Possibilistic quasi–mean value and the risk

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Abstract

This paper studies for the case of discrete possibilistic distributions, two models: one for possibilistic portfolios and one for the possibilistic risk aversion. The mathematical treatment of the two models is based on the notion of quasi–mean value, an indicator attached to a possibilistic distribution with a role similar with the one of the mean value in probability theory.

Keywords: fuzzy number, risk premium, possibility theory, risk aversion

1 Introduction

Usually, phenomena which use the risk are studied by means of probability theory [7]. The theory of probabilistic risk can be adequately applied for events which occur with a sufficiently large frequency.

The probability theory initiated by Zadeh in [11] can be another way of approaching the situations when the risk should be taken into consideration.

The probabilistic risk is studied by probabilistic concepts such as the mean value, dispersion, covariance, etc. In possibility theory, the place of the random variables is taken by possibilistic distributions. The probabilistic concepts of mean value, dispersion, covariance should be replaced by corresponding concepts in possibility theory. For the class of the fuzzy numbers, this desideratum has been successfully achieved in [1], [2], [4], [5], [9].

There are important possibilistic distributions beside the class of the fuzzy numbers (e. g. discrete possibilistic distributions). Therefore one imposes to define some notions of mean value, dispersion, etc. for larger classes of possibilistic distributions.

In [8] there has been introduced a notion of mean value of a normalized possibilistic distribution. For a fuzzy number it does not coincide with the mean value studied in [1] or [5].

At the same time, it does not verify the usual properties of mean values (e. g. linearity). For the case of the discrete possibilistic distributions a very interesting phenomenon appears. To a discrete possibilistic distribution μ one canonically associates a random variable X_μ, whose mean value $E(X_\mu)$ coincides with the mean value of μ in the sense of [8]. This construction causes a possibilistic model related to μ to be associated with a probabilistic model expressed in terms of the random variable X_μ.

For example, a problem of possibilistic optimization can be replaced with a problem of probabilistic optimization, equivalent with the first one. A solution of the second one coincides with a solution of the first one.

This study deals with the mean value introduced in [8] and called by us the quasi–mean value (since it has less properties than the usual mean value).

In Section 2, based on the notions of possibility and necessity measures it is introduced the credibility measure [8]. Each possibility distribution μ on X induces the notion of possibility Pos_μ and the necessity measure Nec_μ on X, by meas of which we define a credibility measure Cr_μ. In this section, we establish the form of Cr_μ for trapezoidal fuzzy numbers and for discrete possibilistic distributions.

Section 3 contains the definition of the possibilistic quasi–mean value and discusses it in the case of discrete possibilistic distributions. To a discrete possibilistic distribution μ one associates a discrete random variable whose mean value coincides with the quasi–mean value of μ.

Section 3 contains the definition of the possibilistic mean value and its discussion in the case of the discrete possibilistic distributions. To a discrete possibilistic distribution μ one associates a discrete random variable, whose mean value coincides with the quasi–mean value of μ.

Section 3 discusses a notion of possibilistic portfolio based on the construction presented in Section 2. To a possibilistic portfolio one associates a probabilistic portfolio such that the risk problematique for the first type of portfolio is reduced to the consideration of a probabilistic risk.

In Section 4 there is proposed a definition of risk premium for discrete possibilistic distributions, based also on the construction from Section 2. There is established a formula for the possibilistic risk premium, expressed in terms of the utility function and of the quasi–mean value.

2 Credibility measures

Credibility measure is a particular fuzzy measure introduced by means of the notions of possibility and necessity measures. Each possibility distribution on a set X defines a credibility measure on X. By means of this one we will define in the next section the notion of quasi–mean value of a possibility distribution on $\mathbf{R}$.

This section contains the definition of the credibility measure and some examples. The references for this section are [11], [8], [4], [2].

Let X be a non–empty set. A *fuzzy measure* on X is a function $m : \mathcal{P}(X) \rightarrow [0, 1]$ such that the following conditions hold:

(M1) $m(\emptyset) = 0$; $m(X) = 1$;

(M2) If $D_1, D_2 \in \mathcal{P}(X)$ then $D_1 \subseteq D_2$ implies $m(D_1) \subseteq m(D_2)$.

A *possibility measure* on X is a function $\Pi : \mathcal{P}(X) \rightarrow [0,1]$ such that the following conditions are verified:

(P1) $\Pi(\emptyset) = 0$; $\Pi(X) = 1$;

(P2) For any family $\{D_i\}_{i \in I}$ of subsets of X, $\Pi(\bigcup_{i \in I} D_i) = \sup_{i \in I} \Pi(D_i)$.

A *necessity measure* on X is a function $N : \mathcal{P}(X) \rightarrow [0,1]$ such that

(N1) $N(\emptyset) = 0$; $N(X) = 1$;

(N2) For any family $\{D_i\}_{i \in I}$ of subsets of X, $N(\bigcap_{i \in I} D_i) = \inf_{i \in I} N(D_i)$.

Proposition 2.1 *Any possibility measure (resp. any necessity measure) on X is a fuzzy measure.*

Proof. Let Π be a possibility measure on X and $D_1 \subseteq D$. Then $D_1 \cup D_2 = D_2$, hence $\Pi(D_2) = \Pi(D_1 \cup D_2) = \Pi(D_1) \vee \Pi(D_2)$ so $\Pi(D_1) \leq \Pi(D_2)$. Analogously we reason for the case of a necessity measure.
■

Remark 2.2 *(i) If Π is a possibility measure then the function $N : \mathcal{P}(X) \rightarrow [0,1]$ defined by $N(D) = 1 - \Pi(X - D)$ for any $D \in \mathcal{P}(X)$ is a necessity measure.*

(ii) If N is a necessity measure then the function $\Pi : \mathcal{P}(X) \rightarrow [0,1]$ defined by

$\Pi(D) = 1 - N(X - D)$ for any $D \in \mathcal{P}(X)$ is a possibilistic measure.

Let Π be a possibility measure on X and N the associated necessity measure (cf. Remark 2.2(i)).

Consider the function $C_r : \mathcal{P}(X) \rightarrow [0,1$ defined by

(i) $Cr(D) = \frac{1}{2}(\Pi(D) + N(D))$ for any $D \in \mathcal{P}(X)$.

Proposition 2.3 *Cr is a fuzzy measure on X.*

Proof. We apply Proposition 2.1. ■

Remark 2.4 *Cr is a self–dual fuzzy measure, i. e. $Cr(D) = 1 - Cr(X - D)$, for each $D \in \mathcal{P}(X)$.*

Cr is called the *credibility measure* associated with Π.

A *possibility distribution* on X is a function $\mu : X \rightarrow [0,1]$ such that $\sup_{x \in X} \mu(X) = 1$.

μ is *normalized* if $\mu(x) = 1$ for some $x \in X$.

If Π is a possibility measure on X then the function $\mu_\Pi : X \rightarrow [0,1]$ defined by $\mu_X(x) = \Pi(\{x\})$ for each $x \in X$ is a possibility distribution.

Let μ be a possibility distribution on X. Let us consider the function Pos_μ : $\mathcal{P}(X) \rightarrow [0,1]$ defined by $\mu_X(x) = \Pi(\{x\})$ for each $x \in X$ is a possibility distribution.

Let μ be a possibility distribution on X. Let us consider the function $Pos_\mu : \mathcal{P}(X) \to [0,1]$ defined by

(2) $Pos_\mu(D) = \sup_{x\in D} \mu(x)$ for any $D \in \mathcal{P}(X)$.

Proposition 2.5 *Pos_μ is a possibility measure on X. The necessity measure Nec_μ associated with (cf. Remark 2.2 (i)) will be given by the formula*

(3) $Nec_\mu(D) = 1 - Pos_\mu(X - D) = 1 - \sup_{x \notin D} \mu(D)$.

3 Expected value and covariance of a fuzzy number

In probability theory the behaviour of the random variables is studied by indicators such as the mean value, dispersion, covariance, etc. In case of possibilistic distributions, one defines similar indicators. For fuzzy numbers, in [1], [2], [5], [9], etc. there have been defined the possibilistic mean value, the possibilistic dispersion, the possibilistic covariance. etc. These concepts have properties very similar with those from the probabilistic case, which allowed the development of the mathematical theory and their applicability.

On the other hand, their definitions are connected to the form of the fuzzy numbers and can be extended to other possibilistic repartitions (e. g. to discrete possibilistic repartitions). One imposes to find some indicators for larger classes of possibilistic repartitions.

This section is dedicated to a concept of mean value for any possibilistic distribution [8]. It is defined by the one introduced in [1], [5] and does not verify such important properties (e. g. linearity). Therefore we shall call it possibilistic quasi–mean value.

For the case of a discrete possibilistic distribution, this quasi–mean value has a very good formula from the point of view of the calculation. At the same time, in the discrete case an interesting phenomenon occurs: to a possibilistic distribution μ one canonically associates a random variable X whose mean value coincides with the quasi mean value of μ. This fact allows that some possibilistic decision making problems should be converted into problems of probabilistic decisions. The solving of the of probabilistic decisions to a solution for the first ones.

Let $\mu : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$ be a normalized possibility distribution.

Definition 3.1 *[8] The possibilistic quasi–mean value $Q(\mu)$* of μ is defined by

(1) $Q(\mu) = \int_0^\infty Cr(\mu \geq r)dr - \int_{-\infty}^0 Cr(\mu \leq r)dr$.

If the right member has the from $\infty - \infty$, then $Q(\mu)$ is not defined.

The definition of $Q(\mu)$has used the *credibility measure* defined in the previous section.

(2) $Cr(D) = \frac{1}{2}[Pos(D) + Nec(D)]$, for any $D \in \mathcal{P}\mathbf{R}$.

(Pos and Nec are the possibility and necessity measures Pos_μ and Nec_μ

associated to μ, according to the formula (2) from the previous section.)

If $f : [0,1] \to [0,1]$ is a function such that $f(1) = 1$ then $f(\mu) = f \circ \mu$ is normalized possibilistic distribution and

(3) $Q(f(\mu)) = \int_0^\infty Cr(f(\mu) \geq r)dr - \int_{-\infty}^0 Cr(\mu \leq r)dr.$

Example 3.2 *[8] If μ is the trapezoidal fuzzy number (a, b, β, α) then $Q(\mu) = \frac{1}{2}(a+b) + \frac{1}{4}(\beta - \alpha)$. By [1], the possibilistic expected value of the fuzzy number μ is $E(\mu) = \frac{a+b}{2} + \frac{1}{4}(\beta - \alpha)$, hence $Q(\mu) \neq E(\mu)$.*

Let us assume that μ is a normalized discrete possibilistic distribution.

(4) $\mu = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ \mu_1 & \mu_2 & \dots & \mu_n \end{vmatrix}.$

where $a_1 < a_2 < \ldots < a_n$.

Recall that $\mu_i = \mu(a_i)$ for $i = 1, \ldots, n$. Let $\mu_0 = \mu = n+1 = 0$. Let us denote

(5) $p_i = \frac{1}{2}[\bigvee_{j=1}^{i} \mu_j - \bigvee_{j=0}^{i-1}] + \frac{1}{2}[\bigvee_{j=1}^{n} \mu_j - \bigvee_{j=i+1}^{n+1} \mu_j]$

for $i = 1, \ldots, n$.

Proposition 3.3 *[8] [8] The numbers $p_1, \ldots, p_n$ satisfy the following properties:*

(i) $p_i \geq 0$ for $i = 1, \ldots, n$;

(ii) $\bigvee_{i=1}^{n} p_i = \bigvee_{i=1}^{n} \mu_i = 1$.

Proposition 3.3 emphasizes a remarkable fact: we can consider a discrete random variable:

(6) $X_\mu = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ p_1 & p_2 & \dots & p_n \end{vmatrix}.$

which takes values $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots, p_n$.

Proposition 3.4 *[8] $Q(\mu) = E(X_\mu) = \sum_{i=1}^{n} a_i p_i$.*

According to Propositions 3.3 and 3.4, to each normalized discrete possibilistic distribution μ one can associate a discrete random variable X_μ, such that the possibilistic quasi–mean $Q(\mu)$ coincides with the probabilistic mean value $E(X_\mu)$. In this way, the possibilistic phenomenon described by μ can be probabilistically modelled by the random variable X_μ. In particular, some possibilistic optimization problems will be treated as probabilistic optimization problems. This idea will be illustrated in the next section.

4 Probabilistic portfolio and possibilistic portfolios

A *portfolio* is a set of financial assets (money, bonds, etc.) and real assets (earth, buildings, gold, etc.) available to be bought. A portfolio is characterized by a return and by a risk connected by actions which will take place in the future.

We shall define two notions of portfolio: probabilistic portfolio and possibilistic portfolio. For the first one, the return is expressed in terms of probabilities and for the second one in terms of possibilities.

In the case of the probabilistic portfolio the risk is given by the dispersion of a random variable. To define the risk of a possibilistic portfolio, we shall make a construction based on the content of the previous section.

To a possibilistic portfolio one canonically associates a portfolio equivalent from the point of view of the revenue. The risk of the possibilistic portfolio will be the dispersion which appears in the framework of the associated probabilistic portfolio.

(I) The probabilistic portfolio ([10], p. 447).

We shall consider that the portfolio consists of m assets $A_1, \ldots, A_m$. By taking into account these m actions is n future time moments by

r_{ij}=the expected return for action A_i at moment j

p_{ij}=the probability of obtaining the expected return r_{ij} for action A_i at moment j,

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

For any $i =, ldots, m$, the probabilities $p_{i1}, \ldots, p_{in}$ satisfy conditions:

$p_{ij} \geq 0$ for any $j = 1, \ldots, n$

$\sum_{j=1}^{n} p_{ij} = 1.$

We define the return of the action i as being the discrete random variable

$R_i := \left| \begin{array}{cccc} r_{i1} & r_{i2} & \ldots & r_{in} \\ p_{i1} & p_{i2} & \ldots & p_{in} \end{array} \right|.$

for $i = 1, \ldots, m$.

For each of these random variables one considers the two main indicators:

(a) *the mean return* of the asset i:

$\bar{R}_i = E(R_i) = \sum_{j=1}^{n} p_{ij} r_{ij}$

(b) *the variance* of the asset i:

$Var(R_i) = \sum_{j=1}^{n} p_{ij}(r_{ij} - \bar{R}_i)^2$

Also we shall consider the *covariance* of two assets A_i and A_2:

$cov(R_i, R_k) = \sum_{s,t=1}^{n} p_{st}(r_{is} - \bar{R}_i)(r_{kt} - \bar{R}_k),$

indicator which shows the relationship between the two assets. We distinguish the cases:

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indicator which shows the relationship between the two assets. We distinguish the cases:

$cov(R_i, R_k) > 0$. Then the returns of the two assets A_i, A_k evolve in the same sense;

$cov(R_i, R_k) = 0$. The two assets are independent;

$cov(R_i, R_k) < 0$. The returns of the two assets A_i, A_k evolve in different senses.

Since the portfolio $\mathcal{P}$ described above is determined by the returns r_{ij} and by the probabilities p_{ij}, we shall use the notation

$\mathcal{P} =< r_{ij}, p_{ij} >$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

We shall denote now the weight of the budget of the investor dedicated for the asset A_i. It is obvious that we have the relations:

$f_1, \ldots, f_m \geq 0$;

$\sum_{i=1}^{m} = 1$.

For the portfolio $\mathcal{P}$ and for the weights $f_1, \ldots, f_m$ we define:

(d) the mean return of the portfolio:

$$E_{\mathcal{P}}(f_1, \ldots, f_m) = \sum_{i=1}^{m} f_i E(R_i).$$

(e) the dispersion of the portfolio

$$V_{\mathcal{P}}(f_1, \ldots, f_m) = \sum_{i,k=1}^{m} f_i f_k cov(R_i, R_k)$$

$$= \sum_{i=1}^{m} f_i^2 Var(R_i) + \sum_{i \neq k} f_i f_k cov(R_i, R_k).$$

The risk of the asset i is given by $Var(R_i)$ and the *total risk* of the portfolio is $V_{\mathcal{P}}(f_1, \ldots, f_m)$.

By establishing a mean value λ of the return of the portfolio, one requires the determination of the numbers $f_1, \ldots, f_m$ such that the total risk of the portfolio to be minim.

One obtains the following optimization problem:

$$(*) \ (20) \begin{cases} \min\limits_{f_1, \ldots, f_m} V_{\mathcal{P}}(f_1, \ldots, f_m) \\ E_{\mathcal{P}}(f_1, \ldots, f_m) = \lambda \\ f_1 + \ldots f_m = 1 \\ f_1, \ldots, f_m \geq 0 \end{cases}$$

(II) The possibilistic portfolio

The portfolio consists of m assets $A_1, \ldots, A_m$ and is defined by the following items:

r_{ij} is the expected return for the asset A_i at time j;

μ_{ij} is the possibility of obtaining the return r_{ij} for asset A_i at time j

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

The difference between the probabilistic and the possibilistic portfolio is essential: for the case of the former one appreciates the probability of obtaining a return and for the latter one appreciates the possibility of obtaining a return.

To this point, the problem is that the probabilistic notions of the probabilistic portfolio (mean values, dispersion, covariance) should be replaced with possibilistic notions.

We emphasized that in the first case we have had discrete random variables; in the second case we shall have discrete possibilistic distributions.

We define *the possibilistic return* corresponding to the asset A_i as being possibilistic distributions:

(f) $\mu_i = \begin{vmatrix} r_{i1} & r_{i2} & \dots & r_{in} \\ \mu_{i1} & \mu_{i2} & \dots & \mu_{in} \end{vmatrix}$

for $i = 1, \dots, m$.

The *possibilistic mean return* of the asset A_i is defined as:

(g) $\bar{\mu}_i = Q(\mu_i)$, for $i = 1, \dots, m$.

In (g) one used the possibilistic quasi–mean value introduced in Section 3. It would follow the introduction of the possibilistic dispersion of an asset, but we do not have a satisfying notion of dispersion of a discrete possibilistic distribution. We will appeal to the construction from the preceding section. One considers the discrete random variable X_i associated to the possibilistic distribution μ_i.

$X_i = \begin{vmatrix} r_{i1} & r_{i2} & \dots & r_{in} \\ p_{i1} & p_{i2} & \dots & p_{in} \end{vmatrix}$

for $i = 1, \dots, m$, where, according to the relation (5) from Section 3,

(h) $p_{ik} = \frac{1}{2}[\bigvee_{j=1}^{k} \mu_{ik} - \bigvee_{j=0}^{k-1}] + \frac{1}{2}[\bigvee_{j=k}^{n} \mu_{ij} - \bigvee_{j=k+1}^{n+1}]$ for any $i = \dots, m$ and $k = 1, \dots, n$.

We recall that in (h) one takes $p_{i0} = p_{in+1} = 0$, $i = 1, \dots, m$.

According to Proposition 3.4, the mean possibilistic return of asset A_i can be expressed by:

(i) $\bar{\mu}_i = Q(\mu_i) = E(X_i) = \sum_{k=1}^{n} r_{ik} p_{ik}$ for $i = 1, \dots, m$.

We denote by $\mathcal{P}' =< r_{ij}, \mu_{ij} >_{i=1,\dots,m}$ the possibilistic portfolio defined at the beginning of (II). Then $\mathcal{P}' =< r_{ij}, p_{ij} >_{i=1,\dots,m,j=1,\dots,n}$ with the probabilities p_{ij} defined by (h) will be a probabilistic portfolio.

The main idea of the following considerations is to reduce the study of the possibilistic portfolio $\mathcal{P}'$ to the study of $\mathcal{P}$.

Let $f_1, \dots, f_m$ be the weights of the revenue of the investor dedicated to the assets $A_1, \dots, A_m$.

The mean possibilistic return of the portfolio $\mathcal{P}'$ will defined as the mean probabilistic return of $\mathcal{P}$:

(j) $E_{\mathcal{P}}(f_1, \dots, f_m) = \sum_{i=1}^{m} f_i E(X_i) = \sum_{i=1}^{m} f_i \bar{\mu}_i$.

The *risk* of the asset A_i will be defined as the dispersion of X_i:

(k) $Var(X_i) = \sum_{j=1}^{n} p_{ij}(r_{ij} - \bar{\mu}_i)^2$, $i = 1, \dots, m$.

The *total risk* of the portfolio $\mathcal{P}'$ will be defined as the total risk of the portfolio $\mathcal{P}$.

(l) $V_{\mathcal{P}(f_1,\dots,f_m)=} \sum_{i,k=1}^{m} f_i f_k cov(x_i, x_k)$

$= \sum_{i=1}^{m} f_i^2 Var(X_i) + \sum_{i \neq k} f_i f_k cov(X_i, X_k).$

In formula (l), $cov(X_i, X_k)$ has the following expression:

(m) $cov(X_i, X_k) = \sum_{s,t=1}^{n} p_{st}(r_{is} - \bar{\mu})(r_{kt-\bar{\mu}_k)}$

By fixing an expected mean value λ of the possibilistic return of the portfolio $\mathcal{P}'$, one reaches the following optimization problem:

$$(**) \begin{cases} \min_{f_1,\ldots,f_m} V_{\mathcal{P}}(f_1, \ldots, f_m) \\ E_{\mathcal{P}}(f_1, \ldots, f_m) = \lambda \\ f_1 + \ldots f_m = 1 \\ f_1, \ldots, f_m \geq 0 \end{cases}$$

We notice that $(**)$ is exactly the optimization problem $(*)$ but the dates offered by the possibilistic portfolio $\mathcal{P}'$. Its reduction means to find the weights $f_1, \ldots, f_m$ for which one achieves the possibilistic expected mean return λ, but with a minimum risk.

5 Possibilistic risk aversion

In the context of the discrete posssibilistic distributions we shall define a notion of risk premium. It will describe the risk aversion of an agent represented by the utility function u. The result of the section is a calculus formula for the risk premium.

Let μ be a discrete possibilistic distribution.

(1) $\mu = \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ \mu_1 & \mu_2 & \ldots & \mu_n \end{vmatrix}.,$

$0 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ $(\mu_i = \mu(a_i))$.

Let us consider the probabilities $p_1, \ldots, p_n$ defined by (5)

$p_i = \frac{1}{2}[\bigvee_{j=1}^{i} \mu_j - \bigvee_{j=0}^{i-1} \mu_j] + \frac{1}{2}[\bigvee_{j=1}^{n} \mu_j - \bigvee_{j=i+1}^{n+1} \mu_j]$

By Proposition 3.3, the possibilistic quasi–mean value $Q(\mu)$ is given by

(2) $\bar{\mu} = Q(\mu) = \bigwedge \sum_{i=1}^{n} a_i p_i.$

Let $u : \mathbf{R}_+ \to \mathbf{R}_+$ be a utility function. shall suppose that μ has the class $\mathbf{C^2}$ on $[0, \infty)$ and that $u' \neq 0$. Let us define

(3) $Q(\mu(u)) = \sum_{i=1}^{n} \mu(a_i) p_i.$

Definition 5.1 *The possibilistic risk premium $\rho = \rho_{\mu,u}$ (associated with the discrete possibilistic distribution μ and with the utility function u) is defined by the equality*

(4) $Q(\mu(u)) = u(\bar{\mu} - \rho)$.

According to (3), relation (4) can be written:

(5) $\sum_{i=1}^{n} \mu(a_i)p_i = u(\bar{\mu} - \rho)$

Proposition 5.2 $\rho = -\frac{1}{2}\frac{u''(\mu)}{u'(\mu)} \sum_{i=1}^{n} (a_i - \bar{\mu})^2 p_i$.

Proof. Under the conditions when $u : \mathbf{R}_+ \to \mathbf{R}+$ has the class C^2 on $[0, \infty)$ and that $u' \neq 0$, we can apply the Taylor formula with the rest of the second degree. By omitting this rest for any $i = 1, \ldots, n$:

$\mu(a_i) = u(\bar{\mu} + a_i - \bar{\mu})$

$= u(\bar{\mu}) + u'(\bar{\mu})(a_i - \bar{\mu}) + \frac{u''(\bar{\mu})}{2}(a_i - \bar{\mu})^2$. ∎

By multiplying these equalities with p_i and then summing after $i = 1, \ldots, n$ (taking into account that $\sum_{i=1}^{n} p_i = 1$):

$\sum_{i=1}^{n} u(a_i)p_i = u(\bar{\mu})$.

$p_i u(a_i) = p_i u(\bar{\mu}) + p_i u'(\bar{\mu})(a_i - \bar{\mu}) + p_i \frac{u''(\bar{\mu})}{2}(a_i - \bar{\mu})^2$, $i = 1, \ldots, n$.

$\sum_{i=1}^{n} p_i \mu(a_i) = \sum_{i=1}^{n} p_i u(\bar{\mu}) + \sum_{i=1}^{n} p_i u'(\bar{\mu}) + \sum_{i=1}^{n} p_i \frac{1}{2} u''(\bar{\mu})(a_i - \bar{\mu})^2$.

$\sum_{i=1}^{n} \mu(a_i)p_i = u(\bar{\mu}) + u'(\bar{\mu}) \sum_{i=1}^{n} (a_i - \bar{\mu})p_i + \frac{1}{2} u''(\bar{\mu}) \sum_{i=1}^{n} p_i (a_i - \bar{\mu})^2$.

We notice that $\sum_{i=1}^{n} (a_i - \bar{\mu})p_i = \sum_{i=1}^{n} a_i p_i - \bar{\mu} \sum_{i=1}^{n} p_i = \bar{\mu} - \bar{\mu} = 0$, therefore

(6) $\sum_{i=1}^{n} u(a_i)p_i = u(\bar{\mu}) + u'(\bar{\mu}) \sum_{i=1}^{n} (a_i - \bar{\mu})p_i + \frac{1}{2} u''(\bar{\mu}) \sum_{i=1}^{n} p_i (a_i - \bar{\mu})^2$.

By applying Taylor's formula

(7) $u(\bar{\mu} - \rho) = u(\bar{\mu}) - u'(\bar{\mu})\rho$.

From (4), $\sum_{i=1}^{u} (a_i)p_i = u(\bar{\mu} - \rho)$. Then, from (6) and (7) it follows:

(8) $u(\bar{\mu}) + \frac{1}{2} u''(\bar{\mu}) \sum_{i=1}^{n} (a_i - \bar{\mu})^2 p_i = u(\bar{\mu}) - u'(\bar{\mu})\rho$.

From that it follows immediately that

$$\rho = -\frac{1}{2} \frac{u''(\bar{\mu}) \sum_{i=1}^{n} (a_i - \bar{\mu})^2 p_i}{u'(\bar{\mu})}.$$

6 Concluding Remarks

Probability theory and possibility theory and are two branches of mathematics which describe the uncertainty. In the field of applicability of possibility theory enter those situations of uncertainty in which we have a reduced databases. The development of possibility theory was done by relating it to the probability theory , by searching some concepts and results which should translate in possibilistic language what was known in case of probability theory.

The transition from probabilities to possibilities is not a simple operation and is not always possible. The finding of some possibilistic concepts corresponding to the mean value, the covariance and the dispersion has been successfully done for the fuzzy numbers [1], [2], [4], [5], [9], etc.

For larger classes of possibilistic distributions the problem seems more difficult. In [8] there has been proposed a notion of mean value of a possibilistic distribution, which does not have any longer the traditional properties of such a concept (e. g. linearity). For this reason, in this paper, the notion introduced in [8] has been named quasi–mean value.

For the case of the discrete possibilistic distributions a very interesting fact happens: to each discrete possibilistic repartition μ one can canonically associates a discrete probabilistic repartition (=discrete random variable), whose mean value is exactly the quasi–mean value of μ. This is the fundamental idea of the two topics studied in this paper: the possibilistic portfolios and the possibilistic risk aversion.

For the case of the first theme, by using the above mentioned construction, to a possibilistic portfolio one associates a probabilistic portfolio. This way the optimization problem associated to the possibilistic portfolio is transformed into an optimization problem corresponding to a probabilistic portfolio.

The second theme refers to the possibilistic risk aversion. In this case the uncertainty situation is described by a possibilistic distribution. For the case when the possibilistic distribution is a fuzzy number, this topic has been treated in [6] by means of possibilistic notions of expected value and variance studied by [1], [2], [4], [5], [9].

We define here a notion of possibilistic risk premium for a discrete possibilistic distribution by using the quasi–mean value and the transition from discrete possibilistic distributions to discrete random variables. This possibilistic risk premium is a measure of the risk aversion of an agent in front of a situation of uncertainty described by a discrete possibilistic distribution. For the calculation of the indicator of risk aversion one proved an analytical formula (see Proposition 5.1).

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