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Abstract

We consider interaction properties of relational periods, where the relation is a compatibility relation on words induced by a relation on letters. By the famous theorem of Fine and Wilf, $p + q - \gcd(p, q) - 1$ is the maximal length of a word having periods p and q but not period $\gcd(p, q)$. Such words of maximal length are called extremal Fine and Wilf words. In this paper we study properties of the corresponding words in a relational variation of the Fine and Wilf theorem.

Keywords: period, compatibility relation, partial word, Fine, Wilf

1 Introduction

Let $w = w_1 \cdots w_n$ be a word of length n. A positive integer p is a period of w if $w_i = w_{i+p}$ for $i = 1, 2, \ldots, n - p$. If w has two periods p and q and n is at least $p + q - \gcd(p, q)$, then the word has also as period the greatest common divisor $\gcd(p, q)$. This result was first proved by Fine and Wilf in 1965 in connection with real functions [13]. The bound on the length of the word is optimal; see [9]. Hence, the maximal length of a non-constant word with coprime periods p and q is $p + q - 2$. Such words are called a *extremal Fine and Wilf words*. In 1994 de Luca and Mignosi [12] showed that the set of all factors of these words coincides with the set of factors of Sturmian words. Furthermore, the extremal words are palindromes and unique up to renaming of letters. The theorem of Fine and Wilf for more that two periods was investigated in several papers [8, 10, 17]. In 2003 Tijdeman and Zamboni [19] gave a fast algorithm to count an extremal word (and its length) for arbitrary number of periods. Moreover, they showed that such word with periods $p_1, \ldots, p_r$ and without period $\gcd(p_1, \ldots, p_r)$ containing a maximal number of distinct letters is uniquely determined up to word isomorphism and is a palindrome.

In this paper we consider *relational Fine and Wilf words*, where the relation is a *similarity relation* on words induced by a compatibility relation on letters. The compatibility relation generalizes that of *partial words* introduced by Berstel and Boasson in 1999 [2]. Combinatorics on partial words has been widely studied in recent years. Motivation for the research of partial words (and words with similarity relations in general) comes partly from the study of biological sequences such as DNA, RNA and proteins [3].

Using similarity relations we introduce *relational periods*. Variations of Fine and Wilf's theorem for these periods were obtained recently by Halava, Harju and Kärki [15, 16]. Optimal bounds for periods' interaction were considered in the cases where a word has one relational period p and one pure period q. A word with relational period p and pure period q but without relational period $\gcd(p, q)$ will be called a relational Fine and Wilf word. We prove that under some natural constraints the structure of such words of maximal length is unique up to renaming of letters. These extremal words are over a ternary alphabet and the relation is necessarily similar to the compatibility relation of partial words. Furthermore, we consider their palindromic properties.

2 Similarity relations

Let $R \subseteq X \times X$ be a relation on a set X. We usually write $x\,R\,y$ instead of $(x, y) \in R$. The identity relation on X is denoted by ι_X. The relation R is a *compatibility relation* if it is both reflexive and symmetric, i.e., $(i)\ \forall x \in X : x\,R\,x$, and $(ii)\ \forall x, y \in X : x\,R\,y \implies y\,R\,x$. In this presentation we consider special kind of relations on words defined in the following way.

Definition 1. Let $\mathcal{A}$ be an alphabet. A relation on words over $\mathcal{A}$ is called a *similarity relation*, if its restriction on letters is a compatibility relation and, for words $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ $(u_i, v_j \in \mathcal{A})$, the relation R satisfies

$$u_1 \cdots u_m \, R \, v_1 \cdots v_n \iff m = n \text{ and } u_i \, R \, v_i \text{ for all } i = 1, 2, \ldots, m.$$

The restriction of R on letters, denoted by $R_\mathcal{A}$, is called the *generating relation* of R. Words u and v satisfying $u \, R \, v$ are said to be R-*similar* or R-*compatible*.

Since a similarity relation R is induced by its restriction on letters, it can be presented by listing all pairs $\{a, b\}$ $(a \neq b)$ such that $(a, b) \in R_\mathcal{A}$. We use the notation

$$R = \langle r_1, \ldots, r_n \rangle,$$

where $r_i = (a_i, b_i) \in \mathcal{A} \times \mathcal{A}$ for $i = 1, 2, \ldots, n$, to denote that R is the similarity relation generated by the symmetric closure of $\iota_\mathcal{A} \cup \{r_1, \ldots, r_n\}$. For example, let $\mathcal{A} = \{a, b\}$ and set $R = \langle (a, b) \rangle$. Then

$$R_\mathcal{A} = \{(a, a), (b, b), (a, b), (b, a)\}$$

Hence, the relation R makes all words over $\mathcal{A}$ with equal length similar to each other. On the other hand, let us consider the ternary alphabet $\mathcal{B} = \{a, b, c\}$ and set $S = \langle (a, b) \rangle$. Then

$$S_\mathcal{B} = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$

and, for example, $abba \, S \, baab$ but, for instance, words $abba$ and $baac$ are not S-similar.

More on properties of similarity relations can be found in [14]. For example, the connection between similarity relations and the compatibility relation of partial words is discussed in detail.

3   Relational periods

For words with compatibility relation on letters, i.e., similarity relation on words we will now define *relational periods*.

Definition 2. Let R be a compatibility relation on an alphabet $\mathcal{A}$. For a word $x = x_1 \cdots x_n \in \mathcal{A}^+$, an integer $p \geq 1$ is an R-*period* of x if, for all $i, j \in \{1, 2, \ldots, n\}$, we have

$$i \equiv j \pmod{p} \implies x_i \, R \, x_j.$$

This definition can be generalized naturally to infinite words. Note that the normal (*pure*) period of words is a relational period where the relation is the identity relation. Note also that, for the relation $R_\uparrow = \langle \{(\diamondsuit, a) \mid a \in \mathcal{A}\} \rangle$, an $R_\uparrow$-period corresponds to a period of partial words, where A is an alphabet and holes are denoted by $\diamondsuit$-symbols.

Example 1. Consider the word $x = babbbcbd$ in the alphabet $\mathcal{A} = \{a, b, c, d\}$. Let $R = \langle \{(a, b), (b, c), (c, d), (d, a)\} \rangle$ be a compatibility relations on $\mathcal{A}$. We consider the minimal R-period of x. Since $(b, d) \notin R$, the smallest R-period must be greater than 5. For example, 2 is not an R-period of x, since $(x_4, x_8) = (b, d)$ and $8 \equiv 4 \pmod 2$. Indeed, the smallest R-period is 6, because of the relation $a\,R\,d$. Note that the minimal pure period of x is 8.

4 Bounds of interaction

In recent years several variations of the theorem of Fine and Wilf have been proved. In particular, there are theorems related to the study of partial words. J. Berstel and L. Boasson gave a variant of the theorem of Fine and Wilf for partial words with one hole in [2]. Generalizations for several holes were considered, for example, by F. Blanchet-Sadri in [4] and F. Blanchet-Sadri and R.A. Hegstrom in [5], where it was shown that local partial periods p and q force a sufficiently long partial word to have a period $\gcd(p, q)$ when certain unavoidable cases (*special words*) are excluded. The bound on the length depends on the number of holes in the word. On the other hand, A.M. Shur and Yu.V. Gamzova found bounds for the length of a word with k holes such that (global) partial periods p and q imply a (global) partial period $\gcd(p, q)$ [18]. These periods' interaction bounds of partial words depend on the number of holes and in this respect show that finding good formulations for periods' interaction in the case of arbitrary relational periods is not possible except for equivalence relations. Namely, any non-transitive compatibility relation R must have letter relations $(x_1, x_2), (x_2, x_3) \in R$, but $(x_1, x_3) \notin R$ for some letters x_1, x_2, x_3. Then the role of the letter x_2 in R is exactly the same as the role of holes in partial words and all binary counter examples of Fine and Wilf's theorem for partial words apply to words with compatibility relation R over the alphabet $\{x_1, x_2, x_3\}$. For instance, we have the following example.

Example 2. Let $R = \langle \{(a, b)(b, c)\} \rangle$. There exists an infinite (not necessarily ultimately periodic) word

$$w = w_1 w_2 w_3 \cdots = acb^{6i_1 - 2} acb^{6i_2 - 2} \cdots ,$$

where the numbers $i_j \geq 1$ are chosen freely. Now w has global R-periods 2 and 3. Namely, $w_1 w_3 w_5 \cdots \in \{a, b\}^*$, $w_2 w_4 w_6 \cdots \in \{b, c\}^*$ and $w_1 w_4 w_7 \cdots \in \{a, b\}^*$, $w_2 w_5 w_8 \cdots \in \{b, c\}^*$, $w_3 w_6 w_9 \cdots \in \{b\}^*$. However, 1 is not a period, not even an R-period of the word w. For example, $(w_1, w_2) = (a, c) \notin R$.

On the other hand, it is possible to get some new interesting variations of the Fine and Wild theorem by assuming that one of the periods is pure while the other is relational by a relation $R \neq \iota$. We define the following bound.

Definition 3. Let $P \geq 2$ and $Q \geq 3$ be positive integers with $\gcd(P, Q) = d$. A positive integer $B = B(P, Q)$ is called the *bound of relational interaction for P and Q*, if it satisfies the following conditions:

(i) The bound B is *sufficient*, i.e., for any similarity relation R and for any word w with length $|w| \geq B$ having a (pure) period Q and an R-period P, the number $\gcd(P, Q) = d$ is an R-period of w.

(ii) The bound is *strict*, i.e., there exist a similarity relation R and a word w with length $|w| = B - 1$ having a (pure) period Q and an R-period P such that $\gcd(P, Q) = d$ is **not** an R-period of w.

Note that in the definition we exclude trivial cases by assuming that $P \geq 2$ and $Q \geq 3$. Namely, if $Q \leq 2$, then the word contains at most two letters and the compatibility relation must be transitive. Furthermore, it is easy to show that it suffices to consider the case where $\gcd(P, Q) = 1$; see [16, Lemma 2]. In [15] Halava, Harju and Kärki obtained the following theorem for the bound B.

Theorem 1. *Let p and q be positive integers with $\gcd(p, q) = 1$. The bound of relational interaction for p and q is $B(p, q)$ given by Table 1.*

$B(p, q)$	$p < q$	$p > q$
p, q odd	$\dfrac{p+1}{2}q$	$q + \dfrac{q-1}{2}p$
p odd, q even	$\dfrac{p+1}{2}q$	$\dfrac{p+1}{2}q$
p even, q odd	$q + \dfrac{q-1}{2}p$	$q + \dfrac{q-1}{2}p$

Table 1: Table of bounds $B(p, q)$

5 Extremal words

Let $p \geq 2$ and $q \geq 3$ be positive integers with $\gcd(p, q) = 1$ and let R be a similarity relation. From here on we consider only words with R-period p and pure period q. If such word is sufficiently long, then it has R-period $\gcd(p, q) = 1$ by Theorem 1. Like in the case of original Fine and Wilf theorem, it seems natural to ask, what properties do those words have which are of maximal length but do not have relational period equal to 1. Hence, let us study the structure of the extremal words mentioned in condition (ii) of Definition 3.

Definition 4. For positive integers $p \geq 2$ and $q \geq 3$ satisfying $\gcd(p, q) = 1$, we define the set of *extremal relational Fine and Wilf words* $FW(p, q)$. A word w is in $FW(p, q)$ if $|w| = B(p, q) - 1$ and there exists a similarity relation R such that w has an R-period p and a (pure) period q but $\gcd(p, q) = 1$ is not an R-period of w. Denote by R_w the similarity relation with minimal number of pairs of letters such that $w \in FW(p, q)$ has R_w-period p.

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Note that the relation R_w is well defined: For each letter a occurring in w, let I_a be the set of positions i such that $w_i = a$. Consider letters $\mathcal{B}_a$ in the positions $\{j \mid \exists i \in I_a : i \equiv j \pmod{p}\}$. The letter a must be R-compatible with the letters in $\mathcal{B}_a$. All other pairs involving a are unnecessary. In other words, $a \, R_w \, b \iff b \in \mathcal{B}_a$.

Note also that by the q periodicity only q different letters can occur in $FW(p,q)$. Moreover, both bounds $\frac{p+1}{2}q$ and $q + \frac{q-1}{2}p$ with $p \geq 2$ and $q \geq 3$ are greater than $p + q - 1$, which implies that the words must have at least three letters. Indeed, words over a binary alphabet $\{a, b\}$ with a relational R-period p and a pure period q and length greater than $p + q - 2$, are either unary by the theorem of Fine and Wilf or $a \, R \, b$. In both cases, $\gcd(p, q)$ is a relational period. Therefore, for $w \in FW(p,q)$, we have

$$3 \leq |\mathrm{Alph}(w)| \leq q,$$

where $\mathrm{Alph}(w)$ denotes the set of all letters occurring in w. Note that, in general, $w \in FW(p,q)$ is not unique, not even up to renaming of letters.

Example 3. Consider the set $FW(3,7)$. For $p = 3$ and $q = 7$, we have the following bound

$$B(p,q) = \frac{p+1}{2}q = 14.$$

Hence, the length of the words in $FW(3,7)$ is 13. For a ternary alphabet $\{a, b, c\}$ and the relation $R = \langle(a,b),(b,c)\rangle$, we notice that $u = babbabcbabbab$ is in FW(3,7). On the other hand, for the alphabet $\{a, b, c, d\}$, we have $v = abcacadabcaca \in FW(3,7)$ with the relation

$$R_v = \langle(a,b),(a,c),(a,d),(b,c),(c,d)\rangle.$$

Even if we restrict our considerations to words having the smallest possible number of different letters we do not have uniqueness. For example, in addition to u, $w = babbbbcbabbbb \in FW(3,7)$.

Despite the previous examples, we show that all words in $FW(p,q)$ share in some sense unique structure. We need the following definitions.

Definition 5. Let R be a similarity relation on $\mathcal{A}^*$. We say that two letters a and b are *relationally isomorphic*, more precisely, *R-isomorphic* if, for each letter $x \in \mathcal{A}$, we have

$$a \, R \, x \iff b \, R \, x.$$

A letter a is *relationally universal*, more precisely, *R-universal* if $a \, R \, x$ for all $x \in \mathcal{A}$.

In the sequel we consider words in $FW(p,q)$ such that they do not have any distinct relationally isomorphic letters and the number of occurrences of a relationally universal letter is minimal. This restriction is justified, since these words are sort of templates for other extremal relational Fine and Wilf words. Namely,

all the words in $FW(p, q)$ can be obtained up to renaming of letters from the word w described in the next theorem by two operations, namely changing some symbols to universal symbols and replacing a letter with two R_w-isomorphic letters. In this respect, $w \in FW(p, q)$ with no distinct R_w-isomorphic letters and with minimal number of occurrences of an R_w-universal letter can be called *minimal*.

We use the notation $[n]_q$ for the least positive residue of an integer $n \pmod q$, i.e., $[n]_q$ is the positive integer m satisfying $1 \leq m \leq q$ and $m \equiv n \pmod q$. For simplicity, denote also $B = B(p, q)$. We have the following theorem.

Theorem 2. *Let w be a word in $FW(p, q)$ with no distinct R_w-isomorphic letters and with minimal number of occurrences of an R_w-universal letter. This word is unique up to renaming of letters. Furthermore, it is of the form uc^{-1}, where*

$$
u = \begin{cases}
\left(\left(b^{[B]_p - 1} a b^{p - [B]_p} \right)^{\lfloor \frac{q}{p} \rfloor} b^{q - 1 - \lfloor \frac{q}{p} \rfloor p} c \right)^{\frac{p+1}{2}} & \text{if } B = \frac{p+1}{2} q \text{ and } p < q, \\
\left(b^{[B]_p - 1} a b^{q - 1 - [B]_p} c \right)^{\frac{p+1}{2}} & \text{if } B = \frac{p+1}{2} q \text{ and } p > q, \\
\left(b^{[B]_q - 1} c b^{q - [B]_q - 1} a \right)^{\frac{B - [B]_q}{q}} b^{[B]_q - 1} c & \text{otherwise,}
\end{cases}
$$

and the relation $R_w = \langle (a, b), (b, c) \rangle$.

Proof. Consider a word u with a pure period q and an R-period p. Hence u is determined by its prefix of length q and the total length of the word. Let m and n be integers in the interval $[1, q]$. Consider solutions (i, j) for the equation

$$
m + iq \equiv n + jq \pmod p. \tag{1}
$$

If there exists a solution such that $\max(m + iq, n + jq) \leq |u|$, then $u_m R u_n$ by the periods p and q. Hence, Equation (1) defines necessary relations on letters. It suffices to consider *minimal* solutions, i.e., solutions where $\max(m + iq, n + jq)$ is as small as possible. Note that if $i > j$ for some solution, then $m + (i - j) \equiv n \pmod p$ gives a smaller solution. Similarly, if $j > i$, then $m \equiv n + (j - i)q \pmod p$ gives a smaller solution. Thus, a minimal solution is of the form where either $i = 0$ or $j = 0$.

Since the relational interaction bound $B = B(p, q)$ is sufficient, there exists a minimal solution satisfying $\max(m + iq, n + jq) \leq B$ for each m and n. On the other hand, for some m' and n', there must be a minimal solution with $\max(m' + iq, n' + jq) = B$, since B is strict. Without loss of generality, we may assume that $j = 0$ and $m' + iq = B$. This implies that

$$
m' = [B]_q \quad \text{and} \quad n' \equiv B \pmod p.
$$

Consider now a word w in $FW(p, q)$ with no distinct R_w-isomorphic letters and with minimal number of occurrences of an R_w-universal letter. The above considerations imply that if a letter in $w_1 \cdots w_q$ is not $w_{[B]_q}$ and not in a position congruent to B modulo p, then it is related to all letters occurring in the word. Let

us denote these positions by U. Note that this set is not empty. The R_w-universal letter is here denoted by b. Hence, for all $i \in U$, we have $w_i = b$.

Let us now consider the position $[B]_q$. If $w_{[B]_q} = b$, then letters in positions $n \equiv B \pmod{p}$ are R_w-compatible with all the letters in w, i.e., with each other and with the universal letter b. Thus $\gcd(p, q) = 1$ is an R_w-period. This is a contradiction. Hence, the letter in position $[B]_q$ is different from b, say $w_{[B]_q} = c$. Since $\gcd(p, q) = 1$ is not an R_w-period, there must exist a letter a in some of the positions $n \equiv B \pmod{p}$ such that $(a, c) \notin R_w$. If a position n is such that the minimal solution of (1) for all $m \in [1, q]$ satisfies $\max(m + iq, n + jq) \leq |w|$, then the letter w_n is related to all the letters in $\mathrm{Alph}(w)$, i.e., $w_n = b$. If this is not the case, then the smallest solution of (1) for $m = [B]_q$ and n must satisfy $\max([B]_q + iq, n + jq) > |w|$. Since in w there is a minimal number of occurrences of the universal letter, this means that $w_n \neq b$. More precisely, $w_n R_w w_m$ for $m \in [1, q] \setminus [B]_q$ and $(w_n, w_{[B]_q}) \notin R_w$. Since w does not have any distinct R_w-isomorphic letters, we may define $w_n = a$. This shows us that all the letters w_l where $l \in [1, q]$ are determined by the minimal solutions of the Equation (1), and the word w is unique.

In order to find out the positions of the letter a more precisely, we must determine which of the positions $1 \leq n \leq q$ satisfying $n \equiv B \pmod{p}$ do not have a solution (i, j) for

$$[B]_q + iq \equiv n + jq \pmod{p} \tag{2}$$

such that $\max([B]_q + iq, n + jq) \leq B - 1$. Again, it suffices to consider minimal solutions. Since $\gcd(p, q) = 1$, we know that $\{[B]_q + iq \mid i = 0, 1, \ldots, p-1\}$ and $\{n + jq \mid j = 0, 1, \ldots, p-1\}$ are complete residue systems modulo p. Hence there exists exactly one $i \in \{0, 1, \ldots, p-1\}$ satisfying $[B]_q + iq \equiv n \pmod{p}$ and exactly one $j \in \{0, 1, \ldots, p-1\}$ satisfying $[B]_q \equiv n + jq \pmod{p}$. Furthermore, for $i \in \{1, 2, \ldots, p-1\}$, we have

$$[B]_q + iq \equiv n \pmod{p} \implies [B]_q \equiv n + (p-i)q \pmod{p},$$

and $p - i \in \{1, 2, \ldots p - 1\}$. Hence, the minimal solution of (2) is either of the form $(i, 0)$ or $(0, p - i)$.

Consider first those cases where $B(p, q) = \frac{p+1}{2}q$ and assume that $n \equiv B \pmod{p}$. For a solution $(i, j) = (\frac{p-1}{2}q, 0)$ we have $[B]_q + iq = q + \frac{p-1}{2}q = B$. For the other solution $(0, p - i)$, we have

$$n + (p - i)q = n + pq - \frac{p-1}{2}q = \frac{p+1}{2}q + n = B + n.$$

This proves that letters in the position $1 \leq n \leq q$ satisfying $n \equiv B \pmod{p}$ are non universal, i.e., all the letters are a's. Note that if $B = \frac{p+1}{2}q$ and $p > q$, then q is even by Table 1 and $B \equiv \frac{q}{2} \pmod{p}$. Hence, $n \in [1, q]$, $n \equiv B \pmod{p}$ really exists.

Consider then the cases where $B(p, q) = q + \frac{q-1}{2}p$. Now $n = q - kp$ for some $k = 0, 1, \ldots, \lfloor \frac{q}{p} \rfloor$. Like above, $(i, j) = (\frac{B - [B]_q}{q}, 0)$ is a solution where

$[B]_q + iq = B$. For the other solution $(0, p - i)$, we have

$$
\begin{aligned}
n + (p - i)q &= n + pq - B + [B]_q = q - kp + pq - q - \frac{q-1}{2}p + [B]_q \\
&= B + [B]_q + (p - q) - kp.
\end{aligned}
$$

If $p > q$, then $k = 0$ and $p - q > 0$. Hence, $n + (p - i)q > B$. If $p < q$, then p is even by Table 1. Hence, $[B]_q = q - \frac{p}{2}$. We get $n + (p - i)q = B + \frac{p}{2} - kp > B$ if and only if $k = 0$. Hence, the only position $n \in [1, q] \setminus [B]_q$ where $w_n = a$ is $n = q$. These calculations imply the words of the statement. $\qquad\square$

We note that the relation $R_w = \langle (a, b), (b, c) \rangle$ in Theorem 2 which was used in defining the minimal extremal words in $FW(p, q)$ corresponds to the compatibility relation of partial words.

Like in the case of normal extremal Fine and Wilf words [12, 19], the minimal extremal relational Fine and Wilf words given in Theorem 2 have nice palindromic properties. A word $w = w_1 \cdots w_n$ is a *palindrome* if $w = \overline{w}$, where $\overline{w} = w_n w_{n-1} \cdots w_1$. A generalization of palindromic words are so called pseudo-palindromic words.

Definition 6. Let $\varphi \colon \mathcal{A} \to \mathcal{A}$ be a morphism satisfying $\varphi^2 = \mathrm{id}$. A word $w = w_1 \cdots w_n$ is a *φ-pseudo-palindrome* if $w = \varphi(\overline{w})$.

For more information on palindromes and pseudo-palindromes, see [1,6,7,11]. In the present paper we prove:

Theorem 3. *Let $w \in \mathcal{A} = \{a, b, c\}$ be a word in $FW(p, q)$ with no distinct R_w-isomorphic letters and with minimal number of occurrences of an R_w-universal letter. Let $R_w = \langle (a, b), (b, c) \rangle$. If $B(p, q) = \frac{p+1}{2}q$, then w is a palindrome. Otherwise, it is a φ-pseudo-palindrome, where $\varphi \colon \mathcal{A} \to \mathcal{A}$ is defined by $\varphi(a) = c$ and $\varphi(b) = b$.*

Proof. The word w is given by the formula of Theorem 2. Consider first $w \in FW(p, q)$ such that $B(p, q) = \frac{p+1}{2}q$. Suppose that $w_m = a$. By Theorem 2, $m = n + iq$ for some i and $1 \le n \le q$ satisfying $n \equiv B \pmod{p}$. Since $B \equiv 0 \pmod{q}$, $w_{B-n-iq} = w_{q-n}$ by the period q. Since $n \equiv B \pmod{p}$, we have $q - n \equiv q - B + pq = \frac{p+1}{2}q = B \pmod{p}$. This means that $w_{B-m} = w_{q-n} = a$.

Then consider occurrences of c in w. Suppose now that $w_m = c$. By Theorem 2, $m \equiv 0 \pmod{q}$. Since $B = \frac{p+1}{2}q$, $B - m \equiv 0 \pmod{q}$. This implies that $w_{B-m} = c$ and we have shown that $w_m = w_{B-m} = w_{|w|+1-m}$ if $w_m = a$ or $w_m = c$. Hence, this is true also for $w_m = b$ and the word w is a palindrome.

Next consider $w \in FW(p, q)$ such that $B(p, q) = q + \frac{q-1}{2}p$. By Theorem 2 we know that if $w_m = a$, then $m \equiv 0 \pmod{q}$. Now $B - m \equiv B \pmod{q}$. Hence, $w_{B-m} = w_{[B]_q} = c$. On the other hand, if $w_m = c$, then $m \equiv B \pmod{q}$ and $B - m \equiv 0 \pmod{q}$. Thus, $w_{B-m} = w_q = a$. This means that $w_m = \varphi(w_{B-m}) = \varphi(w_{|w|+1-m})$, i.e., w is a φ-pseudo-palindrome. $\qquad\square$

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Finally, we note that more relational variations of Fine and Wilf's theorem can be found in [16]. For example, the *local period of partial words* [2], is generalized using similarity relations and new interaction bounds concerning this *local relational period* are proved.

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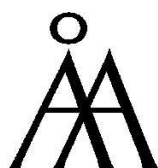[19] R. Tijdeman, L.Q. Zamboni, Fine and Wilf words for any periods. Indag. Math. 14 (2003) 135–147.

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