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Abstract

In this paper we investigate the question whether there exist independent systems of three word equations over three unknowns possessing non-periodic solutions, formulated in 1983 in [4]. In particular, we give a negative answer to this question for a large class of systems. More specifically, the question remains open only for a well specified class of systems. We also investigate what happens when we consider chains of equations such that each time we add a new one, the set of solutions of the whole system strictly decreases. Thus, unlike in the case of independence, now the order in which we choose the equations becomes important. In this context we give some reachable lower bounds for the size of such chains of equations over three and four unknowns, respectively.

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1 Introduction

Word equations constitute a fundamental part of the theory of combinatorics on words. The seminal paper on word equations is that of Makanin, [15], showing the decidability of the satisfiability problem. Another remarkable property of word equations was revealed in the validity of Ehrenfeucht compactness property, see [1] and [6]. More recent interesting achievements of the area are the PSPACE solution for the satisfiability problem, see [17], and tools to show that certain properties are not expressible as solutions of word equations, see [10]. However, despite of them, many simple questions on word equations are still unanswered.

In this paper we consider word equations in a very simple setting, namely assuming that the equations are constant-free and over only three or four unknowns. Even in this simple case problems might be extremely hard. An example of a very involved result of this framework is [9] showing that solutions of word equations over three unknowns are finitely parametrizable, while the same does not hold for equations over four unknowns, as also proved in [9], for a shorter proof see [5]. Another deep result shown in [2] and [19] classifies all maximal sets of equations satisfied by a fixed three-tuple of words. Moreover, the question whether there exist independent systems of three equations over three unknowns possessing non-periodic solutions, formulated by Culik II and Karhumäki in 1983 in [4], is still open.

Word equations can be used to characterize constraints satisfied by a set of words. The Ehrenfeucht's compactness property guarantees that finite sets of words cannot satisfy infinitely many independent relations. But, the question how many such independent constraints we can impose on a finite set of words is still wide open; some non-trivial asymptotic lower bounds were given in [11] and [12]. However, if the number of unknowns is small, then not even such lower bounds are reported for the maximal size of independent systems of equations.

In this paper we tackle another related question, i.e., how large chains of equations we can have such that every time we add a new equation the set of solutions strictly decreases. Thus, now, unlike in the case of independent systems, the order in which we choose the equations becomes very important. When considering only two words, the maximal size of such a chain is three: the first (non-trivial) constraint forces the words to be powers of a common word, the second fixes the ratio of the lengths of the periods, and the third allows only the empty words as the solution. We show here that when we consider equations over three unknowns, a reachable lower bound for the size of such chains is six, while if we increase the number of unknowns to four, then nine becomes a lower bound for the size of such chains.

One of the fundamental results on words is the defect theorem stating that *if a set of n words satisfies a nontrivial relation, then they can be expressed simultaneously as products of at most $n-1$ words*. A natural question is what happens if a set of words satisfies several "different" relations. For instance, whether they impose some *cumulative* defect effect, i.e., if a set of n words satisfies t nontrivial relations, can they be expressed simultaneously as products of at most $n-t$ words? Here, we investigate this problem for sets of three words. First, we formulate "different" as meaning that the system of constraints is *independent*, *every pair of equations is independent*, or *every pair of equations is non-equivalent*. Then, we analyze whether there exists some cumulative defect effect in either of these cases. Moreover, if no such restrictions are used, then we can find an infinite system of "different" equations, such as $\{x^iz = zy^i \mid i \geq 1\}$, which has a non-periodic solution, the equations are graphically pairwise different, but the whole system is equivalent to any single equation of the system.

In the second part of this paper we investigate systems of two and three equations over three unknowns, respectively, from the point of view of possessing also non-periodic solutions. In Section 5 we give some necessary and sufficient conditions for systems of two equations to possess at most *quasi-periodic* solutions, i.e., solutions where the images of at least two unknowns are powers of a common word. In Section 6, we concentrate on the above mentioned open question from [4]. A nontrivial step was achieved in [8], by proving that an independent system of at least two equations over three unknowns possessing a non-periodic solution is composed of *balanced* equations only, i.e., equations where the number of occurrences of each unknown on the left and right hand sides is the same. In this paper, we succeeded to give a negative answer to this question for a large class of systems. More specifically, the question remains open only for systems of the following type (up to the symmetry of x and z):

$$\begin{cases} x^iy\alpha_2(x,z)y\ldots y\alpha_n(x,z) = z\beta_1(x,z)y\beta_2(x,z)y\ldots y\beta_n(x,z) \\ x^iy\gamma_2(x,z)y\ldots y\gamma_m(x,z) = z\delta_1(x,z)y\delta_2(x,z)y\ldots y\delta_m(x,z) \\ x^iy\mu_2(x,z)y\ldots y\mu_p(x,z) = z\nu_1(x,z)y\nu_2(x,z)y\ldots y\nu_p(x,z) \end{cases},$$

with $i \geq 1$ and $\alpha_l(x,z), \beta_l(x,z), \gamma_j(x,z), \delta_j(x,z), \mu_k(x,z), \nu_k(x,z) \in \{x,z\}^*$, for all l, j, and k.

2 Preliminaries

Let Σ be a finite alphabet. We denote by Σ^* the set of all finite words over the alphabet Σ, by 1 the empty word, and by Σ^+ the set of all nonempty finite words over Σ. A word u is a *factor* (resp. *prefix*, *suffix*) of w if there are words x, y such that $w = xuy$ (resp. $w = uy$, $w = xu$). We use the notation $pref_k(w)$ (resp. $suf_k(w)$) to denote the prefix (resp. the suffix) of length k of the word w and $u \wedge v$ to denote the *longest common prefix* of two words $u, v \in \Sigma^*$. For a word $w \in \Sigma^*$ we denote by $Alph_\Sigma(w)$ the set of distinct letters from the alphabet Σ appearing in it, by $|w|$ its *length*, i.e., the number of letters in w, and by $|w|_a$ the number of occurrences of letter a in w for any $a \in Alph_\Sigma(w)$. When no confusion can appear, we write only $Alph(w)$ instead of $Alph_\Sigma(w)$. If $\Sigma = \{a_1, \ldots, a_n\}$ and $w \in \Sigma^*$, then the *Parikh vector associated to* w is defined as $\psi(w) = (|w|_{a_1}, \ldots, |w|_{a_n})$. For more details we refer to [3].

We associate a finite set $X \subseteq \Sigma^+$ with a graph $G_X = (V_X, E_X)$, called *the dependency graph of* X, where the set of vertexes is $V_X = X$ and the set of edges is defined by: $(x, y) \in E_X$ if and only if $xX^* \cap yX^* \neq \emptyset$, with $x, y \in X$. We recall now the following result from [7].

Lemma 1. *For a finite set $X \subseteq \Sigma^+$, let n_c be the number of connected components of the dependency graph associated to it. Then, the elements of X can be simultaneously expressed as products of at most n_c words.*

The following result is an immediate consequence.

Corollary 2. *Two words $w_1, w_2 \in \Sigma^*$ are powers of a common word if and only if they satisfy a nontrivial relation.*

Now, let Σ be a finite alphabet and $X = \{x_1, \ldots, x_n\}$ a set of unknowns, with $\Sigma \cap X = \emptyset$. An *equation* over the alphabet Σ, with X as the set of unknowns is a pair $(u, v) \in (\Sigma \cup X)^* \times (\Sigma \cup X)^*$, usually written as $u = v$. We say that an equation is *constant-free* if both u and v contain only elements from X. An equation $u = v$ is called *reduced* if $\mathit{pref}_1(u) \neq \mathit{pref}_1(v)$ and $\mathit{suf}_1(u) \neq \mathit{suf}_1(v)$ and *balanced* if $|u|_x = |v|_x$ for all unknowns $x \in X$. Throughout this paper we consider only reduced constant-free equations.

A *solution* of an equation $u = v$ is a morphism $\varphi : (X \cup \Sigma)^* \to \Sigma^*$ such that $\varphi(u) = \varphi(v)$ and $\varphi(a) = a$ for every $a \in \Sigma$. Thus, a solution is a $|X|$-tuple of words over the alphabet Σ. We define the *length* of a solution as the sum of lengths $|\varphi(x)|$ for all $x \in X$. We say that a solution φ is *periodic* if there exists a word $u \in \Sigma^*$ such that $\varphi(x) \in u^*$ for any $x \in X$. If $X = \{x, y, z\}$, then we say that φ is *quasi-periodic with respect to x and z* if there exists $u \in \Sigma^*$ such that $\varphi(x), \varphi(z) \in u^*$. We can naturally extend this definition for the case when $X = \{x_1, \ldots, x_n\}$, by saying that φ is quasi-periodic if there exists an index $1 \leq i \leq n$ and some word $u \in \Sigma^*$ such that $\varphi(x) \in u^*$ for all $x \in X \backslash \{x_i\}$. We say that a solution is *purely non-periodic* if the images of no two unknowns are powers of a common word. Note that for equations over three unknowns the sets of periodic, quasi-periodic (which are not periodic), and purely non-periodic solutions form a partition of the solution set.

A *system of equations* is a non-empty set of equations. A *solution* of a system is a morphism $\varphi : (X \cup \Sigma)^* \to \Sigma^*$ satisfying all of its equations. We say that two systems E and E' are *equivalent* if they have the same set of solutions. Moreover, we say that a system E is *independent* if it is not equivalent to any of its proper subsystems. In this paper we also use two weaker conditions: *pairwise independence* and *pairwise non-equivalence*, meaning that any two equations of a system are independent and non-equivalent, respectively.

The basic method of solving word equations uses the idea of eliminating the leftmost (or rightmost) unknowns, see, e.g., [13]. This method, extensively used here, is based on the following lemma, also known as Levi's lemma, see [14].

Lemma 3. *If words u, w, x and y over the alphabet Σ satisfy the relation $uw = xy$, then there exists the unique word t such that either $u = xt$ and $y = tw$, or $x = ut$ and $w = ty$.*

Thus, if we have an equation $xu = yv$ with $x, y \in X$ and $u, v \in (\Sigma \cup X)^*$, then we can write $x = yt$ (or $y = xt$) for some new unknown t. Substituting it into the initial equation, we derive $tu' = v'$, where u' and v' are obtained from u and v, respectively, by replacing every occurrence of x with yt. Hence, the set of unknowns changes from X to $X \cup \{t\} \setminus \{x\}$.

Using this method, we can associate to each equation a graph illustrating a systematic way of searching for solutions. Each vertex of this graph is an equation $xu = yv$, where x and y are either unknowns or constants. From each such vertex we draw edges to three other equations derived from $xu = yv$ by using the transformations $x = yt$, $x = y$, and $y = xt$, respectively. Now, the equation $xu = yv$ has a solution with $|x| > |y|$ if and only if the equation $tu' = v'$ has a solution, and moreover $x = yt$. Also, if we have a solution for the equation $tu' = v'$, then we obtain a solution for the initial equation with $x = yt$. Thus, the set of solutions of $xu = yv$ is found by solving all the equations on the leaves of the graph and applying Levi's lemma in the reverse order. For more details about the construction of these graphs we refer to [16].

We conclude this section by considering a constant free equation with the same number of y's in the left and right hand sides:

$$\alpha_1(x, z)y\alpha_2(x, z)y \dots y\alpha_n(x, z) = \beta_1(x, z)y\beta_2(x, z)y \dots y\beta_n(x, z), \tag{1}$$

where $\alpha_i(x, z), \beta_i(x, z) \in \{x, z\}^*$ for all $1 \leq i \leq n$, $\mathit{pref}_1(\alpha_1(x, z)) = x$, and $\mathit{pref}_1(\beta_1(x, z)) = z$.

Depending on the form of all $\alpha_i(x, z)$ and $\beta_i(x, z)$, $1 \leq i \leq n$, we have the following cases.

Case 1: For every $1 \leq i \leq n$ let

$$|\alpha_i(x, z)|_x = |\beta_i(x, z)|_x \text{ and } |\alpha_i(x, z)|_z = |\beta_i(x, z)|_z,$$

i.e., the Parikh vectors of $\alpha_i(x, z)$ and $\beta_i(x, z)$ coincide. Then, for any $k, l \geq 0$ and $u, y \in \Sigma^*$, (u^k, y, u^l) is a solution of (1). Thus, in this case, we say that *equation (1) admits independently quasi-periodic solutions with respect to x and z.*

Case 2: There exists some $1 \leq i \leq n$ such that the Parikh vectors of $\alpha_i(x, z)$ and $\beta_i(x, z)$ differ and, moreover, for all such i's let

$$|\alpha_i(x, z)|_x = |\beta_i(x, z)|_x \text{ and } |\alpha_i(x, z)|_z \neq |\beta_i(x, z)|_z,$$

or the symmetric case when for all such i's the roles of x and z are interchanged. Then, due to Corollary 2, the only quasi-periodic solutions of (1) with respect to x and z (which are not periodic) are of the form $(u^k, y, 1)$, or symmetrically

$(1, y, u^l)$; other triples, when substituted into (1), do not yield the graphical identity. Throughout this paper, we call triples of the form $(u^k, y, 1)$ or $(1, y, u^l)$, *1-limited quasi-periodic with respect to x and z*. So, in this case, we say that *equation (1) admits only 1-limited quasi-periodic solutions with respect to x and z.*

Case 3: There exist some $i \neq j$ such that

$$|\alpha_i(x,z)|_x \neq |\beta_i(x,z)|_x,\ |\alpha_i(x,z)|_z = |\beta_i(x,z)|_z, \text{ and } |\alpha_j(x,z)|_z \neq |\beta_j(x,z)|_z,$$

or the symmetric case with x and z interchanged. Then, when we substitute a quasi-periodic solution of the form (u^i, y, u^k) in the initial equation we obtain a nontrivial relation on u and y. Thus, due to Corollary 2, any quasi-periodic solution with respect to x and z is actually periodic. So, in this case, we say that *the quasi-periodicity of equation (1) induces periodicity*.

Case 4: Otherwise, for any $1 \leq i \leq n$ we have either

$$|\alpha_i(x,z)|_x \neq |\beta_i(x,z)|_x \text{ and } |\alpha_i(x,z)|_z \neq |\beta_i(x,z)|_z, \text{ or}$$

$$|\alpha_i(x,z)|_x = |\beta_i(x,z)|_x \text{ and } |\alpha_i(x,z)|_z = |\beta_i(x,z)|_z.$$

In this case, for all $1 \leq i \leq n$ such that $\alpha_i(x,z)$ and $\beta_i(x,z)$ have distinct Parikh vectors, let $|\alpha_i(x,z)|_x - |\beta_i(x,z)|_x \neq 0$ be the *i-th exceed of x's* and $|\beta_i(x,z)|_z - |\alpha_i(x,z)|_z \neq 0$ be the *i-th exceed of z's*. For every such $1 \leq i \leq n$, we define the *i-th ratio* of this equation, denoted by R_i, as follows:

$$R_i = |\alpha_i(x,z)|_x - |\beta_i(x,z)|_x \ : \ |\beta_i(x,z)|_z - |\alpha_i(x,z)|_z.$$

If there are two indices $i \neq j$ such that R_i and R_j are defined and $R_i \neq R_j$, then any quasi-periodic solution with respect to x and z is actually periodic since, otherwise, after substituting it in (1) we obtain a non-trivial relation on two words. So, also in this case the quasi-periodicity of equation (1) induces periodicity.

We say that *equation (1) has ratio $R = p : q$* if, for every $1 \leq i \leq n$ for which R_i is defined we have that $R_i = R$. Moreover, the quasi-periodic solutions with respect to x and z are completely characterized by this ratio in the sense that a triple $(x = u^k, y, z = u^l)$ is solution of equation (1) if and only if $kp = lq$.

3 Multiple constraints on three element sets

The defect theorem is one of the fundamental results on words. It is often considered to be folklore, maybe because there are many different formulations, all validating the same defect effect on words; probably the oldest paper where this result is reported is [18].

Theorem 4. *If a set of n words satisfies a nontrivial relation, then they can be expressed simultaneously as products of at most $n-1$ words.*

An important consequence of this theorem is the result on two words formulated in Corollary 2. It is natural to ask what happens if a set of words $X = \{w_1, \ldots, w_n\}$ satisfies two or more "different" relations and whether this imposes a cumulative defect effect on the set X. It is well-known that, in general, the answer to the second question is "no"; there are simple examples of independent systems of two equations admitting non-periodic solutions, see e.g. [3].

In this section we investigate this type of questions for three elements sets $X = \{x, y, z\}$. In Section 5 we present several examples of independent systems of two equations over three unknowns having non-periodic solutions; so two different relations are not enough to impose a cumulative defect effect. However, when considering at least three equations, it is open whether there exist independent systems admitting non-periodic solutions; the following conjecture was implicitly stated in [4] and more explicitly, e.g., in [3].

Conjecture 5. *Any independent system of three equations over a set of three unknowns admits only periodic solutions.*

Here, we try to shed some light on this problem. We approach by considering two different restrictions in addition to the independence of the equations, i.e., the *pairwise independence* and the *pairwise non-equivalence* defined in Section 2.

Theorem 6. *There exit purely non-periodic triples $(x, y, z) \in (\Sigma^+)^3$ satisfying three pairwise independent equations.*

Proof. Consider the following system of three equations over the set of unknowns $X = \{x, y, z\}$:

$$\begin{cases} xyxz = zxyx \\ xyxxz = zxxyx \\ xyzyz = zyzyx \end{cases} .$$

We can check directly that for any $\alpha, \beta \in \Sigma^*$, the words $x = \alpha,\ y = \beta,\ z = \alpha\beta\alpha$ constitute a solution of this system. Thus, for some values of the parameters α and β, the system admits also some purely non-periodic solutions.

Next, we prove that any subsystem of size two is independent. First, we have that $x = a,\ y = baab,\ z = aba$ is a solution for the first equation but not for the second one and $x = a,\ y = baaab,\ z = aba$ is a solution for the second equation but not for the first one. Then, we notice that $x = a,\ z = a,\ y = b$ is a solution for the third equation but not for either of the others. Also, $x = a,\ y = baab,\ z = aba$ is a solution of the first but not the third equation and $x = a,\ y = baaab,\ z = aba$ is a solution of the second but not the third one.

Thus, the equations of the system are pairwise independent and, moreover, they possess purely non-periodic solutions of the form $x = \alpha,\ y = \beta,\ z = \alpha\beta\alpha$. $\square$

So, three pairwise independent relations on a set $X = \{x, y, z\}$ are not enough to impose a cumulative defect effect. Next, we investigate the case when we replace the independence condition with the non-equivalence.

Theorem 7. *There exist purely non-periodic triples $(x, y, z) \in (\Sigma^+)^3$ satisfying four pairwise non-equivalent equations.*

Proof. Consider the following system of four equations:

$$\begin{cases} xyxz = zxyx \\ xyxxz = zxxyx \\ xyzyz = zyzyx \\ zyz = xyzyx \end{cases},$$

for which we can check that the words $x = \alpha,\ y = \beta,\ z = \alpha\beta\alpha$ constitute a solution for any $\alpha, \beta \in \Sigma^*$. Thus, for some values of the parameters α and β, the system admits also some purely non-periodic solutions. Moreover, the proof of Theorem 6 implies that the first three equations are pairwise independent. However, by the length argument, any solution of the fourth equation is also a solution of any of the other three. Let us now take $x = a,\ y = baab,\ z = aba$ a solution of the first equation, $x = a,\ y = baaab,\ z = aba$ a solution of the second equation, and $x = a,\ z = a,\ y = b$ a solution for the third equation. Since none of them is a solution of the fourth equation, we obtain that the equations of the chosen system are pairwise non-equivalent. $\square$

A special type of non-periodic solutions are the quasi-periodic ones. A natural question is how much this restriction influences the bounds given above.

Theorem 8. *The infinite system $\{xy^iz = zy^ix \mid i \geq 1\}$ is pairwise independent and admits quasi-periodic solutions of the form $x = z = \alpha$ and $y = \beta$ for any words $\alpha, \beta \in \Sigma^+$.*

Proof. Consider two arbitrary equations from this system:

$$xy^iz = zy^ix, \quad xy^jz = zy^jx$$

with $i \neq j$. Then $x = (ab^i)^n a,\ y = b,\ z = (ab^i)^m a$ with $n \neq m$ is a solution of the first but not of the second equation and $x = (ab^j)^n a,\ y = b,\ z = (ab^j)^m a$ with $n \neq m$ is a solution of the second but not of the first equation. So, any two equations from the initial system are independent. $\square$

Thus, in this last case not even infinitely many relations on a set $X = \{x, y, z\}$ are enough to impose a cumulative defect effect.

However, Ehrenfeucht compactness property of word equations states that each system over a finite set of unknowns is equivalent to some of its finite subsystems, see for example [3]. In other words, any independent system over a finite

	The equations	The solution set
(1)	$xyzyz = zyzyx$	$\begin{cases} x = (\alpha\beta)^i\alpha \\ y = \beta(\alpha\beta)^j \\ z = (\alpha\beta)^l\alpha \end{cases}$
(2)	$xyxz = zxyx$	$x = \alpha,\ y = \beta,\ z = \alpha\beta\alpha$
(3)	$xzy = zyx$	periodic solutions
(4)	$zyz = xyzyx$	periodic solutions with $\|z\| = \|xyx\|$
(5)	$zyx = x^2z$	periodic solutions with $\|z\| = \|xyx\|$ and $\|x\| = \|y\|$
(6)	$zy = xyz$	$x = y = z = 1$

Table 1: A Chain of equations with strictly decreasing set of solutions

set of unknowns is finite. Thus, it is natural to ask how large such systems can be. However, very little seems to be known about this problem; we refer to [11] and [12] for some non-trivial lower bounds. More specifically, for any $n \geq 1$, one can construct independent systems of n^3 (resp. n^4) equations over $5n$ (resp. $10n$) unknowns admitting non-periodic solutions in free semigroups (resp. free monoids).

Nevertheless, for small numbers of unknowns, no nontrivial lower bounds are reported for the maximal size of independent systems of equations. For instance in the case of three unknowns, it is not even known whether there exist independent systems of three equations admitting non-periodic solutions, see Conjecture 5. Here, we try to tackle this problem in the particular case of equations over three unknowns. Note that an independent system is a set of equations such that in whichever way we order them into a chain, the set of solutions is strictly decreased by each equation. Our approach is to consider, instead of independence, a weaker condition, i.e., we investigate chains of equations such that each time we add a new one the set of solutions strictly decreases. Let us call such sequences *strictly decreasing chains of equations*. Even though by doing this we relax the restrictions imposed on the set of equations, the importance of the problem itself is not diminished.

In Table 1 we give an example of a strictly decreasing chain of word equations of size six. Moreover, since the first three equations are balanced, they impose constraints only on the set of non-periodic solutions. Thus, after adding the third equation to the chain, the set of solutions consists of all periodic triples (u^i, u^j, u^k), for any $u \in \Sigma^*$ and $i, j, k \geq 0$. From this point on, the size of the chain is maximal since on the set of periodic triples (u^i, u^j, u^k) we can impose at most three successive restrictions, each one "limiting" the values of one of the parameters i, j, and k. Hence, six is a reachable lower bound for the size of strictly decreasing chains over three unknowns.

A natural question now is the following.

Problem 9. *Is six also an upper bound for the size of strictly decreasing chains of equations?*

A positive answer for this problem can be given for several types of systems, sometimes obtaining even a smaller upper bound. One such case is obtained from the following well-known result, see for example [3].

Proposition 10. *If a three element set $X = \{x, y, z\} \subseteq \Sigma^+$ satisfies the relations*

$$\begin{cases} x\alpha = z\beta \\ x\gamma = y\delta \end{cases} \quad \textit{with } \alpha, \beta, \gamma, \delta \in X^*,$$

then $x, y,$ and z are powers of a common word.

Thus, two such equations can be extended to a strictly decreasing chain of size at most five since, like in Table 1, we can add at most three more equations each one restricting the length of one of the unknowns.

In Section 6 we prove that independent systems of three equations over three unknowns might have non-periodic solutions only if they are of the form (up to symmetry of x and z):

$$\begin{cases} x^i y\alpha_2(x,z)y\ldots y\alpha_n(x,z) = z\beta_1(x,z)y\beta_2(x,z)y\ldots y\beta_n(x,z) \\ x^i y\gamma_2(x,z)y\ldots y\gamma_m(x,z) = z\delta_1(x,z)y\delta_2(x,z)y\ldots y\delta_m(x,z) \\ x^i y\mu_2(x,z)y\ldots y\mu_p(x,z) = z\nu_1(x,z)y\nu_2(x,z)y\ldots y\nu_p(x,z) \end{cases}$$

with $i \geq 1$ and $\alpha_l(x,z), \beta_l(x,z), \gamma_j(x,z), \delta_j(x,z), \mu_k(x,z), \nu_z(x,z) \in \{x,z\}^*$, for all l, j, and k. In all the other cases, independent systems of three equations possess only periodic solutions, thus making Conjecture 5 true, and hence, like above, we can add at most three more equations in order to obtain a strictly decreasing chain.

Moreover, as explained above, the restriction of considering independent systems is stronger than that of strictly decreasing chains of equations. Thus, if we obtain an upper bound m for the size of strictly decreasing chains of equations, then m is also an upper bound for the size of independent systems.

4 Multiple Constraints on Four Words

In this section we investigate the size of strictly decreasing chains of equations over a set of four unknowns, $Y = \{x, y, z, t\}$. We start by recalling first a result from [8] stating that any independent system over three unknowns with at least two equations and having a non-periodic solution consists of balanced equations only. However, as shown by the following example, this result does not hold anymore when we increase the number of unknowns.

Example 1. *The system*

$$\begin{cases} xyz = zty \\ xy^2z^2 = z^2yty \end{cases}$$

admits nonperiodic solutions of the form $x = \gamma,\ y = \delta\gamma\delta,\ z = \gamma\delta,\ t = \gamma$, for some words $\gamma, \delta \in \Sigma^*$. Moreover, it is independent, since $x = ababa,\ y = bab,\ z = ab,\ t = ababa$ is a solution for the first equation and not the second, and $x = ((ab)^2b)^2aba,\ y = b,\ z = ab,\ t = ((ab)^2b)^2aba$ is a solution for the second equation and not the first.

Hmelevskii proved in [9] that solutions of word equations over three unknowns are finitely parametrizable, i.e., they can be expressed using only a finite number of formulas involving word parameters and numerical parameters. Moreover, the same does not hold for equations over four unknowns, as also proved in [9], for a shorter proof see [5]. Thus, an interesting question is what is the effect of considering equations over four unknowns on the size of chains of equations with strictly decreasing set of solutions.

	The equations	Characterization of the set of solution
(1)	$xytz = ztxy$	$xy = (\alpha\beta)^k\alpha,\ \ t = \beta(\alpha\beta)^j,\ \ z = (\alpha\beta)^i\alpha$
(2)	$xyztz = z^2txy$	$xy = (\alpha\beta)^i\alpha,\ \ t = \beta(\alpha\beta)^j,\ \ z = (\alpha\beta)^i\alpha$
(3)	$xytyz = zytxy$	$xy = (\alpha\beta)^i\alpha,\ \ z = (\alpha\beta)^i\alpha,\ \ t = \gamma^k\ y = \gamma^l$
(4)	$xtyzz = zyzxt$	$xy = (\alpha\beta)^i\alpha,\ \ z = (\alpha\beta)^i\alpha,\ \ t = \gamma^k\ y = \gamma^k$
(5)	$xtzy = ztyx$	periodic solutions
(6)	$x = zyt$	periodic solutions with $\lvert x\rvert = \lvert yzt\rvert$
(7)	$xyt^2yx = zx$	periodic solutions with $\lvert x\rvert = \lvert yzt\rvert$ and $\lvert z\rvert = 2\lvert yt\rvert$
(8)	$xyz = ztx$	periodic solutions with $\lvert x\rvert = \lvert yzt\rvert$, $\lvert z\rvert = 2\lvert yt\rvert$, $\lvert y\rvert = \lvert t\rvert$
(9)	$xy = ztx$	$x = y = z = t = 1$

Table 2: A chain of equations over four unknowns with strictly decreasing set of solutions

As shown by Table 2, the size of the chain increases nontrivially when we switch from three to four unknowns. First, we have four equations such that every time we add a new one the set of solutions strictly decreases, but still includes some non-periodic ones. Then, when we add the fifth equation, the set of solutions includes only periodic triples. But, since up to this point all equations are balanced, they admit as solution any periodic triple (u^i, u^j, u^k, u^l) with $u \in \Sigma^*$ and $i, j, k, l \geq 0$. Then, similar to the case of equations over three unknowns, from this point on we can impose at most four successive restrictions, each one "limiting" the values of one of the parameters i, j, k, and l.

Thus, nine is a reachable lower bound for the size of strictly decreasing chains over four unknowns.

Moreover, note that the system containing the equations (1), (2), and (4) is independent and possesses non-periodic solutions.

5 Systems of two equations over three unknowns

In this section, we investigate systems of two equations over a set of three unknowns, in particular, when they can have non-periodic solutions. Due to Proposition 10, we consider only systems of equations where one side starts with x and the other with z:

$$\begin{cases} \alpha(x,z)y\beta(x,y,z) = \alpha_1(x,z)y\beta_1(x,y,z) \\ \gamma(x,z)y\delta(x,y,z) = \gamma_1(x,z)y\delta_1(x,y,z) \end{cases}$$

where $\alpha(x,z), \alpha_1(x,z), \gamma(x,z), \gamma_1(x,z) \in \{x,z\}^+$, $\beta(x,y,z), \beta_1(x,y,z), \delta(x,y,z)$, $\delta_1(x,y,z) \in \{x,y,z\}^*$, are such that $pref_1(\alpha(x,z)) = pref_1(\gamma(x,z)) = x$, and $pref_1(\alpha_1(x,z)) = pref_1(\gamma_1(x,z)) = z$. We partition the set of such systems depending on the structure of $\alpha(x,z)$, $\alpha_1(x,z)$, $\gamma(x,z)$, and $\gamma_1(x,z)$, that is depending on whether they contain both unknowns x and z or only one of them. Then, for each class, we give some conditions guaranteeing the existence of at most quasi-periodic solutions. Moreover, we show that these conditions are necessary. The following result from [3] is a useful starting point.

Lemma 11. *Let $X = \{x,y\} \subseteq \Sigma^*$ such that $xy \neq yx$. Then, for each words $u, v \in X^*$ we have*

$$u \in xX^+,\ v \in yX^+,\ |u|,|v| \geq |xy \wedge yx|,\ \Rightarrow u \wedge v = xy \wedge yx.$$

As an immediate consequence, we can formulate the following result; an alternative proof was given in [9].

Theorem 12. *An equation of the form $x^i z\alpha(x,y,z) = z^j x\beta(x,y,z)$ admits only solutions where x and z are powers of a common word, i.e. at most quasi-periodic solutions with respect to x and z.*

Thus, any system containing an equation of this type admits at most quasi-periodic solutions with respect to x and z. So, we can restrict to systems where at least on one side of both equations either only x's or only z's appear before the first occurrence of y.

First, we consider systems where, before the first occurrence of y, both equations have on one side only x's while on the other side they have both x's and z's. The case when x and z are interchanged is symmetric.

Theorem 13. *A system of the form*

$$\begin{cases} x^i y\alpha(x,y,z) = z\beta_1(x,z)y\beta_2(x,y,z) \\ x^j y\gamma(x,y,z) = z\delta_1(x,z)y\delta_2(x,y,z) \end{cases}$$

with $i \neq j$ and $x \in Alph(\beta_1(x,z)) \cap Alph(\delta_1(x,z))$ admits at most quasi-periodic solutions with respect to x and z.

Proof. Since $i \neq j$ we can suppose without loss of generality that $i > j \geq 1$. Let $(X, Y, Z) \in (\Sigma^*)^3$ be a solution of the system; the set of solutions is non-empty since $(1, 1, 1)$ is always a solution. Depending on the lengths of X and Z we have three cases.

Case 1: If $|X| = |Z|$, then $X = Z$ and so the solution is of the required form. Moreover, by Corollary 2, the system admits non-periodic solutions of the form (X, Y, X) if and only if after replacing $x = z$ in the initial system we obtain graphical identity in both equations.

Case 2: If $|X| > |Z|$, then we can write $X = ZT$ for some $T \in \Sigma^+$. Now, if we substitute in the first equation of the initial system x by zt, for some new unknown t, then we obtain the equation

$$t(zt)^{i-1}y\alpha(zt, y, z) = \beta_1(zt, z)y\beta_2(zt, y, z)$$

admitting the solution (T, Y, Z). Since $i \geq 2$ and $z, t \in Alph(\beta_1(zt, z))$, Theorem 12 implies that this equation admits only solutions (T, Y, Z) with T and Z powers of a common word. Since $X = ZT$, we also obtain that in the solution (X, Y, Z), X and Z are powers of a common word.

Case 3: If $|X| < |Z|$, then we can write $Z = XT$ for some word $T \in \Sigma^+$. If we substitute in the initial system z by xt for some new unknown t we obtain:

$$\begin{cases} x^{i-1}y\alpha(x, y, xt) = t\beta_1(x, xt)y\beta_2(x, y, xt) \\ x^{j-1}y\gamma(x, y, xt) = t\delta_1(x, xt)y\delta_2(x, y, xt) \end{cases}.$$

But this is of the same type as the initial system only with smaller numerical parameters and, moreover, it admits a solution (X, Y, T) with $|T| < |Z|$. If $j = 1$, then, by Theorem 10, this system admits only periodic solutions implying also that all solutions of the initial system with $|X| < |Z|$ are periodic. Otherwise, i.e. $j \geq 2$, we can repeat the reasoning for this system, every time decreasing the length of the chosen solution. Thus, we can do this reduction only finitely many times and, moreover, from the previous considerations, we always stop with a system admitting solutions as required in the theorem. Since all the applied transformations are of the form $x = zt$, $z = xt$, or $x = z$, we conclude that also the chosen solution (X, Y, Z) is quasi-periodic with respect to X and Z.

So, independently of the lengths of X and Z, the chosen solution (X, Y, Z) is as required. But, since it was chosen arbitrarily, we conclude that systems of this type admit at most quasi-periodic solutions with respect to x and z. □

The next example shows that the condition $i \neq j$ in the above theorem is unavoidable.

Example 2. *The system*

$$\begin{cases} xyxz = zxyx \\ xyxxz = zxxyx \end{cases}$$

is of the type considered in Theorem 13 but with $i = j$. However, it admits purely non-periodic solutions of the form $x = \alpha,\ y = \beta,\ z = \alpha\beta\alpha$, for some words $\alpha, \beta \in \Sigma^+$. Moreover, the system is independent since $x = a,\ y = baab,\ z = aba$ is a solution for the first equation but not for the second one and $x = a,\ y = baaab,\ z = aba$ is a solution for the second equation but not for the first one.

Next, we consider the case when both equations have on one side both x's and z's before the first occurrence of y, while on the other side one equation has only x's and the the other has only z's.

Theorem 14. *A system of the form*

$$\begin{cases} x^i y\alpha(x,y,z) = z\beta_1(x,z)y\beta_2(x,y,z) \\ z^j y\gamma(x,y,z) = x\delta_1(x,z)y\delta_2(x,y,z) \end{cases}$$

with $x \in Alph(\beta_1(x,z))$ and $z \in Alph(\delta_1(x,z))$ admits at most quasi-periodic solutions with respect to x and z. Moreover, if $i = 1$ or $j = 1$, then the system admits only periodic solutions.

Proof. Let $(X,Y,Z) \in (\Sigma^*)^3$ be a solution of this system. We have several cases depending on the values of the parameters i and j and the lengths of X and Z.

We start by considering the case when $i, j \geq 2$.

Case 1: If $|X| = |Z|$, then $X = Z$ and so the solution is of the required form. Moreover, by Corollary 2, the system admits non-periodic solutions of the form (X,Y,X) if and only if after replacing $x = z$ in the initial system we obtain graphical identity in both equations.

Case 2: If $|X| > |Z|$, then we can write $X = ZT$ for some new word $T \in \Sigma^+$. If in the first equation of the system we substitute $x = zt$ for some new unknown t, then we obtain

$$t(zt)^{i-1}y\alpha(zt,y,z) = \beta_1(zt,z)y\beta_2(zt,y,z).$$

Since $i \geq 2$ and $z, t \in Alph(\beta_1(zt,z))$, Theorem 12 implies that this equation admits at most quasi-periodic solutions with respect to t and z. Thus, the solution (X,Y,Z) is such that X and Z are powers of a common word.

Case 3: If $|X| < |Z|$, then we can write $Z = XT$ for some new word $T \in \Sigma^+$. If in the second equation of the system we make the transformation $z = xt$, for some new unknown t, then we obtain the equation

$$t(xt)^{j-1}y\gamma(x,y,xt) = \delta_1(x,xt)y\delta_2(x,y,xt),$$

which by the same reasoning as above admits at most quasi-periodic solutions with respect to x and t. So, the solution (X,Y,Z) is quasi-periodic with respect to X and Z.

Since the initial solution (X,Y,Z) was chosen arbitrarily, the system admits only solutions where x and z are powers of a common word when $i, j \geq 2$.

We continue by proving that if $i = j = 1$, then the system admits only periodic solutions independent of the lengths of X and Z. If $|X| = |Z|$, i.e. $X = Z$, then when substituting $x = z$ in the initial system we do not obtain graphical identity, so, by Corollary 2, it admits only periodic solutions. If $|X| > |Z|$, i.e. $X = ZT$ for some new word $T \in \Sigma^+$, then when substituting in the initial system $x = zt$, for some new unknown t, we obtain:

$$\begin{cases} ty\alpha(zt, y, z) = \beta_1(zt, z)y\beta_2(zt, y, z) \\ y\gamma(zt, y, z) = t\delta_1(zt, z)y\delta_2(zt, y, z) \end{cases}.$$

But then Theorem 10 implies that it admits only periodic solutions. The case when $|X| < |Z|$ is symmetric and again we obtain only periodic solutions.

The only remaining case is when one parameter is 1 and the other is at least 2. Without loss of generality, we take $i = 1$ and $j \geq 2$, the other case being symmetric.

Case 1': If $|X| = |Z|$, then when substituting $x = z$ in the first equation of the initial system we do not obtain graphical identity. So the system admits only periodic solutions.

Case 2': If $|X| < |Z|$, then we can write $Z = XT$ for some new word $T \in \Sigma^+$. If we substitute in the initial system $z = xt$, for some new unknown t, then we obtain:

$$\begin{cases} y\alpha(x, y, xt) = t\beta_1(x, xt)y\beta_2(x, y, xt) \\ t(xt)^{j-1}y\gamma(x, y, xt) = \delta_1(x, xt)y\delta_2(x, y, xt) \end{cases}$$

which admits only periodic solutions. So, the initial system admits only periodic solutions.

Case 3': If $|X| > |Z|$, then we can write $X = ZT$ for some new word $T \in \Sigma^+$. If we substitute in the initial system $x = zt$ for some new unknown t, then we obtain

$$\begin{cases} ty\alpha(zt, y, z) = \beta_1(zt, z)y\beta_2(zt, y, z) \\ z^{j-1}y\gamma(zt, y, z) = t\delta_1(zt, z)y\delta_2(zt, y, z) \end{cases}$$

with $z, t \in Alph(\beta_1(zt, z))$ and $z \in Alph(\delta_1(zt, z))$. But this is of the same type as the initial system, only with a smaller value for the numerical parameter. Moreover, this system admits the solution (T, Y, Z) with $|T| < |X|$. So, we can repeat inductively the transformation step for this system depending on the lengths of T and Z. Since with every transformation we decrease the length of the solution, we have to stop after a finite number of steps and, moreover, from the previous considerations, we stop with a system admitting only periodic solutions. But, since all the transformations applied are of the form $x = z$, $x = zt$, or $z = xt$, we obtain that the initially chosen solution (X, Y, Z) is also periodic.

Thus, since the initial solution (X, Y, Z) was arbitrarily chosen, we proved that if at least one of the parameters i or j is 1, then the system admits only periodic solutions. Otherwise, i.e. $i, j \geq 2$, the system admits at most quasi-periodic solutions with respect to x and z. $\square$

Now, we continue our investigation by considering systems where both equations have on both hand sides only occurrences of one of the unknowns x or z before the first y.

Theorem 15. *A system of the form*

$$\begin{cases} x^i y\alpha(x,y,z) = z^l y\beta(x,y,z) \\ x^j y\gamma(x,y,z) = z^k y\delta_2(x,y,z) \end{cases} \quad \textit{with } i \neq j \textit{ and } l \neq k$$

admits at most quasi-periodic solutions with respect to x and z. Moreover, if $i < j,\ k < l$ or symmetrically $j < i,\ l < k$, then the system admits only periodic solutions.

Proof. We can suppose without loss of generality that $i < j$. Let $(X,Y,Z) \in (\Sigma^*)^3$ be a solution of this system. The idea of this proof is to apply Levi's Lemma for the words X^i and Z^l when $l < k$, or X^i and Z^k when $k < l$, instead of applying it for X and Z as in the case of the previous proofs.

If $l < k$ we have three cases depending on the lengths of X^i and Z^l. If $|X^i| = |Z^l|$, then $X^i = Z^l$ and so, by Corollary 2, the solution is of the required form. If $|X^i| > |Z^l|$, then we can write $X^i = Z^l T$ for some new word $T \in \Sigma^+$ and when we make the transformation $x^i = z^l t$ in the initial system, we obtain:

$$\begin{cases} ty\alpha(x,y,z) = y\beta(x,y,z) \\ tx^{j-i}y\gamma(x,y,z) = z^{k-l}y\delta_2(x,y,z) \end{cases},$$

which, by Theorem 10, possesses only periodic solutions. Thus, the chosen solution (X,Y,Z) is also periodic. Otherwise, i.e. $|X^i| < |Z^l|$, we can write $Z^l = X^i T$ for some new word $T \in \Sigma^+$ and when we apply the transformation $z^l = x^i t$ we obtain the system

$$\begin{cases} y\alpha(x,y,z) = ty\beta(x,y,z) \\ x^{j-i}y\gamma(x,y,z) = tz^{k-l}y\delta_2(x,y,z) \end{cases},$$

which again admits only periodic solutions. Hence, also the chosen solution (X,Y,Z) is periodic. Since the solution (X,Y,Z) was chosen arbitrarily, we obtain that if $l < k$ then the system admits at most quasi-periodic solutions with respect to x and z. Moreover, if $|X^i| \neq |Z^l|$ then all three $X, Y,$ and Z are powers of a common word, i.e. the solution is periodic.

Using similar reasoning we prove next that if $k < l$ then the initial system admits only periodic solutions. Again, we have three cases depending on the lengths of X^i and Z^k. The only different case is when $|X^i| = |Z^k|$ since after making the transformation $x^i = z^l$ we obtain the system

$$\begin{cases} y\alpha(x,y,z) = z^{l-k}y\beta(x,y,z) \\ x^{j-i}y\gamma(x,y,z) = y\delta_2(x,y,z) \end{cases}$$

which, by Theorem 10, admits only periodic solutions. So, in this case the initial solution (X, Y, Z) is periodic. The other two cases, when $|X^i| > |Z^k|$ and $|X^i| < |Z^k|$, are as above.

Thus, the only cases when such a system admits non-periodic solutions (X, Y, Z) (but quasi-periodic with respect to x and z) is when $i < j$, $l < k$, and $|X^i| = |Z^l|$ and the symmetric one, i.e. $j < i$, $k < l$, and $|X^j| = |Z^k|$. $\square$

Again, as shown by the next two examples, the conditions $i \neq j$ and $l \neq k$ in the previous theorem represent the borderline between systems admitting at most quasi-periodic solutions and systems admitting purely non-periodic ones.

Example 3. *The system*

$$\begin{cases} xyzy = zy^2x \\ xyxzy = zy^2x^2 \end{cases}$$

is of the type considered in Theorem 15 but with $i = j$ *and* $k = l$. *However, it admits purely non-periodic solutions of the form* $x = \alpha\beta,\ y = \beta,\ z = \alpha$, *for some words* $\alpha, \beta \in \Sigma^+$. *Moreover, the system is independent since* $x = ab,\ y = b,\ z = abba$ *is a solution for the first equation but not for the second one and* $x = ab,\ y = b,\ z = abbaba$ *is a solution for the second equation but not for the first one.*

Example 4. *The system*

$$\begin{cases} xy^2z = zyxy \\ xyzyz = z^2yxy \end{cases}$$

is another example but with $i = j$ *and* $k \neq l$. *Also for this system we obtain purely non-periodic solutions of the form* $x = \alpha,\ y = \beta,\ z = \alpha\beta$, *for some words* $\alpha, \beta \in \Sigma^+$. *Moreover, the system is independent since* $x = abba,\ y = b,\ z = ab$ *is a solution for the first equation but not for the second one and* $x = ababba,\ y = b,\ z = ab$ *is a solution for the second equation but not for the first one.*

The next theorem investigates the last case of our classification. Now, one equation has on one side only occurrences of x's while on the other side both the unknowns x and z appear before the first y. However, the second equation has on both sides only occurrences of one of the unknowns x or z before the first occurrence of y.

Theorem 16. *A system of the form*

$$\begin{cases} x^i y\alpha(x, y, z) = z\beta_1(x, z)y\beta_2(x, y, z) \\ x^j y\gamma(x, y, z) = z^k y\delta(x, y, z) \end{cases}$$

with $i \neq j$, $k \geq 1$, *and* $x \in Alph(\beta_1(x, z))$ *admits at most quasi-periodic solutions with respect to* x *and* z.

Proof. Let $(X, Y, Z) \in (\Sigma^*)^3$ be a solution of this system. We have three cases depending on the lengths of X and Z.

Case 1: If $|X| = |Z|$, then $X = Z$ and so the solution is of the required form. Moreover, the system admits quasi-periodic solutions of the form (X, Y, X) if and only if after replacing $x = z$ in the initial system we obtain graphical identity in both equations.

Case 2: If $|X| > |Z|$, then we can write $X = ZT$ for some new word $T \in \Sigma^+$. When we substitute in the initial system $x = zt$ for some new unknown t we obtain:

$$\begin{cases} t(zt)^{i-1}y\alpha(zt, y, z) = \beta_1(zt, z)y\beta_2(zt, y, z) \\ t(zt)^{j-1}y\gamma(zt, y, z) = z^{k-1}y\delta(zt, y, z) \end{cases}.$$

If $k = 1$, then, by Theorem 10, this system admits only periodic solutions, and thus also the chosen solution (X, Y, Z) is periodic. Otherwise, if $i \geq 2$, then Theorem 12 implies that the first equation admits only solutions where z and t are powers of a common word. Since $x = zt$, we obtain that also the solution (X, Y, Z) is quasi-periodic with respect to X and Z. If $i = 1$, then Theorem 14 implies that this system admits only periodic solutions, and thus also the triple (X, Y, Z) is periodic.

Case 3: If $|X| < |Z|$, then we can write $Z = XT$ for some new word $T \in \Sigma^+$. When we substitute in the initial system $z = xt$ for some new unknown t we obtain:

$$\begin{cases} x^{i-1}y\alpha(x, y, xt) = t\beta_1(x, xt)y\beta_2(x, y, xt) \\ x^{j-1}y\gamma(x, y, xt) = t(xt)^{k-1}y\delta(x, y, xt) \end{cases}.$$

If $i = 1$ and $j \geq 2$, or symmetrically $j = 1$ and $i \geq 2$, then Theorem 10 implies that this system admits only periodic solutions and so also (X, Y, Z) is periodic. Otherwise, i.e. $i, j \geq 2$, we have two cases depending on the value of parameter k. If $k \geq 2$, then Theorem 13 implies that this system admits only solutions (X, Y, T) where X and T are powers of a common word. Since $Z = XT$, then also the solution (X, Y, Z) is quasi-periodic with respect to X and Z. If $k = 1$, then this is a system of the same type and we can repeat the reasoning, with every transformation reducing the length of the chosen solution. So, we can repeat only finitely many times and moreover, from the previous considerations, we stop with a solution as required in the theorem. Since all the transformations applied are of the form $x = z$, $x = zt$ or $z = xt$, the chosen solution (X, Y, Z) is of the required form.

So, in all cases, the solution (X, Y, Z) has X and Z powers of a common word. Moreover, since it was chosen arbitrarily, we obtain that the initial system admits at most quasi-periodic solutions with respect to x and z. □

Once again the condition $i \neq j$ in the previous theorem proves to be unavoidable.

Example 5. *The system*

$$\begin{cases} xyz = zxy \\ xy^2z = zyxy \end{cases}$$

is of the type considered in Theorem 16 but with $i = j$ and admits purely non-periodic solutions of the form $x = \alpha,\ y = \beta,\ z = \alpha\beta$, for some words $\alpha, \beta \in \Sigma^+$. Moreover, the system is independent since $x = aba,\ y = bab,\ z = ab$ is a solution for the first equation but not for the second one and $x = abba,\ y = b,\ z = ab$ is a solution for the second equation but not for the first one.

6 Systems of three word equations over three unknowns

In this section we tackle the question formulated by Culik II and Karhumäki in [4] asking whether there exits an independent system of three equations over three unknowns admitting a non-periodic solution. We start from the systems analyzed in Section 5 and prove that, in many cases, if we add a third equation, the obtained systems possess only periodic solutions or are not independent.

In the previous section we gave several types of systems of two equations admitting at most quasi-periodic solutions with respect to x and z, i.e., triples of the form (u^i, y, u^k), for some words $u, y \in \Sigma^*$ and $i, k \geq 1$. Moreover, due to Corollary 2, the quasi-periodic solutions (which are not periodic) were obtained if and only if, when substituting a triple (u^i, y, u^k) in the initial system, we obtain graphical identities. But this is possible only if the equations have the same number of y's on both sides. Also, whenever we add a new equation, it has to have the same property; otherwise, by Corollary 2, the quasi-periodic solutions of the initial system are restricted to periodic ones. Thus, in all our future considerations we discuss only equations with equal number of y's in the two sides. Although the following result from [8] enables us to use even a stronger restriction, i.e., all the considered equations are balanced, in the majority of cases we only need equal number of y's in the two sides.

Theorem 17. *An independent system with at least two equations and having a non-periodic solution consists of balanced equations only.*

Consider an equation

$$\alpha_1(x,z)y \ldots y\alpha_n(x,z) = \beta_1(x,z)y \ldots y\beta_n(x,z) \tag{2}$$

having the same number of y's in the two sides, $pref_1(\alpha_1(x,z)) = x$, and $pref_1(\beta(x,z)) = z$. Let (u^i, u^j, u^k) be a periodic solution of this equation; the set of periodic solutions is non-empty since $(1,1,1)$ is solution of any constant-free word equation. Then, when replacing it in the equation (2) we obtain a relation on i and k: $n_1 i + m_1 k = n_2 i + m_2 k$, where n_1, n_2 and m_1, m_2 are the numbers of x's and the numbers of z's in the two sides, respectively. Depending on the values $n_1 - n_2$ and $m_1 - m_2$ the set of periodic solutions of the equation (2) has different characterizations.

If $n_1 = n_2$ and $m_1 = m_2$, then the equation (2) is balanced and thus any periodic triple (u^i, u^j, u^k) with $i, j, k \geq 0$ is a solution.

If $n_1 = n_2$, then we have either $k = 0$ or $m_1 = m_2$. The first situation means that the set of periodic solutions is $\{(u^i, u^j, 1) \mid i, j \geq 0\}$, while the second condition means that the equation is balanced and thus admits as solution any periodic triple (u^i, u^j, u^k) with $i, j, k \geq 0$. The case when $m_1 = m_2$ is similar.

If $n_1 \neq n_2$ and $m_1 \neq m_2$, then the set of periodic solutions admitted by the equation (2) is completely characterized by the ratio $n_1 - n_2 : m_2 - m_1$.

Moreover, depending on the type of the quasi-periodic solutions admitted by (2), we also obtain some restrictions on the set of its periodic solutions. If (2) admits independently quasi-periodic solutions, then it also admits any periodic triple (u^i, u^j, u^k) with $i, j, k \geq 0$ as a solution. If the set of quasi-periodic solutions is characterized by the ratio $R = p : q$, then either any periodic triple is a solution of (2), in the case of balanced equations, or the set of periodic solutions is characterized by the same ratio. Let now equation (2) admit only 1-limited quasi-periodic solutions of the form $(u^i, y, 1)$; the other case is symmetric. Thus, for any $1 \leq l \leq n$ such that $\alpha_l(x, z)$ and $\beta_l(x, z)$ have different Parikh vectors, we have

$$|\alpha_l(x, z)|_x = |\beta_l(x, z)|_x, \text{ and } |\alpha_l(x, z)|_z \neq |\beta_l(x, z)|_z.$$

But this means that the equation has the same number of x's in the two sides, so it admits any periodic triple as a solution, if the equation is balanced, or only periodic solutions of the form $(u^i, u^j, 1)$ for any $i, j \geq 0$, otherwise.

We start our analysis with an equation where in both sides, both x and z appear before the first occurrence of y and investigate what happens with the set of solutions when we add two more equations.

Theorem 18. *Any system of three equations such that one of them is of the form*

$$x^l z\alpha_1(x, z)y \ldots y\alpha_n(x, z) = z^r x\beta_1(x, z)y \ldots y\beta_n(x, z), \tag{3}$$

with $l, r \geq 1$, $\alpha_i(x, z), \beta_i(x, z) \in \{x, z\}^*$ for all $1 \leq i \leq n$, possesses only periodic solutions or it is not independent.

Proof. Let $\mathcal{S}$ be a system of three equations containing equation (3). Theorem 12 implies that equation (3) admits at most quasi-periodic solutions with respect to x and z. Depending on the classification of the set of quasi-periodic solutions, described in Section 2, we have four cases.

Case 1: The equation (3) has ratio $R_1 = p : q$, i.e., the set of quasi-periodic solutions is characterized by this ratio and the set of periodic ones contains either all periodic triples (u^i, u^j, u^k) or only those with $ip = kq$. Then, let

$$\gamma_1(x, z)y\gamma_2(x, z)y \ldots y\gamma_m(x, z) = \delta_1(x, z)y\delta_2(x, z)y \ldots y\delta_m(x, z), \tag{4}$$

with $\gamma_i(x, z), \delta_i(x, z) \in \{x, z\}^*$ for all $1 \leq i \leq m$, $pref_1(\gamma_1(x, z)) = x$, and $pref_1(\delta_1(x, z)) = z$, be the second equation of the system $\mathcal{S}$.

If (4) admits independently quasi-periodic solutions with respect to x and z, then $\mathcal{S}$ is not independent since any solution of (3) is also a solution of (4).

If the quasi-periodicity of (4) implies periodicity, then the system $\mathcal{S}$ admits only periodic solutions.

If equation (4) admits only 1-limited quasi-periodic solutions, then when substituting in it a quasi-periodic solution of (3) (which is not periodic), we do not obtain graphical identity. Thus, by Corollary 2, the system $\mathcal{S}$ possesses only periodic solutions.

Otherwise, let R_2 be the ratio of equation (4). If $R_1 \neq R_2$, then when substituting in (4) a quasi-periodic solution (which is not periodic) characterized by the ratio R_1 we do not obtain graphical identity. So, by Corollary 2, the system $\mathcal{S}$ admits only periodic solutions. If $R_1 = R_2$, then the two equations have exactly the same set of quasi-periodic solutions. Moreover, if the second equation admits all periodic triples (u^i, u^j, u^k) as solutions, then any solution of (3) is also a solution of (4); so the system $\mathcal{S}$ is not independent. The same is true if in both equations the set of periodic solutions is characterized by the ratio R_1. Otherwise, equation (3) admits all periodic triples as solutions while (4) admits only those characterized by the ratio R_1. In this case the system containing equations (3) and (4) admits as periodic solutions only those triples characterized by the ratio R_1. So, we have to consider also the third equation of the system $\mathcal{S}$:

$$\mu_1(x,z)y\mu_2(x,z)y\ldots y\mu_s(x,z) = \nu_1(x,z)y\nu_2(x,z)y\ldots y\nu_s(x,z) \tag{5}$$

with $\mu_i(x,z), \nu_i(x,z) \in \{x,z\}^*$ for all $1 \leq i \leq s$, $pref_1(\mu_1(x,z)) = x$, and $pref_1(\nu_1(x,z)) = z$.

If equation (5) admits only 1-limited quasi-periodic solutions or if quasi-periodicity implies periodicity, then, as above, the system $\mathcal{S}$ possesses only periodic solutions.

If equation (5) admits independently quasi-periodic solutions, then it also admits all periodic triples as solutions. Thus, any solution of the system containing (3) and (4) is also a solution of (5), and so the three equations are not independent.

Otherwise, equation (5) has ratio R_3. If $R_3 \neq R_1$, then, as above, the system $\mathcal{S}$ possesses only periodic solutions. Otherwise, we have $R_1 = R_2 = R_3$ implying that the system containing the three equations is not independent since any solution of (3) and (4) is also a solution of (5).

Thus, in this case any two equations we add we obtain a system which is not independent or possesses only periodic solutions.

Case 2: If the quasi-periodicity of equation (3) implies periodicity, then we see immediately that the system $\mathcal{S}$ possesses only periodic solutions.

Case 3: Let now equation (3) admit independently quasi-periodic solutions with respect to x and z, i.e., for any $u, y \in \Sigma^*$ and any $i, j, k \geq 0$ the triples (u^i, y, u^k) and (u^i, u^j, u^k) characterize completely the sets of quasi-periodic and periodic solutions, respectively. Then, we consider again (4) as the second equation of the system $\mathcal{S}$.

If equation (4) admits independently quasi-periodic solutions with respect to x and z, then (3) and (4) are not independent, since any solution of (3) is also a solution of (4).

If the quasi-periodicity of equation (4) implies periodicity, then $\mathcal{S}$ admits only periodic solutions.

If equation (4) has some ratio $R_2 = p : q$, then when considering (3) and (4) together, they admit at most quasi-periodic solutions characterized by the ratio R_2. Also, the set of periodic solutions of these two equations contains either all periodic triples (u^i, u^j, u^k) with $i, j, k \geq 0$ and $u \in \Sigma^*$ or only those with $ip = kq$. So, we consider again (5) as the third equation of the system $\mathcal{S}$.

If equation (5) admits independently quasi-periodic solutions with respect to x and z, then the system $\mathcal{S}$ is not independent since any solution of (3) and (4) is also a solution of (5). If the quasi-periodicity of equation (5) implies periodicity, then $\mathcal{S}$ admits only periodic solutions. If equation (5) admits only 1-limited quasi-periodic solutions with respect to x and z, then when substituting in (5) a quasi-periodic solution (which is not periodic) characterized by the ratio R_2 we do not obtain graphical identity. So, by Corollary 2, the system $\mathcal{S}$ admits only periodic solutions. Otherwise, let R_3 be the ratio of equation (5). If $R_2 = R_3$, then the system $\mathcal{S}$ is not independent since the set of its solutions can be obtained either from (3) and (4), or from (3) and (5). Otherwise, i.e. $R_2 \neq R_3$, by Corollary 2, $\mathcal{S}$ possesses only periodic solutions since when substituting in (5) a quasi-periodic solution (which is not periodic) characterized by the ratio R_2 we do not obtain graphical identity.

If equation (4) admits only 1-limited quasi-periodic solutions with respect to x and z, then we can suppose without loss of generality that it admits only quasi-periodic solutions with $x = 1$; the other case is symmetric. So, when considering (3) and (4) together, they admit only 1-limited quasi-periodic solutions with $x = 1$. Also, the set of periodic solutions of these two equations contains either all periodic triples (u^i, u^j, u^k) with $i, j, k \geq 0$ and $u \in \Sigma^*$ or only those with $i = 0$. We consider again the third equation (5) as above. If equation (5) admits independently quasi-periodic solutions with respect to x and z, then the system $\mathcal{S}$ is not independent since any solution of (3) and (4) is also a solution of (5). If the quasi-periodicity of equation (5) implies periodicity, then the system $\mathcal{S}$ admits only periodic solutions. If equation (5) admits only 1-limited quasi-periodic solutions with respect to x and z, then $\mathcal{S}$ admits only periodic solutions (if in (5) we have $z = 1$) or it is not independent (if in (5) we have $x = 1$). Otherwise, let R_3 be the ratio of equation (5). Then, the system $\mathcal{S}$ possesses only periodic solutions since any quasi-periodic solution of (3) and (4) (which is not periodic) has $x = 1$ and thus when substituting it in (5) we do not obtain graphical identity.

Thus, also in this case any two equations we add we obtain a system which is not independent or possesses only periodic solutions.

Case 4: The last case to consider is when equation (3) admits only 1-limited quasi-periodic solutions. We can suppose without loss of generality that $x = 1$;

the case when $z = 1$ is symmetric. Moreover, the set of periodic solutions contains either all periodic triples (u^i, u^j, u^k), if the equation is balanced, or only those with $i = 0$, otherwise. Then, we consider again (4) as the second equation of $\mathcal{S}$.

If (4) admits independently quasi-periodic solutions with respect to x and z, then any solution of (3) is also a solution of (4) and so $\mathcal{S}$ is not independent.

If the quasi-periodicity of (4) implies the periodicity, then the system $\mathcal{S}$ admits only periodic solutions.

If equation (4) has ratio $R_2 = p : q$, then when substituting a quasi-periodic solution with $x = 1$ (which is not periodic) we do not obtain graphical identity. So, by Corollary 2, the system $\mathcal{S}$ possesses only periodic solutions.

If equation (4) admits only 1-limited quasi-periodic solutions with $z = 1$ then the system $\mathcal{S}$ admits only periodic solutions. Otherwise, (4) admits only 1-limited quasi-periodic solutions with $x = 1$. Then, the two equations have exactly the same set of quasi-periodic solutions. Moreover, if the second equation admits all periodic triples (u^i, u^j, u^k) as solutions, then any solution of (3) is also a solution of (4); so the system $\mathcal{S}$ is not independent. The same is true if in both equations the set of periodic solutions contains only triples of the form $(1, u^j, u^k)$. Otherwise, equation (3) admits all periodic triples as solutions while (4) admits only those with $x = 1$. Thus, when considering (3) and (4) together, they admit as solutions only triples of the form $(1, y, u^k)$ and $(1, u^j, u^k)$, for any $y, u \in \Sigma^*$ and $j, k \geq 0$. In this case we have to consider again (5), the third equation of the system $\mathcal{S}$.

If (5) admits independently quasi-periodic solutions with respect to x and z, then $\mathcal{S}$ is not independent since any solution of (3) and (4) is also a solution of (5). If the quasi-periodicity of (5) implies periodicity, then $\mathcal{S}$ possesses only periodic solutions. If (5) has some ratio R_3, then when substituting in it a quasi-periodic solution with $x = 1$ (which is not periodic) we do not obtain graphical identity. So, by Corollary 2, the system $\mathcal{S}$ possesses only periodic solutions. If (5) admits only 1-limited quasi-periodic solutions with $z = 1$, then $\mathcal{S}$ admits only periodic solutions. Otherwise, (5) admits only 1-limited quasi-periodic solutions with $x = 1$, but then any solution of (3) and (4) is also a solution of (5). So, the system $\mathcal{S}$ is not independent.

Thus, any system of at least three equations such that one of them is of the form

$$x^l z\alpha_1(x,z)y \dots y\alpha_n(x,z) = z^r x\beta_1(x,z)y \dots y\beta_n(x,z),$$

with $l, r \geq 1$, admits only periodic solutions or it is not independent. $\square$

Note that, just as explained at the beginning of this section, considering only equations with equal numbers of y's in the two sides is not a restriction of generality; otherwise, due to Corollary 2, the equation (3) possesses only periodic solutions, making then Theorem 18 trivial. Moreover, the stronger constraint of Theorem 17, i.e., taking only balanced equations, was not needed anywhere in this proof.

In order to clarify future considerations we make the following observation.

Remark 19. For a system of three equations, we illustrate graphically in Figure 1 the relations between the sets of solutions of all subsystems. First of all, we make a clear distinction between the sets of periodic (denoted by P), quasi-periodic which are not periodic (denoted by QP), and purely non-periodic (denoted by PNP) solutions. The indexes written in the parentheses characterize the subsystem for which we consider the set of solutions. Thus, each region contains all triples of a certain type satisfying the equations of the corresponding subsystem and only those. For instance $PNP(1,3)$ represents the set of all purely non-periodic solutions of both the first and third equation which are not solutions of the second one. These assumptions force all regions to be disjoint. Thus, an independent system of three equations possessing non-periodic solutions imposes two restrictions on the sets illustrated in Figure 1. Firstly, for any $S \subsetneq \{1,2,3\}$, at least one of the sets $P(S)$, $QP(S)$, or $PNP(S)$ is non-empty; in other words the system containing all three equations is not equivalent to any of its subsystems. Secondly, at least one of the sets $PNP(1,2,3)$ or $QP(1,2,3)$ is non-empty; in other words the system possesses also non-periodic solutions.

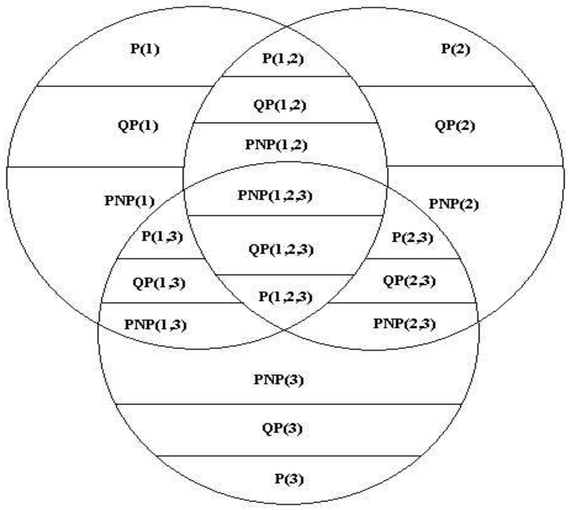

Figure 1: Representation of the set of solutions of a system of three equations

Due to Theorem 18, we can restrict now to systems of equations where at least on one side we have only one unknown before the first occurrence of y, i.e. they start either with $x^l y$ or with $z^r y$ for some $l, r \geq 1$.

Theorem 20. *Any system of three equations such that two of them are*

$$x^l y\alpha_2(x,z)y\ldots y\alpha_n(x,z) = \beta_1(x,z)y\beta_2(x,z)y\ldots y\beta_n(x,z)$$
$$z^r y\gamma_2(x,z)y\ldots y\gamma_m(x,z) = \delta_1(x,z)y\delta_2(x,z)y\ldots y\delta_m(x,z)$$

with $l, r \geq 1$, $pref_1(\beta_1(x,z)) = z$, $pref_1(\delta_1(x,z)) = x$, and $Alph(\beta_1(x,z)) \cap Alph(\delta_1(x,z)) = \{x,z\}$, possesses only periodic solutions or it is not independent.

Proof. Consider first the system containing the two equations from the theorem. Then, Theorem 14 implies that this system admits at most quasi-periodic solutions with respect to x and z and, moreover, if $l = 1$ or $r = 1$ then it admits only periodic ones. So, we can suppose that $l, r \geq 2$.

Since the two sides of the first equation start with $x^l y$ and $\beta_1(x,z)y$ respectively, and $Alph(\beta_1(x,z)) = \{x,z\}$, then it cannot admit independently quasi-periodic solutions with respect to x and z, see Section 2. Also, if it admits only 1-limited quasi-periodic solutions, then they must have $z = 1$; if a quasi-periodic triple with respect to x and z (which is not periodic) has $x = 1$ and $z \neq 1$, then we do not obtain graphical identity when substituting it into the equation. Similarly, the second equation cannot admit independently quasi-periodic solutions with respect to x and z and if it admits only 1-limited quasi-periodic solutions then they must have $x = 1$. So, if both equations admit only 1-limited quasi-periodic solutions, then the system containing them possesses only periodic ones.

If at least in one equation the quasi-periodicity implies periodicity, then any system containing the two equations from the theorem possesses only periodic solutions.

If one equation admits only 1-limited quasi-periodic solutions and the other has some ratio $R = p : q$, then, when substituting a quasi-periodic solution (which is not periodic) of the first equation into the second one, we do not obtain graphical identity. So, also in this case, any system containing these two equations possesses only periodic solutions.

Otherwise, both equations have some ratios; let them be $R_1 = p_1 : q_1$ and $R_2 = p_2 : q_2$, respectively, characterizing completely the sets of quasi-periodic solutions with respect to x and z of the two equations.

If $R_1 \neq R_2$, then when substituting a quasi-periodic solution (which is not periodic) of one of the equations into the other one we do not obtain graphical identity. So, the system containing the two equations admits only periodic solutions. Otherwise, i.e. $R_1 = R_2 = p : q$, the quasi-periodic solutions of the system are completely characterized by this ratio. Moreover, the set of periodic solutions of each equation contains either any periodic triple (u^i, u^j, u^k), or only those satisfying $ip = kq$.

Consider now a third equation, and let $\mathcal{S}$ be the obtained system of three equations. If we look at the graphical representation of the sets of solutions illustrated in Figure 1, then, due to the previous considerations, we already know that the sets $QP(1)$, $QP(2)$, $PNP(1,2)$, and $PNP(1,2,3)$ are empty. We have now four cases depending on the type of quasi-periodic solutions admitted by the third equation.

If the third equation admits independently quasi-periodic solutions with respect to x and z, then it also admits all periodic triples as solutions. Thus, the system $\mathcal{S}$ is not independent since any solution of the initial system is also a solution of the third equation. This case corresponds to the situation when, in Figure 1, both sets $QP(1,2)$ and $P(1,2)$ are empty. Thus, $P(1,2) = QP(1,2) = PNP(1,2) = \emptyset$, meaning that $\mathcal{S}$ is equivalent to its first two equations, see Remark 19.

If in the third equation the quasi-periodicity implies periodicity, then the system $\mathcal{S}$ admits only periodic solutions.

If the third equation admits only 1-limited quasi-periodic solutions, then when we substitute in it a quasi-periodic solution (which is not periodic) of the initial system we do not obtain graphical identity; so, by Corollary 2, the system $\mathcal{S}$ possesses only periodic solutions.

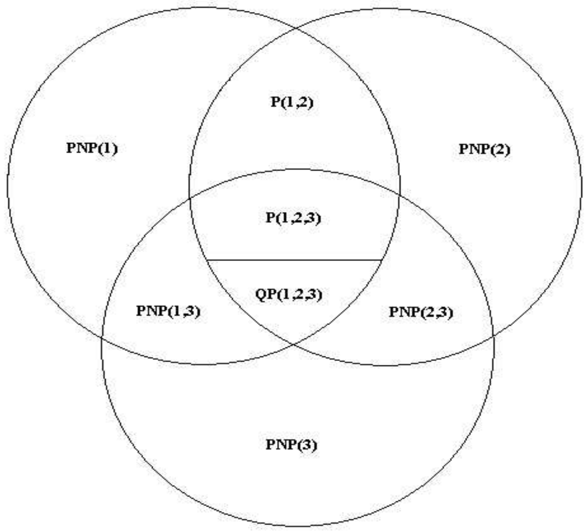

Figure 2: A representation of the sets of solutions

Otherwise, the third equation has some ratio R_3 which characterizes all its quasi-periodic solutions with respect to x and z. If $R_1 = R_2 \neq R_3$, then, by Corollary 2, the system $\mathcal{S}$ admits only periodic solutions since, when substituting in the third equation a quasi-periodic solution (which is not periodic) of the initial system we do not obtain graphical identity. Otherwise, all three equations have the same ratio, i.e. $R_1 = R_2 = R_3 = p : q$. Thus, they all accept the same set of quasi-periodic solutions, i.e., the ones characterized by this ratio. In Figure 1, this means that all the sets $QP(3)$, $QP(1,2)$, $QP(1,3)$, and $QP(2,3)$ are empty. If the third equation admits all periodic triples as solutions, then $\mathcal{S}$ is not independent since any solution of the first two equations is also a solution of the third one. This case corresponds to the situation when, in Figure 1, we have $P(1,2) = \emptyset$ since any periodic solution of the first two equations is also solution of the third one. Thus, again $P(1,2) = QP(1,2) = PNP(1,2) = \emptyset$, meaning that $\mathcal{S}$ is equivalent to its first two equations, see Remark 19. The same is true if the third equation and at least one of the first two admit as periodic solutions only triples (u^i, u^j, u^k) with $ip = kq$. Otherwise, the first two equations of $\mathcal{S}$ admit as solutions any periodic triple (u^i, u^j, u^k), i.e., they are balanced, while the third one admits only those with $ip = kq$, i.e., it is not balanced. This case corresponds to the situation when, in Figure 1, we have that all the sets $P(1)$, $P(2)$, $P(3)$, $P(1,3)$, and $P(2,3)$ are empty; we illustrate this special subcase in Figure 2. Now, if all the sets in Figure 2 are non-empty, then, as explained in Remark 19, this would be an example of an independent system of three equations admitting also non-periodic solutions. However, since one equation of the system is not balanced, this case is not possible due to Theorem 17.

Thus, any system of at least three equations containing the ones in the theorem possesses only periodic solutions or it is not independent. □

Next, we can restrict again to the case when all equations have on one side only occurrences of x's before the first y; the case where x and z are interchanged is symmetric. Using similar reasoning as for the previous theorem, we prove the following result.

Theorem 21. *Consider the following system of two equations:*

$$\begin{aligned} x^l y\alpha_2(x,z)y\dots y\alpha_n(x,z) &= \beta_1(x,z)y\beta_2(x,z)y\dots y\beta_n(x,z)\\ x^{l'} y\gamma_2(x,z)y\dots y\gamma_m(x,z) &= \delta_1(x,z)y\delta_2(x,z)y\dots y\delta_m(x,z)\end{aligned}$$

with $l \neq l'$, $pref_1(\beta_1(x,z)) = pref_1(\delta_1(x,z)) = z$, and if $\beta_1(x,z) = z^r$ and $\delta_1(x,z) = z^{r'}$ then $r \neq r'$. Then, whenever we add a third equation, the obtained system possesses only periodic solutions or it is not independent.

Proof. Let $\mathcal{S}$ be the system of two equations from the theorem. Then, Theorems 13, 15, and 16 imply that $\mathcal{S}$ admits at most quasi-periodic solutions with respect to x and z. Note that the restrictions on the numerical parameters are necessary in order to use the above mentioned theorems.

Notice that since the first equation starts with $x^l y$ and $\beta_1(x,z)y$, respectively, and $z \in Alph(\beta_1(x,z))$, it cannot admit independently quasi-periodic solutions, see Section 2. Similarly, neither can the second equation.

If in either of these two equations the quasi-periodicity implies periodicity, then $\mathcal{S}$ possesses only periodic solutions. Thus, in this case, independently of the third equation we add to $\mathcal{S}$, the obtained system possesses only periodic solutions.

Suppose now that one equation admits only 1-limited quasi-periodic solutions while the other one has some ratio R. Then, when we substitute a quasi-periodic solution (which is not periodic) of the first equation into the second one we do not obtain graphical identity. Thus, by Corollary 2, $\mathcal{S}$ possesses only periodic solutions, and so also in this case, independently of the third equation we add, the obtained system possesses only periodic solutions.

Thus, the only remaining possibilities are that either both equations have some ratios or they both admit only 1-limited quasi-periodic solutions.

Suppose first that both equations have some ratios, completely characterizing their sets of quasi-periodic solutions; let them be R_1 and R_2 respectively. If $R_1 \neq R_2$, then when substituting a quasi-periodic solution (which is not periodic) of the first equation into the second one we do not obtain graphical identity, so by Corollary 2, $\mathcal{S}$ possesses only periodic solutions. Thus, in this case, independently of the equation added to $\mathcal{S}$, the obtained system possesses only periodic solutions. Otherwise, i.e. $R_1 = R_2 = p : q$, the quasi-periodic solutions of the system $\mathcal{S}$ are completely characterized by this ratio. Moreover, the set of periodic solutions of each of the two equations contains either all periodic triples (u^i, u^j, u^k) or only those satisfying $ip = kq$.

Let now $\mathcal{S}_1$ be a system of three equations obtained by adding a third equation to $\mathcal{S}$. If this third equation admits independently quasi-periodic solutions with respect to x and z, then it also admits all periodic triples as solutions. But then, any solution of $\mathcal{S}$ is also a solution of the new equation. Thus, $\mathcal{S}_1$ is not independent. If in the third equation the quasi-periodicity implies periodicity, then the system $\mathcal{S}_1$ possesses only periodic solutions. If the third equation admits only 1-limited quasi-periodic solutions, then when substituting in it a quasi-periodic solution (which is not periodic) of $\mathcal{S}$ we do not obtain graphical identity. So, by Corollary 2, the system $\mathcal{S}_1$ possesses only periodic solutions. Otherwise, the third equation has some ratio R_3 completely characterizing its set of quasi-periodic solutions with respect to x and z. If $R_1 = R_2 \neq R_3$, then, by Corollary 2, the system $\mathcal{S}_1$ possesses only periodic solutions. Otherwise, all three equations have the same ratio, i.e. $R_1 = R_2 = R_3 = p : q$. Thus, the three equations possess exactly the same set of quasi-periodic solutions. If the third equation admits all periodic triples as solutions, then $\mathcal{S}_1$ is not independent since any solution of $\mathcal{S}$ is also a solution of the third equation. The same is true if the third equation and at least one of the first two admit as periodic solutions only triples (u^i, u^j, u^k) with $ip = kq$. Otherwise, the equations of $\mathcal{S}$ are balanced while the third one is not. But, like in the previous theorem, in this case Theorem 17 implies that $\mathcal{S}_1$ is not independent or it possesses only periodic solutions.

Next, we consider the case when both equations of the system $\mathcal{S}$ admit only 1-limited quasi-periodic solutions with respect to x and z. However, if at least one of $\beta_1(x, z)$ or $\delta_1(x, z)$ contains only z's, then, by definition, the corresponding equation cannot admit 1-limited quasi-periodic solutions with respect to x and z, see Section 2. Thus, this case is possible only when $Alph(\beta_1(x, z)) \cap Alph(\delta_1(x, z)) = \{x, z\}$. Moreover, since the left sides of the two equations start with $x^l y$ and $x^{l'} y$, respectively, then the quasi-periodic solutions must be of the form $(u^i, y, 1)$ for some $i \geq 0$ and $u, y \in \Sigma^*$. Also, the sets of periodic solutions of the two equations contain either all periodic triples (u^i, u^j, u^k) or only those with $k = 0$.

Let again $\mathcal{S}_1$ be a system of three equations obtained by adding a third equation to $\mathcal{S}$. As above, if the third equation admits independently quasi-periodic solutions with respect to x and z, then $\mathcal{S}_1$ is not independent. Also, if in the third equation the quasi-periodicity implies periodicity, then $\mathcal{S}_1$ possesses only periodic solutions. If the third equation has some ratio R completely characterizing its set of quasi-periodic solutions, then we do not obtain graphical identity when substituting in it a quasi-periodic solution (which is not periodic) of $\mathcal{S}$. Thus, by Corollary 2, the system $\mathcal{S}_1$ possesses only periodic solutions. Otherwise, this third equation admits only 1-limited quasi-periodic solutions with respect to x and z. If these solutions have $x = 1$, then $\mathcal{S}_1$ possesses only periodic solutions since, again, when substituting in it a quasi-periodic solution (which is not periodic) of $\mathcal{S}$ we do not obtain graphical identity. Otherwise, the three equations of $\mathcal{S}_1$ possess exactly the same set of quasi-periodic solutions with respect to x and z, i.e. triples of the form $(u^i, y, 1)$ for $i \geq 0$ and $u, y \in \Sigma^*$. But then, if the third equation admits all periodic triples as solutions, then $\mathcal{S}_1$ is not independent since any solution of $\mathcal{S}$ is also a solution of the third equation. The same is true if the third equation and at least one of the first two admit as periodic solutions only triples $(u^i, u^j, 1)$. Otherwise, the first two equations of $\mathcal{S}_1$ are balanced while the third one is not. Then, due to Theorem 17, we obtain again that $\mathcal{S}_1$ is not independent or it possesses only periodic solutions.

Thus, independently of the added equation, the obtained system possesses only periodic solutions or it is not independent. □

Remark 22. The proofs of the last two theorems raise the following comment. In some cases of these proofs we needed to use the constraint imposed by Theorem 17, i.e., that a system containing unbalanced equations either possesses only periodic solutions or it is not independent. However, for both theorems, if we consider four equations instead of three, then we can prove that such a system possesses only periodic solutions or it is not independent in the general case of arbitrary equations, i.e., without the help of Theorem 17.

The only remaining case now, up to the symmetry of x and z, is the one when all equations have on one side only occurrences of x's before the first y, and moreover the number of these occurrences is the same in all of them. The theorems in this section reduce Conjecture 5 to this last case.

In order to continue our investigation of this last case we need to introduce a new technique. For an arbitrary word equation over three unknowns, we define inductively a partition of the set of solutions depending on the lengths of the unknowns x, y, and z. For any solution $(X, Y, Z) \in (\Sigma^*)^3$, we have three possibilities: $|X| = |Z|$, $|X| < |Z|$, and $|X| > |Z|$. Depending on these possibilities we can apply to the initial equation three types of transformations: $x = z$, $z = xt$, and $x = zt$, respectively, for a new unknown t. Thus, we first divide the set of solutions into three sets, each one containing triples satisfying only one of the above conditions; let them be $S_{x=z}$, $S_{z=xt}$, and $S_{x=zt}$. While the set $S_{x=z}$, corresponding to the restriction $|X| = |Z|$, remains unchanged, the other two will be modified further on. Let us take now the set $S_{z=xt}$ characterized by the condition $|X| < |Z|$; the considerations for the set characterized by $|X| > |Z|$ are identical. In this case, we can apply to the initial equation the transformation $z = xt$, where t is a new unknown and obtain a new equation admitting a shorter solution, i.e., (X, Y, T) with $|T| < |Z|$. Thus we can repeat inductively the above procedure, this time splitting the set $S_{z=xt}$ into three disjunct parts. Moreover, each of these new subsets is characterized by two constraints: the first one is $|X| < |Z|$ while the second one involves $|X|$, $|Y|$, and $|T|$. Since at each step we reduce the length of the chosen solution (X, Y, Z), we have to stop after finitely many steps; so any solution is included in a unique, clearly defined subset. Thus, when we consider the set of all solutions, we obtain a (possibly infinite) partition $\mathcal{P}$, each class being characterized by a chain of constraints on the lengths of the unknowns. Naturally, such a partition of the set of all solutions can be constructed in the same way for arbitrary systems of equations.

Theorem 23. *Let $\mathcal{P}$ be the above partition of the set of solutions of the following system:*

$$\begin{cases} x^i y\alpha_2(x,z)y\ldots y\alpha_n(x,z) = z\beta_1(x,z)y\beta_2(x,z)y\ldots y\beta_n(x,z) \\ x^i y\gamma_2(x,z)y\ldots y\gamma_m(x,z) = z\delta_1(x,z)y\delta_2(x,z)y\ldots y\delta_m(x,z) \\ x^i y\mu_2(x,z)y\ldots y\mu_p(x,z) = z\nu_1(x,z)y\nu_2(x,z)y\ldots y\nu_p(x,z) \end{cases},$$

where $\beta_1(x,z), \delta_1(x,z), \nu_1(x,z) \in \{x,z\}^*$. Then, on each class of $\mathcal{P}$, the system possesses only periodic solutions or is equivalent to one of its subsystems.

Proof. Since all equations start with $x^i y$ in the left side and with a word of the form $z\{x,z\}^*y$ in the right side, then they cannot admit independently quasi-periodic solutions with respect to x and z, see Section 2.

Let $(X, Y, Z) \in (\Sigma^*)^3$ be a solution of this system; $(1, 1, 1)$ is solution of any constant-free equation.

Case 1: If $|X| = |Z|$ then $X = Z$ and thus the chosen solution (X, Y, X) is quasi-periodic with respect to x and z. If in any of the three equations the quasi-periodicity implies periodicity, then this solution has to be periodic. Also, if any of the three equations admits only 1-limited quasi-periodic solutions with respect to x and z, then the chosen solution is actually $(1, Y, 1)$ and thus periodic. Otherwise, all three equations have some ratios; let them be R_1, R_2, and R_3 respectively which completely characterize the sets of quasi-periodic solutions with respect to x and z of each of the equations. If $R_1 \neq 1:1$, $R_2 \neq 1:1$, or $R_3 \neq 1:1$, then the chosen solution is actually periodic since, by Corollary 2, when replacing in the initial system $x = z$ we do not obtain graphical identity. Otherwise, the three equations have $R_1 = R_2 = R_3 = 1:1$, meaning that they are equivalent to each other on the set of solutions of the form $(X, Y, X) \in (\Sigma^*)^3$. Moreover, if $i = 1$, then this is possible only when $\beta_1(x,z) = \delta_1(x,z) = \nu_1(x,z) = 1$.

Case 2: If $|X| > |Z|$, then we can write $X = ZT$ for some new word $T \in \Sigma^+$. If in the initial system we apply the transformation $x = zt$ for some new unknown t, then we obtain

$$\begin{cases} t(zt)^{i-1}y\alpha_2(zt,z)y\ldots y\alpha_n(zt,z) = \beta_1(zt,z)y\beta_2(zt,z)y\ldots y\beta_n(zt,z) \\ t(zt)^{i-1}y\gamma_2(zt,z)y\ldots y\gamma_m(zt,z) = \delta_1(zt,z)y\delta_2(zt,z)y\ldots y\delta_m(zt,z) \\ t(zt)^{i-1}y\mu_2(zt,z)y\ldots y\mu_p(zt,z) = \nu_1(zt,z)y\nu_2(zt,z)y\ldots y\nu_p(zt,z) \end{cases}.$$

Consider now the case when $i \geq 2$. If in the initial system at least one of $\beta_1(x,z)$, $\delta_1(x,z)$, or $\nu_1(x,z)$ contain also x, then we have at least an equation where in both sides both z and t appear before the first y. But then Theorem 18 implies that the obtained system admits only periodic solutions or it is not independent. Thus either the chosen solution is periodic or the initial system is equivalent to one of its subsystems on the set of solutions of the form $(X,Y,Z) \in (\Sigma^*)^3$ with $|X| > |Z|$.

Otherwise, we have $\beta_1(zt,z), \delta_1(zt,z), \nu_1(zt,z) \in z^*$. If two of them are z^k and respectively z^l with $k \neq l$, then Theorem 21 implies that this system admits only periodic solutions or it is not independent. So, again either the chosen solution is periodic or the initial system is equivalent to one of its subsystems on the set of solutions of the form $(X,Y,Z) \in (\Sigma^*)^3$ with $|X| > |Z|$. If $\beta_1(zt,z) = \delta_1(zt,z) = \nu_1(zt,z) = z^k$, then this is a system of the same type as the initial one for which we have a shorter solution (T,Y,Z) so we can apply inductively the same reasoning.

Now, if $i = 1$, then the obtained system is of the same type as the initial one but admitting a shorter solution (T,Y,Z), so we can repeat inductively the same reasoning. Moreover, if $\beta_1(x,z) = \delta_1(x,z) = \nu_1(x,z) = 1$, then the obtained system is actually of the form

$$\begin{cases} ty\alpha'(t,y,z) = y^{i_1}z\beta'(t,y,z) \\ ty\gamma'(t,y,z) = y^{i_2}z\delta'(t,y,z) \\ ty\mu'(t,y,z) = y^{i_3}z\nu'(t,y,z) \end{cases}.$$

If at least two of i_1, i_2, or i_3 are distinct then Theorem 21 implies that this system admits only periodic solutions or it is not independent. Thus, either the chosen solution is periodic or the initial system is equivalent to one of its subsystems on the set of solutions of the form $(X,Y,Z) \in (\Sigma^*)^3$ with $|X| > |Z|$. Otherwise, i.e. $i_1 = i_2 = i_3$, this is of the same type as the initial system and admits a shorter solution (T,Y,Z) so we can again apply inductively the same reasoning.

Case 3: If $|X| < |Z|$, then we can write $Z = XT$ for some new word $T \in \Sigma^+$.

If $i \geq 2$, then when we apply the transformation $z = xt$ for some new unknown t, we obtain:

$$\begin{cases} x^{i-1}y\alpha_2(x,xt)y\ldots y\alpha_n(x,xt) = t\beta_1(x,xt)y\beta_2(x,xt)y\ldots y\beta_n(x,xt) \\ x^{i-1}y\gamma_2(x,xt)y\ldots y\gamma_m(x,xt) = t\delta_1(x,xt)y\delta_2(x,xt)y\ldots y\delta_m(x,xt) \\ x^{i-1}y\mu_2(x,xt)y\ldots y\mu_p(x,xt) = t\nu_1(x,xt)y\nu_2(x,xt)y\ldots y\nu_p(x,xt) \end{cases},$$

which is of the same type as the initial one and possesses a shorter solution (X, Y, T). So, we can apply inductively for this system the same reasoning as above.

If $i = 1$, then after applying the transformation $z = xt$ for some new unknown t, the obtained system is of the form

$$\begin{cases} y^{i_1} x\alpha'(x, y, t) = t\beta_1(x, xt)y\beta'(x, y, t) \\ y^{i_2} x\gamma'(x, xt) = t\delta_1(x, xt)y\delta'(x, y, t) \\ y^{i_3} x\mu'(x, y, t) = t\nu_1(x, xt)y\nu'(x, y, t) \end{cases}.$$

Suppose first that at least two of i_1, i_2, or i_3 are distinct, e.g., $i_1 \neq i_2$. If at least one of $\beta_1(x, z)$ or $\delta_1(x, z)$ are the empty word, then Theorem 21 implies that this system admits only periodic solutions or it is not independent. Thus, either the chosen solution is periodic or the initial system is equivalent to one of its subsystems on the set of solutions of the form $(X, Y, Z) \in (\Sigma^*)^3$ with $|X| < |Z|$. Otherwise, i.e. $i_1 = i_2 = i_3$, this is of the same type as the initial system and admits a shorter solution (T, Y, Z) so we can apply inductively the same reasoning.

Since with every transformation we reduce the length of the chosen solution, we have to stop after a finite number of steps. But, the previous considerations imply that we stop either with a periodic solution or with a non-independent system. $\square$

Note that if in the previous theorem the equations of the initial system are balanced, then so are all the equations derived throughout the proof. In particular, this implies that all these equations have exactly the same set of periodic solutions, i.e., the set of all periodic triples. In other words, if on one class of the partition the initial system possesses only periodic solutions, then, on that class, it is equivalent with any of its equations. Moreover, if the initial system is always equivalent to exactly the same subsystem or to equivalent subsystems, then it is not independent. This would completely solve the open problem from [4], by giving a negative answer. On the other hand, this theorem gives us some clues on how to look for an example of an independent system of three equations accepting non-periodic solutions, if there exists one. However, searching for such an example seems to be a very difficult task.

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