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Abstract

In this paper, Information Logic of Galois Connections (ILGC) suited for approximate reasoning about knowledge is introduced. Its axiomatization and Kripke-style semantics based on information relations is defined, and its completeness is proved. It is also shown that ILGC is equivalent to the minimal tense logic K_t, and decidability of ILGC follows from this observation. Additionally, relationship of ILGC to the modal logic S4 is studied. Namely, a certain composition of Galois connection mappings forms a lattice-theoretical interior operator, and this motivates us to axiomatize a logic of these compositions. The axiomatization resembles the one of S4, except that our logic is not 'normal' in the sense that axioms N and K of S4 are not included in the set of axioms. Finally, so-called interior model is introduced to define semantics and validity, and completeness of this logic is proved as well.

Keywords: rough sets, fuzzy sets, approximate reasoning, knowledge representation, modal logic

1 Introduction

The theory of rough sets introduced by Pawlak [25] can be viewed as an extension of the classical set theory. Its fundamental idea is that our knowledge about objects of a given universe of discourse U may be inadequate or incomplete. The objects can be then observed only with the accuracy restricted by some indiscernibility relation. According to the Pawlak's original definition, an indiscernibility relation E on U is an equivalence interpreted so that two elements of U are E-related if they cannot be distinguished by their properties. Since there is one-to-one correspondence between equivalences and partitions, each indiscernibility relation induces a partition on U. In this sense, our ability to distinguish objects can be understood to be blurred – we cannot distinguish individual objects, only their equivalence classes.

Each subset X of U can be approximated by two sets: the lower approximation $X^{\blacktriangledown}$ of X consists of E-equivalence classes that are included in X, and X's upper approximation $X^{\blacktriangle}$ contains E-classes intersecting with X. The lower approximation $X^{\blacktriangledown}$ can be viewed as a set of elements that are certainly in X and the upper approximation $X^{\blacktriangle}$ can be considered as a set of elements that possibly belong to X. Note also that approximations can be viewed to be definable or exact in the sense that they are unions of classes of indistinguishable elements. This may be interpreted so that definable sets are describable as the conjunction of the properties of the objects they contain.

The literature, however, contains studies in which rough approximations are defined by relations that are not necessarily equivalences (see e.g. [10] for further details). Note also that Järvinen, Kondo and Kortelainen [11, 12] have studied approximations in a more general setting of complete atomic Boolean lattices. They have also studied definable sets determined by indiscernibility relations of different types in [13, 15].

To be as general as possible, in this paper it is allowed R to be any arbitrary binary relation. We may also define for each subset X of the universe U the rough set approximations by the means of the inverse R^{-1} of R and these sets are denoted by $X^{\triangledown}$ and $X^{\vartriangle}$. Therefore, for every X we may attach two lower approximations, $X^{\blacktriangledown}$ and $X^{\triangledown}$, and two upper approximations $X^{\blacktriangle}$ and $X^{\vartriangle}$. Note also that the studies appearing in the literature usually consider the pair of rough approximation mappings ${}^{\blacktriangle}$ and ${}^{\blacktriangledown}$ that are mutually dual. However, in this work we focus on the pair $({}^{\blacktriangle}, {}^{\triangledown})$ forming a Galois connection. Obviously, even in this more general case, the set $X^{\triangledown}$ can be considered as the set of elements that necessarily are in X, because if an element y is R-related to some element $x \in X^{\triangledown}$, then y must be in X. Similarly, $X^{\blacktriangle}$ may be viewed as the set of elements possibly belonging to X, since if $x \in X^{\blacktriangle}$, then there exists an element y in X to which x is R-related.

In this work we also shortly consider Pawlak's information systems [24]. They consist of a set U of objects and a set of attributes A. Every attribute $a \in A$ attaches the value $a(x)$ of the attribute a to the object x. The key idea in Pawlak's information systems is that each subset $B \subseteq A$ of attributes determines an indiscernibility relation ind(B) which is defined so that two objects x and y of the universe U are B-indiscernible if their values for all attributes in the set B are equal, that is, $a(x) = a(y)$ for all $a \in B$. Orłowska and Pawlak introduced in [23] many-valued information systems as a generalization of Pawlak's original systems. In a many-valued information system each attribute attaches a set of values to objects. Therefore, in many-valued systems it is possible to define several types of information relations reflecting distinguishability or indistinguishability of objects of the system.

Also L-sets introduced by Goguen [8] determine relations reflecting knowledge about objects. The idea presented by Kortelainen [17, 18] is that each L-set φ on U determines a binary relation $\lesssim$ such that $x \lesssim y$ holds, whenever y belongs to the set represented by φ at least at the same extent as x. Now the relation $\lesssim$, or its inverse $\gtrsim$ as well, can be used to determine the approximation mappings. The essential connections between modal-like operators, topologies and fuzzy sets are studied in [14].

In the literature there are several studies on logical foundations of rough sets. Usually these logics have a semantics similar to the one by Kripke [16]. In the paper [26], Pawlak formulated some notions of rough logics. Based on these ideas, Rasiowa and Skowron [27] have introduced first-order predicate logic suited for rough approximations and definability. Orłowska with her coauthors has extensively studied several logics for knowledge representation – see [7, 22, 23], for example. Also Vakarelov [28, 29] has investigated modal logics for information relations of many-valued information systems. Many of these mentioned logics are examined for instance in survey papers [2, 31]. Orłowska has also introduced Kripke models with relative accessibility relations in [21] – these are modifications of the ordinary Kripke structure such that 'accessibility relations' are determined by sets of parameters interpreted as a properties of objects. In addition to this, Demri and Goré [4] have defined cut-free display calculi for knowledge representation logics with relative accessibility relations. It also should be mentioned that Mattila has considered so-called modifier logics closely related to fuzzy logic in several works; see e.g. [19], for further details and references. For example, in [20] a modifier calculi together with relational frame semantics and some ideas for topological semantics is given. Finally, note that von Karger has developed in [30] several temporal logics from the theory of complete lattices, Galois connections, and fixed points.

This paper is presented as follows: In Section 2 we define Galois connections and recall some of their well-known properties. We also introduce generalized rough approximation operations based on information relations. We show how they induce Galois connections and give two examples of approximation operators determined by information relations of information systems and L-sets. Section 3 introduces Information Logic of Galois Connections (ILGC). ILGC is just the standard propositional logic with two modal connectives $\triangledown$ and $\blacktriangle$. Concerning $\triangledown$ and $\blacktriangle$, we have two additional rules of inference mimicking the definition of Galois connection. We also give its semantics and show that the logic is complete. The final section gives some relationships between ILGC, minimal tense logic K_t and modal logic S4. Also the decidability of ILGC is shown there.

2 Galois Connections of Information Relations

We begin our study by recalling Galois connections and their basic properties; these can be found in [6], for example. For two ordered sets P and Q, a pair $({}^{\blacktriangleright}, {}^{\triangleleft})$ of maps ${}^{\blacktriangleright}\colon P \to Q$ and ${}^{\triangleleft}\colon Q \to P$ is called a *Galois connection* between P and Q if for all $p \in P$ and $q \in Q$,

$$p^{\blacktriangleright} \leq q \iff p \leq q^{\triangleleft}.$$

The function ${}^{\blacktriangleright}$ is called a *residuated map* and the function ${}^{\triangleleft}$ is called a *residual map*. The next proposition gives some well-known properties of Galois connections.

Proposition 2.1 *Assume* $({}^{\blacktriangleright}, {}^{\triangleleft})$ *is a Galois connection between ordered sets* P *and* Q*. Let* $p, p_1, p_2 \in P$ *and* $q, q_1, q_2 \in Q$*. Then the following assertions hold:*

(i) $p_1 \leq p_2 \Rightarrow {p_1}^{\blacktriangleright} \leq {p_2}^{\blacktriangleright}$ *and* $q_1 \leq q_2 \Rightarrow {q_1}^{\triangleleft} \leq {q_2}^{\triangleleft}$.

(ii) $p \leq p^{\blacktriangleright\triangleleft}$ *and* $q^{\triangleleft\blacktriangleright} \leq q$.

(iii) $p^{\blacktriangleright} = p^{\blacktriangleright\triangleleft\blacktriangleright}$ *and* $q^{\triangleleft} = q^{\triangleleft\blacktriangleright\triangleleft}$.

(iv) ${}^{\blacktriangleright}$ *preserves all existing joins and* ${}^{\triangleleft}$ *preserves all existing meets.*

(v) *The composite* ${}^{\blacktriangleright\triangleleft}\colon P \to P$ *is a lattice-theoretical closure operator and the composite* ${}^{\triangleleft\blacktriangleright}\colon Q \to Q$ *is a lattice-theoretical interior operator.*

It is known that $({}^{\blacktriangleright}, {}^{\triangleleft})$ is a Galois connection between two ordered sets if and only if ${}^{\blacktriangleright}$ and ${}^{\triangleleft}$ satisfy (i) and (ii). Notice also that Galois connections were originally defined with functions that reverse order. We use the above form since it is more suitable for our purposes.

Proposition 2.1 implies that if P and Q are bounded lattices, then ${}^{\blacktriangleright}$ is a $\vee$-homomorphism and ${}^{\triangleleft}$ is a $\wedge$-homomorphism, that is, $(a \vee b)^{\blacktriangleright} = a^{\blacktriangleright} \vee b^{\blacktriangleright}$ and $(x \wedge y)^{\triangleleft} = x^{\triangleleft} \wedge y^{\triangleleft}$ for all $a, b \in P$ and $x, y \in Q$. Additionally, ${}^{\blacktriangleright}$ is $\bot$-preserving and ${}^{\triangleleft}$ is $\top$-preserving, that is, $\bot^{\blacktriangleright} = \bot$ and $\top^{\triangleleft} = \top$.

Next we consider generalized rough set approximations. Let U be a set, called the *universe of discourse* and let R be a binary relation on U. The *upper approximation* of a set $X \subseteq U$ is

$$X^{\blacktriangle} = \{x \in U \mid (\exists y \in U)\, x\, R\, y \;\&\; y \in X\}$$

and the *lower approximation* of X is

$$X^{\blacktriangledown} = \{x \in U \mid (\forall y \in U)\, x\, R\, y \Rightarrow y \in X\}.$$

Obviously, the maps are dual, that is, for any $X \subseteq U$,

$$X^{c\blacktriangle} = X^{\blacktriangledown c} \text{ and } X^{c\blacktriangledown} = X^{\blacktriangle c},$$

where $X^c = \{x \in U \mid x \notin X\}$ is the complement of X in the universe U.

We may also define an analogous pair of mappings $\wp(U) \to \wp(U)$ by reversing the relation R. For any set $X \subseteq U$, let us define

$$X^{\vartriangle} = \{x \in U \mid (\exists y \in U)\, y\, R\, x \;\&\; y \in X\}$$

and

$$X^{\triangledown} = \{x \in U \mid (\forall y \in U)\, y\, R\, x \Rightarrow y \in X\}.$$

Trivially, $^{\vartriangle}$ and $^{\triangledown}$ also are dual.

The next result is well-known.

Proposition 2.2 *For any binary relation, the pairs* $(^{\blacktriangle}, ^{\triangledown})$ *and* $(^{\vartriangle}, ^{\blacktriangledown})$ *are Galois connections.*

We end this section by considering two more concrete examples of approximation operations.

Information Relations. Many-valued information systems were introduced in [23], and different types of information relations considered here can be found in [5], for instance. A *many-valued information system* is a pair (U, A), where U is a set of objects and A is a set of attributes such that each attribute is a map $a\colon U \to \wp(V_a)$. This means that attributes attach sets of values to objects. For example, if a is the attribute 'knowledge of languages' and a person denoted by x knows English and Finnish, then $a(x) = \{$English, Finnish$\}$.

Objects of an information system may be related in different ways with respect to their values of attributes. We recall some *information relations* reflecting indistinguishability of objects of an information system (U, A). For any $B \subseteq A$, the following relations may be defined:

$$\begin{array}{rcl}
(x,y) \in \mathrm{ind}(B) & \iff & (\forall a \in B)\, a(x) = a(y) \\
(x,y) \in \mathrm{sim}(B) & \iff & (\forall a \in B)\, a(x) \cap a(y) \neq \emptyset \\
(x,y) \in \mathrm{inc}(B) & \iff & (\forall a \in B)\, a(x) \subseteq a(y)
\end{array}$$

These relations are referred to as B*-indiscernibility*, B*-similarity* and B*-inclusion*, respectively.

If a is again the attribute 'knowledge of languages' and R is the a-similarity relation, then two objects x and y are R-related if they have a common language. The similarity relation is obviously symmetric, which gives that $X^{\blacktriangle} = X^{\vartriangle}$ and $X^{\blacktriangledown} = X^{\triangledown}$. Obviously, $x \in X^{\blacktriangle}$ if there exists a person $y \in X$ which has a common language with x. Similarly, $x \in X^{\triangledown}$ if all persons having a common language with x are in X.

Fuzzy Sets. Fuzzy sets were defined first time by Zadeh [32] as mappings from a non-empty set U into the unit interval $[0, 1]$. Then, fuzzy sets were generalized to L-fuzzy sets by Goguen [8] in such a way that an *L-fuzzy set* φ on U is a mapping $\varphi \colon U \to L$, where L is equipped with some ordering structure. However, in this paper, we use the term 'L-set' instead of 'L-fuzzy set'.

Notice that in the literature L is usually assumed to be at least a complete lattice. The motivation for this is that in such a setting it is possible to consider many-valued logics in which some truth values are incomparable. The least element $\perp$ and the greatest element $\top$ of L may be viewed as the 'absolute' truth values *false* and *true*. In the current work, L is always assumed to be a preordered set, that is, the set L is equipped with a reflexive and transitive binary relation $\leq$. Typically, L may consist of attributes such as 'good', 'excellent', 'poor' and 'adequate', for example. Notice that it is natural to assume that the relation $\leq$ is not antisymmetric: if $x, y \in L$ are synonyms, that is, words or expressions that are used with the same meaning, then $x \leq y$ and $x \geq y$, but still x and y are distinct words. This more general setting enables us to move towards the methodology called *computing with words* [33], in which the objects of computation are given by a natural language. Computing with words, in general, is inspired by the human capability to perform a wide variety of tasks without any measurements and any quantizations.

As noted in [17], each L-set $\varphi \colon U \to L$ determines a preorder $\lesssim$ on U by

$$x \lesssim y \iff \varphi(x) \leq \varphi(y).$$

Assume now that $\varphi \colon U \to L$ is an L-set describing the ability of persons in U to speak Japanese. Furthermore, we denote the inverse relation of $\lesssim$ by $\gtrsim$. Then, $x \gtrsim y$ is true if x can speak Japanese at least as well as y.

Let us consider the approximations defined by the relation $\gtrsim$, that is,

$$X^{\blacktriangle} = \{x \in U \mid (\exists y \in U)\, x \gtrsim y \;\&\; y \in X\}$$

and

$$X^{\triangledown} = \{x \in U \mid (\forall y \in U)\, y \gtrsim x \Rightarrow y \in X\}.$$

Now, $x \in X^{\blacktriangle}$ if and only if x can speak Japanese at least as well as some person in X. Furthermore, $x \in X^{\triangledown}$ if and only if $y \gtrsim x$ implies $y \in X$, that is, there cannot be a person outside X speaking Japanese at least as well as x. Thus, approximations have a nice interpretation also in case of fuzzy sets.

Notice that the other pair of approximation maps $({}^{\vartriangle}, {}^{\blacktriangledown})$ also forming a Galois connection is defined by $x \in X^{\vartriangle}$ if and only if there exists $y \in U$ such that $x \lesssim y$ and $y \in X$, and $x \in X^{\blacktriangledown}$ whenever for all $y \in U$, $y \lesssim x$ implies $y \in X$.

3 Information Logic of Galois Connections

In this section, we introduce a simple propositional logic ILGC – an acronym for Information Logic of Galois Connections – with two additional connectives $\blacktriangle$ and

∇. We begin with introducing the syntax and the semantics of this language.

3.1 Syntax and Semantics

Let P be an enumerable set, whose elements are called *propositional variables*. The set of *connectives* consists of logical symbols $\rightarrow$, $\neg$, $\blacktriangle$, and $\triangledown$. A *formula* of ILGC is defined inductively as follows:

(i) Every propositional variable is a formula.

(ii) If A and B are formulae of ILGC, then so are $A \rightarrow B$, $\neg A$, $\blacktriangle A$, and $\triangledown A$.

Let us denote by Φ the set of all formulae of ILGC.

The logical system ILGC has the following three *axioms* of classical propositional logic:

(Ax1) $A \rightarrow (B \rightarrow A)$

(Ax2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(Ax3) $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

Furthermore, ILGC has the following three *rules of inference*:

$$\text{(MP)} \quad \frac{A \qquad A \rightarrow B}{B} \qquad \text{(GC1)} \quad \frac{A \rightarrow \triangledown B}{\blacktriangle A \rightarrow B} \qquad \text{(GC2)} \quad \frac{\blacktriangle A \rightarrow B}{A \rightarrow \triangledown B}$$

The first rule is the classical *modus ponens*, and (GC1) and (GC2) mimic the conditions appearing in the definition of Galois connections.

An ILGC-formula A is said to be *provable*, if there is a finite sequence $A_1, A_2, \ldots, A_n$ of ILGC-formulae such that $A = A_n$ and for every $1 \leq i \leq n$:

(i) either A_i is an axiom of ILGC

(ii) or A_i is the conclusion of some inference rules, whose premises are in the set $\{A_1, \ldots, A_{i-1}\}$.

That A is provable in ILGC is denoted by $\vdash A$.

For the sake of simplicity the following abbreviations for *disjunction*, *conjunction*, *equivalence*, *true*, and *false* are introduced:

$$\begin{aligned} A \vee B &:= (A \rightarrow B) \rightarrow B \\ A \wedge B &:= \neg(\neg A \vee \neg B) \\ A \leftrightarrow B &:= (A \rightarrow B) \wedge (B \rightarrow A) \\ \top &:= A \vee \neg A \\ \bot &:= A \wedge \neg A \end{aligned}$$

Our next proposition presents some provable formulae and additional inference rules of ILGC.

Proposition 3.1 *For all* ILGC-*formulae* A *and* B*, we have:*

(i) $\dfrac{A \to B}{\triangledown A \to \triangledown B}$ *and* $\dfrac{A \to B}{\blacktriangle A \to \blacktriangle B}$

(ii) $\vdash A \to \triangledown\blacktriangle A$ *and* $\vdash \blacktriangle\triangledown A \to A$

(iii) $\vdash \triangledown A \leftrightarrow \triangledown\blacktriangle\triangledown A$ *and* $\vdash \blacktriangle A \leftrightarrow \blacktriangle\triangledown\blacktriangle A$

(iv) $\vdash \triangledown\top \leftrightarrow \top$ *and* $\vdash \blacktriangle\bot \leftrightarrow \bot$

(v) $\dfrac{A}{\triangledown A}$

(vi) $\vdash \triangledown(A \wedge B) \leftrightarrow \triangledown A \wedge \triangledown B$ *and* $\vdash \blacktriangle(A \vee B) \leftrightarrow \blacktriangle A \vee \blacktriangle B$

(vii) $\vdash \triangledown(A \to B) \to (\triangledown A \to \triangledown B)$

Proof. Note that we prove only the first claims of (i)–(iv) and (vi), because their second parts can be proved in an analogous manner.

(i) Suppose that $\vdash A \to B$. Because $\vdash \triangledown A \to \triangledown A$ holds trivially, we obtain $\vdash \blacktriangle\triangledown A \to A$ by (GC1). Hence $\vdash \blacktriangle\triangledown A \to B$, which gives $\vdash \triangledown A \to \triangledown B$ by (GC2).

(ii) Because $\vdash \blacktriangle A \to \blacktriangle A$, we have $\vdash A \to \triangledown\blacktriangle A$ by (GC2).

(iii) Obviously, by (ii), $\vdash \triangledown A \to \triangledown\blacktriangle\triangledown A$. Furthermore, since $\vdash \blacktriangle\triangledown A \to A$, we get $\vdash \triangledown\blacktriangle\triangledown A \to \triangledown A$ by (i).

(iv) It is clear that $\vdash \triangledown\top \to \top$. Conversely, $\vdash \blacktriangle\top \to \top$ implies $\vdash \top \to \triangledown\top$ by (GC2).

(v) Assume $\vdash A$. This means $\vdash \top \to A$ and we get $\vdash \triangledown\top \to \triangledown A$ by (i). Because $\top \to \triangledown\top$ by (iv), we obtain $\vdash \top \to \triangledown A$. Thus, $\vdash \triangledown A$.

(vi) Because $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, we have $\vdash \triangledown(A \wedge B) \to \triangledown A$ and $\vdash \triangledown(A \wedge B) \to \triangledown B$ by (i). Hence, $\vdash \triangledown(A \wedge B) \to \triangledown A \wedge \triangledown B$. On the other hand, $\vdash \triangledown A \wedge \triangledown B \to \triangledown A$ yields $\vdash \blacktriangle(\triangledown A \wedge \triangledown B) \to A$ by (GC2). Similarly, we may show $\vdash \blacktriangle(\triangledown A \wedge \triangledown B) \to B$. This gives that $\vdash \blacktriangle(\triangledown A \wedge \triangledown B) \to A \wedge B$ and $\vdash \triangledown A \wedge \triangledown B \to \triangledown(A \wedge B)$ by (GC2).

(vii) Since $\vdash A \wedge (A \to B) \to B$, we have $\vdash \triangledown(A \wedge (A \to B)) \to \triangledown B$. Furthermore, by (vi), we obtain $\vdash \triangledown A \wedge \triangledown(A \to B) \to \triangledown(A \wedge (A \to B))$. Thus, $\vdash \triangledown A \wedge \triangledown(A \to B) \to \triangledown B$, which is equivalent to $\vdash \triangledown(A \to B) \to (\triangledown A \to \triangledown B)$. □

We may also introduce another pair $\vartriangle$ and $\blacktriangledown$ of connectives. This is done by defining them as the *duals* of $\triangledown$ and $\blacktriangle$. Let us set

$$\vartriangle A := \neg\triangledown\neg A \quad \text{and} \quad \blacktriangledown A := \neg\blacktriangle\neg A.$$

For the connectives $\vartriangle$ and $\blacktriangledown$, we have similar inference rules that we have for the original connectives $\blacktriangle$ and $\triangledown$.

Lemma 3.2 *For all* ILGC-*formulae* A *and* B, *we have*

$$\frac{A \to \blacktriangledown B}{\vartriangle A \to B} \qquad \textit{and} \qquad \frac{\vartriangle A \to B}{A \to \blacktriangledown B}.$$

Proof. We prove the first rule – the second can be prove in an analogous manner. Assume that $\vdash A \to \blacktriangledown B$. By (Ax3), $\vdash (A \to \blacktriangledown B) \to (\neg\blacktriangledown B \to \neg A)$. Therefore, $\vdash \neg\blacktriangledown B \to \neg A$ by (MP) and hence $\vdash \blacktriangle\neg B \to \neg A$. By applying (GC2), we obtain $\vdash \neg B \to \triangledown\neg A$ and $\vdash \neg B \to \neg \vartriangle A$. This implies $\vdash\vartriangle A \to B$ by (Ax3) and (MP). □

Note that Lemma 3.2 means that the connectives $\blacktriangledown$ and $\vartriangle$ have all the properties listed in Proposition 3.1 for $\triangledown$ and $\blacktriangle$.

In the sequel, we introduce the semantics of the language ILGC. A relational structure $\mathcal{F} = (U, R)$, where U is a nonempty set and R is a binary relation on U, is called an ILGC-*frame*. Let v be a function $v \colon P \to \wp(U)$ assigning to each propositional variable p in P a subset $v(p)$ of U. Such functions are called *valuations* and the triple $\mathcal{M} = (U, R, v)$ is called an ILGC-*model*.

For any $x \in U$ and $A \in \Phi$, we define a *satisfiability relation* $\mathcal{M}, x \models A$ according the usual Kripke semantics of the formula A inductively by the following way:

$\mathcal{M}, x \models p$ iff $x \in v(p)$

$\mathcal{M}, x \models \neg A$ iff $\mathcal{M}, x \not\models A$

$\mathcal{M}, x \models A \to B$ iff $\mathcal{M}, x \models A$ implies $x \models B$

$\mathcal{M}, x \models \blacktriangle A$ iff there exists $y \in U$ such that $x\,R\,y$ and $\mathcal{M}, y \models A$

$\mathcal{M}, x \models \triangledown A$ iff for all $y \in U$, $y\,R\,x$ implies $\mathcal{M}, y \models A$

We may extend the valuation function v to all Φ-formulae by setting

$$v(A) = \{x \in U \mid \mathcal{M}, x \models A\}.$$

It is then easy to see that for all $A, B \in \Phi$:

(i) $v(\bot) = \emptyset$ and $v(\top) = U$

(ii) $v(A \vee B) = v(A) \cup v(B)$ and $v(A \wedge B) = v(A) \cap v(B)$

(iii) $v(\neg A) = v(A)^c$ and $v(A \to B) = v(A)^c \cup v(B)$

(iv) $v(\blacktriangle A) = v(A)^{\blacktriangle}$ and $v(\triangledown A) = v(A)^{\triangledown}$

(v) $v(\vartriangle A) = v(A)^{\vartriangle}$ and $v(\blacktriangledown A) = v(A)^{\blacktriangledown}$

An ILGC-formula A is said to be *true* in an ILGC-model $\mathcal{M} = (U, R, v)$, written $\mathcal{M} \models A$, if for all $x \in U$, $\mathcal{M}, x \models A$. Furthermore, if A is true in all ILGC-models based on (U, R), then A is *valid*.

Example 3.3 In classical modal logic necessity and possibility are usually explained by reference to the notion of *possible worlds* in such a way that a valuation gives a truth value to each propositional variable for each of the possible worlds. Hence, the value assigned to a propositional variable p for world w may differ from the value assigned to p for another world w'. Similarly, in temporal logics, the same sentence may have different truth values in different times. The logic ILGC can be interpreted as an information logic in which formulae are viewed to represent properties that objects of a given restricted universe of discourse may have.

For example, let U be some set of human beings and let R be a relation reflecting similarity of people with respect to some suitable attributes – what those properties might be is irrelevant for this consideration. Then, the pair $\mathcal{F} = (U, R)$ is clearly an ILGC-frame. Let $\mathcal{M} = (U, R, v)$ be a model based on the frame $\mathcal{F}$ and let A be an ILGC-formula such that $v(A)$ consists of 'good teachers'. Then, $\mathcal{M}, x \models A$ can be interpreted as a sentence 'x is a good teacher', and $\mathcal{M}, x \models \blacktriangle A$ holds if there exists $y \in U$ such that $x\,R\,y$ and $\mathcal{M}, y \models A$, that is, there is a good teacher y to which x is similar. Analogously, $\mathcal{M}, x \models \triangledown A$ means that $y\,R\,x$ implies $\mathcal{M}, y \models A$, that is, all people similar to x are good teachers.

In case of fuzzy sets, we may consider a situation in which an L-set $\varphi\colon U \to L$ represents how an expert evaluates the suitability of the persons in U to act as a teacher by using some expressions and attributes L of his own language. Let us now consider the relation $\gtrsim$ on U. Then $x \gtrsim y$ means simply that the expert has the opinion that x is at least as good teacher as y. Let B now be an ILGC-formula such that people in $v(B)$ as currently acting as teachers. Then, $\mathcal{M}, x \models \blacktriangle B$ holds if there exists $y \in U$ such that $x \gtrsim y$ and $\mathcal{M}, y \models B$, that is, x is at least as good as one acting teacher, and $\mathcal{M}, x \models \triangledown B$ if $y \gtrsim x$ implies $\mathcal{M}, y \models B$, which may be interpreted so that all persons who have at least as good teaching abilities as x are all acting as teachers.

Note also that being a valid formula has the interpretation that all objects in the universe of discourse U have the property the formula represents.

3.2 Completeness

We conclude Section 3 by showing the completeness of ILGC. We adopt the standard techniques that can be found in [3], for example. First, the soundness theorem of ILGC is presented.

Theorem 3.4 (Soundness Theorem) *Each provable* ILGC-*formula is valid.*

Proof. It is enough to show that every axiom is valid and each rule of inference preserves validity. As an example, we only prove the case that the rule (GC1) of inference preserves validity. Suppose that $A \to \nabla B$ is valid but $\blacktriangle A \to B$ is not. There is an ILGC-model $\mathcal{M} = (U, R, v)$ and an element $x \in U$ such that $\mathcal{M}, x \not\models \blacktriangle A \to B$, that is, $\mathcal{M}, x \models \blacktriangle A$ but $\mathcal{M}, x \not\models B$. This means that there exists $y \in U$ such that $x\,R\,y$ and $\mathcal{M}, y \models A$ by $\mathcal{M}, x \models \blacktriangle A$. Since $A \to \nabla B$ is valid and $\mathcal{M}, y \models A$, we have $\mathcal{M}, y \models \nabla B$. It follows from $x\,R\,y$ that $\mathcal{M}, x \models B$, a contradiction! Thus, the rule (GC1) of inference preserves validity. $\square$

Next we shall show the converse, that is, every valid ILGC-formula is provable. We first recall some notions that will be needed for the proof. A subset Γ of ILGC-formulae is called *inconsistent* if there are formulae $A_1, \ldots, A_n \in \Gamma$ such that $\vdash \neg(A_1 \wedge \cdots \wedge A_n)$; otherwise Γ is *consistent*. We set $\Gamma \vdash A$ to denote that there are formulae $A_1, \ldots, A_n \in \Gamma$ such that $\vdash A_1 \wedge \cdots \wedge A_n \to A$. Additionally, a set Γ of ILGC-formulae is *maximal consistent* if Γ is consistent, and any set of formulae properly containing Γ is inconsistent.

The next lemma presents some important properties of maximal consistent sets.

Lemma 3.5 *Let Γ be a maximal consistent set of* ILGC*-formulae. Then for any $A, B \in \Phi$:*

(i) $\Gamma \vdash A \iff A \in \Gamma \iff \neg A \notin \Gamma.$

(ii) Γ *is closed under modus ponens, that is, if A and $A \to B$ are in Γ, then also B is in Γ.*

(iii) $A \wedge B \in \Gamma \iff A \in \Gamma \text{ and } B \in \Gamma.$

(iv) $A \vee B \in \Gamma \iff A \in \Gamma \text{ or } B \in \Gamma.$

Proof. (i) Suppose that $\Gamma \vdash A$, but $A \notin \Gamma$. Because $\Gamma \subset \Gamma \cup \{A\}$ and Γ is a maximal consistent set, we conclude that $\Gamma \cup \{A\}$ is inconsistent. This means that there are formulae $A_1, \ldots, A_n \in \Gamma$ such that $\vdash \neg(A_1 \wedge \cdots \wedge A_n \wedge A)$, which is equivalent to $\vdash A_1 \wedge \cdots \wedge A_n \to \neg A$. Since each $A_i \in \Gamma$, this implies that $\Gamma \vdash \neg A$, a contradiction! Therefore, $A \in \Gamma$.

Assume $A \in \Gamma$ and $\neg A \in \Gamma$. Then $\{A, \neg A\} \subseteq \Gamma$ and $\vdash \neg(A \wedge \neg A)$, that is, Γ is inconsistent, a contradiction! Hence, $\neg A \notin \Gamma$.

Suppose that $\neg A \notin \Gamma$. In this case, the set $\Gamma \cup \{A\}$ must be consistent. Otherwise, there would be some formulae $A_1, \ldots, A_n$ such that $\vdash \neg(A_1 \wedge \cdots \wedge A_n \wedge A)$. This is equivalent to $\vdash A_1 \wedge \cdots \wedge A_n \to \neg A$. Thus, $\Gamma \vdash \neg A$ and $\neg A \in \Gamma$, a contradiction! Hence $\Gamma \cup \{A\}$ is consistent. Since Γ is a maximal consistent set, we have that $A \in \Gamma$. Trivially, this implies $\Gamma \vdash A$.

(ii) Assume $A \to B \in \Gamma$ and $A \in \Gamma$. Then there exists $A_1, \ldots, A_n \in \Gamma$ such that $\vdash (A_1 \wedge \cdots \wedge A_n) \to (A \to B)$. This is equivalent to $\vdash (A_1 \wedge \cdots A_n \wedge A) \to B$. Since also $A \in \Gamma$, we have $\Gamma \vdash B$, that is, $B \in \Gamma$.

(iii) Let $A \wedge B \in \Gamma$. Since $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, we obtain $A \in \Gamma$ and $B \in \Gamma$. Conversely, let $A, B \in \Gamma$. Because $\vdash A \to (B \to A \wedge B)$, we first obtain $B \to (A \wedge B) \in \Gamma$, which implies $A \wedge B \in \Gamma$ since $B \in \Gamma$.

(iv) Suppose that $A \vee B \in \Gamma$, but $A \notin \Gamma$ and $B \notin \Gamma$. This means $\neg A \in \Gamma$, $\neg B \in \Gamma$, and $\neg A \wedge \neg B \in \Gamma$. Thus, $\neg(A \vee B) \in \Gamma$ and $A \vee B \notin \Gamma$, a contradiction! Conversely, if $A \in \Gamma$ or $B \in \Gamma$, then $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$ imply $A \vee B \in \Gamma$. □

Next we present the result showing that for any consistent set of ILGC-formulae, there exists a maximal consistent set including it.

Lemma 3.6 (Lindenbaum's Lemma) *Let Γ be a consistent set of* ILGC-*formulae. Then there exists a maximal consistent set of formulae Γ^+ such that $\Gamma \subseteq \Gamma^+$.*

Proof. Since the set P of propositional variables is enumerable, also the set Φ of ILGC-formulae is enumerable. Let $A_0, A_1, A_2, \ldots$ be an enumeration of Φ. We define a sequence $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ of ILGC-formulae by setting $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if this is consistent;} \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

It is then easy to prove by the use of Zorn's Lemma that

$$\Gamma^+ = \bigcup_{n \geq 0} \Gamma_n.$$

□

Next we construct the canonical model which proves the valid formulae of ILGC, and only them. The *canonical model* for ILGC is a Kripke model $\mathcal{M}^* = (U^*, R^*, v^*)$, where:

(i) $U^* \subseteq \wp(\Phi)$ is the set of maximal consistent sets

(ii) R^* is a binary relation on U^* defined by

$$x\, R^*\, y \iff (\forall A \in \Phi)\, (A \in y \Rightarrow \blacktriangle A \in x)$$

(iii) $v^*\colon P \to \wp(U^*)$ is the valuation defined by

$$v^*(p) = \{x \in U^* \mid p \in x\}$$

The pair $\mathcal{F}^* = (U^*, R^*)$ is called the *canonical frame*.

Concerning the canonical relation R^*, it is easy to see that the following condition holds.

Lemma 3.7 *Let $\mathcal{M}^* = (U^*, R^*, v^*)$ be the canonical model for* ILGC. *Then for all $x, y \in U^*$:*

$$x\,R^*\,y \iff (\forall A \in \Phi)\,(\triangledown A \in y \Rightarrow A \in x).$$

Proof. Suppose that $x\,R^*\,y$. Then for all $A \in \Phi$, $A \in y$ implies $\blacktriangle A \in x$. Thus, if $\triangledown A \in y$, then we have $\blacktriangle\triangledown A \in x$. Since $\vdash \blacktriangle\triangledown A \to A$ by Proposition 3.1(ii), we must have $A \in x$, because each maximal consistent set is closed under modus ponens.

Conversely, assume that for all $A \in \Phi$, $\triangledown A \in y$ implies $A \in x$. Suppose now that $A \in y$. Because $\vdash A \to \triangledown\blacktriangle A$, we have $\triangledown\blacktriangle A \in y$. This clearly gives $\blacktriangle A \in x$ by our assumption, and therefore $x\,R^*\,y$. $\square$

To prove completeness, we shall also need the following lemma.

Lemma 3.8 (Existence Lemma) *Let $\mathcal{F}^* = (U^*, R^*)$ be the canonical* ILGC-*frame and let $x \in U^*$. Then the following assertions hold for all $A \in \Phi$:*

(i) *If $\triangledown A \notin x$, then there exists a maximal consistent set $y \in U^*$ such that $y\,R^*\,x$ and $A \notin y$.*

(ii) *If $\blacktriangle A \in x$, then there is a maximal consistent set $y \in U^*$ such that $x\,R^*\,y$ and $A \in y$.*

Proof. (i) Let x be a maximal consistent set. Assume $\triangledown A \notin x$. We may now conclude that the set $\Gamma = \{B \mid \triangledown B \in x\} \cup \{\neg A\}$ is consistent. Otherwise, there should be some formulae $B_1, \ldots, B_n$ such that each $\triangledown B_i \in x$ and $\vdash \neg(B_1 \wedge \cdots \wedge B_n \wedge \neg A)$. Therefore, $\vdash B_1 \wedge \cdots \wedge B_n \to A$. From Proposition 3.1(i), we obtain $\vdash \triangledown B_1 \wedge \cdots \wedge \triangledown B_n \to \triangledown A$. Since each $\triangledown B_i$ is in the maximal consistent set x, we get $x \vdash \triangledown A$ and $\triangledown A \in x$, a contradiction! Thus, the set $\Gamma = \{B \mid \triangledown B \in x\} \cup \{\neg A\}$ is consistent, and by Lindenbaum's Lemma, there exists a maximal consistent set y including $\Gamma = \{B \mid \triangledown B \in x\} \cup \{\neg A\}$. By the definition of Γ, it is clear that if $\triangledown B \in x$, then $B \in \Gamma \subseteq y$, which implies $y\,R^*\,x$. Further, $\neg A \in \Gamma \subseteq y$, giving $A \notin y$.

The proof for (ii) is similar. $\square$

The next lemma is essential, showing that maximal consistent sets validate exactly the formulae belonging to them.

Lemma 3.9 (Truth Lemma) *Let $\mathcal{M}^* = (U^*, R^*, v^*)$ be the canonical model for* ILGC. *Then for any maximal consistent set $x \in U^*$ and formula $A \in \Phi$:*

$$\mathcal{M}^*, x \models A \ \textit{ if and only if } \ A \in x.$$

Proof. We show this by induction. If A is proposition variable p, then $\mathcal{M}^*, x \models p$ iff $x \in v^*(p)$ iff $p \in x$. In case A is of the form $\neg B$, we have

$$\begin{aligned} \mathcal{M}^*, x \models \neg B \quad & \text{iff} \quad \mathcal{M}^*, x \not\models B \\ & \text{iff} \quad B \notin x \\ & \text{iff} \quad \neg B \in x \end{aligned}$$

If A is of the form $B \to C$, we have

$$\begin{aligned} \mathcal{M}^*, x \models B \to C \quad & \text{iff} \quad \mathcal{M}^*, x \not\models B \text{ or } \mathcal{M}^*, x \models C \\ & \text{iff} \quad B \notin x \text{ or } C \in x \\ & \text{iff} \quad \neg B \in x \text{ or } C \in x \\ & \text{iff} \quad (\neg B \vee C) \in x \\ & \text{iff} \quad (B \to C) \in x \end{aligned}$$

In case A is of the form $\blacktriangle B$, and B satisfies the required condition, we first suppose that $\mathcal{M}^*, x \models \blacktriangle B$. Then, there exists $y \in U^*$ such that $x\, R^*\, y$ and $\mathcal{M}^*, y \models B$. By the induction hypothesis, we have that $B \in y$. Thus, by the definition of R^*, we obtain $\blacktriangle B \in x$. Conversely, suppose that $\blacktriangle B \in x$. By the Existence Lemma, there is a maximal consistent set y such that $x\, R^*\, y$ and $B \in y$. By the induction hypothesis, $\mathcal{M}^*, y \models B$, and this implies $\mathcal{M}^*, x \models \blacktriangle B$.

The case in which A is of the form $\triangledown B$ can be proved in a similar way. □

We can now show the completeness of ILGC.

Theorem 3.10 (Completeness Theorem) *An* ILGC-*formula is valid if and only if it is provable.*

Proof. Suppose that A is valid, but not provable. Since now the set $\{\neg A\}$ is consistent, there is a maximal consistent set Γ including $\{\neg A\}$ by the Lindenbaums's Lemma. Thus, we get $A \notin \Gamma$. It follows from the Truth Lemma that $\mathcal{M}^*, \Gamma \not\models A$ on the canonical model $\mathcal{M}^*$. This means that A is not valid, a contradiction! Therefore, every valid formula must be provable. The other direction is already proved (Soundness Theorem). □

4 Relationships to Other Logics

In this section we study how our logic relates to other two well-known logics, namely, minimal tense logic K_t and modal logic S4.

4.1 Minimal Tense Logic K_t

Here we show that there are essential connections between ILGC and the minimal tense logic K_t. At first, we present the axiom system of K_t; see [9], for example.

As before, let P be an enumerable set of propositional variables. Now the set of connectives consists of logical symbols $\rightarrow$, $\neg$, $\mathbf{G}$ and $\mathbf{H}$. K_t-formulae are defined inductively as ILGC-formulae, and the set of all K_t-formulae is denoted by Ψ. In distinction, recall that the set of ILGC-formulae is denoted by Φ.

A formula $\mathbf{G}A$ is interpreted as 'it will always be the case that A' and $\mathbf{H}A$ has the meaning 'it has always been the case that A'. Furthermore, their dual connectives $\mathbf{P}$ and $\mathbf{F}$ are defined by

$$\mathbf{F}A := \neg\mathbf{G}\neg A \quad \text{and} \quad \mathbf{P}A := \neg\mathbf{H}\neg A.$$

The logic K_t has the following seven axioms:

(Ax1) $A \rightarrow (B \rightarrow A)$

(Ax2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(Ax3) $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

(Ax4) $A \rightarrow \mathbf{HF}A$

(Ax5) $A \rightarrow \mathbf{GP}A$

(Ax6) $\mathbf{H}(A \rightarrow B) \rightarrow (\mathbf{H}A \rightarrow \mathbf{H}B)$

(Ax7) $\mathbf{G}(A \rightarrow B) \rightarrow (\mathbf{G}A \rightarrow \mathbf{G}B)$

Furthermore, K_t has three rules of inference:

$$\text{(MP)} \quad \frac{A \qquad A \rightarrow B}{B} \qquad\qquad \text{(RH)} \quad \frac{A}{\mathbf{H}A} \qquad\qquad \text{(RG)} \quad \frac{A}{\mathbf{G}A}$$

That a K_t-formula A is provable is defined as in case of ILGC.

Our purpose is to show that ILGC and K_t are equivalent with respect to provability. Indeed, ILGC appears much simpler than K_t, since ILGC has only three axioms (Ax1)–(Ax3) and three rules of inference. Therefore, ILGC can also viewed as a very simple formulation of K_t.

At the first glance the language of our logic ILGC is different from the one of K_t. However, if we replace for an ILGC-formula $A \in \Phi$ every symbol $\blacktriangle$ by $\mathbf{F}$ and every $\triangledown$ by $\mathbf{H}$, we we obtain a K_t-formula $A^\psi \in \Psi$. Similarly, any K_t-formula $B \in \Psi$ can be transformed to an ILGC-formula B^ϕ by replacing the occurrences of $\mathbf{F}$, $\mathbf{G}$, $\mathbf{P}$, and $\mathbf{H}$ by $\blacktriangle$, $\blacktriangledown$, $\vartriangle$, and $\triangledown$, respectively. Therefore, the languages of these languages may be considered to be exactly the same.

It is straightforward to prove the next lemma stating that each provable ILGC-formula A can be translated to a provable K_t-formula A^Ψ.

Lemma 4.1 *If an* ILGC*-formula $A \in \Phi$ is* ILGC*-provable, then the corresponding* K_t*-formula $A^\psi \in \Psi$ is* K_t*-provable.*

Proof. Assume $A \in \Phi$ is a provable ILGC-formula. We prove the claim by induction. If A is an ILGC-axiom, then the assertion holds trivially because the axioms of ILGC are included in the axioms of K_t.

If A is deduced from B and $B \to A$ by (MP), then, by the induction hypothesis B^ψ and $(B \to A)^\psi$ are provable K_t-formulae. Since $(B \to A)^\psi$ is $B^\psi \to A^\psi$, A^ψ is a provable K_t-formula by (MP).

Assume A is equal to $B \to \nabla C$ for some $B, C \in \Phi$, and $B \to \nabla C$ is deduced from $\blacktriangle B \to C$ by (GC2). Because $\mathbf{F}B^\psi \to C^\psi$ is a provable K_t-formula by the induction hypothesis, we have that $\mathbf{HF}B^\psi \to \mathbf{H}C^\psi$ is K_t-provable by (RH). Additionally, $B^\psi \to \mathbf{HF}B^\psi$ is K_t-provable by (Ax4). Thus, we obtain that $B^\psi \to \mathbf{H}C^\psi$ is a provable K_t-formula. This gives that $B^\psi \to (\nabla C)^\psi$ and $(B \to \nabla C)^\psi$ are K_t-provable.

The case involving (GC1) can be proved in an analogous way. □

Our next lemma states that also the converse statement holds.

Lemma 4.2 *If a* K_t*-formula $A \in \Psi$ is* K_t*-provable, then the* ILGC*-formula $A^\phi \in \Phi$ is* ILGC*-provable.*

Proof. The proof is clear by Proposition 3.1 and Lemma 3.2. □

Lemmas 4.1 and 4.2 imply that ILGC and K_t are equivalent with respect to provability. It is well-known that K_t is *decidable*, that is, there exists an algorithm which for every K_t-formula is capable of deciding in finitely many steps whether the formula is provable in the system or not. Therefore, we may give the following theorem.

Theorem 4.3 (Decidability Theorem) ILGC *is decidable.*

4.2 Modal logic S4

Here we study the relationship between ILGC and the well-known modal logic S4. Most of so-called 'normal modal logics' include the *necessitation rule*:

(N) $$\frac{A}{\Box A}\ .$$

Furthermore, the *distribution axiom*

(K) $$\Box(A \to B) \to \Box A \to \Box B$$

is usually included. The weakest normal modal logic, named K in honor of Saul Kripke, is simply the propositional calculus added with an extra connective $\Box$, the rule (N), and the axiom (K). Let us recall also the axioms (T) and (4):

(T) $\Box A \to A$

(4) $\Box A \to \Box\Box A$

The logic S4 is characterized by axioms (T), (4) and (K) together with the rule (N).

If we now come back to ILGC, we may define an additional connective $\Box$ by setting for any ILGC-formula $A \in \Phi$,

$$\Box A := \blacktriangle \triangledown A.$$

By cases (ii) and (iii) of Proposition 3.1, $\vdash \Box A \to A$ and $\vdash \Box A \to \Box\Box A$, that is, the (T) and (4) are provable in ILGC. Similarly, by applying both the rules of Proposition 3.1(i), we may show that $\vdash A \to B$ implies $\vdash \Box A \to \Box B$.

Next, we will formalize the above-described setting. Let the formulae of the logic be built inductively from the connectives $\to$, $\neg$, and $\Box$. The abbreviations for disjunction, conjunction, equivalence, true, and false can be defined as before. The axioms of the system consists of (Ax1)–(Ax3) together with the axioms (T) and (4). Modus Ponens (MP) and Monotonicity

(M) $$\frac{A \to B}{\Box A \to \Box B}$$

are the rules of inference.

So, we may define almost the modal logic S4 in terms of ILGC. Unfortunately, our logic is not normal in the sense that it does not satisfy (N) nor (K). Therefore, it seems clear that we cannot define Kripke-style of semantics for our new connective $\Box$ by the means of a frame $\mathcal{F} = (U, R)$ of just one binary relation in a standard way. Therefore, it is natural to ask what kind of semantics should be determined.

In topological interpretation of a modal logic initiated by Tarski (see e.g. [1], where further references can be found), each propositional variable represents a region of the topological space, and so does every formula. The connectives $\neg$, $\vee$ and $\wedge$ are interpreted as complement, union and intersection, respectively. The modal connectives $\Diamond$ and $\Box$ become the topological closure and interior operators. Topological models $\mathcal{M} = (U, \mathcal{T}, v)$ are topological spaces $(U, \mathcal{T})$ equipped with a valuation function $v \colon P \to \wp(U)$.

Here we may proceed similarly. Let R be any binary relation on U and let the maps ${}^{\triangledown} \colon \wp(U) \to \wp(U)$ and ${}^{\blacktriangle} \colon \wp(U) \to \wp(U)$ be defined as in Section 2. We may now define a mapping ${}^{\Box} \colon \wp(U) \to \wp(U)$ by setting for all $X \subseteq U$,

$$X^{\Box} = X^{\triangledown\blacktriangle}.$$

The map $X \mapsto X^{\Box}$ is a *lattice-theoretical interior operator*, that is,

(Int1) $X^{\Box} \subseteq X$,

(Int2) $X \subseteq Y$ implies $X^{\Box} \subseteq Y^{\Box}$, and

(Int3) $X^{\Box\Box} = X^{\Box}$

for all $X, Y \subseteq U$. The family

$$\mathcal{I} = \{X^{\Box} \mid X \subseteq U\}$$

is closed under arbitrary unions of its elements, and the pair $(U, \mathcal{I})$ is called an *interior system*.

The least element of $\mathcal{I}$ is $\emptyset$ and the greatest element of $\mathcal{I}$ is $U^{\Box}$. Note that possibly $U^{\Box} \neq U$, that is, $U \notin \mathcal{I}$. Interestingly, each $X \in \mathcal{I}$ may be interpreted in such a way that X consists exactly of elements that are 'possibly certainly' in X.

For an interior system $(U, \mathcal{I})$, an *interior model* is a triple $\mathcal{M} = (U, \mathcal{I}, v)$, where $v \colon P \to \wp(U)$ is a valuation function. Validity of formulae can be defined inductively as in Section 3, except that

$$\mathcal{M}, x \models \Box A \text{ iff } (\exists X \in \mathcal{I})\, x \in X \text{ and } \mathcal{M}, y \models A \text{ for all } y \in X.$$

Lemma 4.4 *For any formula A, $v(\Box A) = v(A)^{\Box}$.*

Proof. ($\subseteq$) Suppose that $x \in v(\Box A)$. Then $\mathcal{M}, x \models \Box A$, which means that there exists $X \in \mathcal{I}$ such that $x \in X$ and $\mathcal{M}, y \models A$ for all $y \in X$. Thus, $y \in v(A)$ for all $y \in X$, that is, $X \subseteq v(A)$. This implies $x \in X = X^{\Box} \subseteq v(A)^{\Box}$.

($\supseteq$) If $x \in v(A)^{\Box}$ ($\in \mathcal{I}$), then $v(A)^{\Box} \subseteq v(A)$ implies that for all $y \in v(A)^{\Box}$, $y \in v(A)$ and $\mathcal{M}, y \models A$. Thus, $\mathcal{M}, x \models \Box A$ and $x \in v(\Box A)$ $\Box$

A formula A is said to be *true* in an interior model $\mathcal{M} = (U, \mathcal{I}, v)$, written $\mathcal{M} \models A$, if for all $x \in U$, $\mathcal{M}, x \models A$. Furthermore, if A is true in all models based on the interior system $\mathcal{I}$ on U, then A is *valid*.

Theorem 4.5 (Soundness Theorem) *Each provable formula is valid.*

Proof. We show that axioms (T) and (4) are valid, and that rule (M) preserves validity. That (MP) preserves validity is trivial.

(T) $v(\Box A \to A) = v(\Box A)^c \cup v(A) = (v(A)^{\Box})^c \cup v(A) \supseteq v(A)^c \cup v(A) = U$.

(4) $v(\Box A \to \Box\Box A) = v(\Box A)^c \cup v(\Box\Box A) = (v(A)^{\Box})^c \cup (v(A)^{\Box})^{\Box} = (v(A)^{\Box})^c \cup v(A)^{\Box} = U$.

(M) Assume that $A \to B$ is valid. Then $v(A) \subseteq v(B)$. This implies $v(\Box A) = v(A)^{\Box} \subseteq v(B)^{\Box} = v(\Box B)$. Thus, also $\Box A \to \Box B$ is valid.

$\Box$

Next we construct the canonical interior system and the corresponding canonical model. For that, we denote by U^* the family of all maximal consistent sets of formulae. In addition, for any formula A, we define

$$\widehat{A} = \{\Gamma \in U^* \mid A \in \Gamma\}.$$

The *canonical interior system* $\mathcal{I}^*$ is a subfamily of $\wp(U^*)$ generated by the all unions of the *basic sets*

$$\{\widehat{\Box A} \mid A \text{ is a formula}\}.$$

Clearly, $(U^*, \mathcal{I}^*)$ is an interior system.

The *canonical interior model* is a triple $\mathcal{M}^* = (U^*, \mathcal{I}^*, v^*)$, where

(i) $(U^*, \mathcal{I}^*)$ is the canonical interior system

(ii) $v^*\colon P \to \wp(U^*)$ is the *canonical valuation* defined by

$$v^*(p) = \{\Gamma \in U^* \mid p \in \Gamma\}.$$

Note that $v^*(p) = \widehat{p}$ for all variables $p \in P$. It is clear that for any maximal consistent set $x \in U^*$ and formula A,

$$x \in \widehat{A} \iff A \in x.$$

Lemma 4.6 (Truth Lemma) *Let $\mathcal{M}^* = (U^*, \mathcal{I}^*, v^*)$ be the canonical interior model. Then for any maximal consistent set $x \in U^*$ and formula A,*

$$\mathcal{M}^*, x \models A \textit{ iff } A \in x.$$

Proof. It suffices to the consider the interesting case of the modal operator $\Box$. We show the directions separately.

($\Leftarrow$) Suppose $\Box A \in x$, that is, $x \in \widehat{\Box A}$. By definition, $\widehat{\Box A}$ is a basic set and hence $\widehat{\Box A} \in \mathcal{I}^*$. Furthermore, axiom (T) implies $\widehat{\Box A} \subseteq \widehat{A}$. This means that there exists $X = \widehat{\Box A}$ such that $x \in X \in \mathcal{I}^*$ and for all $y \in X$, $y \in \widehat{A}$. Thus, for all $y \in X$, $A \in y$, and so by the induction hypothesis $\mathcal{M}^*, y \models A$. Thus, $\mathcal{M}^*, x \models \Box A$.

($\Rightarrow$) Assume that $\mathcal{M}^*, x \models \Box A$. Then there exists $X \in \mathcal{I}^*$ such that $x \in X$ and $\mathcal{M}^*, y \models A$ for all $y \in X$. Since X is a union of some basic sets, we have that there is a basic set $\widehat{\Box B}$ for some formula B such that $x \in \widehat{\Box B}$ and for all $y \in \widehat{\Box B}$ $(\subseteq X)$, $\mathcal{M}^*, y \models A$, that is, $A \in y$ and $y \in \widehat{A}$ by the induction hypothesis. This means that $\widehat{\Box B} \subseteq \widehat{A}$. But this implies that we can prove the implication $\Box B \to A$; namely, if not, then there would be some maximal consistent set containing $\Box B$ and $\neg A$, and this would give $\widehat{\Box B} \not\subseteq \widehat{A}$. By rule (M), we can prove also the implication $\Box\Box B \to \Box A$. Therefore, by using axiom (4), we have $\Box B \to \Box A$. This implies $x \in \widehat{\Box B} \subseteq \widehat{\Box A}$, that is, $\Box A \in x$. $\Box$

Completeness is now obvious.

Theorem 4.7 (Completeness Theorem) *A formula is valid if and only if it is provable.*

We conclude the paper by the following remark.

Remark. We may easily include the axiom

(N) $\Box\top$

to our axiom system. Namely, if R is *serial*, that is, for all $x \in U$, there exists $y \in U$ such that $x\,R\,y$, then $U^{\Box} = U$. This means that $U \in \mathcal{I}$ and hence $(U, \mathcal{I})$ becomes so-called *topped interior system*. Note that the assumption of seriality is quite natural – it means simply that each element of the universe is 'comparable' at least with one element.

This modified logic is sound, because $v(\Box\top) = v(\top)^{\Box} = U^{\Box} = U$, that is, the axiom (N) is also valid. Furthermore, the canonical interior system $(U^*, \mathcal{I}^*)$ is now a topped interior system, because (N) implies $\Box\top \in \Gamma$ for all $\Gamma \in U^*$ and

$$\widehat{\Box\top} = \{\Gamma \in U^* \mid \Box\top \in \Gamma\} = U^*,$$

which gives directly $U^* \in \mathcal{I}^*$.

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