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On the Tiling Problem and Reversible Cellular Automata

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Abstract

It is shown that the (infinite) tiling problem is undecidable even if the given tile set is deterministic by two opposite corners, i.e. a tile is uniquely determined by both the colors of the two sides adjacent to some corner and the colors of the sides directly opposite to these sides. The reduction is done from the Turing machine halting problem and uses the 4-way deterministic aperiodic tile set of Kari and Papasoglu.

The tile set construction given here implies also the universality of one-dimensional reversible cellular automata. More specifically, a new proof is given for the result of Dubacq, that any (irreversible) Turing machine can be simulated in real time with a one-dimensional reversible cellular automaton.

Keywords: cellular automata, determinism, reversibility, tiling problem, Wang tiles

TUCS Laboratory
Discrete Mathematics for Information Technology

1 Introduction

A *Wang tile* (or a *tile* in short) is a unit square with colored edges. The edges of a Wang tile are called *north*, *east*, *west* and *south* edges in a natural way. A Wang tile t can be considered also as an ordered 4-tuple $t = (N_t, E_t, W_t, S_t)$ containing the colors in a predefined order. For the given tile t, expressions N_t, E_t, W_t and S_t are used to denote north, east, west and south side colors, respectfully. A *Wang tile set* T (or a *tile set* in short) is a finite set containing Wang tiles. A *tiling* is a mapping $f : \mathbb{Z}^2 \to T$, which assigns a unique Wang tile for each integer pair of the plane. A tiling f is said to be *valid*, if for every pair $(x, y) \in \mathbb{Z}^2$ the tile $f(x, y) \in T$ matches its neighboring tiles (e.g. the south side of tile $f(x, y)$ has the same color as the north side of tile $f(x, y - 1)$ etc.).

A Wang tile set T is said to be *NW-deterministic*, if within the tile set there does not exist two different tiles with the same colors on the north- and west-sides. In general, a Wang tile set is *XY-deterministic*, if the colors of X- and Y-sides uniquely determine a tile in the given Wang tile set. A Wang tile set is *4-way deterministic*, if it is NE-, NW-, SE- and SW-deterministic.

A mapping $f : T_1 \to T_2$ is called a *tile homomorphism* if it respects the colors, i.e. $f(t) = t'$ with $N_{t'} = g(N_t)$, $E_{t'} = g(E_t)$, $W_{t'} = g(W_t)$ and $S_{t'} = g(S_t)$, where g is a mapping from the set of the colors of the tile set T_1 to the set of the colors of the tile set T_2. The *homomorphic image* $f(T)$ of a tile set T is defined in the natural way as the set

$$f(T) = \{f(t) | t \in T\}.$$

A tiling $f : \mathbb{Z}^2 \to T$ is called *periodic* with period (a, b) if $f(x, y) = f(x + a, y + b)$ for all $(x, y) \in \mathbb{Z}^2$ and $(a, b) \neq (0, 0)$. A tile set T is called *aperiodic*, if there exists some tiling with the tile set T, but no tiling with the tile set T is periodic. If the tile set T admits a periodic tiling $f : \mathbb{Z}^2 \to T$ with some period, then it admits also a *doubly periodic* tiling $g : \mathbb{Z}^2 \to T$, that is, there exists such non-zero integers a and b that $g(x, y) = g(x + a, y)$ and $g(x, y) = g(x, y + b)$ for all $(x, y) \in \mathbb{Z}^2$ [7].

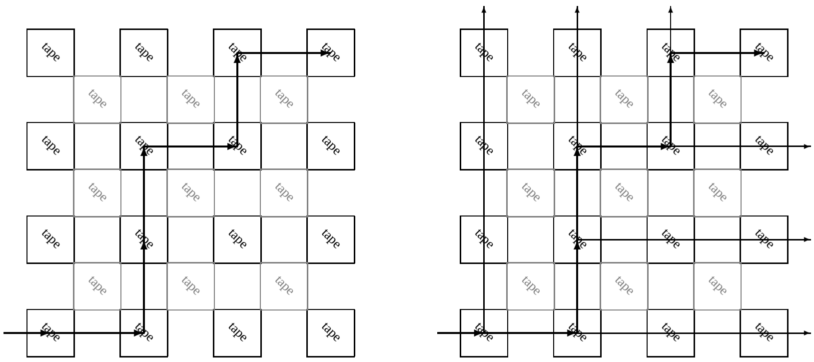
The following question is referred to as the *tiling problem*: "Given a Wang tile set T, does there exist a valid tiling of the plane?" A tiling $f : \mathbb{Z}^2 \to T$ is said to *contain* tile $t \in T$, if for some integers $x, y \in \mathbb{Z}$ equation $f(x, y) = t$ holds. The following question is referred to as the *tiling problem with a seed tile*: "Given a Wang tile set T and a tile $t \in T$, does there exist a valid tiling of the plane that contains the tile t?" If the tiling problem with a seed tile was decidable, then the tiling problem would be decidable. Let T be the tile set of the given instance of the tiling problem. Then the answer for the tiling problem is affirmative, if, and only if, for some tile $t \in T$ the answer for the tiling problem with a seed tile is affirmative considering the tile set T as the tile set of the instance and the tile t as the seed tile of the instance.

It is already known, that the tiling problem is undecidable [7, 2]. Furthermore, it is known to be undecidable even when restricted to tile sets that are deterministic by one corner [4]. In this article it is shown that the tiling problem is undecidable for tile sets that are deterministic by two opposite corners. The proof relies on the 4-way deterministic aperiodic tile set given by Kari and Papasoglu [5].

2 The tiling problem with a seed tile

2.1 The idea for the undecidability proof

The basic idea is to represent the Turing machine tape on diagonal rows as in [4]. It is easy to show, that an arbitrary Turing machine computation can be represented on diagonal rows. The computation on diagonal rows is done in the manner of figure 1(a). Every second diagonal row in the northwest-southeast direction is used to represent the Turing machine configuration at a certain moment. One tile at each diagonal row represents the read-write head and the current symbol to be read. The other tiles of the diagonal row represent the other symbols on the tape located to the left and to the right from the read-write head.



(a) The rough idea of representing Turing machine computation on diagonal rows.

(b) The Turing machine computation with additional information signals of earlier read-write operations.

Figure 1: The general idea of representing the computation on diagonal rows.

Since a Turing machine is a deterministic method of computation, the tile set constructed in this manner is clearly deterministic in (at least) one direction. More specifically, it is the direction to which the computation advances in time. To force determinism also in the opposite direction, some modifications are needed. On every operation of the read-write head, a “signal” is sent to the direction that is opposite to the read-write head movement. This signal contains information about the read-write operation which is currently being conducted and the direction from which the read-write head entered the current cell after the previous move. The computation with signals is represented in figure 1(b). In figure 1(b), if the read-write head moves to the left, the signal is sent towards east, and if the read-write head moves to the right, the signal is sent towards north. In practice, the signal is just a component of a side color which is moved onward unobstructed. These signals containing information about the previous move and the current one are referred to as the *move signals*. The move signals are started on the tiles in figures 2 and 4 (i.e. the tiles that represent the read-write head). The tiles in figure 5 (i.e. the tiles that represent the tape) just move the possible move signals onward. This construction will make the tile set representing the given Turing machine both NE- and SW-deterministic.

2.2 The tile set for the given Turing machine

In this subsection a NE- and SW-deterministic tile set is constructed for the given Turing machine. In what follows, the diagonal rows of tiles are referred to as *diagonals* in short.

In this article, a Turing machine $\mathcal{M}$ is considered to be a four-tuple $\mathcal{M} = (S, T, \delta, q_0)$, where S is the state set, T is the tape alphabet, δ is the transition function and $q_0 \in S$ is the initial state. No "accept"-, "reject"- or "halt"-states are defined explicitly. The tape of a Turing machine is defined to be two-way infinite and symbol ε is used to denote the empty symbol of the Turing machine. The transition function is a mapping

$$\delta : S \times T \to S \times T \times \{L, R\},$$

that is, at every time step the read-write head moves either to the left or to the right. A Turing machine is said to *halt*, if it is in state q reading symbol s and $\delta(q, s)$ in undefined. Transition of the form $\delta(x, y) = (a, b, c)$ can also be written in the form $(x, y) \to (a, b, c)$. The *Turing machine halting problem* is considered to be the following question: "Does the given Turing machine $\mathcal{M}$ halt when started on an empty tape?" The halting problem is known to be undecidable.

The tiles to represent read-write operations For every possible move of the Turing machine, either the tiles in figure 2 and the tile in figure 3(a), or the tiles in figure 4 and the tile in figure 3(b) are added to the tile set.

The tiles for a left move Assume that the Turing machine contains move $(q, a) \to (q', a', L)$. Then the tiles in figure 2 and the tile in figure 3(a) are added to the tile set.

The tile in figure 2(a) is used if the previous move was to the left and the current move is to the left. If the previous move was to the right, then the tile in figure 2(b) is used.

The tiles for a right move Assume that the Turing machine contains move $(q, a) \to (q', a', R)$. Then the tiles in figure 4 and the tile in figure 3(b) are added to the tile set.

The tile in figure 4(a) is used if the previous move was to the left and the current move is to the right. If the previous move was to the right and the current move is to the right, then the tile in figure 4(b) is used.

The tiles that are used to represent the moves of the given Turing machine $\mathcal{M}$ are referred to as *move tiles* or the tile set $M_{\mathcal{M}}$.

(a) The new move is to the left and the previous move was to the left.

(b) The new move is to the left and the previous move was to the right.

Figure 2: The tiles representing the read-write head for move $(q, a) \to (q', a', L)$

(a) The tile for read-write operation $(q, a) \to (q', a', L)$.

(b) The tile for read-write operation $(q, a) \to (q', a', R)$.

Figure 3: The tiles for the read-write operations. The tiles depend on the new state q', the new symbol a' to be written, the move direction and on the new symbol b to be read.

The tiles to represent tape contents For every state q and every element a, b and c of the tape alphabet, the tiles in figure 5 are added to the tile set. The tile in figure 5(a) is used to represent a cell (or the border between two cells if $a \neq b$) of the tape without any information about an earlier read-write operation.

The tiles in figure 5(b) represent tape contents likewise, but contain also information about a read-write operation during which the read-write head moved to the left. That is, the east side and the west side have colors of form $(\cdot, qc, \cdot)$ if, and only if, there exist a move of form $(q, c) \to (\cdot, \cdot, L)$.

The tiles in figure 5(c) are similar to the tiles in figure 5(b) with the exception that they contain information about a read-write operation during which the read-write head moved to the right and not to the left. The north side and the

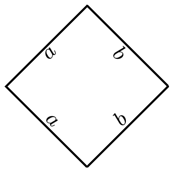
(a) The new move is to the right and the previous move was to the left.

(b) The new move is to the right and the previous move was to the right.

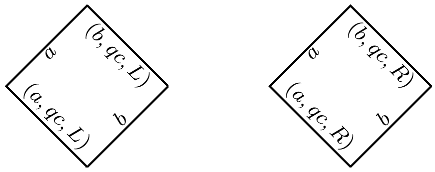
Figure 4: The tiles representing the read-write head for move $(q, a) \to (q', a', R)$

south side have colors of form $(\cdot, qc, \cdot)$ if, and only if, there exist a move of form $(q, c) \rightarrow (\cdot, \cdot, R)$.

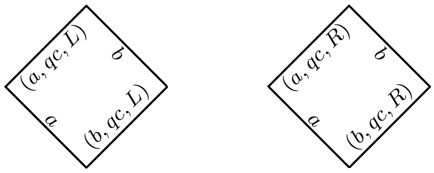
The tile set is being constructed so, that if the seed tile (i.e. the tile in figure 6(c)) is located on an even diagonal, then on every odd diagonal symbols a and b in figure 5 are equal.



(a) A tile without read-write information.



(b) The tiles with information about a move to the left.



(c) The tiles with information about a move to the right.

Figure 5: The tiles to represents the symbols on the tape. Here q denotes an arbitrary state and symbols a, b and c denote arbitrary elements of the tape alphabet.

The tiles that are used to represent the tape contents of the given Turing machine $\mathcal{M}$ are referred to as *symbol tiles* or as the tile set $S_{\mathcal{M}}$.

The auxiliary tiles To force the Turing machine to start on a blank tape only, the tiles in figure 6 are added to the tile set. One of these tiles (namely, the tile in figure 6(c)) is chosen to be the seed tile. If the seed tile is contained within a tiling, then the tiling represents a Turing machine computation. Other tiles in figure 6 force the Turing machine to start on a blank tape. The blank initial configuration of the Turing machine is represented by the tile pattern shown in figure 8. In short, if the seed tile is located in the origin, then the Turing machine simulation is done in one of the quadrants.

For the given Turing machine $\mathcal{M}$, the tiles in figure 6 are referred to as *initialization tiles* or as the tile set $I_{\mathcal{M}}$.

For every Turing machine $\mathcal{M}$, the Wang tile set constructed using the method above is denoted by $T_{\mathcal{M}}$ (i.e. $T_{\mathcal{M}} = M_{\mathcal{M}} \cup S_{\mathcal{M}} \cup I_{\mathcal{M}}$). An example of a Turing machine operation is shown in figure 7.

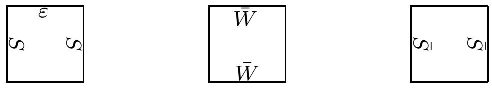
Let (q, a) be any preimage pair for which the transition $\delta(q, a)$ is not defined. Then there will be no tile that would have the color qa on its west side or south side. Therefore, if the Turing machine halts, that is, if at some moment of time the read-write head in state q reads symbol a, then the tiling cannot be completed to cover the entire plane in a valid way.

(a) The blank tile.

(b) The tile to place the read-write head.

(c) The seed tile.

(d) The west border.



(e) The south border.

(f) The west border extension.

(g) The south border extension.

Figure 6: The tiles that are used to start the Turing machine simulation.

Lemma 2.1. *For any given (deterministic) Turing machine $\mathcal{M}$, the tile set $T_{\mathcal{M}}$ is both NE- and SW-deterministic.*

Proof. The tile set is SW-deterministic, since clearly it has no two tiles having same colors on the south sides and the west sides.

Similarly, the tile set is NE-deterministic. No two tiles in figures 2, 3, 4 and 5 have the same colors on the north side and the east side. □

Theorem 2.2. *The following question is undecidable: "Given a Turing machine $\mathcal{M}$, does the tile set $T_{\mathcal{M}}$ admit a valid tiling of the plane containing the tile in figure 6(c)?"*

Proof. The tile set $T_{\mathcal{M}}$ quite obviously corresponds the actions and configurations of the given Turing machine $\mathcal{M}$. Requiring the seed tile to be the tile in figure 6(c), the structure in figure 8 is forced to be tiled on the plane.

The structure in figure 8 obviously corresponds the initial configuration with a blank tape. Therefore, the plane can be tiled correctly if, and only if, the given Turing machine does not halt (when started on a blank tape). Of course, the halting problem with a blank tape is undecidable. □

Since the tiling problem with a seed tile is a generalization of the problem in theorem 2.2, corollary 2.2.1 follows.

Corollary 2.2.1. *The tiling problem with a seed tile is undecidable for tile sets that are both NE-deterministic and SW-deterministic.*

Moreover, the tile set $T_{\mathcal{M}}$ would be NE-deterministic even if the Turing machine $\mathcal{M}$ was nondeterministic. No matter what the state q and symbol a are, the tiles in figures 2, 3 and 4 are uniquely defined by the colors of their north and east sides. Hence, lemma 2.3 follows.

Lemma 2.3. *For any given nondeterministic Turing machine $\mathcal{M}$, the tile set $T_{\mathcal{M}}$ is NE-deterministic.*

A NE- and SW-deterministic tile set can be constructed even for any nondeterministic Turing machine. This tile set is constructed by modifying the tile set $T_{\mathcal{M}}$.

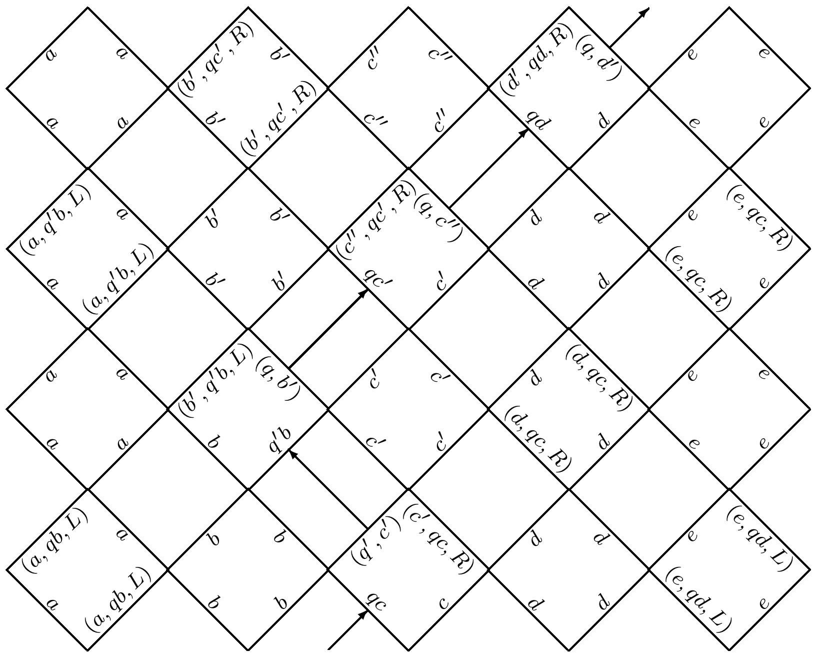


Figure 7: Rewrite operation $abqcd \vdash aq'bc'de \vdash ab'qc'de \vdash ab'c''qde$. For clarity, the tiles in figure 3 are represented by arrows and the tiles in figure 5 on every second row are represented by blanks.

Modification is based on using signals containing information about the particular move that was chosen. These signals are referred to as *decision signals*. The tile in figure 2 is modified so, that it sends a decision signal to the left and backwards in time (i.e. towards west since the computation advances towards northeast). Likewise, the tile in figure 4 is modified to send a decision signal to the right and backwards in time (i.e. towards south). Furthermore, the tiles in figures 5 and 6 are modified to allow crossings with any kinds of decision signals. This new modified tile set is referred to as the tile set $T_{\mathcal{M}}^{N}$ (where N stands for non-determinism). It is quite straightforward to see, that the tile set $T_{\mathcal{M}}^{N}$ is indeed both NE- and SW-deterministic. By using decision signals and lemma 2.3, lemma 2.4 follows.

Lemma 2.4. *For any given nondeterministic Turing machine $\mathcal{M}$, the tile set $T_{\mathcal{M}}^{N}$ is both NE-deterministic and SW-deterministic.*

3 The tiling problem without a seed tile

In this section the tiling problem without a seed tile is shown to be undecidable even for those tile sets that are deterministic by two opposite corners. The argumentation is quite similar to that of earlier proofs [4, 7]. The only difference is the requirement that the final tile set must be both NE- and SW-deterministic.

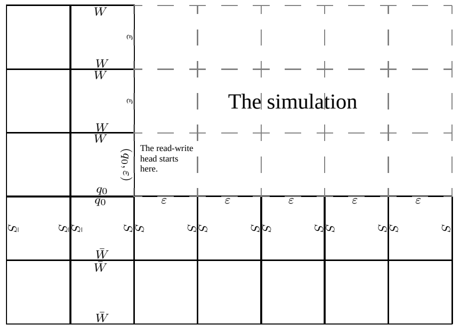


Figure 8: Using the tiles in figure 6 to start the Turing machine simulation on a blank tape.

3.1 A brief outline of the argumentation

The proof for undecidability is similar to that of Kari [4]. The difference is that the tile set construction is more complicated due to the additional requirements of determinism.

The reduction is done from the tiling problem with a seed tile to the tiling problem (when restricted to the instances that are both NE- and SW-deterministic, or course). That is, if there was an algorithm for solving the tiling problem, then there would be also an algorithm for solving the tiling problem with a seed tile.

The idea of the reduction is to construct a more complicated tile set according to the original tile set. For the new tile set, the answer for the tiling problem will be affirmative if, and only if, the answer for the tiling problem with a seed tile is affirmative for the original given tile set and the given seed tile.

The new tile set is such, that on a valid tiling certain areas are used to simulate a tiling with the original tile set. These areas are referred to as free rows and free columns. Identifying the free areas in the earlier case [4] required some modifications to the original proof of Robinson [7]. Now the tile set construction for identifying the free areas is somewhat more complicated.

The construction of the new tile set relies heavily on the use of an aperiodic tile set. By using the square patterns generated by Robinson's tile set, copies of the seed tile are forced to be located at certain points of the plane.

In [4] Kari modified Robinson's tile set resulting a new aperiodic tile set which is deterministic by one corner. Later in [5], Kari and Papasoglu presented an aperiodic 4-way deterministic tile set which can be mapped homomorphically onto the original Robinson's tile set. This 4-way deterministic tile set will be used in the proof instead of Robinson's aperiodic tile set.

The new tile set is constructed in six layers for the given tile set T and a seed tile $t \in T$. The rough outline of the layers is the following:

Layer 1. The tiling forced by the aperiodic tile set of Kari and Papasoglu.

Layer 2. The tiles to identify free areas.

Layer 3. A tiling simulating a tiling by the given tile set T.

Layer 4. The tiles to force a copy of the seed tile $t \in T$ to be located at the center of every red square of layer 1

Layer 5. The tiles to forward the colors of the tile set T at the layer 3 from a free area border to a free area border.

Layer 6. The tiles to forward the colors of the tile set T at the layer 3 from a red border to a red border.

Theorem 3.1. *The tiling problem is undecidable for tile sets that are both NE-deterministic and SW-deterministic.*

Proof (sketch). Subsection 3.2: It is possible to divide the plane into squares of increasing size using the tile set of Kari and Papasoglu. The squares are colored either red or blue. No borders of two squares of the same color can coincide.

Subsection 3.3: Each of the red squares contains free areas that are not between any of the smaller red squares. The free areas can be recognized NE- and SW-deterministically.

Subsection 3.4: A finite area of a tiling by the original tile set is simulated on the free areas within the red squares. The size of the free area inside a red square square is directly proportional to the size of the red square.

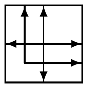
Subsection 3.5: One copy of the seed tile can be forced to be located at the center of the simulation area with a 4-way deterministic construction. Therefore, a tiling by the original tile set is forced to be simulated at arbitrarily long distances from the seed tile to any direction.

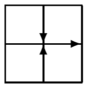
Subsection 3.6: The area consisting of disjoint free areas can be considered as a single continuous square. This is seen by transferring the colors between the free areas using a 4-way deterministic construction (except for the non-4-way determinism caused by layer 2).

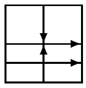
Subsection 3.7: The plane is tiled correctly if, and only if, on every red square the free area is tiled correctly using the original tile set. Any valid tiling by the original tile set can be simulated using the new tile set without a tiling error. This is seen by transferring the outermost colors between the red squares using a 4-way deterministic construction (again, except for the non-4-way determinism caused by layer 2). □

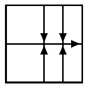
3.2 The aperiodic tile set (layer 1)

A general outline of Robinson's tile set is shown in figure 9. The tile set consists of tiles that are called *crosses* and tiles that are called *arms* as shown in the figure. The colors of the tiles are defined using patterns consisting of *single arrows* and *double arrows*. The arrows are colored either red or blue. In a cross tile all the arrows are of the same color and in an arm tile the intersecting arrows are of different color. Two tiles are considered to match at their abutting sides if an arrow (of some particular type) exiting one of the tiles enters the other tile. Robinson's tile set has also some parity constraints that are not shown in the tiles in figure 9. A more thorough description of Robinson's tile set can be found (naturally) in [7].


(a) A cross.


(b) An arm.


(c) An arm.


(d) An arm.

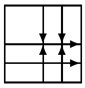

(e) An arm.

Figure 9: The basic tiles of Robinson's tile set (with colors, reflections, rotations and parity constraints omitted).

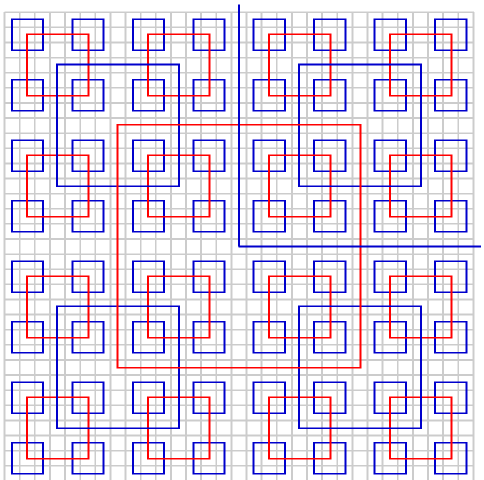


Figure 10: A part of the self-similar pattern generated by the tile set of Robinson (and the tile set of Kari and Papasoglu).

Robinson's tile set forces a self-similar pattern to be tiled, a part of which is shown in figure 10. The tiling forced by the Robinson's tile set is divided into square areas bounded by blue squares or red squares. More specifically, the tiling contains blue squares of height $2^{2n+1} + 1$ and red squares of height $2^{2n} + 1$, for every integer n. Furthermore, the borders of the squares of the same color never coincide. In the center of the area bounded by a blue square there is always some corner of a red square, and likewise in the center of the area bounded by a red square there is always some corner of a blue square.

Kari and Papasoglu have constructed a 4-way deterministic tile set which will be used in this article instead of Robinson's tile set. Formally, the following theorem holds:

Theorem 3.2 (Kari and Papasoglu [5]). *There exists a 4-way deterministic tile set, which*

1. *admits a valid tiling and*

2. *can be mapped homomorphically onto Robinson's tile set.*

An implication of property 2 is the aperiodicity of the tile set. However, the exact structure of the tile set of Kari and Papasoglu is irrelevant. It is sufficient to know that there exists a NE- and SW-deterministic tile set, which can be mapped homomorphically onto Robinson's tile set. In this sense, the tile set of Kari and Papasoglu is more than enough, since it is not only NE- and SW-deterministic, but even 4-way deterministic. Formally, the tile set of and Kari and Papasoglu will be used as layer 1 of the final set of sandwich tiles to remove the requirement of the seed tile.

The aperiodic tile set will be used to admit only such a tiling, that the seed tile is contained in it infinitely many times. Moreover, every instance of the seed tile will be associated with a specific area of the tiling.

3.3 Identifying the free areas (layer 2)

A tile within a red square is said to be located on a *free column* of the red square, if there are no smaller red squares above or below it within the red square. Likewise, a tile within a red square is said to be located on a *free row* of the red square, if there are no smaller red squares within the red square to the right or to the left from its position. A tile within a red square is said to be *free*, if it is located on both a free row and a free column. For example, the free tiles of a $(2^6+1)\times(2^6+1)$ red square are shown in figure 11.

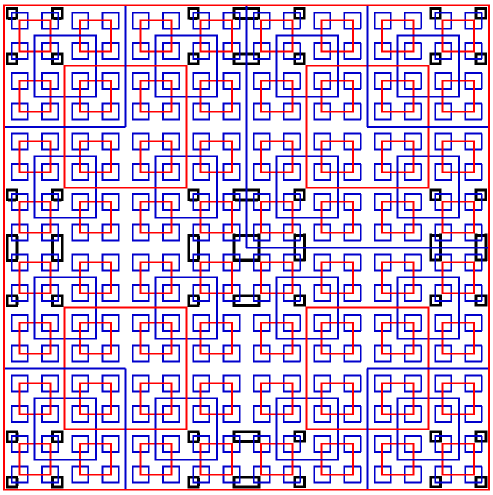


Figure 11: The free area of 9×9 squares within a red square spanning 65×65 squares.

An *identification diagonal* is a diagonal signal advancing from southwest to northeast. If it encounters the lower left corner of a red square, it splits into two *component* signals that move along the borders of this particular red square. As the component signals meet at the upper right corner of a red square, the component signals are merged to form again the diagonal signal. The identification diagonals are used to identify the largest red squares that are centered on the northeast-southwest diagonal between two predefined points. An identification diagonal is shown in figure 12. If the identification diagonal encounters the lower left corner (and then also on the upper right corner) of a red square, then it surrounds the red square with component signals as shown in figure 12.

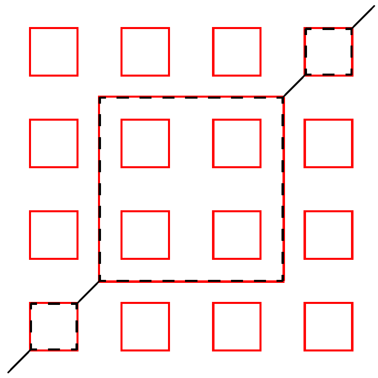


Figure 12: An identification diagonal is used to identify the red squares that are located on a specific northeast-southwest diagonal.

If the end points can be identified using some construction, finite diagonal signals in the northeast-southwest direction can be drawn with a tile set that is both NE- and SW-deterministic. Likewise, the diagonal signals can be split into separate signals (at a specific point) and these signals can be later merged (at a specific point) to form a diagonal signal.

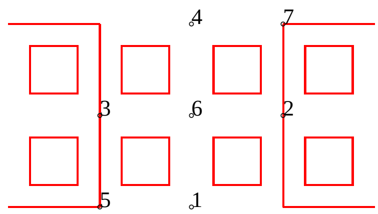


Figure 13: The end points of border diagonals. All the tiles representing the points can be identified 4-way deterministically.

To identify the free areas within the given red square, a set of eight identification diagonals will be drawn in the northeast-southwest direction around the square. Two diagonals will be drawn for each side of the square. These identification diagonals are referred to as *border diagonals* of that particular red square. The end points for the border diagonals are shown in figure 13. In figure 13 it is shown, which specific points of the tiling forced by Robinson's tile set will be used as end points for the border diagonals. The end points correspond the following tiles (enumerated as in figure 13) of Robinson's tile set:

1. The meeting point of red single arrows of the south borders of two red squares.

2. The meeting point of red double arrows of the west border of a red square.

3. The meeting point of red double arrows of the east border of a red square.

4. The meeting point of red single arrows of the north borders of two red squares.

5. The lower right corner of a red square.

6. The meeting point (single or double) horizontal blue arrows.

7. The upper left corner of a red square.

The border diagonals that are drawn between the tiles of types 1 and 2 or 6 and 7 are referred to as *west* border diagonals. Likewise, the border diagonals that are drawn between the tiles of types 3 and 4 or 5 and 6 will be referred to as *east* border diagonals. In a similar way, it is possible to define *north* border diagonals and *south* border diagonals to be located vertically between two red squares.

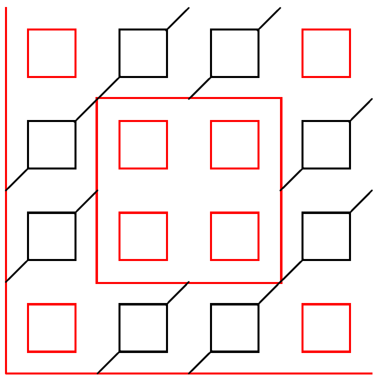
To determine the boundaries of the free areas, signals called *border signals* are drawn between two border diagonals. The border signals act as the borders of the free areas. In short, the border signals are horizontal and vertical signals drawn between some of the corners of the red squares centered at border diagonals. The border diagonals are used to locate the tiles between which the border signals can be drawn to maintain both NE- and SW-determinism.

West border diagonals determine (at the east corners of a red square) the left end points of the horizontal border signals and east border diagonals determine (at the west corners of a red square) the right end points of the horizontal border signals. In a similar way, north border diagonals determine the upper end points points of the vertical border signals and south border diagonals determine the lower end points of the vertical border signals. Figure 14(a) shows the border diagonals for a $(4^2 + 1) \times (4^2 + 1)$ red square. Figure 14(b) shows the border diagonals and the border signals of a $(4^2 + 1) \times (4^2 + 1)$ red square.

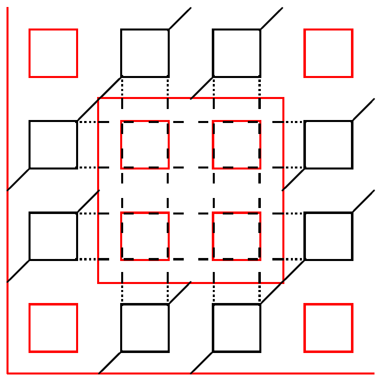
A border signal is defined to be in two different states, either *active* or *inactive*. Only the border signals in the active state are considered to be borders of the free areas. As a border signal is started on a border diagonal, it is in the inactive state. As the signal enters a red square, the state is changed from inactive to active, and as it exists a red square, the state is changed from active to inactive. As the border signal stops at a border diagonal on the other side of the red square, it is again in the inactive state.

The border signals are started on the corners of a red square if a component signal of a particular border diagonal is met at the corner. Horizontal border signals are drawn between the east corners of the red squares at west border diagonals and the west corners of the red squares at east border diagonals. Similarly, vertical border signals are drawn between the south corners of the red squares at north border diagonals and the north corners of the red squares at south border diagonals.

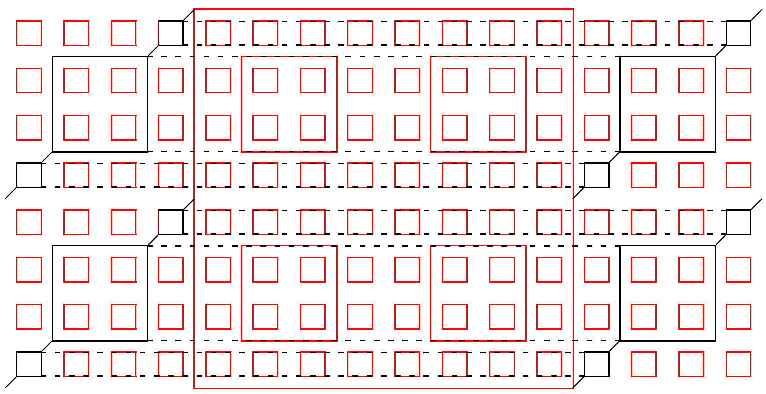
Figure 14(a) shows the border diagonals of a $(4^2+1)\times(4^2+1)$ red square. Figure 14(b) shows the border diagonals along with inactive border signals (dotted lines) and active border signals (dashed lines). The outcome of drawing the east border diagonals and the west borders diagonals for a $(4^3+1)\times(4^3+1)$ red square is shown in figure 14(c). Both inactive and active border signals are drawn with dashed lines in figure 14(c).



(a) The border diagonals of a $(4^2+1)\times(4^2+1)$ red square.



(b) The border diagonals and the border signals of a $(4^2+1)\times(4^2+1)$ red square.



(c) The horizontal border diagonals and the horizontal border signals of a $(4^3+1)\times(4^3+1)$ red square. The border diagonals and signals of the smaller squares are omitted in the figure.

Figure 14: Border diagonals are used to set up end points for border signals.

Theorem 3.3. *The free areas of the tiling forced by Robinson's tile set can be identified simultaneously NE- and SW-deterministically.*

Proof. The end point for the border diagonals can be determined 4-way deterministically using the tile set of Kari and Papasoglu. The tiles that act as the end points are the preimages of the tiles enumerated in figure 13. The tile acting as the end point determines uniquely of which type ("north", "east", "west" or "south") the border diagonal will be. Depending on the type, the border diagonal will draw the border signals to one of the four directions.

There is no ambiguity between the tiles for the end points and the tiles for the middle points of a diagonal, since both sets of tiles are paired with a different, mutually exclusive subset of the (4-way deterministic) aperiodic tile set.

Furthermore, the non-periodic tiling remains valid even if both the border diagonals and the border signals are drawn. □

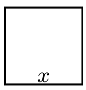
3.4 Simulating the original tile set (layer 3)

Assume that the given instance for the tiling problem with a seed tile is the tile set T and the seed tile t. The goal is to construct a tile set for which the tiling problem has an affirmative answer if, and only if, the given instance of the tiling problem with a seed tile has an affirmative answer.

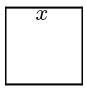
A tiling by the original tile set T is simulated within all the red squares. However, since larger red squares contain smaller red squares, the simulation area cannot be the entire square itself. Instead, the simulation corresponding the particular red square is done on the free areas.

Lemma 3.4 (Robinson [7]). *For every* $(4^n+1)\times(4^n+1)$ *red square, the number of free columns is* 2^n+1 *and the number of free rows is* 2^n+1.

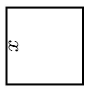
Lemma 3.4 states that the free area within a red square increases with respect to the size of the square. Hence, the tiling by the original tiles (on layer 3) can be arbitrarily large even when restricted to the free rows and free columns. Hence, it is enough to restrict the simulation only to free rows and free columns.



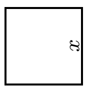
(a) The tiles on red north borders.



(b) The tile on red south borders.



(c) The tiles on red east borders.



(d) The tile on red west borders.

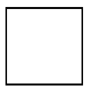
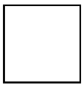
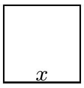

(e) The blank tile on the corners of red squares and non-free areas.

Figure 15: The tiles to restrict the simulation to the inside of a red square. Symbol x denotes an arbitrary color of the tile set T.

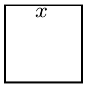
There are four simulations on the smaller red squares and one longer simulation on the free rows and columns of the 9×9 red square itself. From the view of the simulation, the disjoint free rows and columns are not considered to be disjoint at all. For example, the area on both the free rows and the free columns of the 9×9 square is considered as a single continuous 5×5 square. Likewise, the area on both the free rows and the free columns of a 65×65 square is considered as a single continuous 9×9 square. It will be shown (in subsection 3.6), that it is possible to consider the area consisting of the free tiles as a single continuous square.



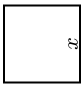
(a) The tile on non-free areas.



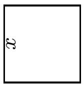
(b) The tiles on north border signals.



(c) The tiles on south border signals.



(d) The tiles on east border signals.



(e) The tiles on west border signals.

Figure 16: The tiles to restrict the simulation to free areas. Symbol x denotes an arbitrary color of the tile set T.

3.5 Forcing the seed tile presence (layer 4)

The seed tile is forced to be contained in the tiling using the construction that is shown in figure 17. The tiles, that are located in the middle of a red border, are used to start signals towards the center of the red square. That is, all the meeting points of red double arrows launch a signal towards the center of the square to which they belong. The tile, on which these signals meet, is paired only with the seed tile $t \in T$ at the layer 3. Hence, the seed tile presence can be forced with a 4-way deterministic construction.

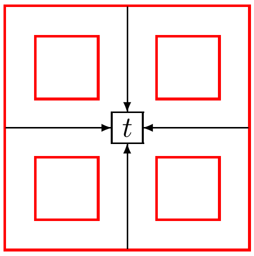
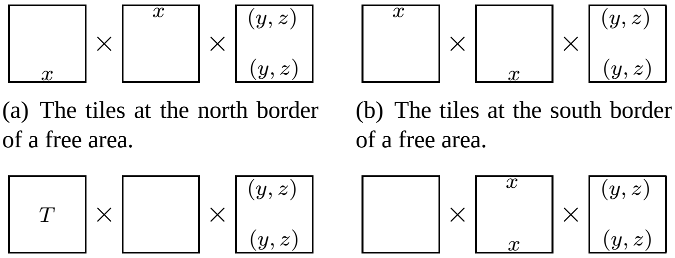


Figure 17: Forcing the seed tile t to be tiled in the center of a red square.

3.6 Joining the free areas (layer 5)

In figure 18 it is shown, how the tiles of layers 3, 5 and 6 are paired to form sandwich tiles for the free areas, the non-free areas and the active border signals of the given red square. In figure 18 symbol x denotes a color (of the given tile set T) encountered at the border of a free area (i.e. on an active border signal) and symbols y and z denote arbitrary colors. For example, at the north border of a free area the last color on layer 3 is erased and raised onto layer 5 to be transferred northwards. At the southern boundary of another free area color x is lowered from layer 5 back to layer 3. The sandwich tiles that are used to transfer colors horizontally between free areas are constructed in a similar way.



(a) The tiles at the north border of a free area.

(b) The tiles at the south border of a free area.

(c) The tiles on free areas.

(d) The tiles on non-free areas.

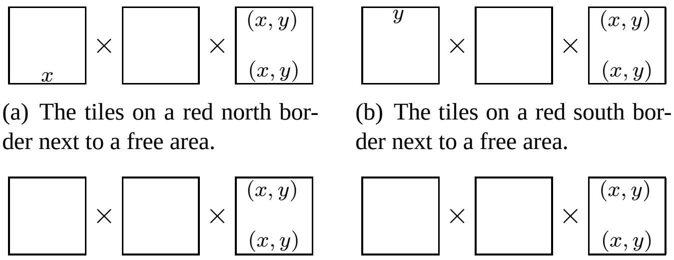
Figure 18: The tile construction at layers 3, 5 and 6 to transfer vertical colors between the free areas inside a red square. Symbols x, y and z denote arbitrary colors of the given tile set T.

The tiles of form 18(a) are used on the northern boundaries of free areas. Likewise, the tiles of form 18(b) are used on the southern boundaries of free areas. On a free area, the sandwich tiles are formed from the tiles of set T according to figure 18(c). On a non-free area, the sandwich tiles are formed according to figure 18(d).

The tiles in figure 18 form a 4-way deterministic tile set when paired properly with the tiles of Kari and Papasoglu. Therefore, the free areas inside inside the given red square can be considered as a continuous square area while maintaining NE- and SW-determinism.

3.7 Allowing arbitrary colors on the red borders (layer 6)

To ensure NE- and SW-determinism of the tile set, the colors of the simulation tiles next to the red borders will be forwarded. In the vertical direction this is done by using the tiles in figure 19.



(a) The tiles on a red north border next to a free area.

(b) The tiles on a red south border next to a free area.

(c) The tiles on a red north border next to a non-free area.

(d) The tiles on a red south border next to a non-free area.

Figure 19: The tile construction at layers 3, 5 and 6 to transfer the uppermost colors on the simulation area of the given red square to the next simulation area of a red square of the same size. Symbols x and y denote arbitrary colors of the given tile set T.

It is shown in figure 19, how the tiles of layers 3, 5 and 6 are paired to form sandwich tiles for the horizontal red borders. Construction for the vertical borders is the same with the exception that the colors are transferred horizontally. In short, at the north and east borders of the red squares the outermost colors are transferred from layer 3 to layer 6.

The parts of the red borders that are located on free rows or columns can be distinguished from the other parts using a NE- and SW-deterministic construction. This can be seen by sending a parity signal from the red crosses along the red double arrows (i.e. the borders of the square). At a point where a border signal enters the square, the parity of the signal is changed to the opposite. If the parity of the signal is initially, say, odd, then every odd run identifies a location of a free row (or column). Hence, it is possible to decide NE- and SW-deterministically, whether a particular location at a red border belongs to a free row or a column. The use of parity signal is in itself a 4-way deterministic construction, but the border signal construction is only NE- and SW-deterministic.

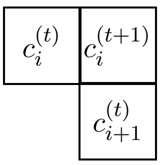
It can be seen straightaway, that the tiles in figure 19 form a 4-way deterministic tile set when restricted to their particular locations. Likewise, there is no ambiguity between the tiles in figure 18 and the tiles in figure 19, since (by definition) the tiles are paired with different tiles of the aperiodic tile set. Hence, the colors next to the right borders can be arbitrary while the final sandwich tile set remains both NE- and SW-deterministic.

4 Corollaries on reversible cellular automata

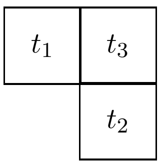
4.1 A tile set as a cellular automaton

Following the presentation of Kari [4], it is possible to regard Wang tile sets (that are deterministic at least in one direction) as one-dimensional cellular automata.

If the given tile set is SW-deterministic, it is possible to consider the tiles as states of a cellular automaton. As shown in figure 20, with a one-dimensional cellular automaton (with neighborhood $\{0, 1\}$) the next state of a cell is determined with a similar procedure as the next tile (to the northeast) in a tiling with a SW-deterministic tile set. With a cellular automaton the new state depends on the old states and in a tiling (with a SW-deterministic tile set) the new tile is determined by the colors of its neighbors.



(a) Cells c_i and c_{i+1} determine the next state of the cell c_i.



(b) Tiles t_1 and t_2 determine tile t_3 SW-deterministically.

Figure 20: The tiles of a NE-deterministic tile set can be considered as states of a cellular automaton.

It should be noted, that the given Wang tile set may not contain all the possible color pairs in the southwest corners of the tiles. If the given tile set T is assumed to be deterministic in only one direction, say, by the southwest corner, it is enough to add a tile to the original tile set for every missing southwest corner color pair. For example, if there is no tile t with $W_t = x$ and $S_t = y$ in the given tile set T, a tile t with $N_t = E_t = z$, $W_t = x$ and $S_t = y$, where z is any color of the tile set T, could be added to the tile set while maintaining SW-determinism.

If the given tile set is assumed to be both NE- and SW-deterministic, equally many colors are missing as northeast corner color pairs. It is trivial to construct (for example, by some ordering method) a one-to-one correspondence between the missing colors in the southwest corners and the missing colors of the northeast corners. This bijection can clearly be considered as a NE- and SW-deterministic set of tiles. Moreover, the union of the initial tile set and this new tile set is both NE- and SW-deterministic tile set containing N^2 tiles, where N is the number of colors in the original tile set.

One of the NE- and SW-deterministic tile sets, in which occur only all the southwest corner color pairs and northeast corner color pairs missing in tile set T, is denoted by expression C_T. Now it is straightforward to see, that all the tile sets $T \cup C_T$ can be considered as cellular automata. It also follows, that the answer for the tiling problem is affirmative for any tile set $T \cup C_T$.

For the NE- and SW-deterministic tile set T, a reversible cellular automaton $\mathcal{A}_T = (1, T \cup C_T, \{0, 1\}, f_T)$ can be defined. The local rule of $\mathcal{A}_T$ is defined as

$$f_T(x, y) = z \quad \text{if } x, y \in T \cup C_T,\ E_x = W_z \text{ and } N_y = S_z.$$

The function $f_T : (T \cup C_T)^2 \to T \cup C_T$ is total and well-defined, since the tile set $T \cup C_T$ is both NE- and SW-deterministic. Expression $G_f(\cdot)$ is used to denote the global function of a cellular automaton with the local rule f.

Theorem 4.1. *Given a reversible cellular automaton $\mathcal{A}_T$, the following question is undecidable: "Does there exist such a configuration c, that $G^i_{f_T}(c)_j \in T$, for all integers $i, j \in \mathbb{Z}$?"*

Proof. Undecidability of the question follows by a reduction from the problem of theorem 3.1.

Assume first, that the given tile set T admits a valid tiling. Then one can choose any northwest-southeast diagonal row of tiles of the valid tiling to be the configuration c. Since the tile set is NE-deterministic and "diagonal" c is part of a valid tiling, $G^i_{f_T}(c)_j \in T$, for all integers $i, j \in \mathbb{Z}$.

Assume second, that the given tile set T does not admit a valid tiling. If for some configuration c (considered again as a northwest-southeast diagonal row of a valid tiling) the condition did hold, then it would be possible to construct a valid tiling. However, this contradicts the assumption.

Hence, a configuration c exists if, and only if, the tile set T admits a valid tiling. □

Corollary 4.1.1. *Let $\mathcal{C}$ be a reversible cellular automaton and set T be a subset of the state set. Then the following question is undecidable: "Does there exist such a configuration c, that $G^i_f(c)_j \in T$, for all integers $i, j \in \mathbb{Z}$?"*

4.2 Universality of reversible cellular automata

It has been shown by Morita and Harao that one-dimensional reversible cellular automata are computationally universal [6]. More precisely, they have shown that any reversible Turing machine can be simulated with some reversible one-dimensional cellular automaton. Since any Turing machine can be simulated with a reversible Turing machine [1], the universality of one-dimensional reversible cellular automata follows.

However, the requirement of reversibility (made in [6]) for the given Turing machine is not necessary for the machine to be simulated with a reversible one-dimensional cellular automaton. In fact, Dubacq has given a construction for a family of reversible cellular automata to simulate any (irreversible) Turing machine in real time [3]. Dubacq's approach was more from the cellular automata point of view. The construction of the family of tile sets $M_{\mathcal{M}} \cup S_{\mathcal{M}}$ gives a different proof for Dubacq's result.

The elements of set $M_{\mathcal{M}} \cup S_{\mathcal{M}}$ can be used to represent a cellular automaton. For example, the colors adjacent to southwest corners of the tiles can be considered as color pairs representing the states of a two-partitioned cellular automaton. Since the tile set $M_{\mathcal{M}} \cup S_{\mathcal{M}}$ is both NE- and SW-deterministic, the tiling procedure can be modelled with the (reversible) local rule of a cellular automaton. The initial configuration of the Turing machine is represented with the tiles of the form shown in figure 5(a). The cellular automaton computes one Turing machine computation step in two of its of computation steps.

Theorem 4.2 (J.-C. Dubacq, 1995 [3]). *Any (deterministic) Turing machine can be simulated using a reversible one-dimensional cellular automaton in real time.*

Proof. The given Turing machine $\mathcal{M}$ can be simulated with the cellular automaton $\mathcal{A}_{M_{\mathcal{M}} \cup S_{\mathcal{M}}}$. However, for every computation step of the Turing machine $\mathcal{M}$ the cellular automaton $\mathcal{A}_{M_{\mathcal{M}} \cup S_{\mathcal{M}}}$ needs to conduct two computation steps. To simulate a given Turing machine in real time, a new family of cellular automata is introduced.

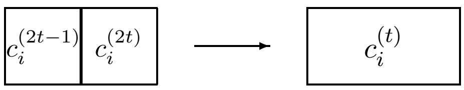


Figure 21: Modifying the cell structure of the cellular automata $\mathcal{A}_T$ to simulate Turing machines in real time.

The new family of cellular automata is constructed by modifying the cell structure of the cellular automata $\mathcal{A}_T$ as shown in figure 21. That is, the computation steps are divided by time into even and odd computation steps. Regrouping the cells of the even and odd computation steps to form "larger" cells (as shown in figure 21), one has a cellular automaton which simulates the given Turing machine in real time. □

Corollary 4.2.1 (K. Morita and M. Harao, 1989 [6]). *Reversible one-dimensional cellular automata are computationally universal.*

5 Conclusions

It was noted that the tiling problem is undecidable even if the tile set was deterministic by two opposite corners. The proof used the aperiodic tile set of Kari and Papasoglu [5].

Open problem: Is the tiling problem undecidable for 4-way deterministic tile sets?

6 Acknowledgements

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