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#### Abstract

We consider several aspects of Wilke's (1996) tree algebra formalism for representing binary labelled trees and compare it with approaches that represent trees as terms in the traditional way. A convergent term rewriting system yields normal form representations of binary trees and contexts, as well as a new completeness proof and a computational decision method for the axiomatization of tree algebras. Varieties of binary tree languages are compared with varieties of tree languages studied earlier in the literature. We also prove a variety theorem thus solving a problem noted by several authors. Syntactic tree algebras are studied and compared with ordinary syntactic algebras. The expressive power of the language of tree algebras is demonstrated by giving equational definitions for some well-known varieties of binary tree languages.


Keywords: Tree automata, Tree languages, Tree algebras, Binary trees, Varieties of tree languages, Syntactic tree algebras.

## 1 Introduction

In algebraic language theory words are usually regarded as elements of the free monoid $X^{*}$ (or the free semigroup $X^{+}$if the empty word is omitted) generated by a given alphabet $X$. In particular, the syntactic monoid (cf. [8, 22]) of a language $L \subseteq X^{*}$ is defined with this interpretation in mind. Similarly, in algebraic treatments of regular tree languages (cf. [7, 31, 13, 14]) trees are often defined as terms, and the syntactic algebra [1, 27, 28] of a tree language is then a quotient algebra of the appropriate term algebra. However, Wilke [33] has proposed a different framework in which trees are not directly viewed as elements of any algebraic structure but are represented by terms over a signature $\Gamma$ with six operation symbols involving the three sorts label, tree and context. The trees thus represented are binary trees over a given label alphabet. A tree algebra is a $\Gamma$-algebra satisfying certain identities that equate some pairs of $\Gamma$-terms representing the same tree or the same context. The component of sort tree of the syntactic tree algebra of a binary tree language $T$ is essentially the syntactic algebra of $T$ in the sense of [1, 27, 28, while its context-component gives the syntactic semigroup of $T$ as defined (as monoids) in [32], and studied further in [26], [21] and [23]. A binary tree language is regular if and only if its syntactic tree algebra is finite [33]. Hence, one may characterize families of binary tree languages by syntactic tree algebras as shown by Wilke [33] in the case of frontier testable (i.e., reverse definite) tree languages.

In this paper we study several aspects of the tree algebra formalism. The theory is formulated in such a way that it
(1) lets us derive the conceptual machinery directly from some general ideas of algebraic language theory,
(2) yields many fundamental results, including the general theorems of [33], in a natural way with transparent algebraic proofs, and
(3) facilitates the comparison with other algebraic approaches to regular tree languages.

A classification theory for binary tree languages based on syntactic tree algebras was called for already in [33], and the lack of an appropriate variety theorem was noted also in [29], [10] and [11]. Here such a theorem is proved. For this, we have to consider varieties of finite tree algebras of a special kind as the direct bijection between varieties of binary tree languages (VBTLs) and all varieties of finite tree algebras fails to hold. We also show that any general variety of tree languages of the kind studied in [29], yields a VBTL when restricted to binary ranked alphabets. That not every VBTL is obtained this way, is due to a subtle difference in the tree homomorphisms used in the definitions of the two kinds of varieties. A similar difference can be noted in the relation between syntactic tree algebras and ordinary syntactic algebras: the syntactic algebra completely determines the syntactic tree algebra, but
the converse is only partially true. Anyway, it seems that mostly the same families of binary tree languages are definable in terms of the two syntactic invariants. On the other hand, the language of tree algebras lends itself better for equational definitions of VBTLs.

Let us now review the contents of the paper section by section. In Section 2 we introduce algebras, terms and trees as well as several related notions, fixing at the same time some general notation to be used throughout the paper. In Section 3 Wilke's tree algebras are introduced, and the representations of binary trees and contexts by Wilke's terms are formalized by homomorphisms from the appropriate term algebras to the corresponding tree algebras of binary trees. In Section 4 we turn Wilke's axioms for tree algebras into a convergent term rewriting system, and describe the corresponding normal form representations of binary trees and contexts. The term rewriting system also yields a completeness theorem for Wilke's axioms, proved differently in [33], as well as a computational method to test the equivalence of two tree or context representations.

In Section 5 we define and survey some basic properties of the syntactic congruences and syntactic algebras of subsets of $\Gamma$-algebras making use of the general many-sorted theory developed in [25]. When these definitions are applied in Section 6 to binary tree languages, regarding these as subsets of sort tree of free tree algebras, we obtain Wilke's syntactic tree algebras as well as some basic facts about them. In particular, by noting some relationships between the syntactic tree algebra $S T A(T)$ of a binary tree language $T$ and its ordinary syntactic algebra $S A(T)$ and syntactic semigroup $S S(T)$, we get in a new way Wilke's theorem stating that $T$ is regular iff $S T A(T)$ is finite. Moreover, we note several general properties of syntactic tree algebras needed in the variety theory.

In Section 7 we introduce varieties of binary tree languages (VBTLs) and varieties of finite tree algebras (VFTAs). However, the natural maps between VBTLs and VFTAs, defined via syntactic algebras, do not yield the complete correspondence one could expect. In Section 8 it is then shown how a Variety Theorem for VBTLs can be obtained by replacing VFTAs with varieties of finite reduced tree algebras; we call a tree algebra reduced if it is generated by its elements of sort label, and no two elements of sort label, or of sort context, are equivalent with respect to the operations of the algebra that yield elements of sort tree. All syntactic tree algebras are reduced in this sense. We also show how any tree algebra $\mathcal{M}$ can be transformed to a reduced tree algebra that is maximal among the reduced tree algebras covered by $\mathcal{M}$.

Varieties of binary tree languages are less general than varieties of [1], [27] and [28] in that they involve binary trees only. On the other hand, they are more general in the sense that the alphabet of labels is not fixed. In this they resemble the general varieties of tree languages (GVTLs) of [29] where tree languages over all ranked alphabets and leaf alphabets appear. In Sec-
tion 9 we show that when a GVTL is restricted to the ranked alphabets of binary tree languages, a VBTL is obtained. Thus the binary parts of many known families of regular tree languages are VBTLs. However, not every VBTL can be obtained this way from a GVTL. This ultimately depends on the fact that in the binary trees of [33], leaves and inner nodes are labelled with the same symbols. A similar subtle difference surfaces when we study connections between the syntactic tree algebra $S T A(T)$, the syntactic algebra $S A(T)$ and the syntactic semigroup $S S(T)$ of a binary tree language $T$. Although $S T A(T)$ is completely determined by $S A(T)$, and we can construct $S T A(T)$ from $S A(T)$, the converse is not completely true. Nevertheless, it appears that essentially the same families of binary tree languages can be characterized by syntactic tree algebras as by syntactic algebras.

In spite of the above conclusion drawn from the results of Section 9, it seems that the language of tree algebras has certain advantages and is very convenient for defining VBTLs by equations. This was first shown by Wilke [33] who gave an elegant equational description of the frontier testable binary tree languages. Wilke also proved that frontier testability is a decidable property for binary tree languages. However, the equational description did not by itself yield a decision method, but a closer analysis of the syntactic tree algebras of frontier testable sets was required. In Section 10 we present, after some relevant general facts, three more examples of equational descriptions of VBTLs.

This paper has been written over a rather long period of time. Hence it both precedes and follows the doctoral dissertation of the first-named author, and some of the results appear already in [24]. However, even in those cases, the presentation may be somewhat different here. The bibliography contains several general references related to the subject matter of this paper. In particular, 30] surveys various algebraic approaches to the classification of regular tree languages, and contains many further relevant references.

## 2 Algebras, terms, trees and contexts

In this section we recall some basic notions mainly to fix our notation for later reference. First a word on notation: we shall frequently write $a:=b$ to indicate that $a$ is defined to be equal to $b$.

Let $\Sigma$ be a ranked alphabet, i.e., a finite set of operation symbols each of which has a given non-negative integer arity. For each $m \geq 0$, let $\Sigma_{m}$ denote the set of $m$-ary symbols in $\Sigma$. A $\Sigma$-algebra $\mathcal{D}=(D, \Sigma)$ consists of a nonempty set $D$ (of the elements of $\mathcal{D}$ ) and a $\Sigma$-indexed family of operations such that if $f \in \Sigma_{m}$, then $f^{\mathcal{D}}: D^{m} \rightarrow D$ is an $m$-ary operation on $D$. In particular, any $c \in \Sigma_{0}$ fixes a constant $c^{\mathcal{D}} \in D$.

Next we recall the usual definition of trees as terms (cf. [7, 31, 13, 14],
for example). Let $X$ be a finite set of symbols disjoint from $\Sigma$, called a leaf alphabet. The set $T_{\Sigma}(X)$ of $\Sigma$-terms over $X$ is defined inductively:
(1) $\Sigma_{0} \cup X \subseteq T_{\Sigma}(X)$;
(2) $f\left(t_{1}, \ldots, t_{m}\right) \in T_{\Sigma}(X)$ if $m>0, f \in \Sigma_{m}$ and $t_{1}, \ldots, t_{m} \in T_{\Sigma}(X)$.

We shall view terms in the usual way as (syntactic representations of) trees labelled with symbols in $\Sigma \cup X$, and call them also $\Sigma X$-trees.

The height $\mathrm{hg}(t)$ of a $\Sigma X$-tree $t$ is defined by setting (1) $\mathrm{hg}(t)=0$ for any $t \in \Sigma_{0} \cup X$, and (2) $\mathrm{hg}(t)=\max \left\{\operatorname{hg}\left(t_{1}\right), \ldots, \mathrm{hg}\left(t_{m}\right)\right\}+1$ for $t=f\left(t_{1}, \ldots, t_{m}\right)$.

Let $\xi$ be a new symbol that does not appear in $\Sigma$ or $X$. A $\Sigma X$-context is a $\Sigma(X \cup\{\xi\})$-tree in which $\xi$ appears exactly once. The set of $\Sigma X$-contexts is denoted by $C_{\Sigma}(X)$. Furthermore, let $C_{\Sigma}^{+}(X)=C_{\Sigma}(X) \backslash\{\xi\}$ be the set of non-unit $\Sigma X$-contexts; $\xi$ is the unit context. In the special case $X=\emptyset$, we get the sets $T_{\Sigma}, C_{\Sigma}$ and $C_{\Sigma}^{+}$of $\Sigma$-trees (or ground $\Sigma$-terms), $\Sigma$-contexts, and non-unit $\Sigma$-contexts, respectively.

The $\xi$-depth $\mathrm{d}_{\xi}(p)$ of $p \in C_{\Sigma}(X)$ is the distance of the $\xi$-labelled node from the root of $p$, that is, (1) $\mathrm{d}_{\xi}(\xi)=0$, and (2) if $p=f\left(t_{1}, \ldots, t_{i-1}, q, t_{i+1}, \ldots, t_{m}\right)$ for some $m>0,1 \leq i \leq m, t_{1}, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{m} \in T_{\Sigma}(X)$ and $q \in C_{\Sigma}(X)$, then $\mathrm{d}_{\xi}(p)=\mathrm{d}_{\xi}(q)+1$.

If $p, q \in C_{\Sigma}(X)$ and $t \in T_{\Sigma}(X)$, then $q \cdot p:=p(q) \in C_{\Sigma}(X)$ and $t \cdot p:=$ $p(t) \in T_{\Sigma}(X)$ are obtained from $p$ by replacing the single occurrence of $\xi$ with $q$ and with $t$, respectively. Obviously, $\xi(p)=p(\xi)=p$ and $\xi(t)=t$ for any context $p$ and any tree $t$. Clearly, $\left(C_{\Sigma}(X), \cdot, \xi\right)$ is a monoid for the product $p \cdot q$. Similarly, $\left(C_{\Sigma}^{+}(X), \cdot\right)$ is a semigroup.

In what follows, we consider especially binary trees in which both the inner nodes and the leaves are labelled with symbols from a given finite non-empty alphabet $A$, the label alphabet. To obtain compatibility with the term formalism, we define them formally as follows. First of all, we associate with $A$ the ranked alphabet $\Sigma^{A}=\Sigma_{0}^{A} \cup \Sigma_{2}^{A}$, where $\Sigma_{0}^{A}=\left\{c_{a} \mid a \in A\right\}$ and $\Sigma_{2}^{A}=\left\{f_{a} \mid a \in A\right\}$. We shall call $\Sigma^{A}$-trees $\Sigma^{A}$-contexts simply $A$-trees and $A$ contexts, respectively, and the notation is simplified correspondingly. Hence the set $T_{A}$ of $A$-trees and the set $C_{A}$ of $A$-contexts are defined inductively:
(1) $c_{a} \in T_{A}$ for every $a \in A$, and $\xi \in C_{A}$;
(2) $f_{a}(s, t) \in T_{A}$ and $f_{a}(p, t), f_{a}(t, p) \in C_{A}$ for all $a \in A, s, t \in T_{A}$ and $p \in C_{A}$.

Moreover, let $C_{A}^{+}=C_{A} \backslash\{\xi\}$ be the set of non-unit $A$-contexts.
The $\Sigma^{A}$-algebra of $A$-trees $\mathcal{T}_{A}=\left(T_{A}, \Sigma^{A}\right)$ is defined by setting
(1) $c_{a}^{\mathcal{T}_{A}}=c_{a}$ for every $a \in A$, and
(2) $f_{a}^{\mathcal{T}_{A}}(s, t)=f_{a}(s, t)$ for every $a \in A$ and all $s, t \in T_{A}$.

Since $\mathcal{T}_{A}$ is the $\Sigma^{A}$-term algebra generated by the empty set, there is for each $\Sigma^{A}$-algebra $\mathcal{D}$ a unique homomorphism $\varphi_{\mathcal{D}}: \mathcal{T}_{A} \rightarrow \mathcal{D}$ defined by
(1) $c_{a} \varphi_{\mathcal{D}}=c_{a}^{\mathcal{D}}$ for $a \in A$, and
(2) $f_{a}(s, t) \varphi_{\mathcal{D}}=f_{a}^{\mathcal{D}}\left(s \varphi_{\mathcal{D}}, t \varphi_{\mathcal{D}}\right)$ for any $a \in A$ and $s, t \in T_{A}$.

Subsets of $T_{A}$ we call $A$-tree languages, and a binary tree language is any set that is an $A$-tree language for some label alphabet $A$.

Let us now introduce Wilke's [33] formalism for representing binary trees by terms over a 3 -sorted ranked alphabet. An overview of the theory of manysorted algebras, as well as many further references, can be found in [19]. In [25] we have developed a general theory of varieties of recognizable subsets of many-sorted algebras, and some of the notions and facts to be presented here could be obtained by a suitable specialization from that theory.

The set of sorts is $S=\{$ label, tree, context $\}$. For the sort names we use the abbreviations $\mathbf{l}=$ label, $\mathbf{t}=$ tree and $\mathbf{c}=$ context. An $S$-sorted set $M$ is a triple $\left\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\right\rangle$ in which $M_{\mathbf{l}}, M_{\mathbf{t}}$ and $M_{\mathbf{c}}$ are the sets of elements of $M$ of sort label, tree and context, respectively. Although this would not be quite necessary, we shall always assume that the sets $M_{1}, M_{\mathrm{t}}$ and $M_{\mathrm{c}}$ are pairwise disjoint, i.e., that the sort of each element of $M$ is uniquely determined.

Now let $\Gamma=\{\iota, \kappa, \lambda, \rho, \eta, \sigma\}$ be the $S$-sorted ranked alphabet where the types of the symbols are as follows:

$$
\iota: \mathbf{l} \rightarrow \mathbf{t} ; \quad \kappa: \mathbf{l} \mathbf{t} \mathbf{t} \rightarrow \mathbf{t} ; \quad \lambda, \rho: \mathbf{l} \mathbf{t} \rightarrow \mathbf{c} ; \quad \eta: \mathbf{c t} \rightarrow \mathbf{t} ; \quad \sigma: \mathbf{c c} \rightarrow \mathbf{c} .
$$

For example, in any $\Gamma$-algebra the $\lambda$-operation forms from an element of sort $\mathbf{l}$ and an element of sort $\mathbf{t}$ an element of sort $\mathbf{c}$.

For the general notion of many-sorted terms we refer the reader to [19] or 25]. Here we introduce just the kind of $\Gamma$-terms to be used in this paper. The $S$-sorted set $\left\langle A, T_{\Gamma}(A), C_{\Gamma}^{+}(A)\right\rangle$ of $\Gamma A$-terms, where $T_{\Gamma}(A)$ is the set of $\Gamma A$-tree terms, and $C_{\Gamma}^{+}(A)$ the set of non-unit $\Gamma A$-context terms, is defined inductively as follows:
(1) if $a \in A$, then $\iota(a) \in T_{\Gamma}(A)$;
(2) if $a \in A$ and $\mathrm{s}, \mathrm{t} \in T_{\Gamma}(A)$, then $\kappa(\mathrm{a}, \mathrm{s}, \mathrm{t}) \in T_{\Gamma}(A)$;
(3) if $a \in A$ and $t \in T_{\Gamma}(A)$, then $\lambda(a, \mathrm{t}) \in C_{\Gamma}^{+}(A)$;
(4) if $a \in A$ and $t \in T_{\Gamma}(A)$, then $\rho(a, \mathrm{t}) \in C_{\Gamma}^{+}(A)$;
(5) if $\mathrm{p} \in C_{\Gamma}^{+}(A)$ and $\mathrm{t} \in T_{\Gamma}(A)$, then $\eta(\mathrm{p}, \mathrm{t}) \in T_{\Gamma}(A)$;
(6) if $\mathrm{p}, \mathrm{q} \in C_{\Gamma}^{+}(A)$, then $\sigma(\mathrm{p}, \mathrm{q}) \in C_{\Gamma}^{+}(A)$.

Hence, the $\Gamma A$-terms are the $\Gamma$-terms over the sorted set $X=\langle A, \emptyset, \emptyset\rangle$ of variables, where $A$ is a the given label alphabet.

Remark 2.1 Note that $C_{\Gamma}^{+}(A)$ does not include the unit context $\xi$. Similarly, the syntactic tree algebras - to be defined later - do not automatically have a unit element of sort context. This means that, in a way, Wilke's [33] theory corresponds to Eilenberg's [8] theory of +-varieties and syntactic semigroups. By adding to $\Gamma$ a constant of sort context one could obtain a variant of the theory that corresponds to the theory of $*$-varieties and syntactic monoids.

Binary $A$-trees and $A$-contexts are represented by $\Gamma A$-tree terms and $\Gamma A$-context terms as follows. For any $\mathrm{t} \in T_{\Gamma}(A)$, let $\hat{\mathrm{t}}$ denote the $A$-tree represented by t. Similarly, $\hat{\mathrm{p}}$ denotes the $A$-context represented by a $\Gamma A$ context term $\mathrm{p} \in C_{\Gamma}^{+}(A)$. The representations are defined by setting for any $a \in A, \mathrm{~s}, \mathrm{t} \in T_{\Gamma}(A)$ and $\mathrm{p}, \mathrm{q} \in C_{\Gamma}^{+}(A)$,
(1) $\iota(a)$ represents the $A$-tree $c_{a}$,
(2) $\kappa(a, \mathrm{~s}, \mathrm{t})$ represents the $A$-tree $f_{a}(\hat{\mathrm{~s}}, \hat{\mathrm{t}})$,
(3) $\lambda(a, \mathrm{t})$ represents the $A$-context $f_{a}(\xi, \hat{\mathrm{t}})$,
(4) $\rho(a, \mathrm{t})$ represents the $A$-context $f_{a}(\hat{\mathrm{t}}, \xi)$,
(5) $\eta(\mathrm{p}, \mathrm{t})$ represents the $A$-tree $\hat{\mathrm{p}}(\hat{\mathrm{t}})$, and
(6) $\sigma(\mathrm{p}, \mathrm{q})$ represents the $A$-context $\hat{\mathrm{p}}(\hat{\mathrm{q}})$.

The following facts are easy to verify by induction on $A$-trees and $A$-contexts.
Lemma 2.2 Let $A$ be any label alphabet. For any $A$-tree $t$ we can find $a$ $\Gamma A$-tree term $\mathrm{t} \in T_{\Gamma}(A)$ such that $\hat{\mathrm{t}}=t$, and for any non-unit $A$-context $p$ a $\Gamma A$-context term $\mathrm{p} \in C_{\Gamma}^{+}(A)$ such that $\hat{\mathrm{p}}=p$.

These representations are usually not unique. For example, the $\{a, b\}$-tree terms $\kappa(b, \kappa(a, \iota(b), \iota(a)), \iota(a))$ and $\eta(\lambda(b, \iota(a)), \kappa(a, \iota(b), \iota(a)))$ both represent the same $\{a, b\}$-tree $f_{b}\left(f_{a}\left(c_{b}, c_{a}\right), c_{a}\right)$. Later on we formulate this representation relation as a homomorphism.

## 3 Tree algebras

А $\Gamma$-algebra $\mathcal{M}=\left(\left\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\right\rangle, \Gamma\right)$ consists of a nonempty set $M_{\mathbf{l}}$ of elements of sort label, a nonempty set $M_{\mathrm{t}}$ of elements of sort tree, and a nonempty set $M_{\mathrm{c}}$ of elements of sort context, and operations
(1) $\iota^{\mathcal{M}}: M_{\mathrm{l}} \rightarrow M_{\mathrm{t}}$
(2) $\kappa^{\mathcal{M}}: M_{\mathrm{l}} \times M_{\mathrm{t}} \times M_{\mathrm{t}} \rightarrow M_{\mathrm{t}}$
(3) $\lambda^{\mathcal{M}}: M_{\mathbf{l}} \times M_{\mathrm{t}} \rightarrow M_{\mathbf{c}}$
(4) $\rho^{\mathcal{M}}: M_{\mathbf{l}} \times M_{\mathrm{t}} \rightarrow M_{\mathrm{c}}$
(5) $\eta^{\mathcal{M}}: M_{\mathrm{c}} \times M_{\mathrm{t}} \rightarrow M_{\mathrm{t}}$
(6) $\sigma^{\mathcal{M}}: M_{\mathbf{c}} \times M_{\mathbf{c}} \rightarrow M_{\mathbf{c}}$,
defined as realizations of the symbols in $\Gamma$. Usually we write simply $\mathcal{M}=$ $(M, \Gamma)$ with the understanding that $M=\left\langle M_{\mathbf{l}}, M_{\mathbf{t}}, M_{\mathbf{c}}\right\rangle$.

The basic algebraic notions, such as subalgebras, congruences, homomorphisms etc., are defined for $\Gamma$-algebras the same way as for many-sorted algebras in general (cf. [19] or [25]). For example, a homomorphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ from a $\Gamma$-algebra $\mathcal{M}=(M, \Gamma)$ to a $\Gamma$-algebra $\mathcal{N}=(N, \Gamma)$ is a sorted mapping $\varphi: M \rightarrow N$, i.e., an $S$-sorted triple of maps $\left\langle\varphi_{1}: M_{1} \rightarrow N_{\mathrm{l}}, \varphi_{\mathrm{t}}: M_{\mathrm{t}} \rightarrow\right.$ $\left.N_{\mathbf{t}}, \varphi_{\mathbf{c}}: M_{\mathbf{c}} \rightarrow N_{\mathbf{c}}\right\rangle$ that preserves all the $\Gamma$-operations between $\mathcal{M}$ and $\mathcal{N}$, that is to say, $\iota^{\mathcal{M}}(a) \varphi_{\mathbf{t}}=\iota^{\mathcal{N}}\left(a \varphi_{\mathbf{1}}\right)$ for every $a \in M_{\mathbf{1}}, \kappa^{\mathcal{M}}(a, s, t) \varphi_{\mathbf{t}}=$ $\kappa^{\mathcal{N}}\left(a \varphi_{\mathbf{1}}, s \varphi_{\mathbf{t}}, t \varphi_{\mathbf{t}}\right)$ for all $a \in M_{1}$ and $s, t \in M_{\mathbf{t}}$, etc. A homomorphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ is an epimorphism if $\varphi$ is surjective, i.e., $\varphi_{\mathrm{i}}: M_{\mathbf{i}} \rightarrow N_{\mathrm{i}}$ is surjective for every $\mathbf{i} \in S$. Similarly, $\varphi$ is a monomorphism if every $\varphi_{\mathbf{i}}: M_{\mathbf{i}} \rightarrow N_{\mathbf{i}}$ is injective. Finally, an isomorphism is a bijective homomorphism. The fact that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic is denoted by writing $\mathcal{M} \cong \mathcal{N}$.

For any label alphabet $A$, the $\Gamma$-algebra of $\Gamma A$-terms

$$
\mathcal{T}_{\Gamma}(A)=\left(\left\langle A, T_{\Gamma}(A), C_{\Gamma}^{+}(A)\right\rangle, \Gamma\right)
$$

is defined by setting
(1) $\iota^{\mathcal{T}_{\Gamma}(A)}(a)=\iota(a), \quad(2) \kappa^{\mathcal{T}_{\Gamma}(A)}(a, \mathrm{~s}, \mathrm{t})=\kappa(a, \mathrm{~s}, \mathrm{t})$,
(3) $\lambda^{T_{\Gamma}(A)}(a, t)=\lambda(a, t)$,
(4) $\rho^{\tau_{\Gamma}(A)}(a, \mathrm{t})=\rho(a, \mathrm{t})$,
(5) $\eta^{\mathcal{T}_{\Gamma}(A)}(\mathrm{p}, \mathrm{t})=\eta(\mathrm{p}, \mathrm{t})$,
(6) $\sigma^{\mathcal{T}_{\Gamma}(A)}(\mathrm{p}, \mathrm{q})=\sigma(\mathrm{p}, \mathrm{q})$,
for all $a \in A, \mathrm{~s}, \mathrm{t} \in T_{\Gamma}(A)$ and $\mathrm{p}, \mathrm{q} \in C_{\Gamma}^{+}(A)$.
Following [33], we call a $\Gamma$-algebra a tree algebra if it satisfies the following set of identities $T A$ :
(TA1) $\sigma(\sigma(\mathrm{p}, \mathrm{q}), \mathrm{r})) \approx \sigma(\mathrm{p}, \sigma(\mathrm{q}, \mathrm{r}))$,
(TA2) $\eta(\sigma(\mathrm{p}, \mathrm{q}), \mathrm{t}) \approx \eta(\mathrm{p}, \eta(\mathrm{q}, \mathrm{t}))$,
(TA3) $\eta(\lambda(\mathrm{a}, \mathrm{s}), \mathrm{t}) \approx \kappa(\mathrm{a}, \mathrm{t}, \mathrm{s})$, and
(TA4) $\eta(\rho(\mathrm{a}, \mathrm{s}), \mathrm{t}) \approx \kappa(\mathrm{a}, \mathrm{s}, \mathrm{t})$.
Here, a is a variable of sort label, s and t are variables of sort tree, and p , $q$ and $r$ variables of sort context. Let TA denote the equational class of all tree algebras.

For each label alphabet $A$, a tree algebra of special interest is the $\Gamma$ algebra of $A$-trees $\mathcal{F}_{\mathbf{T A}}(A)=\left(\left\langle A, T_{A}, C_{A}^{+}\right\rangle, \Gamma\right)$, where for any $a \in A, s, t \in T_{A}$ and $p, q \in C_{A}^{+}$,
(1) $\iota^{\mathcal{F}_{\mathbf{T A}}(A)}(a)=c_{a}$,
(2) $\kappa^{\mathcal{F}_{\mathbf{T A}}(A)}(a, s, t)=f_{a}(s, t)$,
(3) $\lambda^{\mathcal{F}_{\mathbf{T A}}(A)}(a, t)=f_{a}(\xi, t)$,
(4) $\rho^{\mathcal{F}_{\mathrm{TA}}(A)}(a, t)=f_{a}(t, \xi)$,
(5) $\eta^{\mathcal{F}_{\mathrm{TA}}(A)}(p, t)=p(t)$,
(6) $\sigma^{\mathcal{F}_{\mathbf{T A}}(A)}(p, q)=p(q)$.

As shown by Wilke ([33], Proposition 1), and suggested by our notation, $\mathcal{F}_{\mathbf{T A}}(A)$ is the free tree algebra generated by $\langle A, \emptyset, \emptyset\rangle$. This means that $\mathcal{F}_{\mathbf{T A}}(A)$ satisfies the identities $T A$, and that if $\mathcal{M}=(M, \Gamma)$ is any tree algebra, then every mapping $\varphi_{0}: A \rightarrow M_{1}$ can be extended in a unique way to a homomorphism $\varphi=\left\langle\varphi_{\mathbf{l}}, \varphi_{\mathbf{t}}, \varphi_{\mathbf{c}}\right\rangle$ of $\Gamma$-algebras from $\mathcal{F}_{\mathbf{T A}}(A)$ to $\mathcal{M}$. Obviously, then $\varphi_{1}=\varphi_{0}$.

For any label alphabet $A$, an $A$-instance of an identity in $T A$ is any pair of $\Gamma A$-tree or $\Gamma A$-context terms obtained from the identity by assigning each of the variables a, s, $\mathrm{t}, \mathrm{p}, \mathrm{q}$ and r appearing in it a value from the appropriate set $A, T_{A}$ or $C_{A}^{+}$. For example, if $b, c \in A$, then

$$
(\eta(\lambda(c, \iota(b)), \iota(c)), \kappa(c, \iota(b), \iota(c)))
$$

is the $A$-instance of (TA3) obtained by the substitution

$$
\mathrm{a} \mapsto c, \mathrm{~s} \mapsto \iota(b), \mathrm{t} \mapsto \iota(c)
$$

Moreover, let $\equiv{ }^{A}$ be the fully invariant congruence on $\mathcal{T}_{\Gamma}(A)$ generated by the set of all $A$-instances of $T A$, i.e., the equational theory in variables $\langle A, \emptyset, \emptyset\rangle$ defined by $T A$.

It is clear that if $(\mathrm{u}, \mathrm{v})$ is an $A$-instance of an identity in $T A$, then $\hat{\mathrm{u}}=$ $\hat{\mathrm{v}}$. Furthermore, if $(\mathrm{u}, \mathrm{v})$ is obtained from pairs of $\Gamma A$-terms representing the same $A$-tree or the same $A$-context by any inference rule of Birkhoff's equational logic (for the many-sorted version, cf. Section 5.2 in [19]), then again $\hat{\mathrm{u}}=\hat{\mathrm{v}}$. Hence we get at this point the soundness property of Wilke's axiom system $T A$.

Proposition 3.1 Let $A$ be any label alphabet. For any s, t $\in T_{\Gamma}(A)$ and $\mathrm{p}, \mathrm{q} \in C_{\Gamma}^{+}(A)$,
(a) if $\mathrm{s} \equiv_{\mathrm{t}}^{A} \mathrm{t}$, then $\hat{\mathrm{s}}=\hat{\mathrm{t}}$, and
(b) if $\mathrm{p} \equiv_{\mathrm{c}}^{A} \mathrm{q}$, then $\hat{\mathrm{p}}=\hat{\mathrm{q}}$.

As the $\Gamma$-algebras $\mathcal{T}_{\Gamma}(A)$ and $\mathcal{F}_{\mathbf{T A}}(A)$ both are generated by $\langle A, \emptyset, \emptyset\rangle$, the identity mapping $1_{A}: A \rightarrow A$ (of sort label) can be extended in a unique way to an epimorphism $\nu^{A}: \mathcal{T}_{\Gamma}(A) \rightarrow \mathcal{F}_{\mathbf{T A}}(A)$ of $\Gamma$-algebras that we call the canonical $A$-homomorphism. It is the triple of mappings

$$
\left\langle\nu_{\mathbf{l}}^{A}: A \rightarrow A, \nu_{\mathbf{t}}^{A}: T_{\Gamma}(A) \rightarrow T_{A}, \nu_{\mathbf{c}}^{A}: C_{\Gamma}^{+}(A) \rightarrow C_{A}^{+}\right\rangle
$$

such that for all $a \in A, s, t \in T_{\Gamma}(A)$ and $p, q \in C_{\Gamma}^{+}(A)$,
(1) $\nu_{1}^{A}(a)=a$,
(2) $\nu_{\mathbf{t}}^{A}(\iota(a))=c_{a}$,
(3) $\nu_{\mathbf{t}}^{A}(\kappa(a, \mathrm{~s}, \mathrm{t}))=f_{a}\left(\nu_{\mathbf{t}}^{A}(\mathrm{~s}), \nu_{\mathbf{t}}^{A}(\mathrm{t})\right)$,
(4) $\nu_{\mathbf{c}}^{A}(\lambda(a, \mathrm{t}))=f_{a}\left(\xi, \nu_{\mathbf{t}}^{A}(\mathrm{t})\right)$,
(5) $\nu_{\mathbf{c}}^{A}(\rho(a, \mathrm{t}))=f_{a}\left(\nu_{\mathbf{t}}^{A}(\mathrm{t}), \xi\right)$,
(6) $\nu_{\mathrm{t}}^{A}(\eta(\mathrm{p}, \mathrm{t}))=\nu_{\mathrm{c}}^{A}(\mathrm{p})\left(\nu_{\mathrm{t}}^{A}(\mathrm{t})\right)$, and
(7) $\nu_{\mathbf{c}}^{A}(\sigma(\mathrm{p}, \mathrm{q}))=\nu_{\mathbf{c}}^{A}(\mathrm{p})\left(\nu_{\mathbf{c}}^{A}(\mathrm{q})\right)$.

The following lemma is obtained immediately by comparing the above equalities with the clauses defining the $A$-trees and $A$-contexts represented by $\Gamma A$-terms.

Lemma 3.2 For any $\Gamma A$-tree term $\mathrm{t} \in T_{\Gamma}(A)$ and any $\Gamma A$-context term $\mathrm{p} \in C_{\Gamma}^{+}(A)$, we have $\nu_{\mathrm{t}}^{A}(\mathrm{t})=\hat{\mathrm{t}}$ and $\nu_{\mathrm{c}}^{A}(\mathrm{p})=\hat{\mathrm{p}}$.

## 4 Normal form representations

Let us now transform the set of identities $T A$ into a convergent (i.e., terminating and confluent) term rewriting system. For the general theory of term rewriting and the few notions needed here, cf. [3], [4], [6] or [17], for example.

Definition 4.1 Let $\mathcal{R}$ be the term rewriting system consisting of the rules
$(\mathrm{R} 1) \sigma(\sigma(\mathrm{p}, \mathrm{q}), \mathrm{r})) \rightarrow \sigma(\mathrm{p}, \sigma(\mathrm{q}, \mathrm{r}))$,
$(\mathrm{R} 2) ~ \eta(\sigma(\mathrm{p}, \mathrm{q}), \mathrm{t}) \rightarrow \eta(\mathrm{p}, \eta(\mathrm{q}, \mathrm{t}))$,
(R3) $\eta(\lambda(\mathrm{a}, \mathrm{s}), \mathrm{t}) \rightarrow \kappa(\mathrm{a}, \mathrm{t}, \mathrm{s})$, and
(R4) $\eta(\rho(\mathrm{a}, \mathrm{s}), \mathrm{t}) \rightarrow \kappa(\mathrm{a}, \mathrm{s}, \mathrm{t})$.
Proposition 4.2 The system $\mathcal{R}$ is convergent.
Proof. It is clear that $\mathcal{R}$ is compatible with the lexicographic path ordering induced by any order on $\Gamma$ such that $\eta>\kappa$. Hence, $\mathcal{R}$ is terminating. There are just two critical pairs. The pair

$$
\langle\eta(\sigma(\mathrm{p}, \sigma(\mathrm{q}, \mathrm{r})), \mathrm{t}), \eta(\sigma(\mathrm{p}, \mathrm{q}), \eta(\mathrm{r}, \mathrm{t}))\rangle
$$

produced by (R1) and (R2) converges to $\eta(\mathrm{p}, \eta(\mathrm{q}, \eta(\mathrm{r}, \mathrm{t})))$ by applications of (R2), and the other critical pair

$$
\left\langle\sigma\left(\sigma(\mathrm{p}, \sigma(\mathrm{q}, \mathrm{r})), \mathrm{r}^{\prime}\right), \sigma\left(\sigma(\mathrm{p}, \mathrm{q}), \sigma\left(\mathrm{r}, \mathrm{r}^{\prime}\right)\right)\right\rangle
$$

obtained by overlapping (R1) with itself, converges to $\sigma\left(\mathrm{p}, \sigma\left(\mathrm{q}, \sigma\left(\mathrm{r}, \mathrm{r}^{\prime}\right)\right)\right)$ by further applications of (R1). Hence, $\mathcal{R}$ is confluent as well.

Let ${ }_{A} \Rightarrow$ be the ( $S$-sorted) reduction relation defined by $\mathcal{R}$ on the set $\left\langle A, T_{\Gamma}(A), C_{\Gamma}^{+}(A)\right\rangle$ of $\Gamma A$-terms, and let $A^{A} \Leftrightarrow^{*}$ be its equivalence closure. It follows directly from the definitions of $\mathcal{R}$ and $\equiv^{A}$ that ${ }_{A} \Leftrightarrow^{*}=\equiv^{A}$. This suggests the idea to define normal form representations of $A$-trees and $A$ contexts by using $\mathcal{R}$.

Let $\operatorname{IRR}(\mathcal{R}, A)_{\mathbf{1}}, \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$ and $\operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$ be the sets of $\Gamma A$-terms irreducible by $\mathcal{R}$ of sort label, tree and context, respectively. Clearly, $\operatorname{IRR}(\mathcal{R}, A)_{\mathbf{1}}=A$. The other two sets are described in the following proposition.

Proposition 4.3 Let $A$ be any label alphabet.
a. $A \Gamma A$-tree term is irreducible iff it contains the operators $\iota$ and $\kappa$ only, that is to say, $\operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$ is obtained inductively thus:
(1) $\iota(a) \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$ for every $a \in A$, and
(2) if $a \in A$ and $\mathrm{s}, \mathrm{t} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$, then $\kappa(a, \mathrm{~s}, \mathrm{t}) \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$.
b. $\operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$ is the smallest subset of $C_{\Gamma}^{+}(A)$ such that
(1') $\lambda(a, \mathrm{t}), \rho(a, \mathrm{t}) \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$ for all $a \in A$ and $\mathrm{t} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$, and
(2') $\sigma(\lambda(a, \mathrm{t}), \mathrm{p}) \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$ and $\sigma(\rho(a, \mathrm{t}), \mathrm{p}) \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$ for any $a \in A, \mathrm{t} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{t}}$ and $\mathrm{p} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$.

Proof. By considering the rules of $\mathcal{R}$ one sees that clauses (1) and (2) define a set of irreducible $\Gamma A$-tree terms. On the other hand, any $\Gamma A$-tree term with a subterm of the form $\eta(\mathrm{p}, \mathrm{t})$ is reducible because the $\Gamma A$-context term p must begin with $\lambda, \rho$ or $\sigma$. Hence, all irreducible $\Gamma A$-tree terms are obtained by clauses (1) and (2).

It is clear that no rule of $\mathcal{R}$ applies to any $\Gamma A$-context term obtained by rules ( $1^{\prime}$ ) and ( $2^{\prime}$ ). That ( $1^{\prime}$ ) and ( $2^{\prime}$ ) yield all irreducible $\Gamma A$-context terms, is verified by induction on the $\xi$-depth $\mathrm{d}_{\xi}(\hat{\mathrm{p}})$ of the $A$-context represented by $\mathrm{p} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathrm{c}}$. If $\mathrm{d}_{\xi}(\hat{\mathrm{p}})=1$, then p must be a $\Gamma A$-context term given by ( $1^{\prime}$ ). If $\mathrm{d}_{\xi}(\hat{\mathrm{p}})>1$, then $\mathrm{p}=\sigma(\mathrm{q}, \mathrm{r})$ for some $\mathrm{q}, \mathrm{r} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathbf{c}}$, and because of rule (R1), q must be of the form $\lambda(a, \mathrm{t})$ or $\rho(a, \mathrm{t})$ with $\mathrm{t} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathrm{t}}$. Since the inductive assumption applies to r , also p is of the required type.

As noted in Proposition 3.1, any two $\equiv^{A}$-congruent $\Gamma A$-tree terms represent the same $A$-tree. Therefore it follows now from Lemma 2.2 and Proposition 4.2 that any $A$-tree is represented by a unique irreducible $\Gamma A$-tree term. By Proposition 4.3 only the operators $\iota$ and $\kappa$ appear in irreducible
$\Gamma A$-tree terms, and hence it is clear that if $\mathrm{s}, \mathrm{t} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathrm{t}}$ and $\mathrm{s} \neq \mathrm{t}$, then $\hat{\mathrm{s}} \neq \hat{\mathrm{t}}$. Similarly, $A$-contexts are represented by unique irreducible $\Gamma A$-context terms, and since these are of the form

$$
\sigma\left(\mathrm{p}_{1},\left(\sigma\left(\mathrm{p}_{2}, \ldots \sigma\left(\mathrm{p}_{n-1}, \mathrm{p}_{n}\right) \ldots\right),\right.\right.
$$

where $n \geq 1$, and each $\mathrm{p}_{i}$ is of the form $\lambda(a, \mathrm{t})$ or $\rho(a, \mathrm{t})$ with $a \in A$ and $\mathrm{t} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathrm{t}}$, it is again clear that $\hat{\mathrm{p}} \neq \hat{\mathrm{q}}$ for any two distinct $\mathrm{p}, \mathrm{q} \in \operatorname{IRR}(\mathcal{R}, A)_{\mathrm{c}}$. These observations yield the following proposition that completes the picture.

Proposition 4.4 Let $A$ be any label alphabet. Every $A$-tree is represented by a unique $\mathcal{R}$-irreducible $\Gamma$-tree term and hence, if $\hat{\mathrm{s}}=\hat{\mathrm{t}}$ for any $\Gamma A$-tree terms $\mathrm{s}, \mathrm{t} \in T_{\Gamma}(A)$, then $\mathrm{s} \equiv^{A} \mathrm{t}$. Similarly, each $A$-context is represented by a unique $\mathcal{R}$-irreducible $\Gamma A$-context term, and any two $\Gamma A$-context terms that represent the same $A$-context are $\equiv^{A}$-congruent.

By combining this result with Lemma 3.2 and Proposition 3.1, we get the following fact.

Corollary 4.5 For any label alphabet $A$, $\operatorname{ker} \nu^{A}=\equiv^{A}$.
Furthermore, we now get Wilke's 33 Proposition 1 in a new way:
Corollary 4.6 $\mathcal{F}_{\mathbf{T A}}(A)$ is the free tree algebra generated by $\langle A, \emptyset, \emptyset\rangle$.
Proof. Since $\equiv{ }^{A}$ is the fully invariant congruence generated by the $A$-instances of the identities (TA), the quotient algebra $\mathcal{T}_{\Gamma}(A) / \equiv^{A}$ is freely generated by $\langle A, \emptyset, \emptyset\rangle$ (we identify each $a \in A$ with its $\equiv^{A}$-class $\{a\}$ ) over the class TA. On the other hand, $\left.\mathcal{F}_{\mathbf{T A}}(A) \cong \mathcal{T}_{\Gamma}(A)\right) /$ ker $\nu^{A}$ by the Homomorphism Theorem (cf. 19 for the many-sorted version).

By combining Propositions 3.1 and 4.4 , we get as a further corollary the following result.

Proposition 4.7 Let $A$ be any label alphabet.
(a) For any $\mathrm{s}, \mathrm{t} \in T_{\Gamma}(A), \hat{\mathrm{s}}=\hat{\mathrm{t}}$ if and only if $\mathrm{s} \equiv_{\mathrm{t}}^{A} \mathrm{t}$.
(b) For any $\mathrm{p}, \mathrm{q} \in C_{\Gamma}^{+}(A), \hat{\mathrm{p}}=\hat{\mathrm{q}}$ if and only if $\mathrm{p} \equiv_{\mathrm{c}}^{A} \mathrm{q}$.

The proposition may be regarded as a Completeness Theorem for Wilke's axiomatization (TA) with respect to representations of binary trees and contexts. Indeed, it means that any two $\Gamma A$-tree or $\Gamma A$-context terms represent the same $A$-tree or $A$-context, respectively, iff they are $T A$-provably equal.

By Proposition 4.7 the the equational theory $\equiv^{A}$ is trivially decidable: to decide whether $\mathrm{s} \equiv{ }_{\mathrm{t}}^{A} \mathrm{t}$ holds for any given $\mathrm{s}, \mathrm{t} \in T_{\Gamma}(A)$, it suffices to construct the $A$-trees $\hat{\mathrm{s}}$ and $\hat{\mathrm{t}}$ and compare them with each other. Similarly, $\mathrm{p} \equiv_{\mathrm{c}}^{A} \mathrm{q}$ iff $\hat{\mathrm{p}}=\hat{\mathrm{q}}$, for any given $\mathrm{p}, \mathrm{q} \in C_{\Gamma}^{+}(A)$. Of course, this fact is implicit also in [33] since it follows also from Corollary 4.6 (and also from Corollary 4.5, for that matter). However, let us also note that Proposition 4.2 yields also another decision method that does not require forming the trees or contexts: whether any two given $\Gamma A$-tree terms, or two $\Gamma A$-context terms, are $\equiv^{A}$-equivalent can be decided by computing their respective $\mathcal{R}$-normal forms.

## 5 Syntactic 「-algebras

The basic properties of Wilke's [33] syntactic tree algebra congruences and syntactic tree algebras of binary tree languages can be obtained conveniently by considering more generally subsets of arbitrary $\Gamma$-algebras. In [25] we studied these notions for subsets of general many-sorted algebras. Two kinds of subsets were considered, the sorted subsets that have of a component of each sort, and the "pure" subsets consisting of elements of one given sort. Since we eventually apply these notions just to binary tree languages, we will focus here on pure subsets. The general theory will be used here by letting the set of sorts be $S=\{$ label, tree, context $\}$ and the ranked alphabet be $\Gamma=\{\iota, \kappa, \lambda, \rho, \eta, \sigma\}$ as above. In the following section we will then recover Wilke's notions by considering subsets of the free tree algebras $\mathcal{F}_{\mathrm{TA}}(A)$.

A sorted subset of an $S$-sorted set $M$ is a triple $\left\langle L_{1}, L_{\mathbf{t}}, L_{\mathbf{c}}\right\rangle$ such that $L_{1} \subseteq M_{1}, L_{\mathrm{t}} \subseteq M_{\mathrm{t}}$ and $L_{\mathrm{c}} \subseteq M_{\mathrm{c}}$. The inclusion relation and the basic set operations are defined for sorted subsets by the natural sortwise conditions. A subset of sort $\mathbf{i} \in S$ of $M$ is any subset of $M_{\mathbf{i}}$. With a subset $T \subseteq M_{\mathbf{i}}$ of sort $\mathbf{i}$ we associate the sorted subset $\langle T\rangle=\left\langle T_{\mathbf{1}}, T_{\mathbf{t}}, T_{\mathbf{c}}\right\rangle$ such that $T_{\mathbf{i}}=T$ and $T_{\mathbf{j}}=\emptyset$ for $\mathbf{j} \in S, \mathbf{j} \neq \mathbf{i}$. By identifying $T$ with $\langle T\rangle$, we may treat $T$ as a special sorted subset.

Let $\mathcal{M}=(M, \Gamma)$ be a $\Gamma$-algebra. For any $\mathbf{i}, \mathbf{j} \in S$, an elementary $\mathbf{i j}$ translation is any mapping $M_{\mathbf{i}} \rightarrow M_{\mathrm{j}}$ obtained from one of the fundamental operations of sort $\mathbf{j}$ of $\mathcal{M}$ by fixing the values of all arguments save one that is of sort $\mathbf{i}$. Let $\operatorname{ETr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ denote the set of all elementary $\mathbf{i j}$-translations. Thus, for example, $\operatorname{ETr}(\mathcal{M}, \mathbf{l}, \mathbf{t})=\left\{\iota^{\mathcal{M}}\left(\xi_{1}\right)\right\} \cup\left\{\kappa^{\mathcal{M}}\left(\xi_{\mathbf{l}}, u, v\right) \mid u, v \in M_{\mathbf{t}}\right\}$, where $\xi_{1}$ is a variable of sort label. The sets $\operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ of $\mathbf{i j}$-translations ( $\mathbf{i}, \mathbf{j} \in S$ ) are defined inductively by the following:
(1) $\mathrm{ETr}(\mathcal{M}, \mathbf{i}, \mathbf{j}) \subseteq \operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ for all $\mathbf{i}, \mathbf{j} \in S$;
(2) for each $\mathbf{i} \in S$, the identity mapping $1_{\mathbf{i}}: M_{\mathbf{i}} \rightarrow M_{\mathbf{i}}, u \mapsto u$, is in $\operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{i})$;
(3) if $\alpha\left(\xi_{\mathbf{i}}\right) \in \operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ and $\beta\left(\xi_{\mathbf{j}}\right) \in \operatorname{Tr}(\mathcal{M}, \mathbf{j}, \mathbf{k})$ for some $\mathbf{i}, \mathbf{j}, \mathbf{k} \in S$, then $\beta\left(\alpha\left(\xi_{\mathbf{i}}\right)\right) \in \operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{k})$.

Let $\theta=\left\langle\theta_{\mathbf{l}}, \theta_{\mathbf{t}}, \theta_{\mathbf{c}}\right\rangle$ be a sorted equivalence on $M$, i.e., $\theta_{\mathbf{l}}, \theta_{\mathbf{t}}$ and $\theta_{\mathbf{c}}$ are equivalences on $M_{1}, M_{\mathbf{t}}$ and $M_{\mathbf{c}}$, respectively. Then $\theta$ is a congruence on $\mathcal{M}=(M, \Gamma)$ if it is invariant with respect to the operations of $\mathcal{M}$. For example, for the $\kappa$-operation this means that for any $a, a^{\prime} \in M_{1}$ and $s, t, s^{\prime}, t^{\prime} \in M_{\mathbf{t}}$, if $a \theta_{\mathbf{1}} a^{\prime}, s \theta_{\mathbf{t}} s^{\prime}$ and $t \theta_{\mathbf{t}} t^{\prime}$, then $\kappa^{\mathcal{M}}(a, s, t) \theta_{\mathbf{t}} \kappa^{\mathcal{M}}\left(a^{\prime}, s^{\prime}, t^{\prime}\right)$. The congruences of a $\Gamma$-algebra $\mathcal{M}$ enjoy all the general properties the congruences of usual one-sorted algebras. In particular, every congruence of $\mathcal{M}$ is invariant with respect to every translation of $\mathcal{M}$ and, on the other hand, any sorted equivalence on $M$ that is invariant with respect to all elementary translations of $\mathcal{M}$ is a congruence.

Definition 5.1 The syntactic congruence $\approx^{T}$ of a subset $T \subseteq M_{\mathrm{i}}$ of some sort $\mathbf{i} \in S$ is the sorted equivalence $\left\langle\approx_{1}^{T}, \approx_{\mathbf{t}}^{T}, \approx_{\mathbf{c}}^{T}\right\rangle$ on $M$ defined by the condition that for any $\mathbf{j} \in S$ and $u, v \in M_{\mathbf{j}}$,

$$
u \approx_{\mathbf{j}}^{T} v \Leftrightarrow(\forall \alpha \in \operatorname{Tr}(\mathcal{M}, \mathbf{j}, \mathbf{i}))(\alpha(u) \in T \leftrightarrow \alpha(v) \in T) .
$$

and its syntactic algebra is $\mathcal{M} / T:=\mathcal{M} / \approx^{T}$. For an element $u \in M_{\mathbf{j}}$ of any given sort $\mathbf{j} \in S$, let $u / T$ denote the congruence class $u / \approx_{\mathbf{j}}^{T}$ of $u$.

The syntactic congruences and syntactic algebras of (sorted or one-sorted) subsets of $\Gamma$-algebras have the same general properties as the corresponding notions defined for monoids and semigroups [8, 22], for general algebras [1, [27, 28, 29], and for many-sorted algebras [25]. In fact, the following lemmas are special cases of facts presented in [25].

Recall that an equivalence $\theta$ on a set $U$ saturates a subset $L$ of $U$ if $L$ is the union of some $\theta$-classes. Similarly, a sorted equivalence $\theta$ on an $S$-sorted set $M$ saturates a subset $T \subseteq M_{\mathbf{i}}$ of some sort $\mathbf{i} \in S$ if $\theta_{\mathbf{i}}$ saturates $T$.

Lemma 5.2 Let $\mathcal{M}=(M, \Gamma)$ be a $\Gamma$-algebra and $\mathbf{i} \in S$ be a sort. For any subset $T \subseteq M_{\mathrm{i}}, \approx^{T}$ is the greatest congruence on $\mathcal{M}$ that saturates $T$.

Let $\alpha \in \operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ be an $\mathbf{i j}$-translation of a given $\Gamma$-algebra $\mathcal{M}=(M, \Gamma)$ for some $\mathbf{i}, \mathbf{j} \in S$. If $T \subseteq M_{\mathbf{k}}$ is a subset of some sort $\mathbf{k} \in S$, then let $\alpha^{-1}(T):=\left\{u \in M_{\mathbf{i}} \mid \alpha(u) \in T\right\}$ if $\mathbf{k}=\mathbf{j}$, and $\alpha^{-1}(T):=\emptyset$ otherwise. Furthermore, the relation $\varphi \circ \approx^{T} \circ \varphi^{-1}$ appearing in the following lemma denotes the sorted equivalence

$$
\left\langle\varphi_{\mathbf{l}} \circ \approx_{\mathbf{l}}^{T} \circ \varphi_{\mathbf{l}}^{-1}, \varphi_{\mathbf{t}} \circ \approx_{\mathbf{t}}^{T} \circ \varphi_{\mathbf{t}}^{-1}, \varphi_{\mathbf{c}} \circ \approx_{\mathbf{c}}^{T} \circ \varphi_{\mathbf{c}}^{-1}\right\rangle
$$

on $M$, where for all $\mathbf{j} \in S$ and $u, v \in M_{\mathbf{j}}, u \varphi_{\mathbf{j}} \circ \approx_{\mathbf{j}}^{T} \circ \varphi_{\mathbf{j}}^{-1} v$ iff $u \varphi_{\mathbf{j}} \approx_{\mathbf{j}}^{T} v \varphi_{\mathbf{j}}$.
Lemma 5.3 Let $\mathcal{M}=(M, \Gamma)$ and $\mathcal{N}=(N, \Gamma)$ be any $\Gamma$-algebras.
(a) $\approx^{T^{\complement}}=\approx^{T}$ for any subset $T \subseteq M_{\mathbf{i}}$ of any sort $\mathbf{i} \in S$.
(b) $\approx^{T} \cap \approx^{U} \subseteq \approx^{T \cap U}$ and $\approx^{T} \cap \approx^{U} \subseteq \approx^{T \cup U}$ for any subsets $T, U \subseteq M_{\mathbf{i}}$ of any sort $\mathbf{i} \in S$.
(c) If $\alpha \in \operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ is an $\mathbf{i j}$-translation of $\mathcal{M}$ for some $\mathbf{i}, \mathbf{j} \in S$, then $\approx^{T} \subseteq \approx^{\alpha^{-1}(T)}$ for every subset $T \subseteq M_{\mathbf{k}}$ of any sort $\mathbf{k} \in S$.
(d) If $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ is a homomorphism, then $\varphi \circ \approx^{T} \circ \varphi^{-1} \subseteq \approx^{T \varphi_{i}^{-1}}$ for every subset $T \subseteq N_{\mathbf{i}}$ of any sort $\mathbf{i} \in S$. If $\varphi$ is an epimorphism, then $\varphi \circ \approx^{T} \circ \varphi^{-1}=\approx^{T \varphi_{i}^{-1}}$ holds.

Let us now formulate the corresponding facts for syntactic algebras. For this we need a couple of definitions.

Definition 5.4 A $\Gamma$-algebra $\mathcal{N}$ is said to cover a $\Gamma$-algebra $\mathcal{M}$ if $\mathcal{M}$ is an epimorphic image of a subalgebra of $\mathcal{N}$, in notation $\mathcal{M} \preceq \mathcal{N}$.

The covering relation generalizes both

- the subalgebra relation: $\mathcal{M} \subseteq \mathcal{N}$ iff $\mathcal{M}$ is (isomorphic to) a subalgebra of $\mathcal{N}$, and
- the image relation: $\mathcal{M} \leftarrow \mathcal{N}$ iff $\mathcal{M}$ is an epimorphic image of $\mathcal{N}$.

Definition 5.5 A $\Gamma$-algebra $\mathcal{N}$ is said to recognize a subset $T$ of some sort $\mathbf{i} \in I$ of $\mathcal{M}$ if there exist a homomorphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ and subset $F \subseteq N_{\mathbf{i}}$ of sort $\mathbf{i}$ of $\mathcal{N}$ such that $L=F \varphi_{\mathbf{i}}^{-1}$.

Lemma 5.6 Let $\mathcal{M}$ and $\mathcal{N}$ be $\Gamma$-algebras. Then $\mathcal{N}$ recognizes a subset $T \subseteq$ $M_{\mathbf{i}}$ of some sort $\mathbf{i} \in S$ of $\mathcal{M}$ iff $\mathcal{M} / T \preceq \mathcal{N}$.

The lemma expresses the fact that, in a certain sense, the syntactic algebra is the minimal $\Gamma$-algebra recognizing a given subset.

Lemma 5.7 Let $\mathcal{M}=(M, \Gamma)$ and $\mathcal{N}=(N, \Gamma)$ be any $\Gamma$-algebras.
(a) $\mathcal{M} / T^{\complement}=\mathcal{M} / T$ for any subset $T \subseteq M_{\mathbf{i}}$ of any sort $\mathbf{i} \in S$.
(b) $\mathcal{M} / T \cap U \preceq \mathcal{M} / T \times \mathcal{M} / U$ and $\mathcal{M} / T \cup U \preceq \mathcal{M} / T \times \mathcal{M} / U$ for any subsets $T, U \subseteq M_{\mathbf{i}}$ of any sort $\mathbf{i} \in S$.
(c) If $\alpha \in \operatorname{Tr}(\mathcal{M}, \mathbf{i}, \mathbf{j})$ is an $\mathbf{i} \mathbf{j}$-translation of $\mathcal{M}$ for some $\mathbf{i}, \mathbf{j} \in S$, then $\mathcal{M} / \alpha^{-1}(T) \leftarrow \mathcal{M} / T$ for every subset $T \subseteq M_{\mathbf{k}}$ of any sort $\mathbf{k} \in S$.
(d) If $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ is a homomorphism, then $\mathcal{M} / T \varphi_{\mathrm{i}}^{-1} \preceq \mathcal{N} / T$ for every subset $T \subseteq N_{\mathbf{i}}$ and any sort $\mathbf{i} \in S$. If $\varphi$ is an epimorphism, then $\mathcal{M} / T \varphi_{\mathbf{i}}^{-1} \cong \mathcal{N} / T$.

## $6 \quad$ Syntactic tree algebras

We now define the syntactic congruences and syntactic algebras of binary tree languages by regarding these as subsets of sort tree of free tree algebras $\mathcal{F}_{\text {TA }}(A)$, by which the definitions and facts of the previous section get more explicit forms.

Definition 6.1 Let $A$ be a label alphabet. The syntactic tree algebra congruence, the $S T A$-congruence for short, of an $A$-tree language $T$ is its syntactic congruence as a subset of sort tree of the $\Gamma$-algebra $\mathcal{F}_{\mathbf{T A}}(A)$ of $A$-trees. The syntactic algebra $\mathcal{F}_{\mathbf{T A}}(A) / \approx^{T}$ is called the syntactic tree algebra of $T$, and it is denoted by $S T A(T)$.

Since $\mathcal{F}_{\mathbf{T A}}(A)$ is a tree algebra, the syntactic tree algebras of $A$-tree languages are really tree algebras. To show that the above definition agrees with Wilke's [33] definitions, we need a careful analysis of the translations of the free tree algebras $\mathcal{F}_{\mathbf{T A}}(A)$. The relevant parts of such an analysis are presented in the following lemma.

Lemma 6.2 Let $A$ be any label alphabet.
(a) A mapping $\alpha: A \rightarrow T_{A}$ is an $\mathbf{l t}$-translation of $\mathcal{F}_{\mathbf{T A}}(A)$ iff either
(1) there is an $A$-context $p \in C_{A}$ such that $\alpha(a)=p\left(c_{a}\right)$ for every $a \in A$, or
(2) there exist an $A$-context $p \in C_{A}$ and $A$-trees $s, t \in T_{A}$ such that $\alpha(a)=p\left(f_{a}(s, t)\right)$ for every $a \in A$.
(b) A mapping $\alpha: T_{A} \rightarrow T_{A}$ is a $\mathbf{t t}$-translation of $\mathcal{F}_{\mathbf{T A}}(A)$ iff there is an $A$-context $p \in C_{A}$ such that $\alpha(t)=p(t)$ for every $t \in T_{A}$.
(c) A mapping $\alpha: C_{A}^{+} \rightarrow T_{A}$ is a ct-translation of $\mathcal{F}_{\mathbf{T A}}(A)$ iff there exist an $A$-context $r \in C_{A}^{+}$and an $A$-tree $t \in T_{A}$ such that $\alpha(p)=r(p(t))$ for every $p \in C_{A}^{+}$.

Proof. That all translations are expressible as claimed can be proved by induction following the definition of the sets $\operatorname{Tr}\left(\mathcal{F}_{\mathbf{T A}}(A), \mathbf{i}, \mathbf{j}\right)(\mathbf{i}, \mathbf{j} \in S)$ of translations of $\mathcal{F}_{\mathbf{T A}}(A)$. The complete proof presented in the Appendix involves numerous cases and also statements about the missing types of translations. Here we just illustrate the idea by some example cases.

For an elementary lt-translation $\alpha\left(\xi_{1}\right)=\kappa^{\mathcal{F}_{\mathrm{TA}}(A)}\left(\xi_{1}, s, t\right)$, where $s, t \in T_{A}$, we have a case of alternative (2) in statement (a) where $p=\xi$ and $s, t \in T_{A}$ are the given $A$-trees $s$ and $t$. Indeed, $\alpha(a)=\kappa^{\mathcal{F}_{\mathbf{T A}}(A)}(a, s, t)=f_{a}(s, t)=$ $\xi\left(f_{a}(s, t)\right)$ for every $a \in A$.

Consider now an lt-translation $\beta\left(\alpha\left(\xi_{1}\right)\right)$ obtained as the composition of an lt-translation $\alpha$ and a tt-translation $\beta$, and assume that there exist $A$ contexts $p, q \in C_{A}$ such that $\alpha(a)=p\left(c_{a}\right)$ for every $a \in A$, and $\beta(t)=q(t)$ for every $t \in T_{A}$. Then $q(p)$ is an $A$-context such that $q(p)\left(c_{a}\right)=\beta(\alpha(a))$ for every $a \in A$.

Of course, we should also show that all the mappings obtainable by the constructions mentioned in (a)-(c) really are translations of the appropriate types. For example, we have to prove that for any $p \in C_{A}$, the mapping $A \rightarrow T A, a \mapsto p\left(c_{a}\right)$ is an lt-translation of $\mathcal{F}_{\mathbf{T A}}(A)$. This can be done by induction on the $\xi$-depth of $p$.

By using Lemma 6.2 and the observation that the $\mathbf{l}$ - and $\mathbf{c}$-components of $\langle T\rangle$ are empty, we obtain a description of the STA-congruence of an $A$-tree language $T$ that is essentially Wilke's definition.

Proposition 6.3 The $S T A$-congruence $\approx^{T}$ of any $A$-tree language $T \subseteq T_{A}$ is obtained as follows. For any $a, b \in A, s, t \in T_{A}$ and $p, q \in C_{A}^{+}$,
(a) $a \approx_{1}^{T} b$ iff
(1) $\left(\forall p \in C_{A}\right)\left(p\left(c_{a}\right) \in T \leftrightarrow p\left(c_{b}\right) \in T\right)$, and
(2) $\left(\forall p \in C_{A}\right)\left(\forall s, t \in T_{A}\right)\left(p\left(f_{a}(s, t)\right) \in T \leftrightarrow p\left(f_{b}(s, t)\right) \in T\right)$,
(b) $s \approx_{\mathbf{t}}^{T} t \quad$ iff $\quad\left(\forall p \in C_{A}\right)(p(s) \in T \leftrightarrow p(t) \in T)$, and
(c) $p \approx_{\mathbf{c}}^{T} q$ iff $\quad\left(\forall r \in C_{A}\right)\left(\forall t \in T_{A}\right)(r(p(t)) \in T \leftrightarrow r(q(t)) \in T)$.

Let us now show how syntactic tree algebras are related to the usual syntactic algebras [1, 27, 28, 29] and the syntactic semigroups (obtained by a natural modification from the syntactic monoids of [32]). Then we obtain new proofs for Wilke's [33] basic results about syntactic tree algebras and recognizable binary tree languages. The following definitions are restricted directly to binary tree languages.

Definition 6.4 Let $T \subseteq T_{A}$ for some label alphabet $A$.
(a) The syntactic congruence of $T$ is the relation $\theta_{T}$ on $T_{A}$ defined by

$$
s \theta_{T} t \Leftrightarrow\left(\forall p \in C_{A}\right)(p(s) \in T \leftrightarrow p(t) \in T) \quad\left(s, t \in T_{A}\right),
$$

and its syntactic algebra is the $\Sigma^{A}$-algebra $S A(T):=\mathcal{T}_{A} / \theta_{T}$.
(b) The syntactic semigroup congruence of $T$ is the relation $\sigma_{T}$ on $C_{A}^{+}$ defined by the condition that for any $p, q \in C_{A}^{+}$,

$$
p \sigma_{T} q \Leftrightarrow\left(\forall t \in T_{A}\right)\left(\forall r \in C_{A}\right)(r(p(t)) \in T \leftrightarrow r(q(t)) \in T),
$$

and the syntactic semigroup of $T$ is $S S(T):=C_{A}^{+} / \sigma_{T}$, where $C_{A}^{+}$is regarded as a semigroup with respect to the product $p \cdot q=q(p)$.

The usual definition of a recognizable subset of an algebra [20] can be applied to a binary tree language $T \subseteq T_{A}$ either by regarding $T$ as a subset of the $\Sigma^{A}$-algebra $\mathcal{T}_{\Sigma^{A}}=\left(T_{A}, \Sigma^{A}\right)$ or as a subset of sort tree of the tree algebra $\mathcal{F}_{\mathbf{T A}}(A)=\left(\left\langle A, T_{A}, C_{A}^{+}\right\rangle, \Gamma\right)$. However, as shown by Wilke [33], the two definitions are equivalent. We choose the first alternative since it is immediately clear that it means recognizability by a finite tree recognizer (cf. [7, 20, 31, 13, 14], for example).

Definition 6.5 Let $A$ be a label alphabet. An $A$-tree language $T \subseteq T_{A}$ is said to be recognizable, or regular, if there exist a finite $\Sigma^{A}$-algebra $\mathcal{D}$ and a subset $F$ of $\mathcal{D}$ such that $T=F \varphi_{\mathcal{D}}^{-1}$. Let $\operatorname{Rec}_{A}$ denote the set of all recognizable $A$-tree languages.

The above definition can also be expressed by saying that $T \in \operatorname{Rec}_{A}$ iff $T$ is saturated by a congruence on $\mathcal{T}_{A}$ of finite index. The following proposition includes the contents of Wilke's [33] Propositions 2 and 3.

Proposition 6.6 For any binary tree language $T \subseteq T_{A}$ over any label alphabet $A$, the following conditions are equivalent:
(1) $T \in \operatorname{Rec}_{A}$;
(2) $S A(T)$ is a finite $\Sigma^{A}$-algebra;
(3) $S S(T)$ is a finite semigroup;
(4) $S T A(T)$ is a finite tree algebra;
(5) $T$ is recognized by a finite tree algebra.

Proof. That (1)-(3) are equivalent for tree languages quite generally is well known (cf. 13, 14, 28, 30, 32, for example).

Proposition 6.3 shows that $\theta_{T}=\approx_{\mathbf{t}}^{T}$ and $\sigma_{T}=\approx_{\mathbf{c}}^{T}$, and hence (4) implies (1)-(3). The equivalence of (4) and (5) follows from Lemma 5.6.

That (2) implies (4) follows from the fact that the syntactic congruence $\theta_{T}$ determines completely the syntactic semigroup congruence $\sigma_{T}$. Indeed, by comparing the definitions of the two relations, it is easy to see that for any $p, q \in C_{A}^{+}, p \sigma_{T} q$ holds iff $p(t) \theta_{T} q(t)$ for every $t \in T_{A}$. This means, in particular, that if $S A(T)$ is finite, then so is $S S(T)$, and hence also $S T A(T)$ is finite as its l-component is always finite.

The next two lemmas, needed in the variety theory, are also well-known in various other forms, and all of them can be derived from the general manysorted theory of [25]. Here Lemma 6.7 follows from Lemma 5.7 when this is applied to free tree algebras, and Lemma 6.8 follows from Proposition 6.3(b).

Since the $\mathbf{t t}$-translations of a free tree algebra $\mathcal{F}_{\mathbf{T A}}(A)$ are defined by $A$-contexts, we define $p^{-1}(T):=\left\{t \in T_{A} \mid p(t) \in T\right\}$ for any binary tree language $T \subseteq T_{A}$ and any $A$-context $p \in C_{A}$.

Lemma 6.7 Let $A$ and $B$ be label alphabets. For any $A$-tree languages $T, U \subseteq T_{A}$,
(a) $\operatorname{STA}\left(T^{\mathrm{C}}\right)=S T A(T)$,
(b) $S T A(T \cap U), S T A(T \cup U) \preceq S T A(T) \times S T A(U)$,
(c) $S T A\left(p^{-1}(T)\right) \leftarrow S T A(T)$ for every $p \in C_{A}$, and
(d) if $\varphi: \mathcal{F}_{\mathbf{T A}}(B) \rightarrow \mathcal{F}_{\mathbf{T A}}(A)$ is a homomorphism of tree algebras, then $S T A\left(T \varphi_{\mathrm{t}}^{-1}\right) \preceq S T A(T)$.

Lemma 6.8 Let $T \subseteq T_{A}$ for some label alphabet $A$.
(a) $T \in \operatorname{Rec}_{A}$ iff the set $\left\{p^{-1}(T) \mid p \in C_{A}\right\}$ is finite.
(b) The $\approx^{T}$-class $t / T$ of any $A$-tree $t \in T_{A}$ can be given as

$$
\bigcap\left\{p^{-1}(T) \mid p \in C_{A}, p(t) \in T\right\} \backslash \bigcup\left\{p^{-1}(T) \mid p \in C_{A}, p(t) \notin T\right\} .
$$

## $7 \quad$ Varieties of binary tree languages

In this section we introduce varieties of binary tree languages. Although the general many-sorted theory of [25] yielded all the basic properties of syntactic tree algebras, the variety theorems of [25] are not directly applicable here. Firstly, the free algebras $\mathcal{F}_{\mathbf{T A}}(A)$ are always generated by sorted sets of the special form $\langle A, \emptyset, \emptyset\rangle$, not by arbitrary finite sorted sets. Secondly, we are now concerned just with subsets of sort tree while the varieties in [25] consist either of many-sorted sets or one-sorted sets of all possible sorts. In fact, the correspondence one could expect between varieties of binary tree languages and varieties of finite tree algebras fails to hold. The modifications necessary for a true variety theorem are introduced in the following section.

A family of recognizable binary tree languages is a mapping $\mathcal{V}$ that assigns to each label alphabet $A$ a set $\mathcal{V}(A) \subseteq \operatorname{Rec}_{A}$ of regular $A$-tree languages. We write $\mathcal{V}=\{\mathcal{V}(A)\}$ with the understanding that $A$ ranges over all label alphabets. The inclusion relation and various operations on such families are defined in the natural way: if $\mathcal{U}=\{\mathcal{U}(A)\}$ and $\mathcal{V}=\{\mathcal{V}(A)\}$ are families of recognizable binary tree languages, then

- $\mathcal{U} \subseteq \mathcal{V}$ iff $\mathcal{U}(A) \subseteq \mathcal{V}(A)$ for every label alphabet $A$,
- $\mathcal{U} \cap \mathcal{V}$ is the family $\mathcal{W}=\{\mathcal{W}(A)\}$ such that $\mathcal{W}(A)=\mathcal{U}(A) \cap \mathcal{V}(A)$ for every label alphabet $A$, etc.

Definition 7.1 A variety of binary tree languages, a VBTL for short, is a family of recognizable binary tree languages $\mathcal{V}=\{\mathcal{V}(A)\}$ such that for all label alphabets $A$ and $B$,
(1) $\mathcal{V}(A) \neq \emptyset$,
(2) if $T, U \in \mathcal{V}(A)$, then $T^{\complement}, T \cap U \in \mathcal{V}(A)$,
(3) if $T \in \mathcal{V}(A)$, then $p^{-1}(T) \in \mathcal{V}(A)$ for every $p \in C_{A}$, and
(4) if $\varphi: \mathcal{F}_{\mathbf{T A}}(A) \rightarrow \mathcal{F}_{\mathbf{T A}}(B)$ is a homomorphism, then $T \varphi_{\mathbf{t}}^{-1} \in \mathcal{V}(A)$ for every $T \in \mathcal{V}(B)$.

Let VBTL denote the class of all VBTLs.
A variety of finite tree algebras, a VFTA for short, is a nonempty class of finite tree algebras closed under subalgebras, homomorphic images and finite direct products. Let VFTA denote the class of all VFTAs.

In terms of the usual class operators $S$ and $H$ and the operator $P_{\mathrm{f}}$ that forms the class of all direct products with finitely many factors from a given class (cf. [5] and [2], for example), we can define a VFTA as a class $\mathbf{K}$ of finite tree algebras such that $S(\mathbf{K}), H(\mathbf{K}), P_{\mathrm{f}}(\mathbf{K}) \subseteq \mathbf{K}$.

It is clear that $(\mathbf{V B T L}, \subseteq)$ and $(\mathbf{V F T A}, \subseteq)$ are complete lattices. Therefore there is for each family of recognizable binary tree languages $\mathcal{V}$ a least VBTL containing $\mathcal{V}$, the VBTL generated by $\mathcal{V}$. Similarly, for any class $\mathbf{K}$ of finite tree algebras, the VFTA generated by $\mathbf{K}$ is the least VFTA containing $\mathbf{K}$ as a subclass.

The following fact, easy to prove and well-known from other similar situations, is frequently needed. Note that the value $n=0$ yields the trivial tree algebras.

Lemma 7.2 For any class $\mathbf{K}$ of finite tree algebras, the VFTA generated by $\mathbf{K}$ consists of the tree algebras $\mathcal{M}$ such that $\mathcal{M} \preceq \mathcal{M}_{1} \times \cdots \times \mathcal{M}_{n}$ for some $n \geq 0$ and $\mathcal{M}_{1}, \ldots, \mathcal{M}_{n} \in \mathbf{K}$.

Following the general pattern of various variety theorems we define two maps that connect the classes VBTL and VFTA.

Definition 7.3 For any family of recognizable binary tree languages $\mathcal{V}=$ $\{\mathcal{V}(A)\}$, let $\mathcal{V}^{a}$ be the VFTA generated by the class of all syntactic tree algebras $S T A(T)$ where $T \in \mathcal{V}(A)$ for some label alphabet $A$.

For any class $\mathbf{K}$ of finite tree algebras, $\mathbf{K}^{t}$ is the family of recognizable binary tree languages such that $\mathbf{K}^{t}(A)=\left\{T \subseteq T_{A} \mid S T A(T) \in \mathbf{K}\right\}$ for each label alphabet $A$.

In the above definition, and in other similar situations, we tacitly assume that $\mathbf{K}$ is an abstract class of algebras, i.e., it contains every algebra isomorphic to any of its members. The following proposition shows how close to a variety theorem, that would establish an isomorphism between (VBTL, $\subseteq$ ) and (VFTA, $\subseteq$ ), we get with the above definitions.

Proposition 7.4 Let $\mathcal{U}$ and $\mathcal{V}$ be families of recognizable binary tree languages, and let $\mathbf{K}$ and $\mathbf{L}$ be classes of finite tree algebras.
(a) If $\mathcal{U} \subseteq \mathcal{V}$, then $\mathcal{U}^{a} \subseteq \mathcal{V}^{a}$.
(b) If $\mathbf{K} \subseteq \mathbf{L}$, then $\mathbf{K}^{t} \subseteq \mathbf{L}^{t}$.
(c) If $\mathcal{V} \in \mathbf{V B T L}$, then $\mathcal{V}^{a} \in$ VFTA.
(d) If $\mathbf{K} \in \mathbf{V F T A}$, then $\mathbf{K}^{t} \in \mathbf{V B T L}$.
(e) If $\mathcal{V} \in \mathbf{V B T L}$, then $\mathcal{V}^{a t}=\mathcal{V}$.
(f) If $\mathbf{K} \in \mathbf{V F T A}$, then $\mathbf{K}^{t a} \subseteq \mathbf{K}$ but the inclusion may be proper.

Proof. Here we show just that the inclusion in (f) may be proper; the rest can be found in the proof of Proposition 8.7 below.

Let us consider the $\Gamma$-algebra $\mathcal{M}=(\langle\{a, b\},\{t\},\{p\}\rangle, \Gamma)$, where the operations are defined in the only possible way, i.e., $\iota^{\mathcal{M}}(a)=\iota^{\mathcal{M}}(b)=$ $\kappa^{\mathcal{M}}(a, t, t)=\kappa^{\mathcal{M}}(b, t, t)=\eta^{\mathcal{M}}(p, t)=t$ and $\lambda^{\mathcal{M}}(a, t)=\lambda^{\mathcal{M}}(b, t)=\rho^{\mathcal{M}}(a, t)=$ $\rho^{\mathcal{M}}(b, t)=\sigma^{\mathcal{M}}(p, p)=p$. Since the $\mathbf{t}$ - and $\mathbf{c}$-components are singletons, it is clear that $\mathcal{M}$ satisfies the identities $T A$ and is therefore a tree algebra. Let $\mathbf{K}$ be the VFTA generated by $\mathcal{M}$. The $\mathbf{t}$-component of every member of $\mathbf{K}$ is also a singleton, and therefore $\mathbf{K}^{t}(A)=\left\{\emptyset, T_{A}\right\}$ for every label alphabet $A$. This means that $\mathbf{K}^{t a}$ is the class of trivial tree algebras and hence $\mathcal{M} \in \mathbf{K} \backslash \mathbf{K}^{t a}$.

## 8 Reduced tree algebras and a variety theorem

There are natural reasons why a complete correspondence between the classes VBTL and VFTA was not obtained. Firstly, since the algebras $\mathcal{F}_{\mathbf{T A}}(A)$ are generated by their l-components, so are the syntactic tree algebras of all binary tree languages. In fact, Wilke [33] anticipated a variety theorem that would involve varieties of such l-generated finite tree algebras. However, that something more is required, is indicated by the counterexample used in the proof of Proposition 7.4, the tree algebra $\mathcal{M}$ is $\mathbf{l}$-generated. It turns out that we have to focus on tree algebras that do not have pairs of elements of sort label or context that are in a sense equivalent.

Definition 8.1 For any tree algebra $\mathcal{M}=(M, \Gamma)$, let $\mathcal{M}^{l}$ denote the subalgebra of $\mathcal{M}$ generated by $\left\langle M_{1}\right\rangle=\left\langle M_{1}, \emptyset, \emptyset\right\rangle$. If $\mathcal{M}^{l}=\mathcal{M}$, then $\mathcal{M}$ is said to be l-generated. An l-generated tree algebra $\mathcal{M}=(M, \Gamma)$ is reduced if it satisfies the following two additional conditions:
(1) For any $a, b \in M_{\mathrm{l}}$, if $\iota^{\mathcal{M}}(a)=\iota^{\mathcal{M}}(b)$ and $\kappa^{\mathcal{M}}(a, s, t)=\kappa^{\mathcal{M}}(b, s, t)$ for all $s, t \in M_{\mathbf{t}}$, then $a=b$.
(2) For any $p, q \in M_{\mathbf{c}}$, if $\eta^{\mathcal{M}}(p, t)=\eta^{\mathcal{M}}(q, t)$ for every $t \in M_{\mathbf{t}}$, then $p=q$.

Any tree algebra $\mathcal{M}=(M, \Gamma)$ can be reduced as follows. Let $\mathcal{M}^{l}=\mathcal{N}=$ $(N, \Gamma)$, and let $\theta^{\mathcal{M}}$ be the sorted relation on $N$ such that
(1) for any $a, b \in N_{\mathbf{l}}, a \theta_{\mathbf{1}}^{\mathcal{M}} b$ iff

$$
\iota^{\mathcal{N}}(a)=\iota^{\mathcal{N}}(b) \&\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{N}}(a, s, t)=\kappa^{\mathcal{N}}(b, s, t)\right)
$$

(2) for any $s, t \in N_{\mathbf{t}}, s \theta_{\mathbf{t}}^{\mathcal{M}} t$ iff $s=t$, and
(3) for any $p, q \in N_{\mathbf{c}}, p \theta_{\mathbf{c}}^{\mathcal{M}} q$ iff $\left(\forall t \in N_{\mathbf{t}}\right)\left(\eta^{\mathcal{N}}(p, t)=\eta^{\mathcal{N}}(q, t)\right)$.

It is easy to see that $\theta^{\mathcal{M}}$ is a congruence on $\mathcal{N}$, and let $\mathcal{M}^{r}$ denote the quotient algebra $\mathcal{N} / \theta^{\mathcal{M}}$.

Lemma 8.2 For any tree algebra $\mathcal{M}$, the tree algebra $\mathcal{M}^{r}$, as defined above, is reduced. If $\mathcal{M}$ is reduced, then $\mathcal{M}^{r} \cong \mathcal{M}$.

Proof. Let us write $\mathcal{N}=\mathcal{M}^{l}$ and $\theta=\theta^{\mathcal{M}}$. Since $\mathcal{N}$ is l-generated, so is $\mathcal{M}^{r}=\mathcal{N} / \theta$. Assume that for some $a, b \in N_{\mathrm{l}}$,
(A) $\iota^{\mathcal{M}^{r}}\left(a / \theta_{\mathbf{l}}\right)=\iota^{\mathcal{M}^{r}}\left(b / \theta_{\mathbf{1}}\right)$, and
(B) $\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{M}^{r}}\left(a / \theta_{\mathbf{l}}, s / \theta_{\mathbf{t}}, t / \theta_{\mathbf{t}}\right)=\kappa^{\mathcal{M}^{r}}\left(b / \theta_{\mathbf{l}}, s / \theta_{\mathbf{t}}, t / \theta_{\mathbf{t}}\right)\right)$.

Condition (A) is equivalent to $\iota^{\mathcal{M}}(a) / \theta_{\mathbf{t}}=\iota^{\mathcal{M}}(b) / \theta_{\mathbf{t}}$, and hence $\iota^{\mathcal{M}}(a)=$ $\iota^{\mathcal{M}}(b)$ by the definition of $\theta_{\mathbf{t}}$. Similarly, $\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{M}}(a, s, t)=\kappa^{\mathcal{M}}(b, s, t)\right)$ follows from (B). Together (A) and (B) imply $a / \theta_{1}=b / \theta_{1}$ by the definition of $\theta_{\mathbf{l}}$. This means that $\mathcal{M}^{r}$ satisfies (1) of Definition 8.1. Condition (2) follows similarly from the fact that

$$
\left(\forall t \in N_{\mathbf{t}}\right)\left(\eta^{\mathcal{M}^{r}}\left(p / \theta_{\mathbf{c}}, t / \theta_{\mathbf{t}}\right)=\eta^{\mathcal{M}^{r}}\left(q / \theta_{\mathbf{c}}, t / \theta_{\mathbf{t}}\right)\right) \Rightarrow p / \theta_{\mathbf{c}}=q / \theta_{\mathbf{c}}
$$

for all $p, q \in N_{\mathbf{c}}$, and hence $\mathcal{M}^{r}$ is reduced.
If $\mathcal{M}$ is reduced, then $\mathcal{M}^{l}=\mathcal{M}$, and each component of $\theta^{\mathcal{M}}$ is the identity relation on the respective set. Hence, $\mathcal{M}^{r}=\mathcal{M} / \theta^{\mathcal{M}} \cong \mathcal{M}$.

Lemma 8.3 For any tree algebras $\mathcal{M}=(M, \Gamma)$ and $\mathcal{N}=(N, \Gamma)$, if $\mathcal{M} \preceq \mathcal{N}$, then $\mathcal{M}^{r} \preceq \mathcal{N}^{r}$.

Proof. The covering relation is transitive as $\mathcal{M} \preceq \mathcal{N}$ iff $\mathcal{M} \in H S(\{\mathcal{N}\})$, and the well-known properties of the class operators $S$ and $H$ by which $H S H S(\mathbf{K})=H S(\mathbf{K})$ for any class $\mathbf{K}$ of algebras (cf. [5], for example). Therefore it suffices to prove the following special cases of the lemma:
(a) if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M}^{r} \preceq \mathcal{N}^{r}$;
(b) if $\mathcal{M} \leftarrow \mathcal{N}$, then $\mathcal{M}^{r} \preceq \mathcal{N}^{r}$.

If $\mathcal{M} \subseteq \mathcal{N}$, then also $\mathcal{M}^{l} \subseteq \mathcal{N}^{l}$, and therefore we may assume in (a) that $\mathcal{M}$ and $\mathcal{N}$ are l-generated. Let $\mu:=\theta^{\mathcal{M}}$ and let $\nu:=\theta^{\mathcal{N}} \cap(M \times M)$ be the restriction of $\theta^{\mathcal{N}}$ to $M$. Then $\mathcal{M} / \nu \subseteq \mathcal{N}^{r}$, and therefore it is enough to show that $\nu \subseteq \mu$ because then $\mathcal{M}^{r}=\mathcal{M} / \mu \leftarrow \mathcal{M} / \nu$.

For any $a, b \in M_{1}$,

$$
\begin{aligned}
a \nu_{\mathbf{1}} b & \Rightarrow \iota^{\mathcal{N}}(a)=\iota^{\mathcal{N}}(b) \&\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{N}}(a, s, t)=\kappa^{\mathcal{N}}(b, s, t)\right) \\
& \Rightarrow \iota^{\mathcal{M}}(a)=\iota^{\mathcal{M}}(b) \&\left(\forall s, t \in M_{\mathbf{t}}\right)\left(\kappa^{\mathcal{M}}(a, s, t)=\kappa^{\mathcal{M}}(b, s, t)\right) \\
& \Rightarrow a \mu_{\mathbf{1}} b,
\end{aligned}
$$

and hence $\nu_{1} \subseteq \mu_{\mathrm{l}}$. It is obvious that $\nu_{\mathrm{t}}=\mu_{\mathrm{t}}$ and the inclusion $\nu_{\mathrm{c}} \subseteq \mu_{\mathrm{c}}$ is verified similarly as $\nu_{1} \subseteq \mu_{\mathrm{l}}$. Hence $\nu \subseteq \mu$.

To prove (b), let $\varphi: \mathcal{N} \rightarrow \mathcal{M}$ be an epimorphism. Since $N_{1} \varphi=M_{1}$, it is clear that the restriction of $\varphi$ to $N^{l}$ is an epimorphism from $\mathcal{N}^{l}$ onto $\mathcal{M}^{l}$. We may therefore again assume that $\mathcal{M}$ and $\mathcal{N}$ themselves are l-generated. Let $\mu:=\theta^{\mathcal{M}}$ and $\nu:=\theta^{\mathcal{N}}$. We show now that the mapping $\psi: N / \nu \rightarrow M \mu$ defined by

$$
\psi_{\mathbf{1}}: a / \nu_{\mathbf{1}} \mapsto a \varphi_{\mathbf{l}} / \mu_{\mathbf{l}}, \psi_{\mathbf{t}}: t / \nu_{\mathbf{t}} \mapsto t \varphi_{\mathbf{t}} / \mu_{\mathbf{t}}, \psi_{\mathbf{c}}: p / \nu_{\mathbf{c}} \mapsto p \varphi_{\mathbf{c}} / \mu_{\mathbf{c}}
$$

$\left(a \in N_{\mathbf{l}}, t \in N_{\mathbf{t}}, p \in N_{\mathbf{c}}\right)$ is an epimorphism from $\mathcal{N}^{r}$ onto $\mathcal{M}^{r}$.
First we note that $\psi$ is well-defined. For example, for any $a, b \in N_{\mathbf{l}}$,

$$
\begin{aligned}
& a / \nu_{\mathbf{1}}=b / \nu_{\mathbf{1}} \Rightarrow \\
& \iota^{\mathcal{N}}(a)=\iota^{\mathcal{N}}(b) \&\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{N}}(a, s, t)=\kappa^{\mathcal{N}}(b, s, t)\right) \Rightarrow \\
& \iota^{\mathcal{N}}(a) \varphi_{\mathbf{t}}=\iota^{\mathcal{N}}(b) \varphi_{\mathbf{t}} \&\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{N}}(a, s, t) \varphi_{\mathbf{t}}=\kappa^{\mathcal{N}}(b, s, t) \varphi_{\mathbf{t}}\right) \Rightarrow \\
& \iota^{\mathcal{M}}\left(a \varphi_{\mathbf{1}}\right)=\iota^{\mathcal{M}}\left(b \varphi_{\mathbf{1}}\right) \&\left(\forall s, t \in N_{\mathbf{t}}\right)\left(\kappa^{\mathcal{M}}\left(a \varphi_{\mathbf{1}}, s \varphi_{\mathbf{t}}, t \varphi_{\mathbf{t}}\right)=\right. \\
& \left.a \kappa_{\mathbf{1}} / \mu_{\mathbf{1}}=b \varphi_{\mathbf{1}} / \mu_{\mathbf{1}}, \quad \kappa^{\mathcal{M}}\left(b \varphi_{\mathbf{1}}, s \varphi_{\mathbf{t}}, t \varphi_{\mathbf{t}}\right)\right) \Rightarrow
\end{aligned}
$$

where the last equality depends on the assumption that $\varphi$ is surjective. Similarly, $s / \nu_{\mathbf{t}}=t / \nu_{\mathbf{t}}$ implies $s \varphi_{\mathbf{t}} / \mu_{\mathbf{t}}=t \varphi_{\mathbf{t}} / \mu_{\mathbf{t}}$ for any $s, t \in N_{\mathbf{t}}$, and $p / \nu_{\mathbf{c}}=q / \nu_{\mathbf{c}}$ implies $p \varphi_{\mathbf{c}} / \mu_{\mathrm{c}}=q \varphi_{\mathbf{c}} / \mu_{\mathrm{c}}$ for any $p, q \in N_{\mathrm{c}}$.

It is clear that $\psi$ is surjective. Finally, by routine computations it can be verified that $\psi$ is a homomorphism. For example,

$$
\begin{aligned}
\iota^{\mathcal{N} / \nu}\left(a / \nu_{1}\right) \psi_{\mathbf{t}} & =\left(\iota^{\mathcal{N}}(a) / \nu_{\mathbf{t}}\right) \psi_{\mathbf{t}}=\iota^{\mathcal{N}}(a) \varphi_{\mathbf{t}} / \mu_{\mathbf{t}}=\iota^{\mathcal{M}}\left(a \varphi_{\mathbf{1}}\right) / \mu_{\mathbf{t}} \\
& =\iota^{\mathcal{M} / \mu}\left(a \varphi_{\mathbf{1}} / \mu_{\mathbf{1}}\right)=\iota^{\mathcal{M} / \mu}\left(\left(a / \nu_{\mathbf{1}}\right) \psi_{\mathbf{1}}\right) .
\end{aligned}
$$

for every $a \in N_{\mathrm{l}}$.

The following proposition summarizes the main properties of $\mathcal{M}^{r}$ and shows that it is, up to isomorphism, the greatest reduced tree algebra covered by $\mathcal{M}$.

Proposition 8.4 For any tree algebra $\mathcal{M}, \mathcal{M}^{r}$ is a reduced tree algebra such that $\mathcal{M}^{r} \preceq \mathcal{M}$. Moreover, if $\mathcal{N} \preceq \mathcal{M}$ for a reduced tree algebra $\mathcal{N}$, then $\mathcal{N} \preceq \mathcal{M}^{r}$.

Proof. We know already that $\mathcal{M}^{r}$ is reduced and $\mathcal{M}^{r} \preceq \mathcal{M}$ follows directly from the definition. If $\mathcal{N}$ is a reduced tree algebra such that $\mathcal{N} \preceq \mathcal{M}$, then $\mathcal{N} \cong \mathcal{N}^{r} \preceq \mathcal{M}^{r}$ by Lemmas 8.2 and 8.3 .

Let us now note a couple of important facts about reduced tree algebras and syntactic tree algebras.

Lemma 8.5 The syntactic tree algebra of any binary tree language is reduced. On the other hand, for any finite reduced tree algebra $\mathcal{M}$, there exist a label alphabet $A$ and regular $A$-tree languages $T_{1}, \ldots, T_{n} \subseteq T_{A}$, for some $n \geq 1$, such that $S T A\left(T_{j}\right) \preceq \mathcal{M}$ for every $j=1, \ldots, n$, and $\mathcal{M} \subseteq$ $S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)$.

Proof. As a quotient algebra of $\mathcal{F}_{\mathbf{T A}}(A)$, the syntactic tree algebra $S T A(T)$ of a binary tree language $T \subseteq T_{A}$ is naturally l-generated. That $S T A(T)$ satisfies conditions (1) and (2) of Definition 8.1 can be verified by using Proposition 6.3.

To prove the second claim of the proposition, take a label alphabet $A$ such that $|A| \geq\left|M_{1}\right|$. Since $\mathcal{M}$ is $\mathbf{l}$-generated, there is an epimorphism $\varphi: \mathcal{F}_{\mathbf{T A}}(A) \rightarrow \mathcal{M}$. Assume that $M_{\mathbf{t}}=\left\{t_{1}, \ldots, t_{n}\right\}$ for some $n \geq 1$, and let $T_{j}:=t_{j} \varphi^{-1}$ for each $j=1, \ldots, n$. By using Lemma 5.7(d) we obtain for every $j=1, \ldots, n$,

$$
S T A\left(T_{j}\right)=\mathcal{F}_{\mathbf{T A}}(A) / T_{j} \cong \mathcal{M} /\left\{t_{j}\right\} \preceq \mathcal{M} .
$$

To prove that $\mathcal{M} \subseteq S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)$, it suffices to prove that $\mathcal{M}$ is isomorphic to a subalgebra of $\mathcal{M} /\left\{t_{1}\right\} \times \cdots \times \mathcal{M} /\left\{t_{n}\right\}$. To do this, we consider the mapping

$$
\psi: M \rightarrow M /\left\{t_{1}\right\} \times \cdots \times M /\left\{t_{n}\right\}
$$

that maps each element $u=\left\langle u_{\mathbf{l}}, u_{\mathbf{t}}, u_{\mathbf{c}}\right\rangle \in M$ to

$$
\left\langle\left(u_{\mathbf{1}} /\left\{t_{1}\right\}, \ldots, u_{\mathbf{1}} /\left\{t_{n}\right\}\right),\left(u_{\mathbf{t}} /\left\{t_{1}\right\}, \ldots, u_{\mathbf{t}} /\left\{t_{n}\right\}\right),\left(u_{\mathbf{c}} /\left\{t_{1}\right\}, \ldots, u_{\mathbf{c}} /\left\{t_{n}\right\}\right)\right\rangle .
$$

It is clear that $\psi$ is a homomorphism from $\mathcal{M}$ to $\mathcal{M} /\left\{t_{1}\right\} \times \cdots \times \mathcal{M} /\left\{t_{n}\right\}$. Hence, it remains to be shown that $\psi$ is injective.

If $a \psi_{1}=b \psi_{1}$ for some $a, b \in M_{1}$, then $a \approx_{1}^{\left\{t_{j}\right\}} b$ for every $j=1, \ldots, n$. In particular, $a \approx_{1}^{\left\{\iota^{\mathcal{M}}(a)\right\}} b$ which implies that $\iota^{\mathcal{M}}(a)=\iota^{\mathcal{M}}(b)$. Similarly, $a \approx_{1}^{\left\{\kappa^{\mathcal{M}}(a, s, t)\right\}} b$ implies $\kappa^{\mathcal{M}}(a, s, t)=\kappa^{\mathcal{M}}(b, s, t)$, for all $s, t \in M_{\mathrm{t}}$. Since $\mathcal{M}$ is reduced, this means that $a=b$.

If $s \psi_{\mathbf{t}}=t \psi_{\mathbf{t}}$ for some $s, t \in M_{\mathbf{t}}$, then $s \approx^{\{s\}} t$ yields $s=t$.
Finally, if $p \psi_{\mathbf{c}}=q \psi_{\mathbf{c}}$ for some $p, q \in M_{\mathbf{c}}$, then $p \approx_{\mathbf{c}}^{\left\{\eta^{\mathcal{M}}(p, t)\right\}} q$ implies $\eta^{\mathcal{M}}(p, t)=\eta^{\mathcal{M}}(q, t)$ for every $t \in M_{\mathbf{t}}$. Since $\mathcal{M}$ is reduced, this means that $p=q$.

Hence we have shown that $\psi: \mathcal{M} \rightarrow \mathcal{M} /\left\{t_{1}\right\} \times \cdots \times \mathcal{M} /\left\{t_{n}\right\}$ is a monomorphism.

Definition 8.6 A variety of finite reduced tree algebras, an rVFTA for short, is a nonempty class of finite reduced tree algebras $\mathbf{R}$ such that $\mathcal{N} \in \mathbf{R}$ whenever $\mathcal{N}$ is a reduced tree algebra and $\mathcal{N} \preceq \mathcal{M}_{1} \times \cdots \times \mathcal{M}_{n}$ for some $n \geq 1$ and $\mathcal{M}_{1}, \ldots, \mathcal{M}_{n} \in \mathbf{R}$. Let rVFTA denote the class of all rVFTAs.

An rVFTA contains, in particular, all reduced subalgebras and all reduced images of its members. Since the intersection of any collection of rVFTAs is also an rVFTA, we may speak about the rVFTA generated by any given class of finite reduced tree algebras.

We shall now establish an isomorphism between the complete lattices $(\mathbf{r V F T A}, \subseteq)$ and $(\mathbf{V B T L}, \subseteq)$ thus obtaining the desired variety theorem. The mapping $\mathbf{R} \mapsto \mathbf{R}^{t}$ is defined as above but its application is restricted to classes of finite reduced tree algebras. The mapping $\mathcal{V} \mapsto \mathcal{V}^{a}$ is modified as follows: if $\mathcal{V}=\{\mathcal{V}(A)\}$ is any family of recognizable binary tree languages, then $\mathcal{V}^{a}$ is the rVFTA generated by the class of all syntactic tree algebras $S T A(T)$ where $T \in \mathcal{V}(A)$ for some $A$.

Proposition 8.7 (The Variety Theorem) Let $\mathcal{U}$ and $\mathcal{V}$ be families of recognizable binary tree languages, and let $\mathbf{P}$ and $\mathbf{R}$ be classes of finite reduced tree algebras.
(a) If $\mathcal{U} \subseteq \mathcal{V}$, then $\mathcal{U}^{a} \subseteq \mathcal{V}^{a}$.
(b) If $\mathbf{P} \subseteq \mathbf{R}$, then $\mathbf{P}^{t} \subseteq \mathbf{R}^{t}$.
(c) If $\mathcal{V} \in \mathbf{V B T L}$, then $\mathcal{V}^{a} \in \mathbf{r V F T A}$.
(d) If $\mathbf{R} \in \mathbf{r V F T A}$, then $\mathbf{R}^{t} \in \mathbf{V B T L}$.
(e) If $\mathcal{V} \in \mathbf{V B T L}$, then $\mathcal{V}^{a t}=\mathcal{V}$.
(f) If $\mathbf{R} \in \mathbf{r V F T A}$, then $\mathbf{R}^{t a}=\mathbf{R}$.

Proof. Assertions (a) and (b) are completely obvious, (c) follows directly from the definition of $\mathcal{V}^{a}$, and (d) follows from Lemma 6.7.

As to (e), the inclusion $\mathcal{V} \subseteq \mathcal{V}^{\text {at }}$ is also obvious, and the less obvious converse inclusion can be shown by adapting suitably Eilenberg's [8] original proof similarly as, for example, in [28] (Proposition 7.3) or in [25] (Proposition 6.3) where the corresponding fact is proved in the general many-sorted case. For completeness and the reader's convenience we present such a proof for the current case, too.

Assume that $T \in \mathcal{V}^{a t}$ for some $A$. Then $S T A(T) \in \mathcal{V}^{a}$ implies that $S T A(T) \preceq S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)$, where $T_{i} \in \mathcal{V}\left(A_{i}\right)(i=1, \ldots, n)$ for some $n \geq 1$ and label alphabets $A_{1}, \ldots, A_{n}$. For each $i=1, \ldots, n$, let $\varphi^{i}$ denote the syntactic homomorphism $\mathcal{F}_{\mathbf{T A}}\left(A_{i}\right) \rightarrow S T A\left(T_{i}\right)$ that maps each element of $\mathcal{F}_{\mathbf{T A}}\left(A_{i}\right)$ to its $\approx^{T_{i}}$-class. Then there is a homomorphism

$$
\beta: \mathcal{F}_{\mathbf{T A}}\left(A_{1}\right) \times \cdots \times \mathcal{F}_{\mathbf{T A}}\left(A_{n}\right) \rightarrow S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)
$$

such that $\beta \pi^{i}=\tau^{i} \varphi^{i}$ for each $i=1, \ldots, n$, where

$$
\pi^{i}: S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right) \rightarrow S T A\left(T_{i}\right)
$$

and

$$
\tau^{i}: \mathcal{F}_{\mathbf{T A}}\left(A_{1}\right) \times \cdots \times \mathcal{F}_{\mathbf{T A}}\left(A_{n}\right) \rightarrow \mathcal{F}_{\mathbf{T A}}\left(A_{i}\right)
$$

are the respective projections. By Lemma 5.6 there exist a homomorphism $\varphi: \mathcal{F}_{\mathbf{T A}}(A) \rightarrow S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)$ and a subset $H$ of $S T A\left(T_{1}\right)_{\mathbf{t}} \times$ $\cdots \times S T A\left(T_{n}\right)_{\mathbf{t}}$ such that $T=H \varphi_{\mathbf{t}}^{-1}$. Since $\beta$ is an epimorphism, there is a homomorphism $\psi: \mathcal{F}_{\mathbf{T A}}(A) \rightarrow \mathcal{F}_{\mathbf{T A}}\left(A_{1}\right) \times \cdots \times \mathcal{F}_{\mathbf{T A}}\left(A_{n}\right)$ such that $\psi \beta=\varphi$. Because $H$ is finite, $T$ is the union of the finitely many sets $u \varphi_{\mathbf{t}}^{-1}$ with $u=\left(u_{1}, \ldots, u_{n}\right) \in H$. Each such set can be expressed as

$$
u \varphi_{\mathbf{t}}^{-1}=\bigcap\left\{u_{i}\left(\varphi_{\mathbf{t}} \pi_{\mathbf{t}}^{i}\right)^{-1} \mid 1 \leq i \leq n\right\}=\bigcap\left\{u_{i}\left(\varphi_{\mathbf{t}}^{i}\right)^{-1}\left(\psi \tau^{i}\right)_{\mathbf{t}}^{-1} \mid 1 \leq i \leq n\right\} .
$$

It follows from Lemma 6.8 that $u_{i}\left(\varphi_{\mathbf{t}}^{i}\right)^{-1} \in \mathcal{V}\left(T_{i}\right)$ for every $i=1, \ldots, n$, and hence also $T \in \mathcal{V}(A)$.

The inclusion $\mathbf{R}^{t a} \subseteq \mathbf{R}$ in (f) follows from the fact that the syntactic tree algebras that generate $\mathbf{R}^{t a}$ are also in $\mathbf{R}$. Indeed, if $T \in \mathbf{R}^{t}(A)$ for some $A$, then $S T A(T)$ is in $\mathbf{R}$ by the definition of $\mathbf{R}^{t}$.

It remains to show that also $\mathbf{R} \subseteq \mathbf{R}^{t a}$ holds for any rVFTA $\mathbf{R}$. Let us consider an $\mathcal{M} \in \mathbf{R}$. Since $\mathcal{M}$ is a finite reduced tree algebra, there exist by Lemma 8.5 a label alphabet $A$ and recognizable $A$-tree languages $T_{1}, \ldots, T_{n} \subseteq T_{A}(n \geq 1)$ such that $S T A\left(T_{j}\right) \preceq \mathcal{M}(j=1, \ldots, n)$, and $\mathcal{M} \subseteq S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)$. Then $S T A\left(T_{j}\right) \in \mathbf{R}$ for every $j=1, \ldots, n$, and hence $\mathcal{M} \subseteq S T A\left(T_{1}\right) \times \cdots \times S T A\left(T_{n}\right)$ implies that $\mathcal{M} \in \mathbf{R}^{t a}$.

To conclude this section, we shall note that every rVFTA is obtained as the class of reduced members of a VFTA, but that this fact does not
establish a bijection between rVFTA and VFTA because a given rVFTA can be obtained from several VFTAs.

Proposition 8.8 For any VFTA K, the class $\mathbf{K}^{r}$ of all reduced members of $\mathbf{K}$ is an rVFTA. On the other hand, for each rVFTA $\mathbf{R}$, there is a VFTA $\mathbf{K}$ such that $\mathbf{K}^{r}=\mathbf{R}$, but this $\mathbf{K}$ is not necessarily unique for a given $\mathbf{R}$.

Proof. It follows easily from the definitions of VFTAs and rVFTAs that $\mathbf{K}^{r} \in$ rVFTA for any $\mathbf{K} \in$ VFTA, and also that if $\mathbf{R} \in \mathbf{r V F T A}$ and $\mathbf{K}$ is the VFTA generated by $\mathbf{R}$, then $\mathbf{K}^{r}=\mathbf{R}$.

For the last assertion, let $\mathbf{R}$ be the rVFTA of all trivial tree algebras. Then $\mathbf{R}$ is itself a VFTA such that $\mathbf{R}^{r}=\mathbf{R}$. On the other hand, we have $\mathbf{K}^{r}=\mathbf{R}$ also for the VFTA $\mathbf{K}$ of all finite tree algebras $\mathcal{M}=(M, \Gamma)$ such that $\left|M_{\mathbf{t}}\right|=1$.

## 9 VBTLs and general varieties of tree languages

The varieties of binary tree languages considered here are in some sense less general than the varieties of tree languages studied in [28], for example, but at the same time they are in some respect more general. Less general they are because they involve binary trees only and in that there are no separate leaf alphabets. On the other hand, a VBTL is not restricted to one ranked alphabet but contains tree languages over all alphabets of the form $\Sigma^{A}$. In this respect VBTLs resemble the general varieties of tree languages (GVTL) of [29] and the similar varieties studied in [18]. We shall show that each GVTL becomes a VBTL when restricted to the binary ranked alphabets $\Sigma^{A}$ considered here. Since many known families of regular tree languages are indeed GVTLs, this fact yields several natural examples of VBTLs. Such examples include the families of nilpotent, definite, reverse definite, generalized definite, locally testable and non-counting tree languages. For showing the connection between GVTLs and VBTLs we have to recall the definition of a GVTL.

Let $\Sigma$ and $\Omega$ be ranked alphabets. A $g$-morphism from a $\Sigma$-algebra $\mathcal{D}=$ $(D, \Sigma)$ to an $\Omega$-algebra $\mathcal{E}=(E, \Omega)$ is a pair of mappings $\alpha: \Sigma \rightarrow \Omega$ and $\varphi: D \rightarrow E$ such that
(1) $\alpha(f) \in \Omega_{m}$ for any $f \in \Sigma_{m}$ and $m \geq 0$,
(2) $c^{\mathcal{D}} \varphi=\alpha(c)^{\mathcal{E}}$ for every $c \in \Sigma_{0}$, and
(3) $f^{\mathcal{D}}\left(d_{1}, \ldots, d_{m}\right) \varphi=\alpha(f)^{\mathcal{E}}\left(d_{1} \varphi, \ldots, d_{m} \varphi\right)$ for all $m>0, f \in \Sigma_{m}$ and $d_{1}, \ldots, d_{m} \in D$.

It is easy to see (cf. [29]) that a g-morphism $(\alpha, \varphi): \mathcal{T}_{\Sigma}(X) \rightarrow \mathcal{T}_{\Omega}(Y)$ between term algebras is essentially a relabeling of trees that replaces each label from $\Sigma$ with its $\alpha$-image. That leaves labelled with leaf symbols in $X$ may be replaced by any $\Omega Y$-trees, is of no consequence here because in a VBTL all leaf alphabets are empty (and not shown at all).

A general variety of tree languages (GVTL) is a family of regular tree languages $\mathcal{V}=\{\mathcal{V}(\Sigma, X)\}$ such that for all ranked alphabets $\Sigma$ and $\Omega$, and all leaf alphabets $X$ and $Y$,
(G1) $\mathcal{V}(\Sigma, X) \neq \emptyset$,
(G2) if $T, U \in \mathcal{V}(\Sigma, X)$, then $T^{\complement}, T \cap U \in \mathcal{V}(\Sigma, X)$,
(G3) if $T \in \mathcal{V}(\Sigma, X)$ and $p \in C_{\Sigma}(X)$, then $p^{-1}(T) \in \mathcal{V}(\Sigma, X)$, and
(G4) if $(\alpha, \varphi): \mathcal{T}_{\Sigma}(X) \rightarrow \mathcal{T}_{\Omega}(Y)$ is a g -morphism, then $T \varphi^{-1} \in \mathcal{V}(\Sigma, X)$ for every $T \in \mathcal{V}(\Omega, Y)$.

If we restrict ourselves to the ranked alphabets $\Sigma^{A}$ obtained from label alphabets and assume that the leaf alphabets are always empty, then the above definition matches exactly the definition of a VBTL except for the last clauses concerning g-morphisms and homomorphism, respectively. Hence, to determine the relationship between VBTLs and these restricted GVTLs we have to describe the g-morphisms between the term algebras $\mathcal{T}_{A}$ and the t components of homomorphisms between the free tree algebras $\mathcal{F}_{\mathbf{T A}}(A)$. The following two lemmas follow directly from the appropriate definitions.

Lemma 9.1 Let $A$ and $B$ be any label alphabets. If $(\alpha, \varphi): \mathcal{T}_{A} \rightarrow \mathcal{T}_{B}$ is a $g$-morphism, then
(1) $c_{a} \varphi=\alpha\left(c_{a}\right)$ for every $a \in A$, and
(2) $f_{a}(s, t) \varphi=\alpha\left(f_{a}\right)(s \varphi, t \varphi)$ for any $a \in A$ and $s, t \in T_{A}$.

The lemma also shows that the mapping $\varphi: T_{A} \rightarrow T_{B}$ in a g-morphism $(\alpha, \varphi): \mathcal{T}_{A} \rightarrow \mathcal{T}_{B}$ is a relabeling fully determined by $\alpha: \Sigma^{A} \rightarrow \Sigma^{B}$.

Lemma 9.2 Let $A$ and $B$ be label alphabets. If $\varphi: \mathcal{F}_{\mathbf{T A}}(A) \rightarrow \mathcal{F}_{\mathbf{T A}}(B)$ is a homomorphism, then
(1) $c_{a} \varphi_{\mathbf{t}}=c_{a \varphi_{1}}$ for every $a \in A$, and
(2) $f_{a}(s, t) \varphi_{\mathbf{t}}=f_{a \varphi_{1}}\left(s \varphi_{\mathbf{t}}, t \varphi_{\mathbf{t}}\right)$ for all $a \in A$ and $s, t \in T_{A}$.

Hence, homomorphisms between free tree algebras also are just relabelings of binary trees. Moreover, it is clear that for any homomorphism $\varphi: \mathcal{F}_{\mathbf{T A}}(A) \rightarrow \mathcal{F}_{\mathbf{T A}}(B)$ there is a g-morphism $(\alpha, \psi): \mathcal{T}_{A} \rightarrow \mathcal{T}_{B}$, such that $t \varphi_{\mathbf{t}}=t \psi$ for every $t \in T_{A}$; we just define $\alpha$ by setting $\alpha\left(c_{a}\right)=c_{a \varphi_{1}}$ and $\alpha\left(f_{a}\right)=f_{a \varphi_{1}}$ for every $a \in A$, and this is consistent with the idea that $c_{a}$ and $f_{a}$ actually represent the same label $a$. This means that (G4) in the above definition of a GVTL implies the corresponding condition in the definition of a VBTL. The following fact is now obvious.

Proposition 9.3 For any $G V T L \mathcal{V}=\{\mathcal{V}(\Sigma, X)\}$, the family of recognizable binary tree languages $\mathcal{V}^{b}=\left\{\mathcal{V}^{b}(A)\right\}$, where $\mathcal{V}^{b}(A)=\mathcal{V}\left(\Sigma^{A}, \emptyset\right)$ for each label alphabet $A$, is a VBTL.

The relabelings defined by homomorphisms between free tree algebras are somewhat less general than the g-morphisms of Lemma 9.1 because of the bindings between the pairs $c_{a}, f_{a}(a \in A)$ : if $c_{a}$ maps to $c_{b}, f_{a}$ has to map to $f_{b}$, and conversely. This means that (G4) is strictly stronger than clause (4) in Definition 7.1, even when restricted to our binary tree languages, and therefore it is conceivable that not every VBTL is obtained as a restriction of a GVTL. That this is indeed the case, is shown by the following example.

Example 9.4 For each label alphabet $A$, let $\mathcal{V}(A)$ be the set of all regular $A$-tree languages $T \subseteq T_{A}$ such that $f_{a}\left(c_{a}, t\right) \approx_{\mathbf{t}}^{T} t$ for all $a \in A$ and $t \in T_{A}$. It is easy to verify that $\mathcal{V}=\{\mathcal{V}(A)\}$ is a VBTL. Assume that $\mathcal{V}=\mathcal{U}^{b}$ for some GVTL $\mathcal{U}=\{\mathcal{U}(\Sigma, X)\}$. Let $A=\{a, b\}$ and define the $A$-contexts $p_{a}=f_{a}\left(c_{a}, \xi\right)$ and $p_{b}=f_{b}\left(c_{b}, \xi\right)$. Let $T$ be the least $A$-tree language such that $c_{a} \in T$ and $p_{a}(t), p_{b}(t) \in T$ for every $t \in T$. Then $T \in \mathcal{V}=\mathcal{U}\left(\Sigma^{A}, \emptyset\right)$. Consider the g -morphism $(\alpha, \varphi): \mathcal{T}_{A} \rightarrow \mathcal{T}_{B}$ defined by the assignment

$$
\alpha: T_{\Sigma^{A}} \rightarrow T_{\Sigma^{A}}, c_{a} \mapsto c_{a}, c_{b} \mapsto c_{b}, f_{a} \mapsto f_{b}, f_{b} \mapsto f_{a}
$$

and the $A$-tree $t=f_{a}\left(c_{a}, c_{a}\right)$. Now $t \in T \varphi^{-1}$ but $p_{a}(t) \notin T \varphi^{-1}$ because $t \varphi=f_{b}\left(c_{a}, c_{a}\right)=p_{b}\left(c_{a}\right) \in T$ while $p_{a}(t) \varphi=f_{b}\left(c_{a}, f_{b}\left(c_{a}, c_{a}\right)\right) \notin T$. Hence, $T \varphi^{-1} \notin \mathcal{U}\left(\Sigma^{A}, \emptyset\right)$, a contradiction, and we have shown that $\mathcal{V}=\mathcal{U}^{b}$ for no GVTL $\mathcal{U}$.

Whether there are more natural examples of varieties of binary tree languages that cannot be obtained from a GVTL remains to be seen.

The relationship between the two theories can be illuminated also by considering the corresponding syntactic algebras. First we show how the syntactic tree algebra of a binary tree language can be obtained from its syntactic algebra.

The set $\operatorname{Tr}^{+}(\mathcal{D})$ of (non-unit) translations of a $\Sigma$-algebra $\mathcal{D}=(D, \Sigma)$ is the least set of unary operations on $D$ that (1) contains every elementary translation

$$
D \rightarrow D, x \mapsto f^{\mathcal{D}}\left(d_{1}, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_{m}\right) \quad(x \in D)
$$

where $m>0, f \in \Sigma_{m}, 1 \leq i \leq m$ and $d_{1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{m} \in D$ are given, and (2) is closed under composition. Note that $\operatorname{Tr}^{+}(\mathcal{D})$ does not necessarily contain the identity map of $D$.

Definition 9.5 Let $A$ be a label alphabet and $\mathcal{D}=\left(D, \Sigma^{A}\right)$ be any $\Sigma^{A_{-}}$ algebra. Let $\delta_{\mathcal{D}}$ be the equivalence on $A$ defined by

$$
a \delta_{\mathcal{D}} b \Leftrightarrow c_{a}^{\mathcal{D}}=c_{b}^{\mathcal{D}} \& f_{a}^{\mathcal{D}}=f_{b}^{\mathcal{D}} \quad(a, b \in A)
$$

Now the $\Gamma$-algebra $\mathcal{D}^{\bullet}=\left(\left\langle A / \delta_{\mathcal{D}}, D, \operatorname{Tr}^{+}(\mathcal{D})\right\rangle, \Gamma\right)$ is defined by setting for all $a \in A, d, e \in D$ and $p, q \in \operatorname{Tr}^{+}(\mathcal{D})$,
(1) $\iota^{\mathcal{D} \boldsymbol{\bullet}}\left(a / \delta_{\mathcal{D}}\right)=c_{a}^{\mathcal{D}}$,
(2) $\kappa^{\mathcal{D}^{\boldsymbol{0}}}\left(a / \delta_{\mathcal{D}}, d, e\right)=f_{a}^{\mathcal{D}}(d, e)$,
(3) $\lambda^{\mathcal{D}^{\boldsymbol{\bullet}}}\left(a / \delta_{\mathcal{D}}, d\right)=f_{a}^{\mathcal{D}}(\xi, d)$,
(4) $\rho^{\mathcal{D}^{\boldsymbol{\bullet}}}\left(a / \delta_{\mathcal{D}}, d\right)=f_{a}^{\mathcal{D}}(d, \xi)$,
(5) $\eta^{\mathcal{D}^{\boldsymbol{\bullet}}}(p, d)=p(d)$, and
(6) $\sigma^{\mathcal{D}^{\bullet}}(p, q)=p(q)$.

The operations of $\mathcal{D}^{\bullet}$ are well-defined by the definition of $\delta_{\mathcal{D}}$. Moreover, the following holds.

Lemma 9.6 For any label alphabet $A$ and any $\Sigma^{A}$-algebra $\mathcal{D}=\left(D, \Sigma^{A}\right)$, the $\Gamma$-algebra $\mathcal{D}^{\bullet}$ is a tree algebra. Furthermore, if $\mathcal{D}$ is generated by the empty set, then $\mathcal{D}^{\bullet}$ is reduced.

Proof. It is easy to verify that $\mathcal{D}^{\bullet}$ satisfies the identities $T A$. Suppose $\mathcal{D}$ is generated by $\emptyset$. To see that $\mathcal{D}^{\bullet}$ is l-generated, we apply Definition 9.5;
(a) $A / \delta_{\mathcal{D}}$ generates all of $D$ by (1) and (2), and
(b) all elementary translations of $\mathcal{D}^{\bullet}$ are obtained from $A / \delta_{\mathcal{D}}$ and $D$ by (3) and (4), and all of their compositions are obtained by (6).

That $\mathcal{D}^{\bullet}$ satisfies condition (1) of Definition 8.1 follows from the definition of $\delta_{\mathcal{D}}$, and condition (2) follows from the definition of $\eta^{\mathcal{D}^{\bullet}}$.

Proposition 9.7 $S A(T)^{\bullet} \cong S T A(T)$ for any binary tree language $T$.
Proof. Assume that $T \subseteq T_{A}$ for some leaf alphabet $A$. Let us compare

$$
S A(T)^{\bullet}=\left(\left\langle A / \delta_{S A(T)}, T_{A} / \theta_{T}, \operatorname{Tr}^{+}(S A(T))\right\rangle, \Gamma\right)
$$

with

$$
S T A(T)=\mathcal{F}_{\mathbf{T A}}(A) / \approx^{T}=\left(\left\langle A / \approx_{1}^{T}, T_{A} / \approx_{\mathbf{t}}^{T}, C_{A}^{+} / \approx_{\mathbf{c}}^{T}\right\rangle, \Gamma\right) .
$$

First of all, we may replace $\operatorname{Tr}^{+}(S A(T))$ with $C_{A}^{+} / \sigma_{T}$ since by Lemma 6.2, for every $\alpha \in \operatorname{Tr}^{+}(S A(T))$, there is a $p \in C_{A}^{+}$such that $\alpha\left(t / \theta_{T}\right)=p(t) / \theta_{T}$
for every $t \in T_{A}$, and if $p \sigma_{T} q$ for some $p, q \in C_{A}^{+}$, then $p(t) / \theta_{T}=q(t) / \theta_{T}$ for every $t \in T_{A}$.

In the proof of Proposition 6.6 we already noted that $\theta_{T}=\approx_{\mathbf{t}}^{T}$ and $\sigma_{T}=$ $\approx_{\mathrm{c}}^{T}$. That also $\delta_{S A(T)}=\approx_{1}^{T}$ holds, follows from the definitions of $\delta_{S A(T)}$ and $S A(T)$ by repeated use of Proposition 6.3. for any $a, b \in A$,

$$
\begin{aligned}
a \delta_{S A(T)} b & \Leftrightarrow c_{a}^{S A(T)}=c_{b}^{S A(T)} \& f_{a}^{S A(T)}=f_{b}^{S A(T)} \\
& \Leftrightarrow c_{a} \approx_{\mathbf{t}}^{T} c_{b} \&\left(\forall s, t \in T_{A}\right)\left(f_{a}(s, t) \approx_{\mathbf{t}}^{T} f_{b}(s, t)\right) \\
& \Leftrightarrow a \approx_{1}^{T} b .
\end{aligned}
$$

To show that the sorted identity map defines an isomorphism between the tree algebras $S A(T)^{\bullet}$ and $S T A(T)$, we have to verify that the operations of the two $\Gamma$-algebras are the same. This can be done by straightforward, though somewhat tedious, computations directly based on the relevant definitions. As examples, we consider the $\kappa$ - and $\lambda$-operations.

For any $a \in A$ and $s, t \in T_{A}$,

$$
\begin{aligned}
\kappa^{S A(T)}\left(a / \delta_{S A(T)}, s / \theta_{T}, t / \theta_{T}\right) & =f_{a}^{S A(T)}\left(s / \theta_{T}, t / \theta_{T}\right)=f_{a}^{\mathcal{T}_{A}}(s, t) / \theta_{T} \\
& =f_{a}(s, t) / \theta_{T}=f_{a}(s, t) / \approx_{\mathbf{t}}^{T} \\
& =\kappa^{\mathcal{F}_{\mathbf{T A}}(A)}(a, s, t) / \approx_{\mathbf{t}}^{T} \\
& =\kappa^{S T A(T)}\left(a / \approx_{1}^{T}, s / \approx_{\mathbf{t}}^{T}, t / \approx_{\mathbf{t}}^{T}\right) .
\end{aligned}
$$

When considering the operations involving elements of sort context, we identify each translation $T_{A} / \theta_{T} \rightarrow T_{A} / \theta_{T}, t / \theta_{T} \mapsto p(t) / \theta_{T}$ with the $\approx_{\mathbf{c}}^{T}$-class of any $A$-context $p \in C_{A}$ that defines it. For example, $\lambda^{S A(T)^{\bullet}}=\lambda^{S T A}$ is then seen as follows. For any $a \in A$ and $s, t \in T_{A}$,

$$
\begin{aligned}
\lambda^{S A(T)}\left(a / \delta_{S A(T)}, t / \theta_{T}\right)\left(s / \theta_{T}\right) & =f_{a}^{S A(T)}\left(\xi, t / \theta_{T}\right)\left(s / \theta_{T}\right) \\
& =f_{a}^{S A(T)}\left(s / \theta_{T}, t / \theta_{T}\right)=f_{a}(s, t) / \theta_{T} \\
& =f_{a}(s, t) / \approx_{\mathbf{t}}^{T}=\lambda^{\mathcal{F}_{\mathbf{T A}}(A)}(a, t)(s) / \approx_{\mathbf{t}}^{T} \\
& =\lambda^{S T A(T)}\left(a / \approx_{\mathbf{1}}^{T}, t / \approx_{\mathbf{t}}^{T}\right)\left(s / \approx_{\mathbf{t}}^{T}\right) .
\end{aligned}
$$

Corollary 9.8 Let $A$ be any label alphabet. For any $A$-tree languages $T$ and $U$, if $S A(T) \cong S A(U)$, then $S T A(T) \cong S T A(U)$.

Although the syntactic tree algebra of any binary tree language is determined by its ordinary syntactic algebra, there is a subtle point to be observed that explains why not every BVTL is obtained from a GVTL.

In the theory of GVTLs the syntactic invariant used to characterize a tree language $T \subseteq T_{\Sigma}(X)$ is its reduced syntactic algebra $R A(T)$ (cf. [29]). This is obtained from $S A(T)$ by merging equivalent symbols in $\Sigma^{A}$ similarly as we merged label symbols when $\mathcal{M}^{r}$ was constructed from a tree algebra $\mathcal{M}$. However, in the case of a binary tree language $T \subseteq T_{A}$, the construction of $R A(T)$ may merge two symbols $c_{a}$ and $c_{b}$ without merging $f_{a}$ and $f_{b}$, or conversely, and in such a case $a$ and $b$ are not merged in $S T A(T)$.

Example 9.9 Let us consider the $A$-tree language $T=\left\{c_{a}\right\}$ for $A=\{a, b\}$. Clearly, $T_{A} / \theta_{T}=\left\{T, T^{\complement}\right\}$, and

$$
c_{a}^{S A(T)}=T, c_{b}^{S A(T)}=T^{\complement}, f_{a}^{S A(T)}(u, v)=f_{b}^{S A(T)}(u, v)=T^{\complement},
$$

for all $u, v \in T_{A} / \theta_{T}$. Hence, $f_{a}$ and $f_{b}$ are merged when $R A(T)$ is constructed but $c_{a}$ and $c_{b}$ are not. Of course, $a$ and $b$ are not merged in the $\mathbf{l}$-component of $S T A(T)$.

In the GVTL-theory any two algebras $\mathcal{D}=(D, \Sigma)$ and $\mathcal{E}=(E, \Omega)$, possibly over different ranked alphabets, are in effect equivalent if they are $g$-isomorphic, $\mathcal{D} \cong_{g} \mathcal{E}$ in symbols; a $g$-isomorphism is a g-morphism in which both mappings are bijective.

Remark 9.10 The syntactic tree algebras of two binary tree languages may be non-isomorphic even when their syntactic algebras (or even reduced syntactic algebras) are g-isomorphic. More precisely: there exist a leaf alphabet $A$ and two $A$-tree languages $T$ and $U$ such that $S A(T) \cong{ }_{g} S A(U)$ and $R A(T) \cong{ }_{g} R A(U)$, but $S T A(T) \nsubseteq S T A(U)$.

Proof. Let $A=\{a, b\}$ and let us consider the $A$-tree languages
$T=\left\{c_{a}\right\} \cup\left\{f_{a}(s, t) \mid s, t \in T_{A}\right\}$ and $U=\left\{c_{a}\right\} \cup\left\{f_{b}(s, t) \mid s, t \in T_{A}\right\}$.
Now $T_{A} / \theta_{T}=\left\{T, T^{\complement}\right\}$ and $T_{A} / \theta_{U}=\left\{U, U^{\mathrm{C}}\right\}$, and we may let $R A(T)=$ $S A(T)$ and $R A(U)=S A(U)$ because in neither case there are any pairs of equivalent symbols. It is easy to verify that the pair of maps

$$
\begin{gathered}
\alpha: \Sigma^{A} \rightarrow \Sigma^{A}, c_{a} \mapsto c_{a}, c_{b} \mapsto c_{b}, f_{a} \mapsto f_{b}, f_{b} \mapsto f_{a}, \\
\varphi: T_{A} / \theta_{T} \rightarrow T_{A} / \theta_{U}, T \mapsto U, T^{\complement} \mapsto U^{\complement},
\end{gathered}
$$

is a g-isomorphism from $R A(T)$ to $R A(U)$. However, $S T A(T) \nsubseteq S T A(U)$ because $S T A(T)$ satisfies the identity $\iota(\mathrm{a}) \simeq \kappa(\mathrm{a}, \mathrm{s}, \mathrm{t})$ while $S T A(U)$ does not. Indeed, for any $s, t \in T_{A}$,

$$
\iota^{S T A(T)}\left(d / \approx_{1}^{T}\right)=c_{d} / \approx_{\mathfrak{t}}^{T}=f_{d}(s, t) / \approx_{\mathbf{t}}^{T}=\kappa^{S T A(T)}\left(d / \approx_{\mathbf{1}}^{T}, s / \approx_{\mathbf{t}}^{T}, t / \approx_{\mathbf{t}}^{T}\right)
$$

for both $d=a$ and $d=b$, while

$$
\begin{aligned}
\iota^{S T A(U)}\left(a / \approx_{\mathbf{1}}^{U}\right) & =c_{a} / \approx_{\mathbf{t}}^{U} \\
& \neq f_{a}(s, t) / \approx_{\mathbf{t}}^{U}=\kappa^{S T A(U)}\left(a / \approx_{\mathbf{1}}^{U}, s / \approx_{\mathbf{t}}^{U}, t / \approx_{\mathbf{t}}^{U}\right) .
\end{aligned}
$$

Remark 9.10 suggests that by using syntactic tree algebras we can make some distinctions between binary tree languages that cannot be made by using syntactic algebras or reduced syntactic algebras. However, this depends again on the bond between a leaf label $c_{a}$ and an interior node label $f_{a}$ created by the definition of the syntactic tree congruence, and it may be hard to find any natural examples where the difference could be utilized.

To complete the picture, we introduce a partial converse of the construction of Definition 9.5.

Definition 9.11 Let $\mathcal{M}=(M, \Gamma)$ be a tree algebra such that $M_{1}$ is a finite set. Treating $M_{1}$ as a label alphabet and denoting it by $A$, we let $\mathcal{M}^{\circ}=$ $\left(M_{\mathrm{t}}, \Sigma^{A}\right)$ be the $\Sigma^{A}$-algebra such that for any $a \in A$,
(1) $c_{a}^{\mathcal{M}^{\circ}}=\iota^{\mathcal{M}}(a)$, and
(2) $f_{a}^{\mathcal{M}^{\circ}}(u, v)=\kappa^{\mathcal{M}}(a, u, v)$ for all $u, v \in M_{\mathbf{t}}$.

As a general example, we note that $\mathcal{F}_{\mathbf{T A}}(A)^{\circ}=\mathcal{T}_{A}$ for any label alphabet $A$. Consider now any leaf alphabet $A$ and any $A$-tree language $T$, and set $\bar{A}:=A / \approx_{1}^{T}$ and $\bar{a}:=a / \approx_{1}^{T}$ for every $a \in A$. Since $\approx_{\mathbf{t}}^{T}=\theta_{T}$ and $\approx_{\mathbf{c}}^{T}=\sigma_{T}$, we may then write $S T A(T)=\left(\left\langle\bar{A}, T_{A} / \theta_{T}, C_{A}^{+} / \sigma_{T}\right\rangle, \Gamma\right)$. By easy computations, one may verify that for every $a \in A$,
(1) $c_{\bar{a}}^{S T A(T)^{\circ}}=c_{a} / \theta_{T}=c_{a}^{S A(T)}$, and
(2) $f_{\bar{a}}^{S T A(T)^{\circ}}\left(s / \theta_{T}, t / \theta_{T}\right)=f_{a}(s, t) / \theta_{T}=f_{a}^{S A(T)}\left(s / \theta_{T}, t / \theta_{T}\right)$ for all $s, t \in T_{A}$.

Hence, $\alpha: A \rightarrow \bar{A}, a \mapsto \bar{a}$, and the identity map $\varphi: t / \theta_{T} \mapsto t / \theta_{T}$ of $T_{A} / \theta_{T}$, define a g-morphism $(\alpha, \varphi): S A(T) \rightarrow S T A(T)^{\circ}$. Since $\alpha$ is surjective and $\varphi$ the identity map, this means that $S T A(T)^{\circ}$ is very similar to $S A(T)$, the only possible difference being that some identical operations of $S A(T)$ may be replaced by one operation in $S T A(T)^{\circ}$. On the other hand, $R A(T)$ is easily seen to be obtained from $S T A(T)^{\circ}$ by possibly further merging some equally realized operators $c_{\bar{a}}$ and $c_{\bar{b}}$, or $f_{\bar{a}}$ and $f_{\bar{b}}$, for which $a \approx_{1}^{T} b$ does not hold. Any characterization of $T$ in terms of $S A(T)$, or $R A(T)$, is therefore likely to yield a characterization in terms of $S T A(T)$. This is illustrated by some examples in the following section.

## 10 Equational descriptions of VBTLs

Although basically the same families of binary tree languages can be characterized in terms of syntactic tree algebras as by syntactic algebras, or reduced syntactic algebras, in many cases the language of tree algebras appears to
be very convenient for defining the class of tree algebras corresponding to a given VBTL. In [33] Wilke gave an effective characterization of frontier testable, or reverse definite, binary tree languages in terms of syntactic tree algebras and also presented equational definitions of the corresponding class of finite tree algebras. We shall present some further examples of equational descriptions of tree algebras for some well-known families of tree languages. However, first we consider certain special $\Gamma$-terms and identities involving such terms.

Let $\mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{p}_{1}, \mathrm{p}_{2}, \ldots, \mathrm{q}_{1}, \mathrm{q}_{2}, \ldots$ and $\mathrm{s}, \mathrm{t}, \mathrm{s}_{1}, \mathrm{~s}_{2}, \ldots, \mathrm{t}_{1}, \mathrm{t}_{2}, \ldots$ be variables of sort context and tree, respectively. It is a major advantage of the language of tree algebras that it admits such variables. For any $k \geq 0$, let

- $\mathrm{p}_{k} \cdots \mathrm{p}_{1}(\mathrm{t}):=\eta\left(\mathrm{p}_{k}, \eta\left(\mathrm{p}_{k-1}, \ldots, \eta\left(\mathrm{p}_{1}, \mathrm{t}\right) \ldots\right)\right)$, and
- $\mathrm{p}_{k} \cdots \mathrm{p}_{1}:=\sigma\left(\mathrm{p}_{k}, \sigma\left(\mathrm{p}_{k-1}, \ldots, \sigma\left(\mathrm{p}_{2}, \mathrm{p}_{1}\right) \ldots\right)\right)$.

For $k=0$, these expressions stand for t and $\xi$, respectively.
Let $\zeta$ be a valuation of the variables in a given tree algebra $\mathcal{M}=(M, \Gamma)$. If $\zeta\left(\mathrm{p}_{1}\right)=p_{1}, \ldots, \zeta\left(\mathrm{p}_{k}\right)=p_{k}\left(\in M_{\mathbf{c}}\right)$ and $\zeta(\mathrm{t})=t\left(\in M_{\mathbf{t}}\right)$, then $\mathrm{p}_{k} \cdots \mathrm{p}_{1}(\mathrm{t})^{\mathcal{M}}(\zeta)$ denotes the value $\eta^{\mathcal{M}}\left(p_{k}, \ldots, \eta^{\mathcal{M}}\left(p_{1}, t\right) \ldots\right)$ of the term $\mathrm{p}_{k} \cdots \mathrm{p}_{1}(\mathrm{t})$ in $\mathcal{M}$ for the valuation $\zeta$. Similarly, let $\mathrm{p}_{k} \cdots \mathrm{p}_{1}^{\mathcal{M}}(\zeta)$ denote $\sigma^{\mathcal{M}}\left(p_{k}, \ldots, \sigma^{\mathcal{M}}\left(p_{2}, p_{1}\right) \ldots\right)$. In terms of these conventions, we can say that an identity

$$
\mathrm{p}_{h} \cdots \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{q}_{k} \cdots \mathrm{q}_{1}(\mathrm{t})
$$

holds in a tree algebra $\mathcal{M}$, or is satisfied by $\mathcal{M}$, if

$$
\mathrm{p}_{h} \cdots \mathrm{p}_{1}(\mathrm{~s})^{\mathcal{M}}(\zeta)=\mathrm{q}_{k} \cdots \mathrm{q}_{1}(\mathrm{t})^{\mathcal{M}}(\zeta)
$$

for all valuations $\zeta$ of the variables in $\mathcal{M}$.
For any label alphabet $A$, in $\mathcal{F}_{\text {TA }}(A)$ variables of sort context and variables of sort tree range over the set $C_{A}^{+}$of non-unit $A$-contexts and and the set $T_{A}$ of $A$-trees, respectively. Hence, the following useful facts.

Lemma 10.1 Let $A$ be a label alphabet, $t \in T_{A}$ and $q \in C_{A}^{+}$, and let us consider any terms $\mathrm{p}_{h} \cdots \mathrm{p}_{1}(\mathrm{~s})$ and $\mathrm{q}_{k} \cdots \mathrm{q}_{1}$, where $h \geq 0$ and $k \geq 1$. Then
(a) $\operatorname{hg}(t) \geq h$ iff there exists a valuation $\zeta$ of the variables in $\mathcal{F}_{\mathbf{T A}}(A)$ such that $\mathrm{p}_{h} \cdots \mathrm{p}_{1}(\mathrm{~s})^{\mathcal{F}_{\mathrm{TA}}(A)}(\zeta)=t$, and
(b) $\mathrm{d}_{\xi}(q) \geq k$ iff there exists a valuation $\zeta$ of the variables in $\mathcal{F}_{\mathbf{T A}}(A)$ such that $\mathrm{q}_{k} \cdots \mathrm{q}_{1}^{\mathcal{F}_{\mathrm{TA}}(A)}(\zeta)=q$.

In other words,
(a') $\operatorname{hg}(t) \geq h$ iff $t=p_{h}\left(\ldots p_{1}(s) \ldots\right)=s \cdot p_{1} \cdot \ldots \cdot p_{h}$ for some $p_{1}, \ldots, p_{h} \in C_{A}^{+}$ and $s \in T_{A}$, and
(b') $\mathrm{d}_{\xi}(q) \geq k$ iff $q=q_{k}\left(\ldots q_{2}\left(q_{1}\right) \ldots\right)=q_{1} \cdot q_{2} \cdot \ldots \cdot q_{k}$ for some $q_{1}, \ldots, q_{k} \in C_{A}^{+}$.
Consider any label alphabet $A$ and any $\Sigma^{A}$-algebra $\mathcal{D}=\left(D, \Sigma^{A}\right)$, and let $\varphi_{\mathcal{D}}$ be the unique homomorphism from $\mathcal{T}_{A}$ to $\mathcal{D}$. Each $A$-context $p \in C_{A}$ defines a unary operation $p^{\mathcal{D}}: D \rightarrow D$ as follows:
(1) $\xi^{\mathcal{D}}: d \mapsto d$ is the identity map $1_{D}: D \rightarrow D$, and
(2) if $p=f_{a}(q, s)$ or $p=f_{a}(s, q)$, for some $a \in A, q \in C_{A}$ and $s \in T_{A}$, then for every $d \in D, p^{\mathcal{D}}(d)=f_{a}^{\mathcal{D}}\left(q^{\mathcal{D}}(d), s \varphi_{\mathcal{D}}\right)$ or $p^{\mathcal{D}}(d)=f_{a}^{\mathcal{D}}\left(s \varphi_{\mathcal{D}}, q^{\mathcal{D}}(d)\right)$, respectively.

It is clear that each $p^{\mathcal{D}}$ is a translation of $\mathcal{D}$, and if $\varphi_{\mathcal{D}}$ is surjective, then every translation of $\mathcal{D}$ is obtained this way. It is also clear that if $p=q(r)$ for some $q, r \in C_{A}$, then $p^{\mathcal{D}}$ is the composition of $q^{\mathcal{D}}$ and $r^{\mathcal{D}}$, that is to say, $p^{\mathcal{D}}(d)=q^{\mathcal{D}}\left(r^{\mathcal{D}}(d)\right)$ for every $d \in D$.

The following lemma results from the above discussion.
Lemma 10.2 Let $A$ be any leaf alphabet and let $\mathcal{M}=(M, \Gamma)$ be a tree algebra such that $M_{1}=A$. Then the following hold for all $h, k \geq 0$.
(a) $\mathcal{M}$ satisfies $\mathrm{p}_{h} \cdots \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{q}_{k} \cdots \mathrm{q}_{1}(\mathrm{t})$ iff $p^{\mathcal{M}^{\circ}}(u)=q^{\mathcal{M}^{\circ}}(v)$ holds for all $u, v \in M_{\mathrm{t}}$ and all $p, q \in C_{A}$ such that $\mathrm{d}_{\xi}(p) \geq h$ and $\mathrm{d}_{\xi}(q) \geq k$.
(b) $\mathcal{M}$ satisfies $\mathrm{p}_{k} \cdots \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{p}_{k} \cdots \mathrm{p}_{1}(\mathrm{t})$ iff $p^{\mathcal{M}^{\circ}}(u)=p^{\mathcal{M}^{\circ}}(v)$ holds for all $u \in M_{\mathrm{t}}$ and all $p \in C_{A}$ such that $\mathrm{d}_{\xi}(p) \geq k$.
(c) $\mathcal{M}$ satisfies $\mathrm{p}_{h} \cdots \mathrm{p}_{1} \approx \mathrm{q}_{k} \cdots \mathrm{q}_{1}$ iff $p^{\mathcal{M}^{\circ}}=q^{\mathcal{M}^{\circ}}$ holds for all $p, q \in C_{A}$ such that $\mathrm{d}_{\xi}(p) \geq h$ and $\mathrm{d}_{\xi}(q) \geq k$.

Recall that an algebra $\mathcal{D}$ (of any kind) is said to ultimately satisfy (cf. [8]) a sequence of identities $u_{1} \approx v_{1}, u_{2} \approx v_{2}, u_{3} \approx v_{3}, \ldots$ if there is an $n \geq 1$ such that $\mathcal{D}$ satisfies $u_{k} \approx v_{k}$ for every $k \geq n$.

The term function $t^{\mathcal{D}}: D^{X} \rightarrow D$ induced by a term $t \in T_{\Sigma}(X)$ in a given $\Sigma$-algebra $\mathcal{D}=(D, \Sigma)$ is defined as follows. For any assignment $\alpha: X \rightarrow D$ of values to the variables,
(1) $c^{\mathcal{D}}(\alpha)=c^{\mathcal{D}}$ for every $c \in \Sigma_{0}$,
(2) $x^{\mathcal{D}}(\alpha)=\alpha(x)$ for every $x \in X$, and
(3) $t^{\mathcal{D}}(\alpha)=f^{\mathcal{D}}\left(t_{1}^{\mathcal{D}}(\alpha), \ldots, t_{m}^{\mathcal{D}}(\alpha)\right)$ if $t=f\left(t_{1}, \ldots, t_{m}\right)$.

As the first example we consider the GVTL Nil $=\{\operatorname{Nil}(\Sigma, X)\}$ where for each pair $\Sigma$ and $X, \operatorname{Nil}(\Sigma, X)$ is the set of all finite $\Sigma X$-tree languages and their complements in $T_{\Sigma}(X)$. In [12] a finite algebra $\mathcal{D}=(D, \Sigma)$ was defined to be nilpotent if there is an element $d_{0} \in D$ and a number $k>0$ such that for any leaf alphabet $X$, and any $t \in T_{\Sigma}(X)$ such that $\mathrm{hg}(t) \geq k$, $t^{\mathcal{D}}(\alpha)=d_{0}$ for every assignment $\alpha: X \rightarrow D$. In other words, if $\mathcal{D}$ is viewed as a deterministic bottom-up tree automaton, it reaches the root of any tree of height $\geq k$ in state $d_{0}$ for any assignment $\alpha$ of initial states to the leaf symbols. If $\mathcal{D}$ is nilpotent, the minimal value of $k$ is called its degree of nilpotency. In [28] it was noted that for any fixed $\Sigma$, the class $\mathrm{Nil}_{\Sigma}$ of all nilpotent $\Sigma$-algebras is the variety of finite $\Sigma$-algebras that corresponds to
the family $\operatorname{Nil}_{\Sigma}=\{\operatorname{Nil}(\Sigma, X)\}$, where $\Sigma$ is now fixed and $X$ ranges over all leaf alphabets. This fact is easily extended to a correspondence between the GVTL Nil and the generalized variety of finite algebras (GVFA; cf. [29], p. 13) Nil of all nilpotent algebras. This means that for any $\Sigma$ and $X$, a $\Sigma X$-tree language $T$ is in $\operatorname{Nil}(\Sigma, X)$ iff $S A(T) \in \mathbf{N i l}_{\Sigma}$, or equivalently, iff $R A(T) \in$ Nil.

Let $A$ be any leaf alphabet. It is easy to see that a finite $\Sigma^{A}$-algebra $\mathcal{D}=\left(D, \Sigma^{A}\right)$ is nilpotent if there exist a $k \geq 0$ and an element $d_{0} \in D$ such that $p^{\mathcal{D}}(d)=d_{0}$ for every $d \in D$ whenever $p \in C_{A}^{+}$is an $A$-context of height $\geq k$. This means by Lemma 10.2 (a) that for a finite tree algebra $\mathcal{M}=(M, \Gamma)$ such that $M_{1}=A$, the algebra $\mathcal{M}^{\circ}=\left(M_{\mathrm{t}}, \Sigma^{A}\right)$ (defined in the previous section) is nilpotent iff $\mathcal{M}$ satisfies the identity $\mathrm{p}_{k} \cdots \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{q}_{k} \cdots \mathrm{q}_{1}(\mathrm{t})$ for some $k \geq 0$. Furthermore, it is clear that the syntactic algebra $S A(T)$ of any regular $A$-tree language $T$ is nilpotent iff $S T A(T)^{\circ}$ is nilpotent. By collecting together the above observations, we obtain the following description of the VBTL $N i l^{b}$.

Proposition 10.3 If $T$ is any regular $A$-tree language for some label alphabet $A$, then $T \in \operatorname{Nil}^{b}(A)$ iff $S T A(T)$ ultimately satisfies the sequence of identities

$$
\begin{equation*}
\mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{q}_{1}(\mathrm{t}), \mathrm{p}_{2} \cdot \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{q}_{2} \cdot \mathrm{q}_{1}(\mathrm{t}), \mathrm{p}_{3} \cdot \mathrm{p}_{2} \cdot \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{q}_{3} \cdot \mathrm{q}_{2} \cdot \mathrm{q}_{1}(\mathrm{t}), \ldots \tag{N}
\end{equation*}
$$

A couple of remarks are in order here. One can write for any given label alphabet $A$ a sequence of $\Sigma^{A}$-equations that ultimately defines the class of nilpotent $\Sigma^{A}$-algebras. For example, if $A=\{a, b\}$, the class of nilpotent $\Sigma^{A}$-algebras is ultimately defined by a sequence

$$
\begin{aligned}
& x_{1} \approx x_{2}, f_{a}\left(x_{1}, x_{2}\right) \approx f_{b}\left(x_{3}, x_{4}\right), f_{a}\left(f_{a}\left(x_{1}, x_{2}\right), x_{3}\right) \approx f_{a}\left(x_{4}, f_{a}\left(x_{5}, x_{6}\right)\right), \\
& f_{a}\left(f_{a}\left(x_{1}, x_{2}\right), x_{3}\right) \approx f_{a}\left(x_{4}, f_{b}\left(x_{5}, x_{6}\right)\right), \ldots,
\end{aligned}
$$

that for each $k \geq 0$, contains a set of identities defining the class of $\Sigma^{A_{-}}$ algebras of degree of nilpotency $\leq k$. However, this sequence is more complicated than the sequence (Nil) and then it depends on $A$. On the other hand, it has to be noted that a description of a VBTL like the above proposition does not yield automatically a decision method; we still need some bound for the degree of nilpotency of a nilpotent algebra in terms of its size, for example.

As our next example we consider definite tree languages. A tree language is definite if there is a bound $k \geq 0$ such that the membership of a tree in the language can be decided by looking at its root-segment of height $k$. Definite tree languages were first studied by Heuter [15, 16], their variety properties were noted in [28, 29], and in [9] Ésik describes the corresponding algebras and study their structure. The following formal definitions are from [15, 16 as modified in [28].

Let $\Sigma$ be any ranked alphabet, $X$ any leaf alphabet and $k \geq 0$. For any $\Sigma X$-tree $t$, let root $(t)$ denote the label of the root of $t$. The $k$-root $\mathrm{r}_{k}(t)$ of a $\Sigma X$-tree $t$ is now defined as follows:
(1) $\mathrm{r}_{k}(t)=\epsilon$ (the empty root segment) for every $t \in T_{\Sigma}(X)$.
(2) $\mathrm{r}_{1}(t)=\operatorname{root}(t)$ for every $t \in T_{\Sigma}(X)$.
(3) Let $k \geq 2$. If $\operatorname{hg}(t) \leq k$, then $\mathrm{r}_{k}(t)=t$. If $\operatorname{hg}(t)>k$ and $t=$ $f\left(t_{1}, \ldots, t_{m}\right)$, then $\mathrm{r}_{k}(t)=f\left(\mathrm{r}_{k-1}\left(t_{1}\right), \ldots, \mathrm{r}_{k-1}\left(t_{m}\right)\right)$.

A tree language $T \subseteq T_{\Sigma}(X)$ is $k$-definite $(k \geq 0)$ if for all $s, t \in T_{\Sigma}(X)$, if $\mathrm{r}_{k}(s)=\mathrm{r}_{k}(t)$, then $s \in T$ iff $t \in T$. A tree language is definite if it is $k$-definite for some $k \geq 0$. Let $\operatorname{Def}_{k}(\Sigma, X)$ and $\operatorname{Def}(\Sigma, X)$ denote the sets of all $k$-definite and all definite $\Sigma X$-tree languages, respectively. For any $k \geq 0$, $\operatorname{De} f_{k}:=\left\{\operatorname{Def}_{k}(\Sigma, X)\right\}$ is a GVTL, and so is the union $\operatorname{Def}:=\{\operatorname{Def}(\Sigma, X)\}$ of these families (cf. [29]).

A $\Sigma$-algebra $\mathcal{D}=(D, \Sigma)$ is said to be $k$-definite $(k \geq 0)$ if for every $X$ and any $s, t \in T_{\Sigma}(X)$, if $\mathrm{r}_{k}(s)=\mathrm{r}_{k}(t)$, then $s^{\mathcal{D}}(\alpha)=t^{\mathcal{D}}(\alpha)$ for every $\alpha: X \rightarrow D$. An algebra is definite if it is $k$-definite for some $k \geq 0$.

In [9] it was shown that a tree language is ( $k$-)definite iff its syntactic algebra is ( $k$-)definite. To turn this fact into an equational tree algebra characterization, we need one more observation: for any $k \geq 0, \Sigma, X$ and $s, t \in T_{\Sigma}(X)$, $\mathrm{r}_{k}(s)=\mathrm{r}_{k}(t)$ holds iff $s=p_{k}\left(\ldots p_{1}\left(s^{\prime}\right) \ldots\right)$ and $t=p_{k}\left(\ldots p_{1}\left(t^{\prime}\right) \ldots\right)$ for some $p_{1}, \ldots, p_{k} \in C_{\Sigma}^{+}(X)$ and $s^{\prime}, t^{\prime} \in T_{\Sigma}(X)$.

By applying the above definitions and facts to the alphabets $\Sigma^{A}$ and binary tree languages, we get by Lemma 10.2 (b) the following result.

Proposition 10.4 Let $T$ be a regular $A$-tree language for some label alphabet $A$. Then $T \in \operatorname{Def}^{b}(A)$ iff $S T A(T)$ ultimately satisfies the sequence of identities

$$
\begin{equation*}
\mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{p}_{1}(\mathrm{t}), \mathrm{p}_{2} \cdot \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{p}_{2} \cdot \mathrm{p}_{1}(\mathrm{t}), \mathrm{p}_{3} \cdot \mathrm{p}_{2} \cdot \mathrm{p}_{1}(\mathrm{~s}) \approx \mathrm{p}_{3} \cdot \mathrm{p}_{2} \cdot \mathrm{p}_{1}(\mathrm{t}), \ldots \tag{D}
\end{equation*}
$$

Again we can note that one could apply the equational descriptions of definite algebras given by Ésik 9 to obtain, for each $A$, a sequence that ultimately defines the class of definite $\Sigma^{A}$-algebras, but such a sequence is more complicated than (D) and it depends on $A$. As shown by Heuter [15, 16], and by Ésik [9, it is decidable whether a given finite algebra is definite or not, but this does not follow from the equational descriptions alone.

As a further, somewhat different, example, we consider the aperiodic tree languages introduced by Thomas [32]. A $\Sigma X$-tree language $T$ is aperiodic, or non-counting, if there is an $n \geq 0$ such that for all $p, q \in C_{\Sigma}^{+}(X)$ and $t \in T_{\Sigma}(X), t \cdot p^{n} \cdot q \in T$ iff $t \cdot p^{n+1} \cdot q \in T$. If $A p(\Sigma, X)$ denotes the set of all aperiodic $\Sigma X$-tree languages, then $A p:=\{A p(\Sigma, X)\}$ is a GVTL (cf. [29]).

In [32] it is shown that a tree language $T$ is aperiodic iff its syntactic monoid $S M(T)$ is aperiodic, i.e., all of its subgroups are trivial. A semigroup $M$ is known to be aperiodic iff there exists an $n \geq 0$ such that $x^{n+1}=x^{n}$ for every $x \in M$ (cf. [8] or [22], for example).

The c-component $C_{A}^{+} / \approx_{\mathbf{c}}^{T}$ of the syntactic tree algebra of a binary tree language $T \subseteq T_{A}$ forms with respect to the product $p / T \cdot q / T:=p \cdot q / T$ a semigroup isomorphic to the syntactic semigroup $S S(T)$, and this differs from the syntactic monoid $S M(T)$ only in that it does not necessarily have a unit element. Hence, $T$ is aperiodic iff $C_{A}^{+} / \approx_{\mathrm{c}}^{T}$ is an aperiodic semigroup. By Lemma 10.2 (c) we may now turn Thomas' result into the following equational characterization, where $\mathrm{p}^{n}$ stands for the $n$-fold product $\mathrm{p} \cdot \mathrm{p} \cdots \mathrm{p}(n \geq 1)$.

Proposition 10.5 If $T$ is a regular $A$-tree language for some label alphabet $A$, then $T \in A p^{b}(A)$ iff $S T A(T)$ ultimately satisfies the sequence of identities

$$
\begin{equation*}
\mathrm{p}^{2} \approx \mathrm{p}, \mathrm{p}^{3} \approx \mathrm{p}^{2}, \mathrm{p}^{4} \approx \mathrm{p}^{3}, \ldots \tag{A}
\end{equation*}
$$

## 11 Concluding remarks

We have developed the theory of tree algebras and tree algebra representations of binary trees in a systematic algebraic way, and explored the relationships between this formalism and some other approaches to the classification of regular tree languages. The new results include also a Variety Theorem. Of course, many questions remain to be studied. For example, one could try to extend the formalism in such way that the restriction to binary trees could be lifted. Alternatively, one could borrow some ideas from the tree algebra theory to other formalisms to increase their expressive power. One would also like to see further effective characterizations of natural families of binary tree languages in terms of syntactic tree algebras. However, in view of our results, it seems that they would, in most cases, be virtually equivalent to characterizations in terms of ordinary syntactic algebras.

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## Appendix

Here we present proof for the full version of Lemma 6.2. Let us begin by writing out explicitly the set of elementary translations of the free tree algebra $\mathcal{F}_{\mathbf{T A}}(A)$ for a fixed $A$. For simplicity, we write $\mathrm{ETr}_{\mathrm{i}, \mathrm{j}}$ instead of $\operatorname{ETr}\left(\mathcal{F}_{\mathbf{T A}}(A), \mathbf{i}, \mathbf{j}\right)$ where $\mathbf{i}, \mathbf{j} \in S$. We also omit $\mathcal{F}_{\mathbf{T A}}(A)$ as a superscript from the operations $\iota^{\mathcal{F}_{\mathrm{TA}}(A)}, \kappa^{\mathcal{F}_{\mathrm{TA}}(A)}$, etc. Clearly,
(ll) $\mathrm{ETr}_{1, \mathrm{l}}=\emptyset$;
(lt) $\operatorname{ETr}_{1, \mathrm{t}}=\left\{\iota\left(\xi_{1}\right)\right\} \cup\left\{\kappa\left(\xi_{1}, s, t\right) \mid s, t \in T_{A}\right\} ;$
(lc) $\mathrm{ETr}_{\mathbf{l}, \mathbf{c}}=\left\{\lambda\left(\xi_{\mathbf{1}}, t\right) \mid t \in T_{A}\right\} \cup\left\{\rho\left(\xi_{\mathbf{1}}, t\right) \mid t \in T_{A}\right\} ;$
(tl) $\mathrm{ETr}_{\mathrm{t}, \mathrm{l}}=\emptyset$;
(tt) $\mathrm{ETr}_{\mathbf{t}, \mathbf{t}}=\left\{\kappa\left(a, t, \xi_{\mathbf{t}}\right) \mid a \in A, t \in T_{A}\right\}$

$$
\cup\left\{\kappa\left(a, \xi_{\mathbf{t}}, t\right) \mid a \in A, t \in T_{A}\right\} \cup\left\{\eta\left(p, \xi_{\mathbf{t}}\right) \mid p \in C_{A}^{+}\right\}
$$

(tc) $\mathrm{ETr}_{\mathbf{t}, \mathbf{c}}=\left\{\lambda\left(a, \xi_{\mathbf{t}}\right) \mid a \in A\right\} \cup\left\{\rho\left(a, \xi_{\mathrm{t}}\right) \mid a \in A\right\} ;$
(cl) $\mathrm{ETr}_{\mathrm{c}, \mathrm{l}}=\emptyset$;
(ct) $\mathrm{ETr}_{\mathbf{c}, \mathbf{t}}=\left\{\eta\left(\xi_{\mathbf{c}}, t\right) \mid t \in T_{A}\right\} ;$
(cc) $\operatorname{ETr}_{\mathbf{c}, \mathbf{c}}=\left\{\sigma\left(\xi_{\mathbf{c}}, p\right) \mid p \in C_{A}^{+}\right\} \cup\left\{\sigma\left(p, \xi_{\mathbf{c}}\right) \mid p \in C_{A}^{+}\right\} ;$
where $\xi_{1}, \xi_{\mathrm{t}}$ and $\xi_{\mathrm{c}}$ are variables of sort label, tree and context, respectively.
Recall that for contexts $p, q \in C_{A}$ and tree $t \in T_{A}$, the composition of $p$ and $q$ is denoted by $q \cdot p$ which is a context in $C_{A}$ resulted from $p$ by putting $q$ in the place of $\xi$. Similarly, the tree $t \cdot p$ results from $p$ by substituting the $\xi$ by $t$. We note that $(u \cdot q) \cdot p=u \cdot(q \cdot p)$ holds for all $p, q \in C_{A}$, and $u \in T_{A} \cup C_{A}$. Thus we omit the brackets in the expressions like $(u \cdot q) \cdot p$ and simply write $u \cdot q \cdot p$.

For any $\mathbf{i}, \mathbf{j} \in S$ let $\operatorname{Tr}_{\mathbf{i}, \mathbf{j}}$ denote $\operatorname{Tr}\left(\mathcal{F}_{\mathbf{T A}}(A), \mathbf{i}, \mathbf{j}\right)$. The full form of Lemma 6.2 can be written as follows.

## Lemma 6.2

(ll) The only member of $\operatorname{Tr}_{1,1}$ is the identity function $A \rightarrow A$.
(lt) $\alpha: A \rightarrow T_{A}$ belongs to $\operatorname{Tr}_{1, \mathrm{t}}$ iff for some $s, t \in T_{A}$ and $p \in C_{A}$, either $\alpha: a \mapsto c_{a} \cdot p$ for every $a \in A$, or $\alpha: a \mapsto f_{a}(s, t) \cdot p$ for every $a \in A$.
(lc) $\alpha: A \rightarrow C_{A}^{+}$belongs to $\operatorname{Tr}_{1, \mathbf{c}}$ iff for some $b \in A, s, t \in T_{A}$ and some $p, q \in C_{A}$, either

$$
\begin{aligned}
& \alpha: a \mapsto f_{a}(q, t) \cdot p \text { for every } a \in A, \text { or } \\
& \alpha: a \mapsto f_{a}(t, q) \cdot p \text { for every } a \in A, \text { or } \\
& \alpha: a \mapsto f_{b}\left(q, c_{a} \cdot r\right) \cdot p \text { for every } a \in A, \text { or } \\
& \alpha: a \mapsto f_{b}\left(c_{a} \cdot r, q\right) \cdot p \text { for every } a \in A \text {, or } \\
& \alpha: a \mapsto f_{b}\left(q, f_{a}(s, t) \cdot r\right) \cdot p \text { for every } a \in A, \text { or } \\
& \alpha: a \mapsto f_{b}\left(f_{a}(s, t) \cdot r, q\right) \cdot p \text { for every } a \in A .
\end{aligned}
$$

(tl) $\operatorname{Tr}_{\mathbf{t}, \mathrm{l}}=\emptyset$.
( $\mathbf{t t}$ ) $\alpha: T_{A} \rightarrow T_{A}$ belongs to $\operatorname{Tr}_{\mathbf{t}, \mathrm{t}}$ iff for some $p \in C_{A}$,
$\alpha: t \mapsto t \cdot p$ for every $t \in T_{A}$.
(tc) $\alpha: T_{A} \rightarrow C_{A}^{+}$belongs to $\operatorname{Tr}_{\mathbf{t}, \mathbf{c}}$ iff for some $a \in A$ and $p, q, r \in C_{A}$, either
$\alpha: t \mapsto f_{a}(q, t \cdot r) \cdot p$ for every $t \in T_{A}$, or
$\alpha: t \mapsto f_{a}(t \cdot r, q) \cdot p$ for every $t \in T_{A}$.
(cl) $\operatorname{Tr}_{\mathrm{c}, 1}=\emptyset$.
(ct) $\alpha: C_{A}^{+} \rightarrow T_{A}$ belongs to $\operatorname{Tr}_{\mathbf{c}, \mathbf{t}}$ iff for some $q \in C_{A}$ and $t \in T_{A}$,
$\alpha: p \mapsto t \cdot p \cdot q$ for every $p \in C_{A}$.
(cc) $\alpha: C_{A}^{+} \rightarrow C_{A}^{+}$belongs to $\operatorname{Tr}_{\mathbf{c}, \mathbf{c}}$ iff for some $a \in A, t \in T_{A}$ and some $q, r, d \in C_{A}$, either
$\alpha: p \mapsto r \cdot p \cdot q$ for every $p \in C_{A}$, or
$\alpha: p \mapsto f_{a}(r, t \cdot p \cdot d) \cdot q$ for every $p \in C_{A}$, or
$\alpha: p \mapsto f_{a}(t \cdot p \cdot d, r) \cdot q$ for every $p \in C_{A}$.

Proof. For each pair $\mathbf{i}, \mathbf{j} \in S$, let $\mathrm{T}(\mathbf{i}, \mathbf{j})$ denote the set of maps specified in clauses ( $\mathbf{i j}$ ) of the lemma. It is then enough to show that
(A) $E T_{\mathbf{i}, \mathbf{j}} \subseteq T(\mathbf{i}, \mathbf{j}) \subseteq \operatorname{Tr}_{\mathbf{i}, \mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in S$,
(B) for any $\mathbf{i} \in S, \mathrm{~T}(\mathbf{i}, \mathbf{i})$ contains the identity function of sort $\mathbf{i}$, and
(C) the collection $\{\mathrm{T}(\mathbf{i}, \mathbf{j})\}_{\mathbf{i}, \mathrm{j} \in S}$ is closed under composition of mappings.
(A) The claimed inclusions follow from the following observations. We always assume that $a, b \in A, s, t, u \in T_{A}$ and $p, q, r, d \in C_{A}$.
(lt) • $\iota(a)=c_{a} \cdot \xi, \kappa(a, s, t)=f_{a}(s, t) \cdot \xi$;

- $c_{a} \cdot p=\eta(p, \iota(a)), f_{a}(s, t) \cdot p=\eta(p, \kappa(a, s, t))$;
(lc) - $\lambda(a, t)=f_{a}(\xi, t) \cdot \xi, \rho(a, t)=f_{a}(t, \xi) \cdot \xi$;
- $f_{a}(q, t) \cdot p=\sigma(p, \sigma(\lambda(a, t), q))$, $f_{a}(t, q) \cdot p=\sigma(p, \sigma(\rho(a, t), q))$, $f_{b}\left(q, c_{a} \cdot r\right) \cdot p=\sigma(p, \sigma(\lambda(a, \eta(r, \iota(a))), q))$, $f_{b}\left(c_{a} \cdot r, q\right) \cdot p=\sigma(p, \sigma(\rho(a, \eta(r, \iota(a))), q))$, $f_{b}\left(q, f_{a}(s, t) \cdot r\right) \cdot p=\sigma(p, \sigma(\lambda(a, \eta(r, \kappa(a, s, t))), q))$, $f_{b}\left(f_{a}(s, t) \cdot r, q\right) \cdot p=\sigma(p, \sigma(\rho(a, \eta(r, \kappa(a, s, t))), q))$;
$(\mathbf{t t}) \quad \kappa(a, s, t)=t \cdot f_{a}(s, \xi), \kappa(a, t, s)=t \cdot f_{a}(\xi, s), \eta(p, t)=t \cdot p$;
- $t \cdot p=\eta(p, t)$;
$(\mathbf{t c}) \quad-\lambda(a, t)=f_{a}(\xi, t \cdot \xi) \cdot \xi, \rho(a, t)=f_{a}(t \cdot \xi, \xi) \cdot \xi ;$
- $f_{a}(q, t \cdot r) \cdot p=\sigma(p, \sigma(\lambda(a, \eta(r, t)), q))$, $f_{a}(t \cdot r, q) \cdot p=\sigma(p, \sigma(\rho(a, \eta(r, t)), q)) ;$
(ct) - $\eta(p, t)=t \cdot p \cdot \xi$;
- $t \cdot p \cdot q=\eta(q, \eta(p, t))$;
(cc) - $\sigma(p, q)=q \cdot p \cdot \xi, \sigma(q, p)=\xi \cdot p \cdot q$;
- $r \cdot p \cdot q=\sigma(q, \sigma(p, r))$,

$$
f_{a}(r, t \cdot p \cdot d) \cdot q=\sigma(q, \sigma(\lambda(a, \eta(d, \eta(p, t))), r)),
$$

$$
f_{a}(t \cdot p \cdot d, r) \cdot q=\sigma(q, \sigma(\rho(a, \eta(d, \eta(p, t))), r)) .
$$

(B) It is straightforward to see that $\mathrm{T}(\mathbf{i}, \mathbf{i})$ contains the identity function for each $\mathbf{i} \in S$.
(C) We now show that for each $\alpha \in \mathrm{T}(\mathbf{i}, \mathbf{j})$ and $\beta \in \mathrm{T}(\mathbf{j}, \mathbf{k})(\mathbf{i}, \mathbf{j}, \mathbf{k} \in S)$ their composition $\alpha \cdot \beta$ belongs to $T(\mathbf{i}, \mathbf{k})$. We always assume that $a, b, o \in A$, $s, t, u \in T_{A}$ and $p, q, r, d, e, g, h \in C_{A}$.

- If $\alpha \in \operatorname{Tr}_{\mathbf{l}, \mathbf{1}}$ then $\alpha$ is the identity function, thus for any $\beta \in \mathrm{T}(\mathbf{l}, \mathbf{k})$ we have $\alpha \cdot \beta=\beta \in \mathrm{T}(\mathbf{l}, \mathbf{k})$.
- For $\alpha \in \mathrm{T}(\mathbf{l}, \mathbf{t})$ and $\beta \in \mathrm{T}(\mathbf{t}, \mathbf{t})$, if $\beta: t \mapsto t \cdot p$ then
if $\alpha: a \mapsto c_{a} \cdot q$ then $\alpha \cdot \beta: a \mapsto c_{a} \cdot q \cdot p \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$; and if $\alpha: a \mapsto f_{a}(s, t) \cdot q$ then $\alpha \cdot \beta: a \mapsto f_{a}(s, t) \cdot q \cdot p \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$.
- For $\alpha \in \mathrm{T}(\mathbf{l}, \mathbf{t})$ and $\beta \in \mathrm{T}(\mathbf{t}, \mathbf{c})$, first assume $\alpha: a \mapsto c_{a} \cdot p$. Then if $\beta: t \mapsto f_{b}(r, t \cdot d) \cdot q$, then $\alpha \cdot \beta: a \mapsto f_{b}\left(r, c_{a} \cdot p \cdot q\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$; and
if $\beta: t \mapsto f_{b}(t \cdot d, r) \cdot q$, then $\alpha \cdot \beta: a \mapsto f_{b}\left(c_{a} \cdot p \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$.
Now, assume $\alpha: a \mapsto f_{a}(s, t) \cdot p$. Then
if $\beta: t \mapsto f_{b}(r, t \cdot d) \cdot q$, then $\alpha \cdot \beta: a \mapsto f_{b}\left(r, f_{a}(s, t) \cdot p \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$; and if $\beta: t \mapsto f_{b}(t \cdot d, r) \cdot q$, then $\alpha \cdot \beta: a \mapsto f_{b}\left(f_{a}(s, t) \cdot p \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$.
- For $\alpha \in \mathrm{T}(\mathbf{l}, \mathbf{c})$ and $\beta \in \mathrm{T}(\mathbf{c}, \mathbf{t})$, if $\beta: p \mapsto t \cdot p \cdot q$ then
if $\alpha: a \mapsto f_{a}(d, s) \cdot r$ then $\alpha \cdot \beta: a \mapsto f_{a}(t \cdot d, s) \cdot r \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$;
if $\alpha: a \mapsto f_{a}(s, d) \cdot r$ then $\alpha \cdot \beta: a \mapsto f_{a}(s, t \cdot d) \cdot r \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$;
if $\alpha: a \mapsto f_{b}\left(d, c_{a} \cdot e\right) \cdot r$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(t \cdot d, c_{a} \cdot e\right) \cdot r \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$;
if $\alpha: a \mapsto f_{b}\left(c_{a} \cdot e, d\right) \cdot r$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(c_{a} \cdot e, t \cdot d\right) \cdot r \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$;
if $\alpha: a \mapsto f_{b}\left(d, f_{a}(s, u) \cdot e\right) \cdot r$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(t \cdot d, f_{a}(s, u) \cdot e\right) \cdot r \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$;
if $\alpha: a \mapsto f_{b}\left(f_{a}(s, u) \cdot e, d\right) \cdot r$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(f_{a}(s, u) \cdot e, t \cdot d\right) \cdot r \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{t})$.
- For $\alpha \in \mathrm{T}(\mathbf{l}, \mathbf{c})$ and $\beta \in \mathrm{T}(\mathbf{c}, \mathbf{c})$, we distinguish three different cases of $\beta$ :
(1) if $\beta: p \mapsto r \cdot p \cdot q$, then
if $\alpha: a \mapsto f_{a}(e, t) \cdot d$ then $\alpha \cdot \beta: a \mapsto f_{a}(r \cdot e, t) \cdot d \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{a}(t, e) \cdot d$ then $\alpha \cdot \beta: a \mapsto f_{a}(t, r \cdot e) \cdot d \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{b}\left(e, c_{a} \cdot g\right) \cdot d$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(r \cdot e, c_{a} \cdot g\right) \cdot d \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{b}\left(c_{a} \cdot g, e\right) \cdot d$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(c_{a} \cdot g, r \cdot e\right) \cdot d \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{b}\left(e, f_{a}(s, t) \cdot g\right) \cdot d$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(r \cdot e, f_{a}(s, t) \cdot g\right) \cdot d \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{b}\left(f_{a}(s, t) \cdot g, e\right) \cdot d$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(f_{a}(s, t) \cdot g, r \cdot e\right) \cdot d \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$.
(2) if $\beta: p \mapsto f_{b}(r, t \cdot p \cdot d) \cdot q$, then
if $\alpha: a \mapsto f_{a}(g, s) \cdot e$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(r, f_{a}(t \cdot g, s) \cdot e \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{a}(s, g) \cdot e$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(r, f_{a}(s, t \cdot g) \cdot e \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{o}\left(g, c_{a} \cdot h\right) \cdot e$ then
$\alpha \cdot \beta: a \mapsto f_{b}\left(r, f_{o}\left(t \cdot g, c_{a} \cdot h\right) \cdot e \cdot d\right) \cdot q=f_{b}\left(r, c_{a} \cdot f_{o}(t \cdot g, h) \cdot e \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) ;$
if $\alpha: a \mapsto f_{o}\left(c_{a} \cdot h, g\right) \cdot e$ then
$\alpha \cdot \beta: a \mapsto f_{b}\left(r, f_{o}\left(c_{a} \cdot h, t \cdot g\right) \cdot e \cdot d\right) \cdot q=f_{b}\left(r, c_{a} \cdot f_{o}(h, t \cdot g) \cdot e \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) ;$
if $\alpha: a \mapsto f_{o}\left(g, f_{a}(s, u) \cdot h\right) \cdot e$ then

$$
\begin{aligned}
& \alpha \cdot \beta: a \mapsto f_{b}\left(r, f_{o}\left(t \cdot g, f_{a}(s, u) \cdot h\right) \cdot e \cdot d\right) \cdot q= \\
& \quad f_{b}\left(r, f_{a}(s, u) \cdot f_{o}(t \cdot g, h) \cdot e \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) ;
\end{aligned}
$$

if $\alpha: a \mapsto f_{o}\left(f_{a}(s, u) \cdot h, g\right) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: a \mapsto & f_{b}(r, \\
& \left.f_{o}\left(f_{a}(s, u) \cdot h, t \cdot g\right) \cdot e \cdot d\right) \cdot q= \\
& f_{b}\left(r, f_{a}(s, u) \cdot f_{o}(h, t \cdot g) \cdot e \cdot d\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) .
\end{aligned}
$$

(3) if $\beta: p \mapsto f_{b}(t \cdot p \cdot d, r) \cdot q$, then
if $\alpha: a \mapsto f_{a}(g, s) \cdot e$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(f_{a}(t \cdot g, s) \cdot e \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{a}(s, g) \cdot e$ then $\alpha \cdot \beta: a \mapsto f_{b}\left(f_{a}(s, t \cdot g) \cdot e \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c})$;
if $\alpha: a \mapsto f_{o}\left(g, c_{a} \cdot h\right) \cdot e$ then
$\alpha \cdot \beta: a \mapsto f_{b}\left(f_{o}\left(t \cdot g, c_{a} \cdot h\right) \cdot e \cdot d, r\right) \cdot q=f_{b}\left(c_{a} \cdot f_{o}(t \cdot g, h) \cdot e \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) ;$
if $\alpha: a \mapsto f_{o}\left(c_{a} \cdot h, g\right) \cdot e$ then
$\alpha \cdot \beta: a \mapsto f_{b}\left(f_{o}\left(c_{a} \cdot h, t \cdot g\right) \cdot e \cdot d, r\right) \cdot q=f_{b}\left(c_{a} \cdot f_{o}(h, t \cdot g) \cdot e \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) ;$
if $\alpha: a \mapsto f_{o}\left(g, f_{a}(s, u) \cdot h\right) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: a \mapsto & f_{b}\left(f_{o}\left(t \cdot g, f_{a}(s, u) \cdot h\right) \cdot e \cdot d, r\right) \cdot q= \\
& f_{b}\left(f_{a}(s, u) \cdot f_{o}(t \cdot g, h) \cdot e \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) ;
\end{aligned}
$$

if $\alpha: a \mapsto f_{o}\left(f_{a}(s, u) \cdot h, g\right) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: a \mapsto & f_{b}\left(f_{o}\left(f_{a}(s, u) \cdot h, t \cdot g\right) \cdot e \cdot d, r\right) \cdot q= \\
& f_{b}\left(f_{a}(s, u) \cdot f_{o}(h, t \cdot g) \cdot e \cdot d, r\right) \cdot q \in \mathrm{~T}(\mathbf{l}, \mathbf{c}) .
\end{aligned}
$$

- For $\alpha \in \mathrm{T}(\mathbf{t}, \mathbf{t})$ and $\beta \in \mathrm{T}(\mathbf{t}, \mathbf{t})$, if $\alpha: t \mapsto t \cdot p$ and $\beta: t \mapsto t \cdot q$, then $\alpha \cdot \beta: t \mapsto t \cdot p \cdot q \in \mathrm{~T}(\mathbf{t}, \mathbf{t})$.
- For $\alpha \in \mathrm{T}(\mathbf{t}, \mathbf{t})$ and $\beta \in \mathrm{T}(\mathbf{t}, \mathbf{c})$, if $\alpha: t \mapsto t \cdot p$ then
if $\beta: t \mapsto f_{a}(r, t \cdot d) \cdot q$ then $\alpha \cdot \beta: t \mapsto f_{a}(r, t \cdot d \cdot p) \cdot q \in \mathrm{~T}(\mathbf{t}, \mathbf{c})$; and if $\beta: t \mapsto f_{a}(t \cdot d, r) \cdot q$ then $\alpha \cdot \beta: t \mapsto f_{a}(t \cdot p \cdot d, r) \cdot q \in \mathrm{~T}(\mathbf{t}, \mathbf{c})$.
- For $\alpha \in \mathrm{T}(\mathbf{c}, \mathbf{t})$ and $\beta \in \mathrm{T}(\mathbf{t}, \mathbf{t})$, if $\alpha: p \mapsto t \cdot p \cdot q$ and $\beta: t \mapsto t \cdot r$, then $\alpha \cdot \beta: p \mapsto t \cdot p \cdot q \cdot r=t \cdot p \cdot(q \cdot r) \in \mathrm{T}(\mathbf{c}, \mathbf{t})$.
- For $\alpha \in \mathrm{T}(\mathbf{c}, \mathbf{t})$ and $\beta \in \mathrm{T}(\mathbf{t}, \mathbf{c})$, if $\alpha: p \mapsto s \cdot p \cdot q$ then
if $\beta: t \mapsto f_{a}(d, t \cdot e) \cdot r$ then
$\alpha \cdot \beta: p \mapsto f_{a}(d, s \cdot p \cdot q \cdot e) \cdot r=f_{a}(d, s \cdot p \cdot(q \cdot e)) \cdot r \in \mathrm{~T}(\mathbf{c}, \mathbf{c}) ;$ and
if $\beta: t \mapsto f_{a}(t \cdot e, d) \cdot r$ then

$$
\alpha \cdot \beta: p \mapsto f_{a}(s \cdot p \cdot q \cdot e, d) \cdot r=f_{a}(s \cdot p \cdot(q \cdot e), d) \cdot r \in \mathrm{~T}(\mathbf{c}, \mathbf{c}) .
$$

- For $\alpha \in \mathrm{T}(\mathbf{c}, \mathbf{c})$ and $\beta \in \mathrm{T}(\mathbf{c}, \mathbf{t})$, if $\beta: p \mapsto t \cdot p \cdot q$ then if $\alpha: p \mapsto d \cdot p \cdot r$ then $\alpha \cdot \beta: p \mapsto t \cdot d \cdot p \cdot r \cdot q=(t \cdot d) \cdot p \cdot(r \cdot q) \in \mathrm{T}(\mathbf{c}, \mathbf{t})$; if $\alpha: p \mapsto f_{a}(d, s \cdot p \cdot e) \cdot r$ then $\alpha \cdot \beta: p \mapsto f_{a}(t \cdot d, s \cdot p \cdot e) \cdot r \in \mathrm{~T}(\mathbf{c}, \mathbf{c})$; and if $\alpha: p \mapsto f_{a}(s \cdot p \cdot e, d) \cdot r$ then $\alpha \cdot \beta: p \mapsto f_{a}(s \cdot p \cdot e, t \cdot d) \cdot r \in \mathrm{~T}(\mathbf{c}, \mathbf{c})$.
- For $\alpha \in \mathrm{T}(\mathbf{c}, \mathbf{c})$ and $\beta \in \mathrm{T}(\mathbf{c}, \mathbf{c})$, we distinguish three different cases of $\alpha$ :
(1) if $\alpha: p \mapsto r \cdot p \cdot q$, then
if $\beta: p \mapsto e \cdot p \cdot d$ then $\alpha \cdot \beta: p \mapsto(e \cdot r) \cdot p \cdot(q \cdot d) \in \mathrm{T}(\mathbf{c}, \mathbf{c})$;
if $\beta: p \mapsto f_{a}(e, t \cdot p \cdot g) \cdot d$ then $\alpha \cdot \beta: p \mapsto f_{a}(e,(t \cdot r) \cdot p \cdot(q \cdot g)) \cdot d \in \mathrm{~T}(\mathbf{c}, \mathbf{c})$;
if $\beta: p \mapsto f_{a}(t \cdot p \cdot g, e) \cdot d$ then $\alpha \cdot \beta: p \mapsto f_{a}((t \cdot r) \cdot p \cdot(q \cdot g), e) \cdot d \in \mathrm{~T}(\mathbf{c}, \mathbf{c})$.
(2) if $\alpha: p \mapsto f_{a}(r, t \cdot p \cdot d) \cdot q$, then
if $\beta: p \mapsto g \cdot p \cdot e$ then $\alpha \cdot \beta: p \mapsto f_{a}(g \cdot r, t \cdot p \cdot d) \cdot q \cdot e \in \mathrm{~T}(\mathbf{c}, \mathbf{c})$;
if $\beta: p \mapsto f_{b}(g, s \cdot p \cdot h) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: p \mapsto & f_{b}\left(g, f_{a}(s \cdot r, t \cdot p \cdot d) \cdot q \cdot h\right) \cdot e \\
& f_{b}\left(g, t \cdot p \cdot f_{a}(s \cdot r, d) \cdot q \cdot h\right) \cdot e
\end{aligned}
$$

if $\beta: p \mapsto f_{b}(s \cdot p \cdot h, g) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: p \mapsto & f_{b}\left(f_{a}(s \cdot r, t \cdot p \cdot d) \cdot q \cdot h, g\right) \cdot e= \\
& f_{b}\left(t \cdot p \cdot f_{a}(s \cdot r, d) \cdot q \cdot h, g\right) \cdot e \in \mathrm{~T}(\mathbf{c}, \mathbf{c}) .
\end{aligned}
$$

(3) if $\alpha: p \mapsto f_{a}(t \cdot p \cdot d, r) \cdot q$, then
if $\beta: p \mapsto g \cdot p \cdot e$ then $\alpha \cdot \beta: p \mapsto f_{a}(t \cdot p \cdot d, g \cdot r) \cdot q \cdot e \in \mathrm{~T}(\mathbf{c}, \mathbf{c})$;
if $\beta: p \mapsto f_{b}(g, s \cdot p \cdot h) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: p \mapsto & f_{b}\left(g, f_{a}(t \cdot p \cdot d, s \cdot r) \cdot q \cdot h\right) \cdot e= \\
& f_{b}\left(g, t \cdot p \cdot f_{a}(d, s \cdot r) \cdot q \cdot h\right) \cdot e \in \mathrm{~T}(\mathbf{c}, \mathbf{c}) ;
\end{aligned}
$$

if $\beta: p \mapsto f_{b}(s \cdot p \cdot h, g) \cdot e$ then

$$
\begin{aligned}
\alpha \cdot \beta: p \mapsto & f_{b}\left(f_{a}(t \cdot p \cdot d, s \cdot r) \cdot q \cdot h, g\right) \cdot e \\
& = \\
f_{b}\left(t \cdot p \cdot f_{a}(d, s \cdot r) \cdot q \cdot h, g\right) \cdot e & \in \mathrm{~T}(\mathbf{c}, \mathbf{c}) .
\end{aligned}
$$

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