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Abstract

We give two recursive expressions for both MacWilliams and Chebyshev matrices. The expressions give rise to simple recursive algorithms for constructing the matrices. In order to derive the second recursion for the Chebyshev matrices we find out the Krawtchouk coefficients of the Discrete Chebyshev polynomials, a task interesting on its own.

Keywords: Orthogonal polynomials, Krawtchouk polynomials, Discrete Chebyshev polynomials, MacWilliams transform, MacWilliams matrices, Chebyshev matrices, Recursion, Krawtchouk coefficients

1 Introduction

The Krawtchouk polynomials play an important role in the theory of error-correcting codes [2]. The importance of the Krawtchouk polynomials emerges from the fact that the weight spectrum of the characters of $\mathbb{F}_2^N$ consists of the values of the Krawtchouk polynomials, and consequently the weight spectrum of a function and that of its Hadamard transform are connected via the values of Krawtchouk polynomials (see the next section for the definitions of notions mentioned in this section).

The Krawtchouk polynomials are also interesting from a geometric viewpoint. They are an example of *orthogonal polynomials*, and so are the Discrete Chebyshev polynomials (see [7] for a general treatise on orthogonal polynomials). For the purposes of this article it is sufficient to treat inner products with discrete weight functions, and hence we consider the vector space $\mathcal{P}_N$ of polynomials having degree at most N as a general reference frame when speaking about the orthogonal polynomials. The sum and the scalar product in $\mathcal{P}_N$ are defined pointwise, and the inner product is defined as

$$\langle p, q \rangle_w = \sum_{i=0}^{N} w_i p(i) q(i).$$

The Krawtchouk polynomials $K_0^{(N)}$, $K_1^{(N)}$, …, $K_N^{(N)}$ (of order N) are orthogonal with respect to weight function $w_i = \binom{N}{i}$ and the discrete Chebyshev polynomials $D_0^{(N)}$, $D_1^{(N)}$, …, $D_N^{(N)}$ of order N with respect to weight function $w_i = 1$ for each i (see [7]).

As orthogonal polynomials, the Krawtchouk polynomials (and the discrete Chebyshev polynomials) form a basis of $\mathcal{P}_N$, and hence any polynomial p of degree at most N can be uniquely represented as

$$p = c_0 K_0^{(N)} + c_1 K_1^{(N)} + \ldots + c_N K_N^{(N)}, \tag{1}$$

where $c_i \in \mathbb{C}$, and a similar representation

$$p = d_0 D_0^{(N)} + d_1 D_1^{(N)} + \ldots + d_N D_N^{(N)} \tag{2}$$

can be found for the discrete Chebyshev polynomials. Coefficients c_i in (1) are called the *Krawtchouk coefficients* of p, and the coefficients d_i are the *discrete Chebyshev coefficients*. Since the discrete Chebyshev polynomials are orthogonal with respect to constant weight function, they have the following property important in the approximation theory: With respect to norm $||p - q||^2 = \sum_{i=0}^{N}(p(i) - q(i))^2$, the best approximation of p in $\mathcal{P}_M$ can be found by simply taking $M + 1$ first summands of (2) (see [6], for instance).

For a fixed N, it is sometimes interesting and useful to include all the values $K_i^{(N)}(j)$ (or $D_i^{(N)}(j)$) in the same study, and this leads naturally to a matrix formalism. We call $(N+1) \times (N+1)$-matrices $\mathbf{M}_N = (K_i^{(N)}(j))$ and $\mathbf{D}_N = (D_i^{(N)}(j))$ *MacWilliams* and *Chebyshev* matrices, respectively. In this article, we give two recurrence relations useful for computing both $\mathbf{M}_N$ and $\mathbf{D}_N$. To do so, we derive

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recursion formulas relating the values of Nth order Krawtchouk (resp. discrete Chebyshev) polynomials to the values of $(N-1)$th order Krawtchouk (resp. discrete Chebyshev) polynomials. This kind of recurrences seem to appear very infrequently in the pure theoretical literature, whereas the recurrences relating $K_n^{(N)}$ (resp. $D_n^{(N)}$) to $K_{n-1}^{(N)}$ and $K_{n-2}^{(N)}$ (resp. $D_{n-1}^{(N)}$ and $D_{n-2}^{(N)}$) are well-known [7].

Also, to derive the latter recurrence relation for $\mathbf{D}_N$, we compute the Krawtchouk coefficients of the discrete Chebyshev polynomials. A previous occurrence of the Krawtchouk coefficients of the discrete Chebyshev polynomials in the literature is not known to the authors.

2 Preliminaries

2.1 Notations and Basic Definitions

Notation $\mathbf{1}$ stands for the identity matrix, and $\mathbf{O}$ means the zero matrix, whereas we use $\mathbf{0}$ to denote the zero (column) vector. The dimensions of $\mathbf{1}$, $\mathbf{O}$, and $\mathbf{0}$ will be clear by the context. By $\mathrm{diag}(c_1, \ldots, c_n)$ we understand $n \times n$ diagonal matrix with c_1, ..., c_n as diagonal entries. We use expressions $\underset{1,\ldots,N-1}{\mathrm{Rows}}(\mathbf{A}) = \mathbf{B}$ and $\mathrm{Row}_N(\mathbf{A}) = \boldsymbol{c}$ to define an $N \times N$ matrix $\mathbf{A}$ in two stages: the rows 1, ..., $N-1$ of $\mathbf{A}$ consists of an $(N-1) \times N$-matrix $\mathbf{B}$, and the last row of $\mathbf{A}$ consists of an N-dimensional row vector $\boldsymbol{c}$.

For any matrix $\mathbf{A}$, $\overline{|\mathbf{A}}$ stands for the zero-padding of $\mathbf{A}$ from its left and upper sides, i.e,

$$\overline{|\mathbf{A}} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{A} \end{pmatrix},$$

and the notations $\underline{\mathbf{A}}|$, $\overline{\mathbf{A}}|$, $\underline{|\mathbf{A}}$, $|\mathbf{A}$ and $\mathbf{A}|$ have an analogous meaning.

For $N \geq 1$ let $\mathbb{F}_2^N$ be an N-dimensional vector space over the binary field $\mathbb{F}_2 = \{0,1\}$ and let V_N be the 2^N-dimensional complex group algebra of all functions $f : \mathbb{F}_2^N \to \mathbb{C}$ with the usually defined multiplication (*dyadic convolution*)

$$(f * g)(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{F}_2^N} f(\boldsymbol{y}) g(\boldsymbol{x} + \boldsymbol{y}), \quad f, g \in V_n, \ \boldsymbol{x} \in \mathbb{F}_2^N \tag{3}$$

equipped with the inner product (also called the scalar product)

$$(f * g)(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{F}_2^N} f(\boldsymbol{y})^* g(\boldsymbol{x}), \tag{4}$$

where c^* stands for the complex conjugate of $c \in \mathbb{C}$.

Every *character* of algebra V_n can be written in a form

$$\chi_{\boldsymbol{y}}(\boldsymbol{x}) = \chi_{\boldsymbol{x}}(\boldsymbol{y}) = (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}}, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_2^N, \tag{5}$$

where $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \ldots + x_N y_N$. When useful, we interpret $\boldsymbol{x}$ and $\boldsymbol{y}$ as subsets of indices $\{1, 2, \ldots, N\}$ in the obvious way. Interpreting so, clearly

$$\boldsymbol{x} \cdot \boldsymbol{y} = |\boldsymbol{x} \cap \boldsymbol{y}| \tag{6}$$

where $|\cdots|$ denotes the cardinality. In accordance to this interpretation, we often use notation $|\boldsymbol{x}| = \mathrm{wt}(\boldsymbol{x})$ for the Hamming weight [2] of $\boldsymbol{x} \in \mathbb{F}_2^N$ if there is no danger of confusion. Note also that in this interpretation $\boldsymbol{x} + \boldsymbol{y}$ becomes the symmetric difference of subsets $\boldsymbol{x}$ and $\boldsymbol{y}$. When $\boldsymbol{x} \in \mathbb{F}_2^N$ is interpreted as a subset of $\{1, 2, \ldots, N\}$, we also use notation $\overline{\boldsymbol{x}}$ for the *complement* of $\boldsymbol{x}$, i.e., $\overline{\boldsymbol{x}}_i = 1$ if $\boldsymbol{x}_i = 0$ and vice versa.

It is easy to verify that

$$\langle \chi_{\boldsymbol{x}}, \chi_{\boldsymbol{y}} \rangle = 2^N T_{\boldsymbol{y}}(\boldsymbol{x}), \ \text{ where } T_{\boldsymbol{y}}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} = \boldsymbol{y} \\ 0, & \text{if } \boldsymbol{x} \neq \boldsymbol{y}, \end{cases} \tag{7}$$

and it follows that the characters form an orthogonal basis of V_N. For any $f \in V_N$ we define the *(discrete) Fourier transform* $\mathcal{F}(f) = \widehat{f} \in V_N$ (also called *Hadamard transform*) as

$$\mathcal{F}(f)(\boldsymbol{y}) = \widehat{f}(\boldsymbol{y}) = \sum_{\boldsymbol{x} \in \mathbb{F}_2^N} f(\boldsymbol{x}) \chi_{\boldsymbol{x}}(\boldsymbol{y}) = \langle \chi_{\boldsymbol{y}}, f \rangle, \quad \boldsymbol{y} \in \mathbb{F}_2^N. \tag{8}$$

The following properties of the Hadamard transform are well-known and easy to check:

$$\widehat{\widehat{f}} = 2^N f, \quad \langle \widehat{f}, \widehat{g} \rangle = 2^N \langle f, g \rangle \tag{9}$$

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g), \quad \mathcal{F}(f \cdot g) = 2^{-N} \mathcal{F}(f) * \mathcal{F}(g), \tag{10}$$

where $f \cdot g$ means the usual pointwise product of two functions.

2.2 Weight Spectrum, Krawtchouk Polynomials and MacWilliams Matrices

For any $r \in [0, N]$ let

$$S_r^{(N)} \equiv S_r = \{\boldsymbol{x} \in \mathbb{F}_2^N \mid \mathrm{wt}(\boldsymbol{x}) = r\} \tag{11}$$

be the rth *Hamming sphere* in $\mathbb{F}_2^N$ and let $\psi_r^{(N)} (\equiv \psi_r)$ be its characteristic function

$$\psi_r(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in S_r^{(N)} \\ 0, & \boldsymbol{x} \notin S_r^{(N)}. \end{cases} \tag{12}$$

Furthermore, for any function $f \in V_N$ we define

$$A_r(f) = \langle \psi_r, f \rangle = \sum_{\boldsymbol{x} \in S_r^{(N)}} f(\boldsymbol{x}), \quad 0 \leq r \leq N \tag{13}$$

and call $(A_0(f), A_1(f), \ldots, A_N(f))^T \in \mathbb{C}^{N+1}$ the *weight spectrum* of f (cf. [2]). According to formulas (9)

$$A_r(\widehat{f}) = \langle \psi_r, \widehat{f} \rangle = 2^{-N} \langle \widehat{\psi_r}, \widehat{\widehat{f}} \rangle = \langle \widehat{\psi_r}, f \rangle, \tag{14}$$

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but on the other side, according to (8)

$$\widehat{\psi}_r(\boldsymbol{x}) = \langle \chi_{\boldsymbol{x}}, \psi_r \rangle = \sum_{\boldsymbol{y} \in S_r} \chi_{\boldsymbol{y}}(\boldsymbol{x}) = K_r^{(N)}(x), \tag{15}$$

where $x = |\boldsymbol{x}|$ and

$$K_r^{(N)}(x) = K_r(x) = \sum_{i=0}^{r} (-1)^i \binom{N-x}{r-i} \binom{x}{i} \tag{16}$$

is the rth *Krawtchouk polynomial of order N* (cf. [1], [2]).[1]

Now we can easily rewrite (14) to obtain the famous *MacWilliams formula for the dual spectra*:

$$A_r(\widehat{f}) = \sum_{i=0}^{N} K_r(i) A_i(f). \tag{17}$$

We define the *MacWilliams matrix* of order N by

$$(\mathbf{M}_N)_{ij} = K_i^{(N)}(j) \tag{18}$$

for $0 \leq i, j \leq N$. Then formula (17) can be rewritten in a matrix form: $(N+1)$-dimensional column vectors $\boldsymbol{a}_i = A_i(f)$ and $\mathcal{M}_N(\boldsymbol{a})_i = A_i(\widehat{f})$ are connected through the matrix equality

$$\mathcal{M}_N(\boldsymbol{a}) = \mathbf{M}_N \boldsymbol{a}, \tag{19}$$

and this formula defines the *MacWilliams transform* of order N for an arbitrary $(N+1)$-dimensional vector $\boldsymbol{a}$ (cf. [2]).

To conclude this section we list some useful formulas for the Krawtchouk polynomials and MacWilliams matrices. All of them (except formula (29)) can be found, for example, in [1] [2], and [3]:

1. The generating function (see [2]):

$$(1+t)^{N-z}(1-t)^z = \sum_{k=0}^{\infty} K_k^{(N)}(z) t^k. \tag{20}$$

2. Explicit expressions:

$$\begin{aligned} K_r^{(N)}(z) &= \sum_{l=0}^{r} (-1)^l \binom{N-z}{r-l} \binom{z}{l} \\ &= \sum_{l=0}^{r} (-1)^l 2^{r-l} \binom{N-r+l}{l} \binom{N-z}{r-l} \\ &= \sum_{l=0}^{r} (-2)^l \binom{N-l}{r-l} \binom{z}{l}. \end{aligned} \tag{21}$$

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[1] By definition $K_0^{(0)}(0) = 1$.

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The leading coefficient

$$\text{Coeff}_r\left(K_r^{(N)}(z)\right) = \frac{(-2)^r}{r!}. \tag{22}$$

3. Orthogonality:

Let $\mathbf{C} = \text{diag}\left(\binom{N}{0}, \binom{N}{1}, \ldots, \binom{N}{N}\right)$. Then the following identities hold:

$$\sum_{i=0}^{N} \binom{N}{i} K_r(i) K_s(i) = 2^N \binom{N}{r} \delta_{r,s} \quad \text{i.e.} \quad \mathbf{M}_N \mathbf{C} \mathbf{M}_M^T = 2^N \mathbf{C}. \tag{23}$$

$$\sum_{i=0}^{N} K_r(i) K_i(s) = 2^N \delta_{r,s} \quad \text{i.e.} \quad \mathbf{M}_N^2 = 2^N \mathbf{I}. \tag{24}$$

4. Symmetry:

$$K_r^{(N)}(z) = (-1)^r K_r^{(N)}(N - z) \quad \text{for any } z, \text{ and}$$
$$K_r^{(N)}(z) = (-1)^z K_{N-r}^{(N)}(z) \quad \text{for } z \in \{0, 1, \ldots, N\} \tag{25}$$

$$\binom{N}{r} K_s(r) = \binom{N}{s} K_r(s) \quad \text{i.e.} \quad \mathbf{M}_N^T = \mathbf{C}^{-1} \mathbf{M}_n \mathbf{C}$$
(the reciprocity formula). $\tag{26}$

5. Recurrence relations (see [2], [5]):

$$(r + 1) K_{r+1}(z) = (N - 2z) K_r(z) - (N - r + 1) K_{r-1}(z),$$
$$K_0(z) = 1, \ K_1(z) = N - 2z. \tag{27}$$

$$(N - r) K_l(r + 1) = (N - 2l) K_l(r) - r K_l(r - 1),$$
$$K_l(0) = \binom{N}{l}, \ K_l(1) = \binom{N}{l}\left(1 - \frac{2l}{N}\right). \tag{28}$$

$$K_i^{(N)}(j) = d_N(j)\left(K_i^{(N-1)}(j) + K_{i-1}^{(N-1)}(j)\right.$$
$$+ \ K_i^{(N-1)}(j - 1) - K_{i-1}^{(N-1)}(j - 1)\left.\right), \quad N \geq 1, \tag{29}$$

where $0 \leq i, j \leq N$, and

$$d_N(j) = \begin{cases} 1, & \text{if } j \in \{0, N\} \\ \frac{1}{2}, & \text{if } j \notin \{0, N\}, \end{cases} \tag{30}$$

with the understanding that $K_i^{(N-1)}(N) = K_N^{(N-1)}(j) = 0$, and $K_i(j) = 0$ for $i = -1$ or $j = -1$ (or both).

Proof of formula (29). Let for a while $\mathbf{X} = \mathbf{M}_{N-1}$, and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the first and the last columns in $N \times N$-matrix $\boldsymbol{X}$, respectively, so that

$$\boldsymbol{X} = (\boldsymbol{a} \mid \mathbf{Z} \mid \boldsymbol{b}), \tag{31}$$

where $N \times (N-2)$-matrix $\mathbf{Z}$ contains all columns of $\mathbf{X}$ indexed from 1 to $N - 2$.

Then the system of equalities (29) can be rewritten as a matrix equality

$$\mathbf{M}_N = \left(\begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0}^T & 0 \\ \mathbf{X} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{X} \\ 0 & \mathbf{0}^T \end{pmatrix} - \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \right) \Omega, \tag{32}$$

where $\Omega = \operatorname{diag}(1, \frac{1}{2}, \ldots, \frac{1}{2}, 1)$ is a diagonal matrix.

Let's now rewrite formula (20) for $j \neq N$ as follows:

$$(1 + t)\big((1 + t)^{N-1-j}(1 - t)^j\big) = \sum_{i=0}^{N} K_i^{(N)}(j)t^i, \tag{33}$$

which gives us a relation

$$K_i^{(N)}(j) = K_i^{(N-1)}(j) + K_{i-1}^{(N-1)}(j), \quad j \neq N, \tag{34}$$

whereas for $j = N$ one has $(1 - t)^N = (1 - t)(1 - t)^{N-1}$, which implies that

$$K_i^{(N)}(N) = K_i^{(N-1)}(N - 1) - K_{i-1}^{(N-1)}(N - 1), \tag{35}$$

if $1 \leq i \leq N - 1$. Evidently one can write (34) and (35) into a matrix form:

$$\mathbf{M}_N = \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0}^T & 0 \\ \mathbf{X} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{O} & \boldsymbol{b} \\ \mathbf{0}^T & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{0}^T & 0 \\ \mathbf{O} & \boldsymbol{b} \end{pmatrix}. \tag{36}$$

In a similar way, from trivial equalities

$$(1 + t)^{N-j}(1 - t)^j = \big((1 + t)^{(N-1)-(j-1)}(1 - t)^{j-1}\big)(1 - t)$$

for $j \neq 0$ and $(1 + t)^N = (1 + t)(1 + t)^{N-1}$ for $j = 0$ we get an analogous formula

$$\mathbf{M}_N = \begin{pmatrix} \mathbf{0} & \mathbf{X} \\ 0 & \mathbf{0}^T \end{pmatrix} - \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{X} \end{pmatrix} + \begin{pmatrix} \boldsymbol{a} & \mathbf{O} \\ 0 & \mathbf{0}^T \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{0}^T \\ \boldsymbol{a} & \mathbf{O} \end{pmatrix}. \tag{37}$$

Summing (36) to (37) and recalling decomposition $\mathbf{X} = (\boldsymbol{a}|\mathbf{Z}|\boldsymbol{b})$ one gets

$$\begin{aligned} 2\mathbf{M}_N &= \begin{pmatrix} 2\boldsymbol{a}|\mathbf{Z}|\boldsymbol{b} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0}^T & 0 \\ 2\boldsymbol{a}|\mathbf{Z}|\boldsymbol{b} & \mathbf{0} \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{0} & \boldsymbol{a}|\mathbf{Z}|2\boldsymbol{b} \\ 0 & \mathbf{0}^T \end{pmatrix} - \begin{pmatrix} 0 & \mathbf{0}^T \\ 0 & \boldsymbol{a}|\mathbf{Z}|2\boldsymbol{b} \end{pmatrix}, \end{aligned}$$

which shows that $\mathbf{M}_N$ can be obtained from

$$\mathbf{Y} = \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0}^T & 0 \\ \mathbf{X} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{X} \\ 0 & \mathbf{0}^T \end{pmatrix} - \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \tag{38}$$

by multiplying the first and the last column by two. Hence $\mathbf{M}_N$ can be obtained from $\mathbf{Y}$ by dividing the columns from 2 to $N - 2$ by 2, or equivalently, by multiplying $\mathbf{Y}$ (from the right) by Ω. □

By using the zero-padding notations, formula (32) can be written as

$$\mathbf{M}_{N+1} = (\underline{\mathbf{M}_N}| + \overline{\mathbf{M}_N}| + \underline{|\mathbf{M}_N} - \overline{|\mathbf{M}_N}) \cdot \mathrm{diag}(1, \frac{1}{2}, \dots, \frac{1}{2}, 1), \qquad (39)$$

which gives rise to a simple algorithm for constructing the MacWilliams matrices recursively:[2]

Example 1.

$$\mathbf{M}_0 = (1),$$

$$
\begin{aligned}
\mathbf{M}_1 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_2 &= \left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \right. \\
&\left. \quad - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix},
\end{aligned}
$$

and so on.

3 The Discrete Chebyshev Polynomials and Chebyshev Matrices

For any fixed $\boldsymbol{y} \in \mathbb{F}_2^N$ we define a function $B\boldsymbol{y} : \mathbb{F}_2^N \to \mathbb{C}$ by

$$B_{\boldsymbol{y}}(\boldsymbol{x}) = \begin{pmatrix} |\boldsymbol{y}| \\ |\boldsymbol{x} \cap \boldsymbol{y}| \end{pmatrix} \cdot \chi_{\boldsymbol{y}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{F}_2^N. \qquad (40)$$

Functions $B_{\boldsymbol{y}}$ were first introduced in [1], where it was shown that $B_{\boldsymbol{y}}$ form a basis (which we refer as to H-basis) of V_N and that for any $r \in \{0, 1, \dots, N\}$, sum

$$\sum_{\boldsymbol{y} \in S_r} B\boldsymbol{y}(\boldsymbol{x}) = \sum_{i=0}^{r} (-1)^i \begin{pmatrix} r \\ i \end{pmatrix} \begin{pmatrix} N - |\boldsymbol{x}| \\ r - i \end{pmatrix} \begin{pmatrix} |\boldsymbol{x}| \\ i \end{pmatrix}, \qquad (41)$$

depends obviously only of $x = |\boldsymbol{x}|$. Equation (41) defines the *rth discrete Chebyshev polynomial of order* N (cf. [1], [3]):

$$D_r^{(N)}(x) = D_r(x) = \sum_{i=0}^{r} (-1)^i \begin{pmatrix} r \\ i \end{pmatrix} \begin{pmatrix} N - x \\ r - i \end{pmatrix} \begin{pmatrix} x \\ i \end{pmatrix}. \qquad (42)$$

---

[2]Of course, formulas (36) and (37) would also yield such an algorithm, but (39) has a bit more uniform appearance.

Similarly to (18), an $(N+1) \times (N+1)$-matrix

$$(\mathbf{D}_N)_{ij} = D_i^{(N)}(j), \quad 0 \leq i, j \leq N \tag{43}$$

is called the *Chebyshev matrix of order* N.

Below we give a list of some useful formulas for the Chebyshev polynomials and matrices, most of which (except formula (55)) can be found in [1], [2], and [7]:

1. Difference formula;

$$D_n^{(N)}(x) = (-1)^n \Delta^n \left(\binom{x}{n} \binom{x-N-1}{n} \right), \tag{44}$$

where Δ^n is the nth power of the difference operator defined as $\Delta f(x) = f(x+1) - f(x)$.

2. Explicit expressions:

$$\begin{aligned}
D_n^{(N)}(x) &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{N-x}{n-i} \binom{x}{i} \\
&= \sum_{i=0}^{n} (-1)^i \binom{n+i}{n} \binom{N-i}{n-i} \binom{x}{i} \\
&= \sum_{i=0}^{n} \binom{n-N-1}{i} \binom{n+N+1}{n-i} \binom{x+i}{n}.
\end{aligned} \tag{45}$$

3. Orthogonality:

$$\sum_{i=0}^{N} D_r(i) D_s(i) = \gamma_r \delta_{rs}, \text{ where } \gamma_r = \binom{2r}{r} \binom{N+1+r}{2r+1}, \tag{46}$$

which means that

$$\mathbf{D}_N \mathbf{D}_N^T = \mathrm{diag}(\gamma_0, \gamma_1, \ldots, \gamma_N). \tag{47}$$

4. Symmetry and special values:

$$D_n(N-x) = (-1)^n D_n(x), \tag{48}$$

$$D_n^{(N)}(0) = \binom{N}{n}, \quad D_N^{(N)}(m) = (-1)^m \binom{N}{m}. \,^3 \tag{49}$$

For even N:

$$D_n^{(N)}\left(\tfrac{N}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^m \binom{\frac{N}{2}}{m} \binom{\frac{N}{2}+m}{m}, & \text{if } n = 2m \end{cases} \tag{50}$$

---

3 $\quad D_0^{(0)}(0) = 1$ by definition.

The leading coefficient:

$$\operatorname{Coeff}_n(D_n^{(N)}) = \frac{(-1)^n}{n!} \binom{2n}{n} \qquad (51)$$

$$D_0(x) = 1, \; D_1(x) = -2x + N, \; D_2(x) = 3x^2 - 3Nx + \binom{N}{2}. \qquad (52)$$

5. Recurrence formulas:

$$n^2 D_n = (2n-1)D_1 D_{n-1} - (N+n)(N-n+2)D_{n-2}, \qquad (53)$$

$$\Delta\big((x+1)(x-N)\Delta D_n(x)\big) = n(n+1)D_n(x+1),$$

$$D_n(0) = \binom{N}{n}, \quad D_n(1) = \binom{N-1}{n} - n\binom{N-1}{n-1}$$

$$\text{(Difference equation for } D_n(x)\text{).} \qquad (54)$$

$$\begin{aligned}
nD_n^{(N)}(x) &= nD_n^{(N-1)}(x-1) + (N-x)D_{n-1}^{(N-1)}(x) \\
&\quad - (N+n-x)D_{n-1}^{(N-1)}(x-1), \quad N \geq 1. \qquad (55)
\end{aligned}$$

Proof of Formula (55). For $n \neq 0$ we get:

$$\begin{aligned}
&(-1)^n D_n^{(N)}(x) \\
=\; &\Delta^n\left(\binom{x}{n}\binom{x-N-1}{n}\right) \\
=\; &\Delta^n\left(\left(\binom{x-1}{n} + \binom{x-1}{n-1}\right)\binom{(x-1)-N}{n}\right) \\
=\; &\Delta^n\left(\binom{x-1}{n}\binom{(x-1)-N}{n}\right) + \Delta^n\left(\binom{x-1}{n-1}\binom{(x-1)-N}{n}\right) \\
=\; &(-1)^n D_n^{(N-1)}(x-1) + \Delta^n\left(\binom{x-1}{n-1}\frac{x-(N+n)}{n}\binom{(x-1)-N}{n-1}\right) \\
=\; &(-1)^n D_n^{(N-1)}(x-1) - \frac{N+n}{n}\Delta\left(\Delta^{n-1}\left(\binom{x-1}{n-1}\binom{(x-1)-N}{n-1}\right)\right) \\
&+ \frac{1}{n}\Delta^n\left(x\binom{x-1}{n-1}\binom{(x-1)-N}{n}\right).
\end{aligned}$$

By using $\Delta^n\big(x \cdot f(x)\big) = (x+n)\Delta^n\big(f(x)\big) + n\Delta^{n-1}\big(f(x)\big)$, which is easy to

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prove by induction, we learn that

$$
\begin{aligned}
&(-1)^n D_n^{(N)}(x) \\
={} &(-1)^n D_n^{(N-1)}(x-1) - (-1)^{n-1}\frac{N+n}{n}\Delta\big(D_{n-1}^{(N-1)}(x-1)\big) \\
+{} &\frac{1}{n}\left((x+n)\Delta^n\left(\binom{x-1}{n-1}\binom{(x-1)-N}{n-1}\right)\right. \\
+{} &\left. n\Delta^{n-1}\left(\binom{x-1}{n-1}\binom{(x-1)-N}{n-1}\right)\right) \\
={} &(-1)^n D_n^{(N-1)}(x-1) - (-1)^{n-1}\frac{N+n}{n}\big(D_{n-1}^{(N-1)}(x) - D_{n-1}^{(N-1)}(x-1)\big) \\
+{} &(-1)^{n-1}\frac{x+n}{n}\Delta\big(D_{n-1}^{(N-1)}(x-1)\big) + (-1)^{n-1}D_{n-1}^{(N-1)}(x-1),
\end{aligned}
$$

which gives the desired recurrent formula (for $n \neq 0$):

$$
nD_n^{(N)}(x) = nD_n^{(N-1)}(x-1) + (N-x)D_{n-1}^{(N-1)}(x) - (N+n-x)D_{n-1}^{(N-1)}(x-1).
$$

Evidently this formula remains valid also for $n = 0$, supposing $D_{-1}(x) \equiv 0$. $\square$

Similarly to the formula (39) for MacWilliams matrices, formula (55) can be written in matrix form as

$$
\begin{aligned}
\mathrm{diag}(0,\ldots,N) \cdot \mathbf{D}_N ={} &\mathrm{diag}(0,\ldots,N)|\underline{\mathbf{D}_{N-1}} \\
-{} &\big(|\overline{\mathbf{D}_{N-1}} - \overline{\mathbf{D}_{N-1}}|\big) \cdot \mathrm{diag}(N,\ldots,0) - \mathrm{diag}(0,\ldots,N)|\overline{\mathbf{D}_{N-1}}. \quad (56)
\end{aligned}
$$

The first row in $\big(|\overline{\mathbf{D}_{N-1}} - \overline{\mathbf{D}_{N-1}}|\big)$ consists of zeros, whereas the first rows of $\mathbf{D}_N$ and $\mathbf{D}_{N-1}$ consist of ones, and the first row of $|\underline{\mathbf{D}_{N-1}}$ consists of zero followed by ones. Hence formula (56) can be rewritten as

$$
\begin{aligned}
\mathbf{D}_N ={} &\big(|\underline{\mathbf{D}_{N-1}} - \overline{\mathbf{D}_{N-1}}\big) \\
-{} &\mathrm{diag}(0,1,\ldots,1/N)\big(|\overline{\mathbf{D}_{N-1}} - \overline{\mathbf{D}_{N-1}}|\big)\mathrm{diag}(N,\ldots,0) + \mathbf{e}_{00}, \quad (57)
\end{aligned}
$$

where $\mathbf{e}_{00}$ is a matrix with 1 as its left upper corner and zeroes in all the other entries. This gives rise to a simple recurrent algorithm for constructing the Chebyshev matrices, illustrated in the following example.

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Example 2.

$$\mathbf{D}_0 = (1)$$

$$\mathbf{D}_1 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$- \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{D}_2 = \left(\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \right)$$

$$- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \right) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{pmatrix},$$

and so on.

4 The Krawtchouk Coefficients of The Discrete Chebyshev Polynomials

The strategy is to first find the Hadamard transform of the function defined by formula (40) and then to apply the MacWilliams formula (17).

Lemma 1. *Let*

$$\Phi_{\boldsymbol{y}}(\boldsymbol{x}) = \binom{|\boldsymbol{y}|}{|\boldsymbol{x} \cap \boldsymbol{y}|} \tag{58}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_2^N$, ($\boldsymbol{y}$ is a fixed vector). Then the Hadamard transform of $\Phi_{\boldsymbol{y}}$ is given as follows:

$$\widehat{\Phi}_{\boldsymbol{y}}(\boldsymbol{z}) = 2^{N-y} K_y^{(2y)}(|\boldsymbol{z}|) \cdot X_{\boldsymbol{z}}(\boldsymbol{y}), \tag{59}$$

where $\boldsymbol{z} \in \mathbb{F}_2^N$, $y = |\boldsymbol{y}|$, and

$$X_{\boldsymbol{z}}(\boldsymbol{y}) = \begin{cases} 1, & \text{if } \boldsymbol{z} \subseteq \boldsymbol{y} \\ 0, & \text{if } \boldsymbol{z} \not\subseteq \boldsymbol{y} \end{cases} \tag{60}$$

Proof. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{F}_2^N$, and $\boldsymbol{x}_1 = \boldsymbol{x} \cap \boldsymbol{y}$, $\boldsymbol{x}_2 = \boldsymbol{x} \cap \overline{\boldsymbol{y}}$, $\boldsymbol{z}_1 = \boldsymbol{z} \cap \boldsymbol{y}$, and

$z_2 = z \cap \overline{y}$. Then a direct calculation gives

$$
\begin{aligned}
\widehat{\Phi}_{\boldsymbol{y}}(\boldsymbol{z}) &= \sum_{\boldsymbol{x} \in V_n} (-1)^{|\boldsymbol{x} \cap \boldsymbol{z}|} \binom{|\boldsymbol{y}|}{|\boldsymbol{x} \cap \boldsymbol{y}|} \\
&= \sum_{\boldsymbol{x}_1 \subseteq \boldsymbol{y}} \sum_{\boldsymbol{x}_2 \subseteq \overline{\boldsymbol{y}}} (-1)^{|(\boldsymbol{x}_1 + \boldsymbol{x}_2) \cap (\boldsymbol{z}_1 + \boldsymbol{z}_2)|} \binom{|\boldsymbol{y}|}{|\boldsymbol{x} \cap \boldsymbol{y}|} \\
&= \sum_{\boldsymbol{x}_2 \subseteq \overline{\boldsymbol{y}}} (-1)^{|\boldsymbol{x}_2 \cap \boldsymbol{z}_2|} \sum_{\boldsymbol{x}_1 \subseteq \boldsymbol{y}} (-1)^{|\boldsymbol{x}_1 \cap \boldsymbol{z}_1|} \binom{|\boldsymbol{y}|}{|\boldsymbol{x}_1|}.
\end{aligned}
$$

In the inner sum, we split the bits of $\boldsymbol{x}_1$ into two parts: those contained in $\boldsymbol{z}_1$ and those outside of $\boldsymbol{z}_1$ (there are $|\boldsymbol{y}| - |\boldsymbol{z}_1|$ of them) to get

$$
\begin{aligned}
\widehat{\Phi}_{\boldsymbol{y}}(\boldsymbol{z}) &= \sum_{\boldsymbol{x}_2 \subseteq \overline{\boldsymbol{y}}} (-1)^{|\boldsymbol{x}_2 \cap \boldsymbol{z}_2|} \sum_{a=0}^{|\boldsymbol{z}_1|} (-1)^a \binom{|\boldsymbol{z}_1|}{a} \sum_{b=0}^{|\boldsymbol{y}| - |\boldsymbol{z}_1|} \binom{|\boldsymbol{y}| - |\boldsymbol{z}_1|}{b} \cdot \binom{|\boldsymbol{y}|}{a+b} \\
&= \sum_{\boldsymbol{x}_2 \subseteq \overline{\boldsymbol{y}}} (-1)^{|\boldsymbol{x}_2 \cap \boldsymbol{z}_2|} \sum_{a=0}^{|\boldsymbol{z}_1|} (-1)^a \binom{|\boldsymbol{z}_1|}{a} \binom{2|\boldsymbol{y}| - |\boldsymbol{z}_1|}{|\boldsymbol{y}| - a} \\
&= K_{|\boldsymbol{y}|}^{(2|\boldsymbol{y}|)}(|\boldsymbol{z}_1|) \cdot \sum_{\boldsymbol{x}_2 \subseteq \overline{\boldsymbol{y}}} (-1)^{|\boldsymbol{x}_2 \cap \boldsymbol{z}_2|}.
\end{aligned}
$$

The equality between the first and the second line is due to Vandermonde's convolution (see [1], for instance). The inner sum in the last line can naturally be interpreted as a sum over all characters of a $|\overline{\boldsymbol{y}}|$-dimensional vector space over $\mathbb{F}_2$, and hence it equals to 0, if $\boldsymbol{z}_2 \neq \boldsymbol{0}$, and to $2^{|\overline{\boldsymbol{y}}|}$, if $\boldsymbol{z}_2 = \boldsymbol{0}$. Because $\boldsymbol{z}_2 = \boldsymbol{0}$ is equivalent to $\boldsymbol{z} \subseteq \boldsymbol{y}$ (which is equivalent to $\boldsymbol{z} = \boldsymbol{z}_1$), we have

$$
\widehat{\Phi}_{\boldsymbol{y}}(\boldsymbol{z}) = \begin{cases} 2^{|\overline{\boldsymbol{y}}|} K_{|\boldsymbol{y}|}^{(2|\boldsymbol{y}|)}(|\boldsymbol{z}|), & \boldsymbol{z} \subseteq \boldsymbol{y} \\ 0, & \boldsymbol{z} \not\subseteq \boldsymbol{y} \end{cases} = 2^{|\overline{\boldsymbol{y}}|} K_{|\boldsymbol{y}|}^{(2|\boldsymbol{y}|)}(|\boldsymbol{z}|) \cdot X_{\boldsymbol{z}}(\boldsymbol{y}), \quad (61)
$$

as claimed. $\qquad\qquad\square$

Lemma 2.
$$
\widehat{B}_{\boldsymbol{y}}(\boldsymbol{u}) = 2^{N-y} K_y^{(2y)}(y - |\boldsymbol{u}|) X_{\boldsymbol{u}}(\boldsymbol{y}), \quad \boldsymbol{u} \in \mathbb{F}_2^N \qquad (62)
$$

Proof. Since evidently $\widehat{T}_{\boldsymbol{y}} = \chi_{\boldsymbol{y}}$ (see formulae (5), (7) and (8)), and due to (9) and (10), we can easily find the Hadamard transform of $B_{\boldsymbol{y}} = \Phi_{\boldsymbol{y}} \cdot \chi_{\boldsymbol{y}}$. Since $(f * T_{\boldsymbol{y}})(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{y})$ for any $f \in V_N$, we first get

$$
\widehat{B}_{\boldsymbol{y}}(\boldsymbol{u}) = 2^{-N}(\widehat{\Phi}_{\boldsymbol{y}} * 2^N T_{\boldsymbol{y}})(\boldsymbol{u}) = \widehat{\Phi}_{\boldsymbol{y}}(\boldsymbol{u} + \boldsymbol{y}),
$$

and Lemma 1 then gives

$$
\begin{aligned}
\widehat{B}_{\boldsymbol{y}}(\boldsymbol{u}) &= 2^{N-y} K_y^{(2y)}(|\boldsymbol{u} + \boldsymbol{y}|) X_{\boldsymbol{u}+\boldsymbol{y}}(\boldsymbol{y}) \\
&= 2^{N-y} K_y^{(2y)}(y - |\boldsymbol{u}|) X_{\boldsymbol{u}}(\boldsymbol{y}).
\end{aligned}
$$

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The last equality follows from the easily verifiable equality $X_{\boldsymbol{u}+\boldsymbol{y}}(\boldsymbol{y}) = X_{\boldsymbol{u}}(\boldsymbol{y})$. $\qquad\square$

Proposition 1 (The Krawtchouk coefficients of the discrete Chebyshev polynomials). *For $0 \leq n \leq N$*

$$
\begin{aligned}
D_n^{(N)}(m) &= 2^{-n} \cdot \sum_{i=0}^{N} \binom{N-i}{n-i} K_n^{(2n)}(n-i) K_i^{(N)}(m) \\
&= 2^{-n} \cdot \sum_{i=0}^{n} \binom{N-n+i}{i} K_n^{(2n)}(i) K_{n-i}^{(N)}(m).
\end{aligned}
\tag{63}
$$

Proof. The ith coefficient of the weight spectrum of $\widehat{B}_{\boldsymbol{y}}$ can be calculated by the previous lemma to get

$$
\begin{aligned}
A_i(\widehat{B}_{\boldsymbol{y}}) &= \sum_{\boldsymbol{u} \in S_i} \widehat{B}_{\boldsymbol{y}}(\boldsymbol{u}) = \sum_{\boldsymbol{u} \in S_i} 2^{N-y} K_y^{(2y)}(y - |\boldsymbol{u}|) X_{\boldsymbol{u}}(\boldsymbol{y}) \\
&= 2^{N-y} \binom{y}{i} K_y^{(2y)}(y-i).
\end{aligned}
\tag{64}
$$

By the MacWilliams formula (17) and by (9) we get

$$
2^N \cdot A_m(B_{\boldsymbol{y}}) = \sum_{i=0}^{N} K_m^{(N)}(i) A_i(\widehat{B}_{\boldsymbol{y}}) = 2^{N-y} \sum_{i=0}^{N} K_m^{(N)}(i) \binom{y}{i} K_y^{(2y)}(y-i),
\tag{65}
$$

and hence

$$
A_m(B_{\boldsymbol{y}}) = 2^{-y} \sum_{i=0}^{N} K_m^{(N)}(i) \binom{y}{i} K_y^{(2y)}(y-i).
\tag{66}
$$

On the other side, according to formulas (41) and (42) we have

$$
D_r^{(N)}(x) = \sum_{\boldsymbol{y} \in S_r} B_{\boldsymbol{y}}(\boldsymbol{x}), \quad x = |\boldsymbol{x}|
\tag{67}
$$

and summing up left and right sides of the equality (67) over each $\boldsymbol{x} \in \mathbb{F}_2^N$ with $m = |\boldsymbol{x}|$ and applying formula (66) we find that

$$
\begin{aligned}
\binom{N}{m} D_r(m) &= \sum_{\boldsymbol{y} \in S_r} \sum_{\boldsymbol{x} \in S_m} B_{\boldsymbol{y}}(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in S_r} A_m(B_{\boldsymbol{y}}) \\
&= \binom{N}{r} 2^{-r} \sum_{i=0}^{N} K_m^{(N)}(i) \binom{r}{i} K_r^{(2r)}(r-i).
\end{aligned}
\tag{68}
$$

So we get the following expression of the discrete Chebyshev polynomials by the Krawtchouk polynomials:

$$
D_n^{(N)}(m) = 2^{-n} \frac{\binom{N}{n}}{\binom{N}{m}} \sum_{i=0}^{N} \binom{n}{i} K_n^{(2n)}(n-i) K_m^{(N)}(i).
\tag{69}
$$

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Applying the reciprocity formula (26)

$$\binom{N}{i} \cdot K_m^{(N)}(i) = \binom{N}{m} \cdot K_i^{(N)}(m)$$

we get the claim by direct calculations. The latter form can be obtained by replacing the summation index i with $n - i$. $\square$

Lemma 3.

1. *For $K_n^{(2n)}$ we have*

$$K_n^{(2n)}(z) = \frac{(-2)^n}{n!} \prod_{q=0}^{n-1} (z - (2q+1)) = (-4)^n \binom{\frac{z-1}{2}}{n}. \tag{70}$$

2. *For $s \in \{0, 1, \ldots, 2n\}$*

$$K_n^{(2n)}(s) = \begin{cases} 0, & \text{if } s = 2r+1 \\ (-1)^r \binom{2n}{n} \frac{\binom{n}{r}}{\binom{2n}{r}}, & \text{if } s = 2r. \end{cases} \tag{71}$$

Proof. From the symmetry formula (25) we get

$$K_n^{(2n)}(2r+1) = (-1)^{2r+1} K_{2n-n}^{(2n)}(2r+1) = -K_n^{(2n)}(2r+1),$$

so $K_n^{(2n)}(2r+1) = 0$ and it is exactly the first equality of item 2.

Since $K_n^{(2n)}$ has degree n and because we know all of its zeros by item 2, we only need to recall formula (22) for the leading coefficient of Krawtchouk polynomials to get the first equality of item 1. The second equality is evidently just a reformulation of the first one.

Now the claim in expression (71) for $K_n^{(2n)}(2r)$ can be obtained straightforwardly by substitution $z = 2r$ in formula (70). $\square$

Remark 1. From formula (70) we get a formula for generating function of $K_n^{(2n)}(z)$:

$$(1 - 4t)^{\frac{z-1}{2}} = \sum_{n=0}^{\infty} K_n^{(2n)}(z) t^n. \tag{72}$$

Recalling a formula for the generating function of the Catalan numbers C_n (cf. [4])

$$\frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n=0}^{\infty} C_n \cdot t^n \tag{73}$$

we find that

$$K_n^{(2n)}(2) = -2 \cdot C_{n-1}, \quad n \geq 1. \tag{74}$$

For the next proposition, which is a matrix reformulation of (63), let now

$$\mathbf{B}_N = \text{diag}(1, 2^{-1}, \ldots, 2^{-N}), \quad \mathbf{L}_N = \left(\binom{N-i}{n-i} K_n^{(2n)}(n-i) \right)_{0 \leq n, i \leq N}.$$

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Proposition 2.

$$\mathbf{D}_N = \mathbf{Q}_N \mathbf{M}_N, \tag{75}$$

where $\mathbf{Q}_N = \mathbf{B}_N \mathbf{L}_N$.

Since $\mathbf{M}_N^2 = 2^N \mathbf{I}$, (75) can be also rewritten as

$$\mathbf{D}_N \mathbf{M}_N = 2^N \mathbf{Q}_N = \mathrm{diag}(2^N, 2^{N-1}, \dots, 1) \cdot \mathbf{L}_N \tag{76}$$

to get an expression where all the matrix entries are integers.

Example 3. For $N = 4$ we have

$$\mathbf{M}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \mathbf{L}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -12 & 0 & 6 & 0 & 0 \\ 0 & -12 & 0 & 20 & 0 \\ 6 & 0 & -10 & 0 & 70 \end{pmatrix},$$

and

$$\mathrm{diag}(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}) \cdot \mathbf{L}_4 \mathbf{M}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & -3 & -6 & -3 & 6 \\ 4 & -8 & 0 & 8 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} = \mathbf{D}_4$$

According to formula (71) we can conclude that also in the general case the matrix $\mathbf{L}_N$ has an "even sub-diagonal" structure:

$$(\mathbf{L}_N)_{n,i} = \begin{cases} 0, & \text{if } n - i \text{ is odd} \\ (-1)^{\frac{n-i}{2}} \binom{N-i}{n-i} \binom{2n}{n} \frac{\binom{n}{\frac{n-i}{2}}}{\binom{2n}{n-i}}, & \text{if } n - i \text{ is even.} \end{cases} \tag{77}$$

Proposition 3. *For $N \geq 1$, the discrete Chebyshev polynomials satisfy the following recurrence relation:*

$$(N - n)D_n^{(N)}(m) = (N - m)D_n^{(N-1)}(m) + mD_n^{(N-1)}(m - 1), \tag{78}$$

where $0 \leq m, n \leq N$.

Proof. To obtain this recurrence relation we are going to apply the recurrence relation

$$K_i^{(N)}(z) - K_i^{(N-1)}(z) = K_{i-1}^{(N-1)}(z), \ 0 \leq z \leq N - 1 \tag{79}$$

for Krawtchouk polynomials. The above formula follows easily from a simple equality $(1 + t)^{N-z}(1 - t)^z = (1 + t)\big((1 + t)^{N-1-z}(1 - t)^z\big)$. If $m \leq N - 1$, we

get, by using (69) and (79) that

$$2^n \frac{\binom{N}{m}}{\binom{N}{n}} D_n^{(N)}(m) - 2^n \frac{\binom{N-1}{m}}{\binom{N-1}{n}} D_n^{(N-1)}(m) = \binom{n}{N} K_n^{(2n)}(n-N) K_m^{(N)}(N)$$

$$+ \sum_{i=0}^{N-1} \binom{n}{i} K_n^{(2n)}(n-i) \left(K_m^{(N)}(i) - K_m^{(N-1)}(i) \right)$$

$$= (-1)^m \binom{N}{m} \cdot \binom{2N}{N} \cdot \delta_{N,n} + \sum_{i=0}^{N-1} \binom{n}{i} K_n^{(2n)}(n-i) K_{m-1}^{(N-1)}(i)$$

$$= (-1)^m \binom{N}{m} \cdot \binom{2N}{N} \cdot \delta_{N,n} + 2^n \frac{\binom{N-1}{m-1}}{\binom{N-1}{n}} D_n^{(N-1)}(m-1),$$

and multiplying both sides by $2^{-n} \cdot \frac{m!(N-m)!}{(n-1)!(N-n-1)!}$ we get for $n \geq 0$

$$n(N-n)D_n^{(N)}(m) = n(N-m)D_n^{(N-1)}(m) + nmD_n^{(N-1)}(m-1)$$

$$+ \quad (-1)^m 2^{-n} N(N-1) \binom{N-2}{n-1} \cdot \binom{2N}{N} \cdot \delta_{N,n} \tag{80}$$

The last term in (80) is equal to zero for $n \in \{0, 1, \ldots, N\}$, so for $n \geq 0$ and $m \leq N-1$ we get

$$(N-n)D_n^{(N)}(m) = (N-m)D_n^{(N-1)}(m) + mD_n^{(N-1)}(m-1).$$

Since $D_0^{(N)}(m) \equiv 1$ and $D_n^{(N)}(N) = (-1)^n \binom{N}{n}$ (see formulae (48) and (49)), this equality holds for each $0 \leq m, n \leq N$. $\qquad \square$

For $n = N$ formula (78) becomes a trivial identity $0 = 0$, and similarly to recurrence formula (55), formula (78) gives rise to an algorithm for calculating the Chebyshev matrices. The algorithm works as follows: Given a matrix $\mathbf{D}_{N-1}$ we first construct the "upper part" of the matrix $\mathbf{D}_N$ (i.e. all its rows except the Nth one) by formula

$$\operatorname*{Rows}_{1,\ldots,N-1}(\mathbf{D}_N) \quad = \quad \operatorname{diag}(\tfrac{1}{N}, \tfrac{1}{N-1}, \ldots, 1) \big((|\mathbf{D}_{N-1}) \cdot \operatorname{diag}(0, 1, \ldots, N)$$

$$+ \quad (\mathbf{D}_{N-1}|) \cdot \operatorname{diag}(N, \ldots, 1, 0) \big). \tag{81}$$

By (49) the last row of the Chebyshev matrices consists of the binomial coefficients with alternating signs, and hence we find the Nth row of $\mathbf{D}_N$ as

$$\operatorname{Row}_N(\mathbf{D}_N) = \big(\operatorname{Row}_{N-1}(\mathbf{D}_{N-1}), 0 \big) - \big(0, \operatorname{Row}_{N-1}(\mathbf{D}_{N-1}) \big). \tag{82}$$

To get $\mathbf{D}_N$, we concatenate $\operatorname*{Rows}_{1,\ldots,N-1}(\mathbf{D}_N)$ and $\operatorname{Row}_N(\mathbf{D}_{N-1})$.

Example 4.

$$\mathbf{D}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

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$$\underset{1,\ldots,3}{\mathrm{Rows}}(\mathbf{D}_3) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \right.$$

$$+ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -3 & -3 & 3 \end{pmatrix}.$$

and

$$\mathrm{Row}_4(\mathbf{D}_3) = \begin{pmatrix} 1 & -2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \end{pmatrix},$$

hence

$$\mathbf{D}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -3 & -3 & 3 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

References

[1] M. Hirvencalo: *Studies on Boolean Functions Related to Quantum Computing*. Ph.D. thesis, University of Turku (2003).

[2] F.J. MacWilliams and N.J.A. Sloane: *The Theory of Error-Correcting Codes*. North-Holland (1977).

[3] H. Bateman and A. Erdelyi: *Higher Transcendental Functions*. Vol. 2, McGraw-Hill (1953).

[4] I.M. Gessel and R. P. Stanley: *Algebraic enumeration*. In R. L. Graham, M. Grötschel, L. Lovász (eds.): *Handbook of Combinatorics Vol II*. Elsevier Science & MIT Press (1995).

[5] Н. Гогин и А. Мюлляри: Последовательность Фибоначчи-Падована и матрицы преобразования Мак-Вилвямс. Программирование №2, с. 1–6 (2007). English variant:
N.D. Gogin and A.A. Mylläri: *The Fibonacci-Padovan Sequence and MacWilliams Transform Matrices*. Programming and Computing Software Vol 33:2, pp. 74–79 (2007).

[6] A.N. Kolmogorov and S.V. Fomin: *Introductory Real Analysis*. Dover (1975).

[7] G. Szegö: *Orthogonal Polynomials*. Providence, Rhode Island: American Mathematical Society, Colloquium Publications, Volume XXIII (1975).
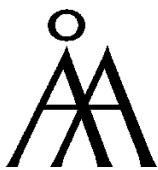
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