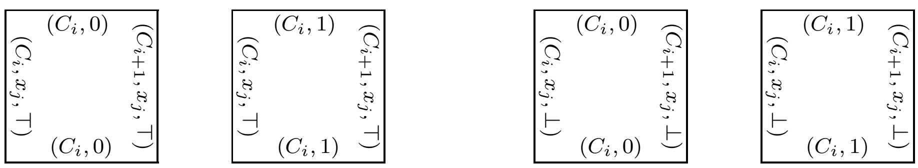

Figure 10: The tiles for literals x_j not in clause C_i.

It is quite straightforward to see that a rectangle can be tiled with this new tile set if, and only if, the given formula has a positive solution. Furthermore, also this new tile set is 4-way deterministic. The only minor difference is that the possible tiling error may occur also on other rows instead of only the uppermost row in the rectangle.

3 Determinism by opposite sides

In this section it is shown that the square tiling problem is NP-complete even if a tile is uniquely determined in the given tile set by any two of its sides.

A *corner tile* is a unit square, which is divided into four colored corners as in figure 11. Corner tiles are also Wang tiles, since the color of an edge is determined by corner colors of that particular edge. The notions concerning determinism can be extended to corner tile sets by considering them as Wang tile sets in the sense in figure 11. The concept of corner tiles was introduced in [7].

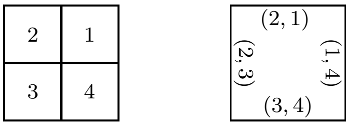

Figure 11: A corner tile and the corresponding Wang tile.

Every Wang tile set can be converted into a corner tile set, which admits a valid tiling of a square if, and only if, the original Wang tile set admits a valid tiling of a square. This can be done by dividing each Wang tile into four corner tiles (as quadrants) as in figure 12. The corners are colored so that the neighboring corner tiles match if, and only if, they correspond to the same original Wang tile or their corresponding original Wang tiles match. It is done simply by introducing a new color uniquely identifying the original tile and coloring the corners according to the side colors of the original tile. The technique of figure 12 has been used earlier in converting arbitrary aperiodic Wang tile set to corner tile sets [7].

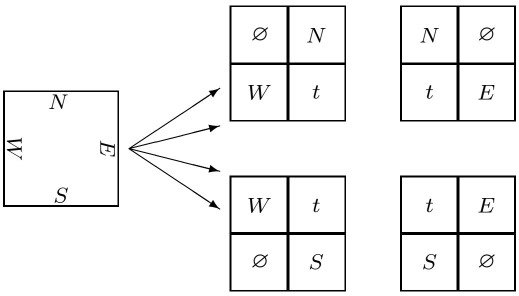

Figure 12: From Wang tiles to corner tiles. Color t is a unique new color representing the old Wang tile.

Lemma 3.1. *If the given Wang tile set T is XY-deterministic, then also the corner tile set (considered as a Wang tile set in the sense of figure 11), which is constructed from set T by the operation in figure 12, is XY-deterministic.*

Proof. A corner tile set is always deterministic with respect to any two opposite sides and therefore it is enough to consider only determinism by adjacent sides. Assume that the given tile set is SW-deterministic.

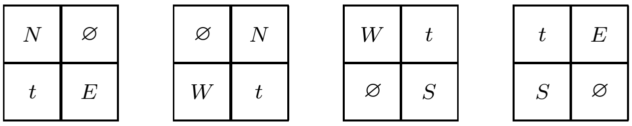

Figure 13: The new corner tiles for Wang tile t.

The first one of the new tiles in figure 13 is uniquely defined by both its south-side and its west-side, since no other corner tile has color t in its SW-corner. The second tile is uniquely defined by its south-side, since no other corner tile has color t in its SE-corner. The third tile is uniquely defined by its south-side and west-side, since the original Wang tile set was assumed to be SW-deterministic. The fourth tile is uniquely defined by its west-side, since no other corner tile has color t in its NW-corner.

Argumentation for NW-, NE- and SE-determinism is similar. $\square$

Theorem 3.2. *The square tiling problem is NP-complete even if the given tile set is deterministic with respect to any two sides.*

Proof. A tile set that is deterministic by any two sides can be constructed by applying the operation depicted in figure 12 to the tile set of theorem 2.2. This new tile set can tile a $2(m+1) \times 2(n+1)$-rectangle if, and only if, the given instance of 3SAT with m clauses and n variables has a positive solution. $\square$

4 Conclusions

It was noted that the so-called square tiling problem is NP-complete even when any two sides uniquely determine a tile in the given tile set. The reduction was done from 3SAT and used a small observation concerning corner tiles. Also an alternative method of reduction from one-in-three 3SAT was described.

Open question: Is the (infinite) tiling problem undecidable for 4-way deterministic tile sets?

Open question: What kind of patterns can be tiled with a 4-way deterministic tile set when a unique color is assigned to each of the tiles?

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