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A Possibilistic Pratt Theorem

Irina Georgescu
Åbo Akademi University, Department of Information Technologies
Joukahaisenkatu 3–5, 4th floor, Turku, Finland
irina.georgescu@abo.fi

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Abstract

The possibilistic risk premium is defined in a possibilistic context, made by a utility function, a fuzzy number and a weighting function. This notion measures the aversion to the possibilistic risk. The main result of the paper is a Pratt–type theorem on the possibilistic risk aversion. This result, combined with the Pratt theorem leads to a surprising conclusion: the aversion to the probabilistic risk is equivalent with the aversion to the possibilistic risk.

Keywords: fuzzy number, risk premium, possibilistic theory, risk aversion

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1 Introduction

Risk aversion is traditionally treated in probabilistic terms. An agent, represented by a utility function is confronted with a random phenomenon (e. g. a lottery), represented by a random variable. Therefore the framework in which the risk aversion is studied consists of a utility function u and a random variable. In this context we define the notions of risk premium which expresses the agent's u aversion towards X. The probabilistic risk premium is expressed with help of the Arrow–Pratt index associated with a utility function u and a random variable. Therefore the framework in which the risk aversion is studied consists of a utility function u and a random variable. In this context, we define the notion of risk premium, which expresses the agent's u aversion towards X. The probabilistic risk premium is expressed with help of the Arrow–Pratt index associated with a utility function u ([1], [10], [12]).

A Pratt theorem [12] shows that the comparison of the risk aversion represented by the utility functions u_1, u_2 is equivalent with the comparison of the Arrow–Pratt indices of u_1, u_2.

The comparison of the Arrow–Pratt indices is an analytical condition which appears both in the Pratt theorem and also in our result. Then, by combining the two results, one obtains a fact which has a remarkable significance: the probabilistic risk aversion is equivalent with the possibilistic risk aversion.

In Section 2 there is recalled the expected value $E_f(A)$ and the variance $Var_f(A)$ of a fuzzy number A with respect to a weighting function f. Besides them there is considered the possibilistic indicator $Var_f^*(A)$ related with $Var_f(A)$ but different from it. A possibilistic form of Jensen inequality is proved.

Section 3 presents definitions and the calculation of the possibilistic risk premium and the Pratt theorem.

In Section 3 there is proved a possibilistic form of the Pratt theorem, by using the possibilistic Jensen inequality. Theorem 4.5 obtained by combining this result with the Pratt theorem establishes the equivalence of four properties: the probabilistic risk aversion, the possibilistic risk aversion, a concavity condition and analytical condition of the Arrow–Pratt indices.

2 Possibilistic indicators of fuzzy numbers

In this section we shall recall the definition and some properties of some possibilistic indicators of fuzzy numbers [2], [3], [4], [5], [7], [11]. At the same time, we shall prove a possibilistic version of the probabilistic Jensen inequality (see [9], p. 201).

Let A be a fuzzy number. Let $\gamma \in [0, 1]$. The γ–level set of A is defined by

$$[A]^\gamma = \begin{cases} \{x \in \mathbf{R} | \mathbf{A}(\mathbf{x}) \geq \gamma\} & \text{if} \gamma > 0 \\ cl\{x \in \mathbf{R} | \mathbf{A}(\mathbf{x}) > \mathbf{0}\} & \text{if} \gamma = 0 \end{cases}$$

The $cl\{x \in \mathbf{R}|\mathbf{A}(\mathbf{x}) > \mathbf{0}\}$ means the closure of $\{x \in \mathbf{R}|\mathbf{A}(\mathbf{x}) > \mathbf{0}\}$ means the closure of $\{x \in \mathbf{R}|\mathbf{A}(\mathbf{x}) > \mathbf{0}\}$ in $\mathbf{R}$. $[A]^\gamma$ is a compact and convex subset of $\mathbf{R}$. If we denote $a_1(\gamma) = \min[A]^\gamma$ and $a_2(\gamma) = \max[A]^\gamma$ then $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0, 1]$.

We fix a fuzzy number A.

A non–negative and monotone increasing function $f : [0, 1] \to \mathbf{R}$ is a *weighting function* if $\int_0^1 f(\gamma)d\gamma = 1$.

The *expected value* of A with respect to a weighting function f is defined by

(1) $E_f(A) = \int_0^1 \frac{a_1(\gamma)+2(\gamma)}{2} f(\gamma)d\gamma$.

If $f(\gamma) = 2\gamma$ then $E_f(A)$ coincides with the possibilistic mean value introduces in [2]. By considering a real number r as a degenerate fuzzy number, one notices that $E(r) = r$.

Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function.Then $g(A) : \mathbf{R} \to \mathbf{R}$ is defined by using the Zadeh extension principle:

$$g(A)(y) = \begin{cases} \sup A(x)_{f(x)=y} & \text{if there existsx} \in \mathbf{R} \text{ such that } f(x) = y \\ 0 & \text{otherwise} \end{cases}$$

The expected value of $g(A)$ with respect to a weighting function f is introduced by

(2) $E_f(g(A)) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx] f(\gamma)d\gamma$.

The variance of A with respect to a weighting function f is defined by

(3) $Var_f(A) = \int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))^2}{12} f(\gamma)d\gamma$.

Remark 2.1 *Consider the possibilistic variance* $\sigma_A = \frac{1}{2}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2 \gamma d\gamma$ *of the fuzzy number A introduced in [2]. If we take* $f(\gamma) = 2\gamma$ *for all* $\gamma \in [0, 1]$ *then* $Var_f(A) = \frac{1}{3}\sigma_A$.

Remark 2.2 *In Definition 3 of [7], the variance of A with respect to f is defined by* $Var_f(A) = \int_0^1 \frac{(a_2(\gamma)-a_1(\gamma)^2)}{4} f(\gamma)d\gamma$. *The form of $Var_f(A)$ given in (3) can be found in [11], p. 87. In this paper we shall use for $Var_f(A)$ the form (3).*

Proposition 2.3 *[7] Let A, B be two fuzzy numbers and λ, $\mu \in \mathbf{R}$. Then*
$E_f(\lambda A + \mu B) = \lambda E_f(A) + \mu E_f(B)$.

Lemma 2.4 *Let* $g : \mathbf{R} \to \mathbf{R}$, $h : \mathbf{R} \to \mathbf{R}$ *be two continuous functions. If* $g \le h$ *then* $E_f(g(A)) \le E_f(h(A))$.

Proof. Assume $g \le h$. Then, for any $\gamma \in [0, 1]$, we have

$\int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx \le \int_{a_1(\gamma)}^{a_2(\gamma)} h(x)dx$.

Since $a_1(\gamma) \le a_2(\gamma)$ and $f \ge 0$, the following inequality holds:

$[\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx] f(\gamma) \le [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} h(x)dx] f(\gamma)$

for each $x \in [0, 1]$. By taking into account (2), one obtains $E_f(g(A)) \le E_f(h(A))$. ∎

The following result is a possibilistic version of the probabilistic Jensen inequality. Its proof will be an adaptation of the proof in [9], p. 201, but it will use the properties of the possibilistic expected value.

Proposition 2.5 *If the function* $g : \mathbf{R} \to \mathbf{R}$ *is convex and continuous, then* $g(E_f(A)) \leq E_f(g(A))$.

Proof. From the real analysis we know that if g is convex then there exist two sequences of real numbers (a_n) and (b_n) such that

(a) $g(x) = \sup_n (a_n x + b_n)$, for any $x \in \mathbf{R}$.

Let $n \in \mathbf{N}$. Then $a_n x + b_n \leq g(x)$ for any $x \in \mathbf{R}$. By Lemma 2.4 we get $E_f(a_n A + b_n) \leq E_f(g(A))$.

Applying Proposition 2.3, $E_f(a_n A + b_n) = a_n E_f(A) + b_n$, hence

(b) $a_n E_f(A) + b_n \leq E_f(g(A))$ for all $n \in \mathbf{N}$.

From (a) and (b) it follows

$g(E_f(A)) = \sup_n (a_n E_f(A) + b_n) \leq E_f(g(A))$. ∎

Corollary 2.6 *If the function* $g : \mathbf{R} \to \mathbf{R}$ *is concave and continuous then* $g(E_f(A)) \geq E_f(g(A))$.

Proof. It follows from Proposition 2.5 and from the fact that g is concave iff $-g$ is convexe. We consider the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = (x - E_f(A))^2$ for any $x \in \mathbf{R}$. We denote

(4) $Var_f^*(A) = E_f(g(A)) = E_f[(A - E_f(A))^2]$

Cf. (2) we have

(5) $Var_f^*(A) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A)^2) dx] f(\gamma) d\gamma$.

$Var_f^*(A)$ can be considered a possibilistic indicator inspired by one of the forms of the probabilistic variance. We shall observe that $Var_f(A)$ and $Var_f^*(A)$ are different. It is easy to notice that $Var_f^*(A) \geq 0$. ∎

Proposition 2.7 *[8]* $Var_f^*(A)$ *can be written under the following form:*

(i) $Var_f^*(A) = \frac{1}{3} \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma) a_2(\gamma)] f(\gamma) d\gamma - E_f^2(A)$

(ii) $Var_f^*(A) = 4Var_f(A) - E_f^2(A) + \int_0^1 a_1(\gamma) a_2(\gamma) f(\gamma) d\gamma$.

3 Probabilistic risk aversion

In this section we shall recall some notions and results regarding the ways the risk aversion is evaluated probabilistically. They will be a starting point in the treatment of the probabilistic risk aversion.

Let Ω be a set of states endowed with a probabilistic space (Ω, K, P). Assume that $X : \Omega \to \mathbf{R}$ is a random variable and $u : \mathbf{R} \to \mathbf{R}$ is a continuous function.

Then $u(X) = u \circ X$ is a random variable and $E(u(X))$ denotes the expected value of $u(X)$. If X has a density function $f : \mathbf{R} \to \mathbf{R}$ then

$E(u(X)) = \int_{\infty}^{\infty} u(X) f(X) dX.$

Throughout this section we shall assume that $\Omega = \mathbf{R}$ and K is the Σ–algebra B of Borelian subsets of $\mathbf{R}$.

We consider an agent represented by a utility function $u : \mathbf{R} \to \mathbf{R}$ continuous and strictly increasing.

Definition 3.1 *Let X be a random variable with respect to $(\mathbf{R}, \mathbf{B})$. The probabilistic risk premium $\rho_{X,u}$* (associated with X and u) is defined by the identity :

(1) $E(u(X)) = u(E(X) - \rho_{X,u})$

Due to the injectivity of u, the *probabilistic risk premium $\rho_{X,u}$* is uniquely determined by relation (1). The probabilistic risk premium $\rho_{X,u)}$ expresses the risk aversion of the agent represented by u with respect to the random variable X. The bigger $\rho_{X,u}$ is, the more risk–prone the agent is.

Proposition 3.2 *[1] Assume that u is twice differentiable, strictly concave and strictly increasing. Then*

(2) $\rho_{X,u} = -\frac{1}{2}\sigma_X^2 \frac{u''(E(X))}{u'(E(X))}$,

where σ_X^2 is the variance of X.

In the following we shall assume that the utility function verifies the properties of Proposition 3.2. The Arrow–Pratt index associated with the utility function u is defined by

(3) $r_n(x) = -\frac{u''(x)}{u'(x)}$ for each $x \in \mathbf{R}$.

Let u_1, u_2 be two utility functions. Let us denote by $r_1(x) = r_{u_1}(x)$ and $r_2(x) = r_{u_2}(x)$ the Arrow–Pratt indices of u_1 and u_2.

Theorem 3.3 *[12] The following assertions are equivalent:*

(1) $r_1(x) \geq r_2(x)$ for each $x \in \mathbf{R}$;

(2) $u_1 \circ u_2^{-1}$ is concave;

(3) $\rho_{X,u_1} \geq \rho_{X,u_2}$ for any random variable X with respect to $(\mathbf{R}, \mathbf{B})$.

Remark 3.4 *Suppose that agents 1 and 2 are represented by the utility functions u_1 and u_2. By taking into account the interpretation of the probabilistic risk premium, condition (3) of Theorem 3.3 means that the agent 1 is more risk–prone than the agent 2, with respect to any random variable X. In this case we shall denote $u_1 \succeq_{probab} u_2$ iff the Arrow–Pratt index of u_1 is bigger than the Arrow–Pratt index of u_2. It follows that the Arrow–Pratt index is a measure of the probabilistic risk aversion.*

4 Possibilistic risk aversion

In this section we shall prove a Pratt–type theorem [12] for the possibilistic risk aversion premium associated with a fuzzy number, a utility function and a weighting function (see [8]). By combining this result with the Pratt theorem for the possibilistic risk one reaches a surprising result: the aversion to the probabilistic risk is equivalent with the aversion to the possibilistic risk.

We consider an agent represented by a utility function $u : \mathbf{R} \to \mathbf{R}$ continuous and strictly increasing. Then we can consider its inverse $u^{-1} : Im(u) \to \mathbf{R}$ where $Im(f) = \{u(x)|x \in \mathbf{R}\}$.

We fix a weighting function $f : [0, 1] \to \mathbf{R}$.

Definition 4.1 *[8] Let A be a fuzzy number. The possibilistic risk premium $\rho_A = \rho_{A,f,u}$* (associated with the fuzzy number A, the weighting function f and the utility function u) is defined by the following equality:

(1) $u(E_f(A) - \rho_A) = E_f(u(A))$.

Since function u is injective, the possibilistic risk premium ρ_A is uniquely determined by the equality (1). For the rest of the section we shall assume that the utility function is twice differentiable, strictly concave and strictly increasing.

Proposition 4.2 *[8] Let u be a utility function and A a fuzzy number. Then the possibilistic risk premium ρ_A has the form:*

(2) $\rho_A = -\frac{1}{2}Var_f^*(A)\frac{u''(E_f(A))}{u'(E_f(A))}$

Remark 4.3 *The possibilistic risk premium ρ_A expresses the risk aversion of the agent represented by u with respect to the probabilistic distribution given by a fuzzy number a and the weighting function f. The bigger ρ_A is, the more prone to the possibilistic risk the agent is.*

Recall from the previous section the form of the Arrow–Pratt index associated with the utility function u:

(3) $r_n(x) = -\frac{u''(x)}{u'(x)}$, for any $x \in \mathbf{R}$.

Then the relation (2) becomes:

(4) $\rho_A = \frac{1}{2}Var_f^*(A)r(E_f(A))$.

Relation (4) shows that the possibilistic risk premium can be expressed function of the Arrow–Pratt index and the possibilistic indicators $E_f(A)$ and $Var_f^*(A)$.

The following result is the possibilistic aversion of the Pratt theorem:

Theorem 4.4 *Let u_1, u_2 be two utility functions and $r_1 = r_{u_1}$, $r_2 = r_{u_2}$ be the Arrow-indices of u_1 and u_2. The following assertions are equivalent:*

(1) $r_1(x) \leq r_2(x)$, for any $x \in \mathbf{R}$;

(2) $u \circ u_2^{-1}$ is concave;

(3) For all fuzzy numbers A and weighting functions f, we have

$\rho_{A,f,u_1} \geq \rho_{A,f,u_2}$.

Proof. (1) $\Leftrightarrow$ (2) By Pratt's theorem:

(2) $\Rightarrow$ (3) let $\rho_i = \rho_{A,f,u_i}$, $i = 1, 2$. Cf (1)

$u_1(E_f(A) - \rho_1) = E_f(u_1(A))$

$u_2(E_f(A) - \rho_2) = E_f(u_2(A))$.

By applying to these equalities the inverses u_1^{-1}, u_2^{-1} of u_1 and u_2 one deduces:

$\rho_1 = E_f(A) - u_1^{-1}(E_f(u_1(A))$

$\rho_2 = E_f(A) - u_2^{-1}(E_f(u_2(A))$

By subtracting these two inequalities one obtains:

(a) $\rho_1 - \rho_2 = u_2^{-1}(E_f(u_2(A)) - u_1^{-1}(E_f(u_1(A))$.

Since $u_1 \circ u_2^{-1}$ is concave, by applying Corollary 2.6 we have:

$E_f(u_1(A)) = E_f((u_1 \circ u_2^{-1})(u_2(A)) \leq (u_1 \circ u_2^{-1})(E_f(u_2(A)))$.

But u_1^{-1} is increasing, therefore:

$u_1^{-1}(E_f(u_1(A))) \leq u_1^{-1}((u_1 \circ u_2 - 1)(E_f(u_2(A))) = u_2^{-1}(E_f(u_2(A)))$.

By taking into account (a) and the preceding inequality, it follows $\rho_1 \geq \rho_2$.

(3) $\Rightarrow$ (1) Let $x \in \mathbf{R}$. We consider a fuzzy number A and a weighting function such that $x = E_f(A)$. Cf. (4) we have:

$\rho_{A,f,u_1} = \frac{1}{2} Var_f^*(A) r_1(x)$

$\rho_{A,f,u_2} = \frac{1}{2} Var_f^*(A) r_2(x)$

Since $Var_f^*(A) \geq 0$ and $\rho_{A,f,u_1} \geq \rho_{A,f,u_2}$ it follows $r_1(x) \geq r_2(x)$. ∎

By combining Theorems 3.3 and 4.4 we obtain:

Theorem 4.5 *Let u_1, u_2 be two utility functions and $r_1 = r_{u_1}$, $r_2 = r_{u_2}$ the Arrow–Pratt indices of u_1 and u_2. The following assertions are equivalent:*

(1) $r_1(x) \geq r_2(x)$, for any $x \in \mathbf{R}$;

(2) $u_1 \circ u_2^{-1}$ is concave;

(3) For any random variable X with respect to $(\mathbf{R}, \mathbf{B})$, $\rho_{X,u_1} \geq \rho_{X,u_2}$;

(4) For all fuzzy numbers A and weighting functions f, we have:

$\rho_{A,f,u_1} \geq \rho_{A,f,u_2}$.

If condition (4) of Theorem 4.5 holds, then we have $u_1 \geq_{posib} u_2$ and we will say that the agent represented by u_1 is more risk–prone to the possibilistic risk than the agent represented by u_2.

Remark 4.6 *The equivalence (3) $\Leftrightarrow$ (4) of Theorem 4.5 gets the form:*

$u_1 \geq_{probab} u_2$ *iff* $u_1 \geq_{possib} u_2$.

The significance of this inequality is remarkable: the probabilistic risk aversion is equivalent with the possibilistic risk aversion.

Remark 4.7 *The binary relation $u_1 \geq_{probab} u_2$ is defined with respect to the set of the fuzzy numbers.*

An open problem is if the equivalence of Remark 4.6 still holds when instead of fuzzy numbers we consider another class of possibilistic distributions.

5 Concluding Remarks

This paper continues the study of the possibilistic risk aversion stated in [8]. By establishing a possibilistic Pratt-type theorem, we can measure the possibilistic risk aversion by the Arrow–Pratt index.

The proof of such a result was due to the fact that the indicators associated with a fuzzy number (mean value, variance, etc.) have properties very similar to the probabilistic case.

We consider the following directions in which this topic might be continued:

(1) By keeping the framework offered by a fuzzy number to obtain the possibilistic version of some results of probabilistic risk aversion (e. g. Ross theory [13]).

(2) The treatment of the risk aversion of an agent with respect to a family of possibilistic distributions (in particular the study of the dynamic possibilistic risk aversion).

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University of Turku

- Department of Information Technology
- Department of Mathematics

Åbo Akademi University

- Department of Computer Science
- Institute for Advanced Management Systems Research

Turku School of Economics and Business Administration

- Institute of Information Systems Sciences

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