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Abstract

Many modal connectives may be considered either complete join- or meet-morphism on a complete lattice. Therefore, they induce Galois connections. In this work we introduce a negationless logic which can be characterized by the class of upper-bounded distributive lattices equipped with a Galois connection.

Keywords: Galois correspondences, Reasoning under uncertainty, Fuzzy sets (logic)

1 Background

In this section, we briefly recall some notions and results from existing literature. We consider Galois connections, relatively pseudocomplemented lattices, fuzzy sets, L-fuzzy sets, rough sets, and rough L-sets. We also describe so-called information logic of Galois connections introduced earlier by the authors.

1.1 Rough Sets and Galois Connections

Rough set theory introduced by Pawlak [12] is a mathematical formalism dealing with reasoning under uncertainty. To some extent rough set theory overlaps with fuzzy set theory presented by Zadeh [14]. In fuzzy set theory vagueness is expressed by *membership functions*, as rough set theory is based on *approximations* determined by *indiscernibility relations*.

Originally, indiscernibility relations were assumed to *equivalences* – reflexive, symmetric and transitive binary relations – interpreted so that two elements are equivalent if we cannot distinguish them by their properties. The *lower approximation* of a set X, denoted by $X^{\blacktriangledown}$ consists of elements such that their indiscernibility classes are included in X, and the *upper approximation* $X^{\blacktriangle}$ of X contains the elements of which indiscernibility classes have at least one common element with X.

Here we consider generalized rough approximations. Let R be a binary relation on U and $X \subseteq U$. The *upper approximation* $X^{\blacktriangle}$ is defined by

$$x \in X^{\blacktriangle} \iff (\exists y \in U)\, x\, R\, y \text{ and } y \in X$$

and the *lower approximation* $X^{\blacktriangledown}$ is specified by the condition:

$$x \in X^{\blacktriangledown} \iff (\forall y \in U)\, x\, R\, y \text{ implies } y \in X.$$

For instance, if R is a relation describing the similarity of objects in U, then $x \in X^{\blacktriangle}$ if there exists an object in X similar to x, and $x \in X^{\blacktriangledown}$ if all objects that are similar to x are in X. Therefore, $X^{\blacktriangle}$ and $X^{\blacktriangledown}$ can be viewed as sets of elements *possibly* and *certainly* belonging to X.

For two ordered sets P and Q, a pair (f, g) of maps $f\colon P \to Q$ and $g\colon Q \to P$ is called a *Galois connection* between P and Q if for all $p \in P$ and $q \in Q$,

$$f(p) \leq q \iff p \leq g(q).$$

If (f, g) is a Galois connection, we say that f has an *adjoint* g, and g has a *co-adjoint* f.

It is clear by the definition that if (f, g) is a Galois connection between bounded ordered sets P and Q, then f is bottom-preserving and g is top-preserving, that is, $f(0) = 0$ and $g(1) = 1$. We may also give the following alternative but equivalent definition: the pair (f, g) is a Galois connection if and only if

(GC1) $p \leq g(f(p))$ for all $p \in P$ and $f(g(q)) \leq q$ for all $q \in Q$;

(GC2) the maps f and g are order-preserving.

We know that if L and K are complete lattices, then a map $f \colon L \to K$ has an adjoint if and only if

$$f\Big(\bigvee S\Big) = \bigvee\{f(x) \mid x \in S\}$$

for all $S \subseteq L$. Let us denote by $\wp(U)$ the family of all subsets of U. Since ${}^{\blacktriangle} \colon \wp(U) \to \wp(U)$ distributes over arbitrary unions, that is, for all $\mathcal{S} \subseteq \wp(U)$,

$$\Big(\bigcup \mathcal{S}\Big)^{\blacktriangle} = \bigcup\{X^{\blacktriangle} \mid X \in \mathcal{S}\},$$

the map ${}^{\blacktriangle}$ induces a Galois connection. Next we will describe the adjoint of ${}^{\blacktriangle}$. This is done by defining another pair of approximation maps ${}^{\vartriangle}$ and ${}^{\triangledown}$ in terms of the inverse of R:

$$x \in X^{\vartriangle} \iff (\exists y \in U)\, y \, R \, x \text{ and } y \in X$$

and

$$x \in X^{\triangledown} \iff (\forall y \in U)\, y \, R \, x \text{ implies } y \in X.$$

It is known that the pairs $({}^{\blacktriangle}, {}^{\triangledown})$ and $({}^{\vartriangle}, {}^{\blacktriangledown})$ are Galois connections (see e.g. [6, 8] for further details and references).

1.2 Relatively Pseudocomplemented Lattices and Fuzzy Sets

Notions and basic results for relatively pseudocomplemented lattices may be found in [1, 13], for example.

Suppose that L is a lattice and $a, b \in L$. If there is a largest element $x \in L$ such that $a \wedge x \leq b$, then this is denoted by $a \to b$ and is called the *relative pseudocomplement of a with respect to b*. In this work, we often call the operation $\to$ simply as *implication*. It is obvious that $a \to b$ is the relative pseudocomplement of a with respect to b, if for all $x \in L$,

$$a \wedge x \leq b \iff x \leq a \to b.$$

Albeit $\bigvee\{x \in L \mid a \wedge x \leq b\}$ exists in a complete lattice for any pair of elements $a, b \in L$, it may fail to be the relative pseudocomplement $a \to b$. Note that if L has only two elements, say $0 < 1$, then $x \to y = 1$ if and only if

$x = 0$ or $y = 1$. Hence, the implication $\to$ can be viewed as a generalization of the classical Boolean implication.

In a *relatively pseudocomplemented lattice* $a \to b$ exists for all $a, b \in L$. The existence of $a \to b$ for each pair of elements $a, b \in L$ has surprising consequences. Namely, a relatively pseudocomplemented lattice has always the greatest element $1 = x \to x$, where x is any element of L. Furthermore, a relatively pseudocomplemented lattice is always distributive; if L is itself a complete lattice, then for all $S \subseteq L$ and $a \in L$,

$$a \wedge \Big(\bigvee S\Big) = \bigvee \{a \wedge b \mid b \in S\}.$$

In the following is listed some key properties of implication operation (see [13], for instance):

(i) $a \leq b$ if and only if $a \to b = 1$

(ii) if $a_1 \leq a_2$, then $a_1 \to b \geq a_2 \to b$

(iii) if $b_1 \leq b_2$, then $a \to b_1 \leq a \to b_2$

(iv) $a \wedge (a \to b) \leq b$

(v) $a \to (a \wedge b) \geq b$

(vi) $a \to (b \to c) = (a \wedge b) \to c = b \to (a \to c)$

(vii) $(a \to b) \leq (b \to c) \to (a \to c)$

If P is an ordered set and X is any set, then we can order the set P^X of all maps from X to P *pointwise* by defining

$$f \leq g \iff (\forall x \in X)\, f(x) \leq g(x).$$

If L is a lattice, then L^X is a lattice such that

$$(f \vee g)(x) = f(x) \vee g(x) \quad \text{and} \quad (f \wedge g)(x) = f(x) \wedge g(x).$$

In addition, if L is a complete lattice, then L^X is a complete lattice such that for all $\{f_i\} \subseteq L^X$ and $x \in X$,

$$\Big(\bigvee_{i \in I} f_i\Big)(x) = \bigvee_{i \in I} f_i(x)$$

and

$$\Big(\bigwedge_{i \in I} f_i\Big)(x) = \bigwedge_{i \in I} f_i(x).$$

The least element of L^X is $\mathbf{0}\colon x \mapsto 0$ and the greatest element of L^X is $\mathbf{1}\colon x \mapsto 1$, where 0 and 1 are the least and the greatest elements of L. Furthermore, if L is relatively pseudocomplemented, then L^X is relatively pseudocomplemented in such a way that

$$(f \to g)(x) = f(x) \to g(x)$$

for all $f, g \in L^X$ and $x \in X$.

Let $\mathbb{I}$ denote the unit interval $[0, 1]$ with its usual order. Then $\mathbb{I}$ is a complete lattice such that for all $S \subseteq \mathbb{I}$,

$$\bigvee S = \sup S \quad \text{and} \quad \bigwedge S = \inf S.$$

In particular,

$$x \vee y = \max\{x, y\} \quad \text{and} \quad x \wedge y = \min\{x, y\}$$

for all $x, y \in \mathbb{I}$. Trivially, 0 and 1 are the least and the greatest elements of $\mathbb{I}$. It is also easy to notice that $\mathbb{I}$ is relatively pseudocomplemented lattice in which for all $x, y \in \mathbb{I}$,

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Let U be a nonempty set. The elements of $\mathbb{I}^U$ are called *fuzzy sets* on U. Each fuzzy set φ associates with each element $x \in U$ a real number $\varphi(x)$ representing the grade of membership of x in the fuzzy set φ.

It is now obvious that $\mathbb{I}^U$ is a complete relatively pseudocomplemented distributive lattice such that

$$\Big(\bigvee \varphi_i\Big)(x) = \sup\{\varphi_i(x) \mid i \in I\}$$

and

$$\Big(\bigwedge \varphi_i\Big)(x) = \inf\{\varphi_i(x) \mid i \in I\}$$

for all $\{\varphi_i\}_{i \in I} \subseteq \mathbb{I}^U$ and $x \in U$. In particular,

$$(\varphi \vee \psi)(x) = \max\{\varphi(x), \psi(x)\}$$

and

$$(\varphi \wedge \psi)(x) = \min\{\varphi(x), \psi(x)\}$$

for all $\varphi, \psi \in \mathbb{I}^U$ and $x \in U$. The relative pseudocomplement of φ with respect to ψ is defined pointwise by setting

$$(\varphi \to \psi)(x) = \varphi(x) \to \psi(x)$$

for all $x \in X$.

Goguen generalized fuzzy sets to L-fuzzy sets in [5]. An *L-fuzzy set* φ on U is a mapping $\varphi\colon U \to L$, where L is a complete lattice. The motivation for this generalization is that in this kind of setting it is possible to consider sets for which membership values may be incomparable. The least element 0 and the greatest element 1 of L may be viewed as the 'absolute' membership values.

For any nonempty set U, the set L^U is the set of all *L-fuzzy sets* on U. As noted by Goguen, the set of all L-fuzzy sets on a set U can be equipped whatever operations L has, and these inherited operations obey any law valid in L which extends pointwise. Thus, as we already noted, with respect to the pointwise order, L^U is a complete lattice such that

$$\Big(\bigvee \varphi_i\Big)(x) = \bigvee_{i \in I} \varphi_i(x)$$

and

$$\Big(\bigwedge \varphi_i\Big)(x) = \bigwedge_{i \in I} \varphi_i(x)$$

for all $\{\varphi_i\}_{i \in I} \subseteq L^U$ and $x \in U$. If addition, if L is relatively pseudocomplemented lattice, then L^U is relatively pseudocomplemented and

$$(\varphi \to \psi)(x) = \varphi(x) \to \psi(x)$$

for all $\varphi, \psi \in L^U$ and $x \in U$.

It should also be noted that Järvinen, in [7], has introduced set operations of union, intersection and complement for L-fuzzy sets in cases L is just a preordered set, which means that joins, meets, and complements are not defined in L. The presented approach even handles the union and the intersection of an L_1-fuzzy set φ and an L_2-fuzzy set ψ on the same universe U, but not necessarily on the same preordered set. This means that we can, for example, combine with 'or', 'and', and 'not' judgements of evaluators all wanting to use their own words and expressions.

1.3 Rough L-Sets

Dubois and Prade introduced in [4] *rough fuzzy sets*. The idea is that the objects to be approximated are fuzzy instead of classical sets, and the approximations are determined by means of fuzzy relations. Here we study *rough L-sets*, which means that approximations of L-sets are determined by L-fuzzy relations.

Let L be a relatively pseudocomplemented complete lattice. This means that L is now so-called *Heyting algebra*. In particular, L is distributive. Let $\varphi \in L^U$ be an L-fuzzy set and let R be an L-fuzzy relation on U, that is, R

is a mapping $R\colon U \times U \to L$. Then we may define the L-fuzzy sets $\varphi^{\blacktriangle}$ and $\varphi^{\blacktriangledown}$ on U by setting

$$\begin{aligned}
\varphi^{\blacktriangle}(x) &= \bigvee_{y \in U} \{R(x,y) \wedge \varphi(y)\} \\
\varphi^{\blacktriangledown}(x) &= \bigwedge_{y \in U} \{R(x,y) \to \varphi(y)\}
\end{aligned}$$

for all $x \in U$. The L-sets $\varphi^{\blacktriangle}$ and $\varphi^{\blacktriangledown}$ and called the *upper* and the *lower approximations* of φ, and these sets can be viewed as 'coarsened L-sets'.

Naturally, we can define another pair of mappings in terms of the 'inverse' of R by setting:

$$\begin{aligned}
\varphi^{\vartriangle}(x) &= \bigvee_{y \in U} \{R(y,x) \wedge \varphi(y)\} \\
\varphi^{\triangledown}(x) &= \bigwedge_{y \in U} \{R(y,x) \to \varphi(y)\}
\end{aligned}$$

for all $x \in U$. It is clear that if φ is a 'classical' two-valued set on U and R is a two-valued binary relation on U, then the operations ${}^{\blacktriangle}$, ${}^{\blacktriangledown}$, ${}^{\vartriangle}$, and ${}^{\triangledown}$ coincide with the operations defined in Section 1.1.

Lemma 1. *The pair* $({}^{\blacktriangle}, {}^{\triangledown})$ *is a Galois connection on* L^U.

Proof. Suppose φ and ψ are L-fuzzy sets such that $\varphi \leq \psi$. Then for all $y \in U$, $R(x,y) \wedge \varphi(y) \ \leq \ R(x,y) \wedge \psi(y)$ and this implies

$$\varphi^{\blacktriangle}(x) = \bigvee_{y \in U} \{R(x,y) \wedge \varphi(y)\} \leq \bigvee_{y \in U} \{R(x,y) \wedge \psi(y)\} = \psi^{\blacktriangle}(x)$$

Similarly, $R(y,x) \to \varphi(y) \ \leq \ R(y,x) \to \psi(y)$ for all $y \in U$. Thus,

$$\varphi^{\triangledown}(x) = \bigwedge_{y \in U} \{R(y,x) \to \varphi(y)\} \leq \bigwedge_{y \in U} \{R(y,x) \to \psi(y)\} = \psi^{\triangledown}(x).$$

We have now shown that ${}^{\blacktriangle}$ and ${}^{\triangledown}$ are order-preserving.

By definition, for all $x \in U$,

$$\begin{aligned}
\varphi^{\triangledown\blacktriangle}(x) &= \bigvee_{y \in U} \{R(x,y) \wedge \varphi^{\triangledown}(y)\} = \bigvee_{y \in U} \Big\{ R(x,y) \wedge \bigwedge_{z \in U} \{R(z,y) \to \varphi(z)\} \Big\} \\
&\leq \bigvee_{y \in U} \{R(x,y) \wedge (R(x,y) \to \varphi(x)\,)\,\} \leq \bigvee_{y \in U} \{\varphi(x)\} = \varphi(x).
\end{aligned}$$

This means that $\varphi^{\blacktriangle\triangledown} \leq \varphi$. Similarly, for all $x \in U$,

$$\begin{aligned}
\varphi^{\blacktriangle\triangledown}(x) &= \bigwedge_{y \in U} \{R(y,x) \to \varphi^{\blacktriangle}(y)\} = \bigwedge_{y \in U} \Big\{ R(y,x) \to \bigvee_{z \in U} \{R(y,z) \wedge \varphi(z)\} \Big\} \\
&\geq \bigwedge_{y \in U} \{R(y,x) \to (\,R(y,x) \wedge \varphi(x)\,)\} \geq \bigwedge_{y \in U} \{\varphi(x)\} = \varphi(x).
\end{aligned}$$

Thus, also $\varphi \leq \varphi^{\blacktriangle\triangledown}$. $\square$

It is also interesting to notice that if the fuzzy relation R is *reflexive*, which means that $R(x,x) = 1$ for all $x \in U$, then for any $\varphi \in L^U$,

$$\varphi^{\blacktriangle}(x) = \bigvee_{y \in U} \{R(x,y) \wedge \varphi(y)\} \geq R(x,x) \wedge \varphi(x) = \varphi(x),$$

that is, $\varphi \leq \varphi^{\blacktriangle}$ and $\varphi \to \varphi^{\blacktriangle} = \mathbf{1}$. Hence, ${}^{\blacktriangle}$ can be considered as a *weakening modifier* in the sense of [10]. Modifiers of fuzzy sets are self-maps on the set of all fuzzy sets. Their idea is that they modify every fuzzy set to another fuzzy set. Similarly,

$$\varphi^{\blacktriangledown}(x) = \bigwedge_{y \in U} \{R(x,y) \to \varphi(y)\} \leq R(x,x) \to \varphi(x) = \varphi(x),$$

and hence ${}^{\blacktriangledown}$ may be viewed as a *substantiating* modifier. Since in case of classical rough sets, $X^{\blacktriangle}$ and $X^{\blacktriangledown}$ are considered as sets of elements that certainly and possibly are in X, the modifiers ${}^{\blacktriangledown}\colon L^U \to L^U$ and ${}^{\blacktriangle}\colon L^U \to L^U$ may also be viewed as soft of linguistic modifiers, called hedges [11]. Similar observation hold, of course, for ${}^{\vartriangle}$ and ${}^{\triangledown}$.

1.4 Information Logic of Galois Connections

In [9], Information Logic of Galois Connections (ILGC) suited for approximate reasoning about knowledge is introduced. ILGC is just the standard propositional logic with two modal connectives $\blacktriangle$ and $\triangledown$ mimicking the performance of Galois connection maps.

The set of *connectives* consists of logical symbols $\to$, $\neg$, $\blacktriangle$, and $\triangledown$. In addition of the three classical propositional logic axioms and the inference rule of modus ponens, ILGC contains only two rules of inference:

$$\frac{A \to \triangledown B}{\blacktriangle A \to B} \qquad \text{and} \qquad \frac{\blacktriangle A \to B}{A \to \triangledown B}$$

We introduced another pair $\vartriangle$ and $\blacktriangledown$ of connectives by defining them as the *duals* of $\triangledown$ and $\blacktriangle$ by setting

$$\vartriangle A := \neg \triangledown \neg A \quad \text{and} \quad \blacktriangledown A := \neg \blacktriangle \neg A.$$

and showed that $\vartriangle$ and $\blacktriangledown$ we have similar inference rules that we have for the original connectives $\blacktriangle$ and $\triangledown$.

At the first glance the language of our logic ILGC is different from the one of minimal tense logic K_t (see e.g. [2] for further details), because ILGC has only three axioms and three inferences rules, meanwhile K_t has seven axioms and three rules of inference. However, we showed in [9] that in fact ILGC is with respect to provability equivalent to the minimal tense logic K_t. This means that ILGC can be viewed as a simple formulation of K_t. Additionally, since K_t is known to be decidable, decidability of ILGC follows from this.

For ILGC, a Kripke-style semantics may be defined. If R is a binary relation representing knowledge about objects U and $x \in U$, then

$$\begin{array}{lcl} \blacktriangle A \text{ is true for } x & \iff & A \text{ is true for some object } y \text{ such that } x\,R\,y; \\ \triangledown A \text{ is true for } x & \iff & A \text{ is true for all objects } y \text{ such that } y\,R\,x. \end{array}$$

For instance, let U be some set of human beings and let R be a relation reflecting similarity of people with respect to some suitable attributes – what those properties might be is irrelevant for this consideration. Then, the pair (U, R) is so-called ILGC-frame. Let (U, R, v) be a model based on the ILGC-frame (U, R) , where v is the valuation mapping from the set of all ILGC-formulas to $\wp(U)$. If A is an ILGC-formula such that $v(A)$ consists of 'good teachers', then $\blacktriangle A$ is true for x if there is a good teacher y to which x is similar. This means that x is *possibly* a good teacher. Similarly, $\triangledown A$ is true for x if all people similar to x are good teachers. Thus, x is *certainly* a good teacher. In addition, if B is a formula such that $v(B)$ is the set of 'patient persons', then the implication $B \to \blacktriangle A$ is true, since patient persons are usually good teachers.

Note also that for the minimal tense logic K_t, validity of formulas can be defined in a similar manner. This implies that ILGC is a complete logic, since K_t is known to be complete.

In [9], also a survey of the literature on related modal logics is presented. In this work our purpose is to generalize the logic ILGC. In the next section, we present a negationless logic and present for it a lattice-valued semantics.

2 Logic LGC

In this section, we define the logic LGC which can be proved to be determined by the class of all upper-bounded distributive lattices with a pair (f, g) of Galois connection. LGC is suitable for dealing with, for instance, rough L-sets, but also other application areas exist. Note that LGC is a generalization of ILGC since its language does not include negation of any kind.

2.1 Language and Syntax

Let P be an enumerable set, whose elements are called *propositional variables*. The set of *connectives* consists of logical symbols $\to$, $\vee$, $\wedge$, $\blacktriangle$, and $\triangledown$. *Formulas* of LGC are defined inductively as follows:

(i) Every propositional variable is a formula.

(ii) If A and B are formulas, then so are $A \to B$, $A \vee B$, $A \wedge B$, $\blacktriangle A$, and $\triangledown A$.

We agree that in formulas implication has weaker precedence than conjunction and disjunction, which in turn have weaker precedence than the modal connectives $\blacktriangle$ and $\triangledown$. Let us denote by Φ the set of all LGC-formulas. The logical system LGC has the following eight axioms:

(Ax1) $A \to A$

(Ax2) $A \wedge B \to A$

(Ax3) $A \wedge B \to B \wedge A$

(Ax4) $A \to A \vee B$

(Ax5) $A \vee B \to B \vee A$

(Ax6) $(A \wedge B \to C) \to (A \to (B \to C))$

(Ax7) $(A \to (B \to C)) \to (A \wedge B \to C)$

(Ax8) $(A \to B) \to ((B \to C) \to (A \to C))$

Furthermore, LGC has the following five *rules of inference*:

(MP) $\dfrac{A \qquad A \to B}{B}$

($\mathrm{R}_\wedge$) $\dfrac{A \to B \qquad A \to C}{A \to B \wedge C}$ ($\mathrm{R}_\vee$) $\dfrac{A \to C \qquad B \to C}{A \vee B \to C}$

(GC1) $\dfrac{A \to \triangledown B}{\blacktriangle A \to B}$ (GC2) $\dfrac{\blacktriangle A \to B}{A \to \triangledown B}$

An LGC-formula A is said to be *provable*, if there is a finite sequence $A_1, A_2, \ldots, A_n$ of LGC-formulas such that $A = A_n$ and for every $1 \leq i \leq n$:

(i) either A_i is an axiom of LGC

(ii) or A_i is the conclusion of some inference rules, whose premises are in the set $\{A_1, \ldots, A_{i-1}\}$.

That A is provable is denoted by $\vdash A$.

In the next proposition we present some provable formulas and additional inference rules of LGC.

Proposition 1. *For all* LGC*-formulas $A, B, C \in \Phi$, the following assertions hold.*

(i) $\dfrac{A \to B \quad B \to C}{A \to C}$

(ii) $\dfrac{A \to (B \to C)}{B \to (A \to C)}$

(iii) $\vdash A \wedge (A \to B) \to B$

(iv) $\dfrac{A \to B}{A \wedge C \to B \wedge C}$ *and* $\dfrac{A \to B}{A \vee C \to B \vee C}$

(v) $\dfrac{A \to B}{(C \to A) \to (C \to B)}$ *and* $\dfrac{A \to B}{(B \to C) \to (A \to C)}$

(vi) $\vdash B \to (A \to A)$

(vii) $\vdash B$ *if and only if* $\vdash (A \to A) \to B$

Proof. (i) By (Ax8), $\vdash (A \to B) \to ((B \to C) \to (A \to C))$. Assume $\vdash A \to B$. Then $\vdash (B \to C) \to (A \to C)$ by (MP). If also $\vdash B \to C$, then $\vdash A \to C$.

(ii) If $\vdash A \to (B \to C)$, then $\vdash A \wedge B \to C$ and $\vdash B \wedge A \to C$ by (Ax7) and (Ax3). This gives $\vdash B \to (A \to C)$ by (Ax6).

(iii) By (Ax1), $\vdash (A \to B) \to (A \to B)$. We get $\vdash A \to ((A \to B) \to B)$ using (ii). This gives $\vdash A \wedge (A \to B) \to B$ by (Ax7).

(iv) By (Ax2), $\vdash A \wedge C \to A$. If $\vdash A \to B$, then also $\vdash A \wedge C \to B$. Obviously, $\vdash A \wedge C \to C$. Hence, $\vdash A \wedge C \to B \wedge C$ by ($\mathrm{R}_\wedge$). For the other case, assume that $\vdash A \to B$. Because $\vdash B \to B \vee C$, we have $\vdash A \to B \vee C$. Clearly, $\vdash C \to B \vee C$. These imply $\vdash A \vee C \to B \vee C$ by ($\mathrm{R}_\vee$).

(v) By (iii), $\vdash C \wedge (C \to A) \to A$. Suppose $\vdash A \to B$. Then, $\vdash C \wedge (C \to A) \to B$. This is equivalent to $\vdash (C \to A) \wedge C \to B$. By (Ax6), we obtain $\vdash (C \to A) \to (C \to B)$. The other part is obvious by (Ax8).

(vi) By (Ax2), $\vdash A \wedge B \to A$. This implies $\vdash A \to (B \to A)$ by (Ax6). By (ii), we obtain $\vdash B \to (A \to A)$.

(vii) By (vi), $\vdash (A \to A) \to (B \to B)$. This gives $\vdash (A \to A) \wedge B \to B$ and $\vdash B \wedge (A \to A) \to B$. Thus, if $\vdash B$, then $\vdash (A \to A) \to B$. Conversely, if $(A \to A) \to B$ is provable, then clearly $\vdash B$, because $A \to A$ is provable by (Ax1). $\square$

Hereafter, we denote $A \to A$ simply by $\top$. Therefore, $\vdash \top$ and $A \to \top$ for all $A \in \Phi$. Additionally, $\vdash A$ if and only if $\vdash \top \to A$.

In the next proposition we present some essential provable formulas involving the modal connectives $\blacktriangle$ and $\triangledown$.

Proposition 2. *For all* LGC*-formulas* $A, B, C \in \Phi$*, the following assertions hold.*

(i) $\dfrac{A \to B}{\triangledown A \to \triangledown B}$ *and* $\dfrac{A \to B}{\blacktriangle A \to \blacktriangle B}$.

(ii) $\vdash A \to \triangledown\blacktriangle A$ *and* $\vdash \blacktriangle\triangledown A \to A$.

(iii) $\dfrac{A}{\nabla A}$.

(iv) $\vdash \nabla(A \wedge B) \to \nabla A \wedge \nabla B$ *and* $\vdash \nabla A \wedge \nabla B \to \nabla(A \wedge B)$.

(v) $\vdash \blacktriangle(A \vee B) \to \blacktriangle A \vee \blacktriangle B$ *and* $\vdash \blacktriangle A \vee \blacktriangle B \to \blacktriangle(A \vee B)$.

(vi) $\vdash \nabla(A \to B) \to (\nabla A \to \nabla B)$.

Proof. (i) Suppose $\vdash A \to B$. Since $\vdash \nabla A \to \nabla A$ holds trivially, we obtain $\vdash \blacktriangle\nabla A \to A$ by (GC1). Hence $\vdash \blacktriangle\nabla A \to B$, which gives $\vdash \nabla A \to \nabla B$ by (GC2). For the other part, suppose that $\vdash A \to B$. Since $\vdash \blacktriangle B \to \blacktriangle B$, we have $\vdash B \to \nabla\blacktriangle B$ by (GC2). Thus, $\vdash A \to \nabla\blacktriangle B$. This implies $\vdash \blacktriangle A \to \blacktriangle B$ by (GC1).

(ii) Because $\vdash \blacktriangle A \to \blacktriangle A$, we have $\vdash A \to \nabla\blacktriangle A$ by (GC2). Similarly, $\vdash \nabla A \to \nabla A$ gives $\vdash \blacktriangle\nabla A \to A$ by (GC1).

(iii) Assume $\vdash A$. This means $\vdash \top \to A$ and we get $\vdash \nabla\top \to \nabla A$ by (i). Because $\blacktriangle\top \to \top$ is provable, we have $\vdash \top \to \nabla\top$. Thus, $\vdash \top \to \nabla A$ and $\vdash \nabla A$.

(iv) Since $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, we have $\vdash \nabla(A \wedge B) \to \nabla A$ and $\vdash \nabla(A \wedge B) \to \nabla B$. Hence, $\vdash \nabla(A \wedge B) \to \nabla A \wedge \nabla B$ by $(\mathrm{R}_\wedge)$. On the other hand, $\vdash \nabla A \wedge \nabla B \to \nabla A$ yields $\vdash \blacktriangle(\nabla A \wedge \nabla B) \to A$ by (GC1). Similarly, we may show $\vdash \blacktriangle(\nabla A \wedge \nabla B) \to B$. This gives that $\vdash \blacktriangle(\nabla A \wedge \nabla B) \to A \wedge B$ and $\vdash \nabla A \wedge \nabla B \to \nabla(A \wedge B)$ by (GC2).

(v) Because $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$, we have $\vdash \blacktriangle A \to \blacktriangle(A \vee B)$ and $\vdash \blacktriangle B \to \blacktriangle(A \vee B)$. These give $\vdash \blacktriangle A \vee \blacktriangle B \to \blacktriangle(A \vee B)$ by $(\mathrm{R}_\vee)$. Conversely, $\vdash \blacktriangle A \to \blacktriangle A \vee \blacktriangle B$ and $\vdash \blacktriangle B \to \blacktriangle A \vee \blacktriangle B$. imply $\vdash A \to \nabla(\blacktriangle A \vee \blacktriangle B)$ and $\vdash B \to \nabla(\blacktriangle A \vee \blacktriangle B)$. We get $\vdash A \vee B \to \nabla(\blacktriangle A \vee \blacktriangle B)$ and $\vdash \blacktriangle(A \vee B) \to \blacktriangle A \vee \blacktriangle B$.

(vi) Since $\vdash A \wedge (A \to B) \to B$, we have $\vdash \nabla(A \wedge (A \to B)) \to \nabla B$. Furthermore, by (iv), we obtain $\vdash \nabla A \wedge \nabla(A \to B) \to \nabla(A \wedge (A \to B))$. Thus, $\vdash \nabla A \wedge \nabla(A \to B) \to \nabla B$, which is equivalent to $\vdash \nabla(A \to B) \to (\nabla A \to \nabla B)$.

□

2.2 Semantics and Completeness

In this section, we define an algebraic semantics for LGC. A *GC-algebra* $(L, \wedge, \vee, \to, f, g, 1)$ is an algebra of type $(2, 2, 2, 1, 1, 0)$ such that

(GCA1) $(L, \wedge, \vee, 1)$ is a lattice with the greatest element 1.

(GCA2) For all $x, y, z \in L$, $x \wedge y \leq z$ if and only if $x \leq y \to z$.

(GCA3) For all $x, y \in L$, $f(x) \leq y$ if and only if $x \leq g(y)$.

In other words, GC-algebras are relatively pseudocomplemented lattices equipped with a Galois connection. Note that this means that they are always distributive. Additionally, $f(x) \to y = 1$ if and only if $x \to g(y) = 1$.

Let $(L, \wedge, \vee, \to, f, g, 1)$ be a GC-algebra. Recall that we denote by P the set of all propositional variables. Let v be a function $v\colon P \to L$ assigning to each propositional variable p in P an element $v(p)$ of the lattice L. Such functions are called *valuations*. The valuation v can be extended uniquely to the set Φ of all formulas inductively by the following way:

$$v(A \wedge B) = v(A) \wedge v(B)$$

$$v(A \vee B) = v(A) \vee v(B)$$

$$v(A \to B) = v(A) \to v(B)$$

$$v(\blacktriangle A) = f(v(A))$$

$$v(\triangledown A) = g(v(A))$$

We say that an LGC-formula $A \in \Phi$ is *valid* if $v(A) = 1$ for any valuation $v\colon \Phi \to L$ on any GC-algebra $(L, \wedge, \vee, \to, f, g, 1)$.

Theorem 1 (Soundness). *Every provable* LGC-*formula is valid.*

Proof. Let v be a valuation from the set of all formulas Φ to some GC-algebra L. We show first that every axiom is valid.

(Ax1): $v(A \to A) = v(A) \to v(A) = 1$.

(Ax2): $v(A \wedge B) = v(A) \wedge v(B) \leq v(A)$ implies $v(A \wedge B \to A) = v(A \wedge B) \to v(A) = 1$

(Ax3): $v(A \wedge B) = v(A) \wedge v(B) = v(B) \wedge v(A) = v(B \wedge A)$. This implies $v(A \wedge B \to B \wedge A) = v(A \wedge B) \to v(B \wedge A) = 1$.

(Ax4): $v(A) \leq v(A) \vee v(B) = v(A \vee B)$ implies $v(A \to A \vee B) = v(A) \to v(A \vee B) = 1$.

(Ax5): $v(A \vee B) = v(A) \vee v(B) = v(B) \vee v(A) = v(B \vee A)$. Therefore, $v(A \vee B \to B \vee A) = v(A \vee B) \to v(B \vee A) = 1$.

(Ax6–7): By the properties of relative pseudocomplements:

$$\begin{aligned} v(A \wedge B \to C) &= v(A) \wedge v(B) \to v(C) \\ &= v(A) \to (v(B) \to v(C)) \\ &= v(A \to (B \to C)) \end{aligned}$$

The validity of the axioms follows from this equality.

(Ax8): In a similar way:

$$\begin{aligned} v(A \to B) &= v(A) \to v(B) \\ &\leq (v(B) \to v(C)) \to (v(A) \to v(C)) \\ &= v((B \to C) \to (A \to C)). \end{aligned}$$

Next we show that each rule of inference preserves validity.

(MP): If $v(A) = 1$ and $v(A \to B) = v(A) \to v(B) = 1$, then $v(B) \geq v(A) = 1$.

($R_\wedge$): If $v(A \to B) = v(A) \to v(B) = 1$ and $v(A \to C) = v(A) \to v(C) = 1$, then $v(A) \leq v(B)$ and $v(A) \leq v(C)$, which gives $v(A) \leq v(B) \wedge v(C) = v(B \wedge C)$. Thus, $v(A \to B \wedge C) = v(A) \to v(B \wedge C) = 1$.

($R_\vee$): If $v(A \to C) = v(A) \to v(C) = 1$ and $v(B \to C) = v(B) \to v(C) = 1$, then $v(A) \leq v(C)$ and $v(B) \leq v(C)$. So, $v(A \vee B) = v(A) \vee v(B) \leq v(C)$. and $v(A \vee B \to C) = v(A \vee B) \to v(C) = 1$.

(GC1–2): By the definition of Galois connections,

$$\begin{aligned} v(A \to \triangledown B) = 1 &\iff v(A) \to v(\triangledown B) = 1 \iff v(A) \leq v(\triangledown B) \\ &\iff v(A) \leq f(v(B)) \iff g(v(A)) \leq v(B) \\ &\iff v(\blacktriangle A) \leq v(B) \iff v(\blacktriangle A) \to v(B) = 1 \\ &\iff v(\blacktriangle A \to B) = 1. \end{aligned}$$

$\square$

To obtain completeness, we show that valid formulas are provable by applying so-called Lindenbaum-Tarski algebras. Firstly, we define an equivalence relation $\equiv$ on the set Φ of LGC-formulas. Let $A, B \in \Phi$. Then:

$$A \equiv B \iff \vdash A \to B \text{ and } \vdash B \to A.$$

By applying Proposition 1, we may show that the equivalence $\equiv$ is a congruence on Φ, as done in the next lemma.

Lemma 2. *The equivalence relation $\equiv$ satisfies the substitution property, that is, if $A \equiv B$ and $C \equiv D$, then the following assertions hold.*

(i) $A \wedge C \equiv B \wedge D$,

(ii) $A \vee C \equiv B \vee D$,

(iii) $A \to C \equiv B \to D$,

(iv) $\blacktriangle A \equiv \blacktriangle B$,

(v) $\nabla A \equiv \nabla B$.

Proof. Assume that $A \equiv B$ and $C \equiv D$. We prove only claims (i), (iii) and (iv); cases (ii) and (v) can be proved in an analogous manner.

(i) By (Ax2), $\vdash A \wedge C \to A$ and $\vdash A \wedge C \to C$. Because $\vdash A \to B$ and $\vdash C \to D$, we have $\vdash A \wedge C \to B$ and $\vdash A \wedge C \to D$. This gives by $(\mathrm{R}_\wedge)$, $\vdash A \wedge C \to B \wedge D$. In a similar way, we can prove $\vdash B \wedge D \to A \wedge C$. Thus, $A \wedge C \equiv B \wedge D$.

(iii) By assumption, $\vdash B \to A$ and $\vdash C \to D$. These imply by Proposition 1(v) that $\vdash (A \to C) \to (B \to C)$ and $\vdash (B \to C) \to (B \to D)$. Therefore, $\vdash (A \to C) \to (B \to D)$. Similarly, we can show $\vdash (B \to D) \to (A \to C)$. Hence, $A \to C \equiv B \to D$.

(iv) Now $\vdash A \to B$ and $\vdash B \to A$ give $\vdash \blacktriangle A \to \blacktriangle B$ and $\vdash \blacktriangle B \to \blacktriangle A$, that is, $\blacktriangle A \equiv \blacktriangle B$. $\square$

For an LGC-formula $A \in \Phi$, we denote by $[A]$ the equivalence class of A, that is,

$$[A] = \{B \in \Phi \mid A \equiv B\}.$$

In addition, the set of all Φ-classes $\{\,[A] \mid A \in \Phi\}$ is denoted by $\Phi/{\equiv}$.

Because the equivalence $\equiv$ is a congruence on the set Φ, we may now define its *quotient algebra* (cf. [3]) by introducing the following operations on the quotient set $\Phi/{\equiv}$. Let $A, B \in \Phi$. Then:

$$\begin{aligned} [A] \vee [B] &= [A \vee B], \\ [A] \wedge [B] &= [A \wedge B], \\ [A] \to [B] &= [A \to B], \\ f([A]) &= [\blacktriangle A], \\ g([A]) &= [\nabla A], \\ \mathbf{1} &= [\top]. \end{aligned}$$

We now consider the quotient algebra $(\Phi/{\equiv}, \vee, \wedge, \to, \mathbf{1}, f, g)$ more carefully.

Lemma 3. *The algebra* $(\Phi/{\equiv}, \vee, \wedge)$ *is a lattice.*

Proof. An algebra is a lattice if $\vee$ and $\wedge$ are idempotent, commutative, associative and satisfy the absorption law. For instance, for the absorption law $A \equiv A \wedge (A \vee B)$ we have that by (Ax2), $\vdash A \wedge (A \vee B) \to A$. For the converse, by (Ax1) and (Ax4), $\vdash A \to A$ and $\vdash A \to A \vee B$. These yield $\vdash A \to A \wedge (A \vee B)$ by $(\mathrm{R}_\wedge)$. $\square$

Let us define an order on $\Phi/{\equiv}$ by setting

$$[A] \leq [B] \iff [A] \vee [B] = [B] \iff [A] \wedge [B] = [A].$$

Then, the ordered set $(\Phi/{\equiv}, \leq)$ is a lattice in which joins and meets agree with the join and meet operations of the quotient algebra.

The order of $\Phi/{\equiv}$ is closely connected to the implication relation of the logic, as shown in the next lemma.

Lemma 4. *For all* LGC*-formulas* $A, B \in \Phi$,

$$\vdash A \to B \iff [A] \leq [B].$$

Proof. Let $A, B \in \Phi$. Then,

$$\begin{aligned} [A] \leq [B] &\iff [A \wedge B] = [A] \wedge [B] = [A] \\ &\iff \vdash A \wedge B \to A \ \text{ and } \ \vdash A \to A \wedge B \\ &\iff \vdash A \to A \wedge B \end{aligned}$$

By (Ax8), $\vdash (A \to A \wedge B) \to ((A \wedge B \to B) \to (A \to B))$. This means that $\vdash A \to A \wedge B$ implies $\vdash (A \wedge B \to B) \to (A \to B)$. Because $\vdash A \wedge B \to B$ by (Ax2), we have $\vdash A \to B$.

On the other hand, assume that $\vdash A \to B$. Because $\vdash A \to A$ by (Ax1), we have $\vdash A \to A \wedge B$ by rule ($\mathrm{R}_\wedge$). □

Proposition 3. *The algebra* $(\Phi/{\equiv}, \vee, \wedge, \to, \mathbf{1}, f, g)$ *is a GC-algebra.*

Proof. It is clear that the algebra is a lattice with the greatest element $\mathbf{1} = [\top]$. Let $A, B, X \in \Phi$. Then

$$\begin{aligned} [A] \wedge [X] = [A \wedge X] \leq [B] &\iff \vdash (A \wedge X) \to B \\ &\iff \vdash X \to (A \to B) \\ &\iff [X] \leq [A \to B] = [A] \to [B]. \end{aligned}$$

The equivalence $\vdash (A \wedge X) \to B \iff \vdash X \to (A \to B)$ follows from (Ax6) and (Ax7). Thus, the lattice is relatively pseudocomplemented. For all $A, B \in \Phi$,

$$\begin{aligned} f([A]) \leq [B] &\iff [\blacktriangle A] \leq [B] \iff \vdash \blacktriangle A \to B \\ &\iff \vdash A \to \triangledown B \iff [A] \leq [\triangledown B] \\ &\iff [A] \leq g([B]). \end{aligned}$$

Thus, the pair (f, g) is a Galois connection. □

The GC-algebra $(\Phi/{\equiv}, \vee, \wedge, \to, \mathbf{1}, f, g)$ is referred to the *Lindenbaum-Tarski LGC-algebra*. We may now define a valuation $v^*\colon P \to \Phi/{\equiv}$ by

$$v^*(p) = [p].$$

It can be easily verified by a straightforward formula induction that with this definition, we have

$$v^*(A) = [A]$$

for all LGC-formulas $A \in \Phi$. Now we may prove the following lemma.

Lemma 5. *For any* LGC-*formula* $A \in \Phi$,

$$\vdash A \iff v^*(A) = \mathbf{1}.$$

Proof. Assume that $\vdash A$. Then $\vdash \top \to A$. The inverse $\vdash A \to \top$ holds always. Thus, $A \equiv \top$ and $v^*(A) = [A] = [\top] = \mathbf{1}$.

Conversely, if $v^*(A) = [A] = \mathbf{1}$, then $\vdash \top \to A$. This means $\vdash A$. □

We can now conclude the work by showing the completeness of LGC.

Theorem 2 (Completeness). *An* LGC-*formula is valid if and only if it is provable.*

Proof. Suppose that A is valid. Then $v(A) = 1$ for every valuation v on any GC-algebra. In particular, we have $v^*(A) = \mathbf{1}$ in the Lindenbaum-Tarski GC-algebra. From Lemma 5 we obtain that A must be provable. The other direction is already proved (Soundness Theorem). □

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