Arrow index of a fuzzy choice function

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Abstract

The Arrow index of a fuzzy choice function C is a measure of the degree to which C satisfies the Fuzzy Arrow Axiom, a fuzzy version of the classical Arrow Axiom. The main result of this paper shows that $\mathcal{A}(C)$ characterizes the degree to which C is full rational. We also obtain a method for computing $\mathcal{A}(C)$. The Arrow index allows to rank the fuzzy choice functions with respect to their rationality. Thus, if for solving a decision problem several fuzzy choice functions are proposed, by the Arrow index the most rational one will be chosen.

Keywords: fuzzy choice function, revealed preference indicator, congruence indicator, similarity

1 Introduction

One of the themes of classical economic theory is the rationality of the consumer behaviour. Revealed preference is a concept introduced by Samuelson in 1938 [26], aiming to postulate the rationality of a consumer's behaviour in terms of a preference relation associated with a demand function. Uzawa [35] and Arrow [1] have developed a revealed preference theory in an abstract framework. The work of Uzawa and Arrow was continued by Richter [24], Sen [27], [28], [29], Suzumura [30], [31] and many others.

Because of the insufficient information and human subjectivity the preferences of the individuals are often fuzzy. For this reason instead of saying that alternative x is better than alternative y, it is better to evaluate the degree of preference of x to y, this will be always a number in $[0, 1]$. The idea of mathematical modeling of vague preferences is obvious: given a set X of alternatives, the preferences will be represented by a binary fuzzy relation on X [22], [8], [9], [10].

Even if the preference is ambiguous, the choice can be exact or vague. Some authors [3], [4], [5], [20] study various crisp choice functions based on a fuzzy

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preference relation. There are cases (e. g. negotiations on electronic market-places) when the decision maker cannot make a definitive choice [33], [17], [21], [34]. In this process of decision making, the choice is potential [2]. Therefore, the act of choice will be modelled by a fuzzy choice function.

Fuzzy choice functions have been studied by several authors [2], [7], [10], [20], [22], [25], [34]. In particular, Banerjee developed in [2] a theory of revealed preference for a class of fuzzy choice functions. A Banerjee choice function has the domain consisting of crisp sets and the range consisting of fuzzy sets.

In [7] Dasgupta and Deb study properties of rationality for a class of fuzzy choice functions. The authors consider that:

" Even when one is analyzing *precise* choice when preferences are fuzzy, this approach is useful. Fuzzy choice sets provide an important intermediate step analogous to the "substitution effect" in the theory of consumer demand. Thus, precise choice with fuzzy preference may be viewed as taking place in two steps: (a) fuzzy choice; (b) fuzzy choice being "made" precise in some "natural" way".

In [11], [12], [13], [14], [15], [16] we worked with a more general definition of fuzzy choice functions. Banerjee fuzzifies only the range of a choice function, in our approach both the domain and the range of a choice function are made of fuzzy subsets of a universe of alternatives.

Works [12], [13], [14], [15], [16] are an attempt to build a theory of revealed preference for these fuzzy choice functions. In [15] we have introduced the Fuzzy Arrow Axiom FAA and we have proved that a fuzzy choice function satisfies FAA iff it is full rational. This result extends a classical result of [1] which establishes the equivalence between the Arrow Axiom and the full rationality of crisp choice functions (see also [31]). In this paper there is also defined the Arrow index $\mathcal{A}(C)$ of a fuzzy choice function C. $\mathcal{A}(C)$ is a number which represents the degree to which C satisfies the Fuzzy Arrow Axiom.

The main result of this paper is a characterization theorem for the Arrow index $\mathcal{A}(C)$. Intuitively, this theorem shows that $\mathcal{A}(C)$ characterizes the degree to which the fuzzy choice function C is full rational. At the same time, this theorem refines the mentioned result from [15] which is obtained as a particular case.

The main theorem of [14] established the equivalence between FAA and the full rationality of fuzzy choice functions. By measuring by the Arrow index the degree of full rationality of a fuzzy choice function, the results of this paper allow to compare them from the point of view of their rationality. Then by using the Arrow index one can obtain a hierarchy of fuzzy choice functions with respect to their full rationality.

2 Preliminaries

In this section we present some basic matter on the residuated structure of the interval $[0, 1]$ as well as some basic notions and facts on fuzzy relations. The background is given in [6], [10], [18], [19].

For any $\{a_i\}_{i \in I} \subseteq [0,1]$ we shall denote $\bigvee_{i \in I} a_i = \sup\{a_i | i \in I\}$ and $\bigwedge_{i \in I} a_i = \inf\{a_i | i \in I\}$. In particular, for any $a, b \in [0,1]$, $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. Then the interval $[0,1]$ becomes a complete distributive lattice.

We define a new binary relation $\rightarrow$ on $[0,1]$ called implication or residuum:

$$a \rightarrow b = \bigvee\{c \in [0,1] | a \wedge c \leq b\} = \begin{cases} 1 & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$$

Another operation on $[0,1]$ is the biresiduum $\leftrightarrow$ defined by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. The following two lemmas contain some useful properties of the residuum.

Lemma 2.1 *[6], [10], [18],[19] For any $a, b, c \in [0,1]$ the following properties are true:*

(1) $a \wedge b \leq c \Leftrightarrow a \leq b \rightarrow c$;

(2) $a \wedge (a \rightarrow b) = a \wedge b$;

(3) $a \leq b \Leftrightarrow a \rightarrow b = 1$;

(4) $a = 1 \rightarrow a$;

(5) $1 = a \rightarrow a$;

(6) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c = b \rightarrow (a \rightarrow c)$.

Lemma 2.2 *[6], [10], [18],[19] For any $\{a_i\}_{i \in I} \subseteq [0,1]$ and $a \in [0,1]$ the following properties hold:*

(1) $(\bigvee_{i \in I} a_i) \wedge a = \bigvee_{i \in I} (a_i \wedge a)$;

(2) $a \rightarrow (\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} (a \rightarrow a_i)$;

(3) $(\bigwedge_{i \in I} a_i) \rightarrow a = \bigwedge_{i \in I} (a \rightarrow a_i)$.

Let X be a non-empty set. A *fuzzy subset* of X is a function $A : X \rightarrow [0,1]$. We denote by $\mathcal{P}(X)$ the family of crisp subsets of X and by $\mathcal{F}(X)$ the family of fuzzy subsets of X. As a crisp subset of X is defined by its characteristic function, then we have $\mathcal{P}(X) \subseteq \mathcal{F}(X)$. For any $A, B \in \mathcal{F}(X)$, by $A \subseteq B$ we mean that $A(x) \leq B(x)$ for each $x \in X$. A fuzzy subset A of X is *non-zero* if $A(x) \neq 0$ for some $x \in X$; A is *normal* if $A(x) = 1$ for some $x \in X$.

If $x_1, \ldots, x_n \in X$ then $[x_1, \ldots, x_n]$ will denote the characteristic function of $\{x_1, \ldots, x_n\}$:

$$[x_1, \ldots, x_n](y) = \begin{cases} 1 & \text{if} \quad y \in \{x_1, \ldots, x_n\} \\ 0 & \text{if} \quad y \notin \{x_1, \ldots, x_n\} \end{cases}$$

For $A, B \in \mathcal{F}(X)$ let us denote

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$$I(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)) \text{ and } E(A, B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)).$$

It is clear that $A \subseteq B$ iff $I(A, B) = 1$ and $A = B$ iff $E(A, B) = 1$. For any $x \in X$ we have:

$I(A, B) \leq A(x) \to B(x)$ and $E(A, B) \leq A(x) \leftrightarrow B(x)$.

$I(A, B)$ is called the *subsethood degree* of A and B and $E(A, B)$ the *degree of equality* (degree of similarity) of A and B. Intuitively $I(A, B)$ expresses the truth value of the statement "A is included in B" and $E(A, B)$ the truth value of the statement "A and B contain the same elements" (see [6], p. 82).

A *fuzzy preference relation* Q on X is a function $Q : X^2 \to [0, 1]$, i. e. a fuzzy subset of X^2. The elements of X are interpreted as alternatives. If $x, y \in X$ then the real number $Q(x, y)$ represents the degree to which the alternative x is at least as good as y.

A fuzzy preference relation Q is said to be:

reflexive if $Q(x, x) = 1$ for all $x \in X$;

transitive if $Q(x, y) \wedge Q(y, z) \leq Q(x, z)$ for all $x, y, z \in X$;

strongly total if $Q(x, y) = 1$ or $Q(y, x) = 1$ for all distinct $x, y \in X$;

Following [6], p. 190, for any fuzzy preference relation Q let us denote

$$Ref(Q) = \bigwedge_{x \in X} R(x, x);$$

$$Trans(Q) = \bigwedge_{x,y,z \in X} [(R(x, y) \wedge R(y, z)) \to R(x, z)];$$

$$ST(Q) = \bigwedge_{x \neq y} (Q(x, y) \vee Q(y, x)).$$

$Ref(Q)$ is called the degree of reflexivity of Q. $Trans(Q)$ is called the degree of transitivity of Q and $ST(Q)$ the degree of strong totality of Q.

Lemma 2.3 *[6] For any fuzzy preference relation Q the following equivalences hold:*

(i) Q is reflexive iff $Ref(Q) = 1$;

(ii) Q is transitive if $Trans(Q) = 1$;

(iii) Q is strongly total iff $ST(Q) = 1$.

These indicators allow to compare two fuzzy preference relations with respect to a property; for example, if $Trans(Q_1) \geq Trans(Q_2)$, then the preference relation Q_1 is "more transitive" than Q_2.

Lemma 2.4 *[6] Let Q be a fuzzy preference relation on X and $x, y, z \in X$. Then*

$Trans(Q) \wedge Q(x, y) \wedge Q(y, z) \leq Q(x, z)$.

Proof. By the definition of $Trans(Q)$ we have $Trans(Q) \leq (Q(x, y) \wedge Q(y, z)) \to Q(x, z))$, then we apply Lemma 2.1 (1). ∎

Let us denote by $\mathcal{R}$ the family of all fuzzy preference relations on X.

3 Fuzzy Choice Functions

A vast literature has been dedicated to fuzzy choice functions and fuzzy prefer-
ence relations (see [10], [20]). Most authors build their results on the thesis that
social choice is governed by fuzzy preferences (hence modeled through fuzzy bi-
nary relations) but the act of choice is exact (hence choice functions are crisp)
(see [3], [4], [5], [20]). They study crisp choice functions by fuzzy preference
relations.

On the other hand, several situations in the world require to consider fuzzy
choice functions ([2], [10], [20], [22], [24], [34]). The first fuzzy choice function
was emphasized by Orlovsky [22].

In [2] Banerjee admits the vagueness of the act of choice and develops a
theory of revealed preference for choice functions with a fuzzy behaviour. He
studies choice functions whose domain is the family of non–empty finite subsets
of a universe of alternatives X and whose range consists of non–zero fuzzy
subsets of X. Banerjee's thesis is that "If preferences are permitted to be fuzzy,
it seems natural to permit the choice functions to be fuzzy as well. This also
tallies with experience". He studies in [2] a revealed preference for these fuzzy
choice functions.

In [12], [13], [14], [15] we have considered fuzzy choice functions C for which
the domain and the range are made of fuzzy subsets of X. Banerjee fuzzifies
only the range of a choice function; we use a fuzzification of both the domain and
the range of a choice function. In our case, the available sets of alternatives are
fuzzy subsets of X. This leads to the introduction of the notion of availability
degree of an alternative x with respect to an available set S. The availability
degree might be useful when the decision maker possess partial information
on the alternative x or when a criterion limits the possibility of choosing x.
Therefore the available sets can be considered criteria in decision making.

We shall recall some notions and results in [12], [13], [14], [15], [16].

A fuzzy choice space is a pair $(X, \mathcal{B})$ where X is a non–empty set of alter-
natives and $\mathcal{B}$ is a non–empty family of non–zero fuzzy subsets of X. A fuzzy
choice function on $(X, \mathcal{B})$ is a function $C : \mathcal{B} \to \mathcal{F}(X)$ such that for any $S \in \mathcal{B}$,
$C(S)$ is a non–zero fuzzy subset of X and $C(S) \subseteq S$.

A lot of results in [12], [13], [14], [15], [16] require the following hypotheses:

H_1 Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X;

H_2 $\mathcal{B}$ includes the fuzzy sets $[x_1, \ldots, x_n]$ for any $n \geq 1$ and $x_1, \ldots, x_n \in X$.

For the crisp case ($\mathcal{B} \subseteq \mathcal{P}(X)$), H_1 is automatically fulfilled in accordance
with the definition of a (crisp) choice function, for the same case, H_2 asserts
that $\mathcal{B}$ includes all non–empty finite subsets of X.

Throughout this paper we shall assume that H_1, H_2 are verified.

If C is a fuzzy choice function on $(X, \mathcal{B})$ then we define two fuzzy preference
relations R and $\bar{R}$ on X by:

$R(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y))$

$\bar{R}(x, y) = C([x, y])(x)$

for all $x, y \in X$.

The fuzzy preference relations R and $\bar{R}$ defined above are fuzzy versions of crisp preference relations which have an important role in the classical revealed preference theory (see [26], [1], [24], [27], etc.). For example, the real number $R(x,y)$ is the degree of truth of the statement "there exists a criterion S such that alternative x is chosen with respect to S and alternative y verifies S".

Lemma 3.1 *[12] If C is a fuzzy choice function on $(X,\mathcal{B})$ then $\bar{R} \subseteq R$ and $R, \bar{R}$ are reflexive and strongly total.*

Let $(X,\mathcal{B})$ be a fuzzy choice space and Q be a fuzzy preference relation on X. For any $S \in \mathcal{B}$ let us define the fuzzy subset $G(S,Q)$ of X:

$$G(S,Q)(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \to Q(x,y)]$$

for any $x \in X$. In this way one obtains a function $G(.,Q) : \mathcal{B} \to \mathcal{F}(X)$.

Lemma 3.2 *[14] If C is a fuzzy choice function on $(X,\mathcal{B})$ then $C(S) \subseteq G(S,R)$ for any $S \in \mathcal{B}$.*

In general, $G(.,Q)$ is not a fuzzy choice function: it might exist an $S \in \mathcal{B}$ such that $G(S,Q)$ is the zero set. According to Lemma 3.2, $G(.,R)$ is a fuzzy choice function called the image of C.

A fuzzy choice function C on $(X,\mathcal{B})$ is said to be rational if $C = G(.,Q)$ for some fuzzy preference relation Q on X. If Q is reflexive, transitive and strongly total, then C is called full rational.

Let C_1, C_2 be two fuzzy choice functions on $(X,\mathcal{B})$. Following [15], [16] the degree of similarity of C_1 and C_2 is defined by

$$E(C_1,C_2) = \bigwedge_{S \in \mathcal{B}} \bigwedge_{x \in X} [C_1(S)(x) \leftrightarrow C_2(S)(x)].$$

Lemma 3.3 *[16] For any fuzzy choice functions C_1, C_2 on $(X,\mathcal{B})$ we have:*

(i) $E(C_1,C_2) = 1$ iff $C_1 = C_2$; (ii) $E(C_1,C_2) = E(C_2,C_1)$; (iii) $E(C_1,C_2) \wedge E(C_2,C_3) \leq E(C_1,C_3)$.

By this lemma, the function $(C_1,C_2) \mapsto E(C_1,C_2)$ is a similarity relation on the set of fuzzy choice functions defined on $(X,\mathcal{B})$.

Similarity of fuzzy choice functions is a concept analogous to the similarity of fuzzy preference relations ([6], [10], [18], [19]). If we interpret X as a universe of alternatives and C_1, C_2 as the "choices" of two agents, then the real number $E(C_1,C_2)$ expresses how "similar" these choices are.

Lemma 3.4 *[16] Let C_1, C_2 be a fuzzy choice function on $(X,\mathcal{B})$. For any $S \in \mathcal{B}$ and $x \in X$ the following inequality holds:*

$$E(C_1,C_2) \wedge C(S_1)(x) \leq C(S_2)(x).$$

Lemma 3.5 *[16] Let C be a fuzzy choice function on $(X,\mathcal{B})$ and Q be a fuzzy preference relation on X. Then $E(C,G(.,Q)) \wedge C(S)(s) \wedge S(t) \leq Q(s,t)$ for all $S \in \mathcal{B}$ and $s,t \in X$.*

Proof. By Lemma 3.4 we have

$$E(C, G(., Q)) \wedge C(S)(s) \leq G(S, Q)(x) \leq S(t) \rightarrow Q(s, t)$$

hence according to Lemma 2.1 (1) we get the desired inequality. ■

Following [14], we say that a fuzzy choice function C on $(X, \mathcal{B})$ verifies the Fuzzy Arrow Axiom (FAA) if for all $S_1, S_2 \in \mathcal{B}$ and $x \in X$ we have

$$I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \leq E(S_1 \cap C(S_2), C(S_1)).$$

FAA is a fuzzy version of the Arrow Axiom AA in the theory of crisp choice functions (see [31], p. 25).

Intuitively, FAA says: "The maximum degree that, for some x, S_1 and S_2, x is chosen from S_2, x belongs to S_1 and S_1 is included in S_2, is less than or equal to the degree that the set of alternatives chosen from S_2, and also in S_1 is equal to the set of alternatives chosen from S_1" (cf. [16], p. 136).

4   The Arrow Index

The Arrow axiom AA was introduced in [1] in order to characterize the rationality of choice functions (for some information see also [31], pp. 20–30).

These two properties have been later integrated by Sen [27] in a general result regarding the characterization of the rationality of a crisp choice function by various conditions (the axioms of revealed preference $WARP$, $SARP$, the congruence axioms WCA, SCA, etc.). This result is known as the Arrow–Sen theorem.

In [15] we introduced the Fuzzy Arrow Axiom FAA, a generalization of AA to fuzzy choice functions. We extended the classical result by showing that FAA is equivalent with the full rationality of fuzzy choice functions.

A new concept defined in [14] is the Arrow index $\mathcal{A}(C)$ of a fuzzy choice function C. $\mathcal{A}(C)$ is the real number that expresses the degree to which C verifies the Fuzzy Arrow Axiom.

The main theorem of this section shows that $\mathcal{A}(C)$ characterizes the degree to which C is full rational. This type of result changes the perspective of studying the rationality of fuzzy choice functions. The attention is no longer focused on rational fuzzy choice functions, but the entire class of fuzzy choice functions has been taken into account with respect to their rationality. By the Arrow index, one can appreciate how rational any fuzzy choice function is. This fact has direct consequences in concrete problems. More often, the choices of an agent do not fulfill conditions of rationality or full rationality. By using the Arrow index, from a set of fuzzy choice functions we can select the ones with the maximum degree of rationality.

The results of this section offer a method for the calculation of the Arrow index in terms of the fuzzy preference relations R and $\bar{R}$.

Let $(X, \mathcal{B})$ be a fuzzy choice space.

Definition 4.1 *Let C be a fuzzy choice function on $(X, \mathcal{B})$. The Arrow index $\mathcal{A}(C)$ of C is defined by*

$$\mathcal{A}(C) = \bigwedge_{S_1,S_2 \in \mathcal{B}} \bigwedge_{x \in X} [(I(S_1,S_2) \wedge S_1(x) \wedge C(S_2)(x)) \to E(S_1 \cap C(S_2), C(S_1))].$$

We remark that $\mathcal{A}(C) = 1$ iff C verifies FAA. Intuitively $\mathcal{A}(C)$ represents the degree of the statement "The fuzzy choice function C verifies the Fuzzy Arrow Axiom".

The following proposition expresses, in the language of indicators, the fact that the transitivity of the fuzzy preference relation $\bar{R}$ is ensured by the verification of the Arrow Axiom.

Proposition 4.2 *If C is a fuzzy choice function on $(X, \mathcal{B})$, then $\mathcal{A}(C) \leq Trans(\bar{R})$.*

Proof. Let $x, y \in X$. We shall prove that:
(a) $\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \leq C([x,y])(z)$.
Denote $S = [x,y,z]$. We shall establish the inequality
(b) $\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \leq C(S)(x)$.
Assume by absurdum that
(c) $\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \not\leq C(S)(x)$.
We shall prove that (c) implies the condition
(d) $\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \not\leq C(S)(y)$.
Assume that (d) does not hold, i. e.
(e) $\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \leq C(S)(y)$.
Using Lemma 2.1 (2) we have
$\mathcal{A}(C) \wedge C(S)(y) = \mathcal{A}(C) \wedge I([x,y],S) \wedge [x,y](y) \wedge C(S)(y) \leq$
$\leq I([x,y],S) \wedge [x,y](y) \wedge C(S)(y) \wedge [I([x,y],S) \wedge [x,y](y) \wedge C(S)(y) \to E([x,y] \cap C(S), C([x,y]))].$
$I([x,y],S) \wedge x,y](y) \wedge C(S)(y) \wedge E([x,y] \cap C(S), C([x,y])) \leq$
$\leq E([x,y] \cap C(S), C([x,y])) \leq ([x,y](x) \wedge C(S)(x)) \leftrightarrow C([x,y])(x) =$
$= C(S)(x) \leftrightarrow C([x,y])(x) \leq C([x,y])(x) \to C(S)(x).$
By Lemma 2.1 (1) it follows that
$\mathcal{A}(C) \wedge C(S)(y) \wedge C([x,y])(x) \leq C(S)(x).$
In accordance with (e)
$\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \leq \mathcal{A}(C) \wedge C(S)(y) \wedge C([x,y])(x) \leq C(S)(x)$
contradicting (c). Then we obtain (d). Similarly we get
(f) $\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([x,y])(y) \not\leq C(S)(z).$
From (c), (d) and (f) we infer that $C(S)(x) \neq 1$, $C(S)(y) \neq 1$ and $C(S)(z) \neq 1$, contradicting that $C(S)$ is a normal fuzzy subset of X (cf. H_1). This contradiction shows that (b) holds. We remark that
$\mathcal{A}(C) \wedge C(S)(x) = \mathcal{A}(C) \wedge I([x,z],S) \wedge [x,y](x) \wedge C(S)(x) \leq$
$\leq I([x,z],S) \wedge [x,z](x) \wedge C(S)(x) \wedge [(I(x,z),S) \wedge [x,z](x) \wedge C(S)(x)) \to$
$E([x,z] \cap C(S), C([x,z]))] \leq E([x,z] \cap C(S)) \leq ([x,y] \cap C(S))(x) \to$
$\to C([x,z])(x) \leq C(S)(x) \to C([x,z])(x)$
hence by Lemma 2.1 (1) we obtain
$\mathcal{A}(C) \wedge C(S)(x) \leq C([x,z])(x).$
Using (b) it follows that

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$\mathcal{A}(C) \wedge C([x,y])(x) \wedge C([y,z])(y) \leq \mathcal{A}(C) \wedge C(S) \leq C([x,z])(x).$
Then for all $x, y, z \in X$ we have
$\mathcal{A}(C) \leq (C([x,y])(x) \wedge C([y,z])(y)) \rightarrow C([x,z])(x) = (\bar{R}(x,y) \wedge \bar{R}(y,z)) \rightarrow \bar{R}(x,z)$
therefore
$$\mathcal{A}(C) \leq \bigwedge_{x,y,z \in X} [(\bar{R}(x,y) \wedge \bar{R}(y,z)) \rightarrow \bar{R}(x,z)) \leq Trans(\bar{R}). \blacksquare$$

Corollary 4.3 *If C satisfies FAA then $\bar{R}$ is transitive.*

Proof. By Proposition 4.2, $\mathcal{A}(C) = 1$ implies $Trans(\bar{R}) = 1$. $\blacksquare$

Recall that $\mathcal{R}$ is the family of all fuzzy preference relations on X.

Theorem 4.4 *If C is a fuzzy choice function on $(X, \mathcal{B})$ then*
$$\mathcal{A}(C) = \bigvee_{Q \in \mathcal{R}} [E(C, G(.,Q)) \wedge Trans(Q) \wedge Ref(Q) \wedge ST(Q)] =$$
$$= E(C, G(., \bar{R})) \wedge Trans(\bar{R}).$$

Proof. Let $Q \in \mathcal{R}$. We shall prove that
(a') $E(C, G(.,Q)) \wedge Trans(Q) \wedge Ref(Q) \wedge ST(Q) \leq \mathcal{A}(C)$.
Let $S_1, S_2 \in \mathcal{B}$ and $x \in X$. We shall establish the inequality
(b') $E(C, G(.,Q)) \wedge Trans(Q) \wedge Ref(Q) \wedge ST(Q) \leq$
$\leq (I(S_1, S_2)) \wedge S_1(x) \wedge C(S_2))(x) \rightarrow E(S_1 \cap C(S_2)), C(S_1)).$
Let us denote
$\alpha = E(C, G(.,Q)) \wedge Trans(Q) \wedge Ref(Q) \wedge ST(Q) \wedge I(S_1, S_2)) \wedge S_1(x) \wedge C(S_2)(x).$
By Lemma 2.1 (1), (b') is equivalent to the inequality
(c') $\alpha \leq E(S_1 \cap C(S_2), C(S_1)).$
Let $z, v \in X$. Then by Lemmas 3.5 and 2.4 we have
$\alpha \wedge I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge C(S_1)(z) \wedge S_2(v) \leq$
$\alpha \wedge [E(C, G(.,Q)) \wedge C(S_1)(z) \wedge S_1(x)] \wedge [E(C, G(.,Q)) \wedge C(S_1)(x) \wedge S_2(v)] \leq$
$\alpha \wedge Q(z,x) \wedge Q(z,v) \leq$
$\leq Trans(Q) \wedge Q(z,x) \wedge Q(x,v) \leq Q(z,v).$
By Lemma 2.1 (1) we get
$\alpha \wedge C(S_1)(z) = \alpha \wedge I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge C(S_1)(z) \leq S_2(v) \rightarrow Q(z,v)$
for all $v \in X$, hence
$\alpha \wedge C(S_1)(z) \wedge \bigwedge_{v \in X} [S_2(v) \rightarrow Q(z,v)].$
On the other hand,
$\alpha \wedge C(S_1)(z) \leq I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge C(S_1)(z) \leq C(S_1)(z) \wedge I(S_1, S_2) \leq$
$\leq S_1(z) \wedge [S_1(z) \rightarrow S_2(z)] \leq S_2(z).$
Therefore
$\alpha \wedge C(S_1)(z) \leq S_2(z) \wedge \bigwedge_{v \in X} [S_2(v) \rightarrow Q(z,v)] = G(S_2, Q)(z).$

Since $\alpha \wedge C(S_1)(z) \leq \alpha \leq E(C, G(., Q))$
by Lemma 3.4 we obtain
$\alpha \wedge C(S_1)(z) \wedge E(C, G(., Q)) \wedge G(S_2, Q)(z) \leq C(S_2)(z)$,
hence by Lemma 2.1 (1), (3) and Lemma 2.2 (2) it follows that
(d') $\alpha \leq C(S_1)(z) \rightarrow C(S_2)(z) = [C(S_1)(z) \rightarrow S_1(z)] \wedge [C(S_1)(z) \rightarrow C(S_2)(z)] =$
$= C(S_1)(z) \rightarrow (S_1(z) \wedge C(S_2)(z))$.
Let $y, z \in X$. According to Lemma 2.1 (2), Lemma 2.5 and Lemma 2.4 the following inequality holds:
$\alpha \wedge S_1(z) \wedge C(S_2)(z) \wedge S_1(y) =$
$= \alpha \wedge I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge S_1(z) \wedge C(S_2)(z) \wedge S_1(y) \leq$
$\leq S_1(y) \wedge S_2(y) \wedge C(S_2)(x) \wedge C(S_2)(z) =$
$= \alpha \wedge S_1(y) \wedge S_2(y) \wedge C(S_2)(x) \wedge C(S_2)(z) \leq$
$\alpha \wedge [E(C, G(., Q)) \wedge C(S_2)(z) \wedge S_2(x)] \wedge [E(C, G(., Q))] \wedge C(S_2)(x) \wedge S_2(y)] \leq$
$\leq \alpha \wedge Q(z, x) \wedge Q(x, y) \leq$
$Trans(Q) \wedge Q(z, x) \wedge Q(x, y) \leq Q(z, y)$.
By Lemma 2.1 (1) we obtain:
$\alpha \wedge S_1(z) \wedge C(S_2)(z) \leq S_1(y) \rightarrow Q(z, y)$
for all $y \in X$, hence
$$\alpha \wedge S_1(z) \wedge C(S_2)(z) \leq S_1(z) \wedge \bigwedge_{y \in X} [S_1(y) \rightarrow Q(z, y)] = G(S_1, Q)(z).$$

Since $\alpha \leq E(C, G(., Q))$, by Lemma 3.4 we get
$\alpha \wedge S_1(z) \wedge C(S_2)(z) \leq E(C, G(.Q)) \wedge G(S_1, Q)(z) \leq C(S_1)(z)$.
By Lemma 2.1 (1) we get for all $z \in X$:
(e') $\alpha \leq (S_1(z) \wedge C(S_2)(z)) \rightarrow C(S_1)(z)$.
Combining (d') and (e') it follows that
$$\alpha \leq \bigwedge_{z \in X} [(S_1(z) \wedge C(S_2)(z)) \leftrightarrow C(S_1)(z)] = E(S_1 \cap C(S_2), C(S_1)).$$
Thus (e') was proved, hence (b') is verified. Since (b') holds for all $S_1, S_2 \in \mathcal{B}$ and $x \in X$ we obtain (a'). From (a') we can derive the inequality:
(f') $\bigvee_{Q \in \mathcal{R}} [E(C, G(., Q)) \wedge Trans(Q) \wedge Ref(Q) \wedge ST(Q)] \leq \mathcal{A}(C)$.
Now we shall prove the inequality
(g') $\mathcal{A}(C) \leq E(C, G(., \bar{R}))$.
Let $S \in \mathcal{B}$ and $x \in X$. We shall establish the following two inequalities:
(h') $\mathcal{A}(C) \leq C(S)(x) \rightarrow G(S, \bar{R})$
(i') $\mathcal{A}(C) \leq G(S, \bar{R})(x) \rightarrow C(S)(x)$.
Let $z \in X$. According to Lemma 2.1 (1) we have
$\mathcal{A}(C) \wedge C(S)(x) \wedge S(z) = \mathcal{A}(C) \wedge (I([x, z], S) \wedge [x, z](x) \wedge C(S)(x)) \leq$
$\leq I([x, z], S) \wedge [x, z](x) \wedge C(S)(x) \wedge [(I([x, z], S) \wedge [x, z](x) \wedge C(S)(x)) \rightarrow E([x, z] \cap C(S), C([x, z]))] =$
$= I([x, z], S) \wedge [x, z](x) \wedge C(S)(x) \wedge E([x, z] \cap C(S), C([x, z])) \leq$
$\leq E([x, z] \cap C(S), C([x, z]) \leq ([x, z](x) \wedge C(S)(x) \leftrightarrow C([x, z])(x)) =$
$= C(S)(x) \leftrightarrow \bar{R}(x, z) \leq C(S)(x) \rightarrow \bar{R}(x, z)$,
hence by Lemma 2.1 (1)

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$\mathcal{A}(C) \wedge C(S)(x) \wedge S(z) \leq \bar{R}(x, z)$.

Applying again Lemma 2.1 (1), $\mathcal{A}(C) \wedge C(S)(x) \leq S(z) \to \bar{R}(x, z)$ for each $z \in X$, hence

$\mathcal{A}(C) \wedge C(S)(x) \leq S(x) \wedge \bigwedge_{z \in X} [S(z) \to \bar{R}(x, z)] = G(S, \bar{R})(x)$.

Thus $\mathcal{A}(C) \wedge C(S)(x) \to G(S, \bar{R})(x)$, so (h') holds.

In order to prove (i') we consider an element $y \in X$ such that $C(S)(y) = 1$ (because $C(S)$ is a normal fuzzy subset of X), then $S(y) = 1$. Thus

$G(S, \bar{R})(x) \leq S(y) \to \bar{R}(x, y) = 1 \to \bar{R}(x, y) = \bar{R}(x, y) = C([x, y])(x)$.

We remark that

$G(S, \bar{R})(x) \leq S(x) = S(x) \wedge S(y) \wedge C(S)(y) = I([x, y], S) \wedge [x, y](y) \wedge C(S)(y)$

therefore

$\mathcal{A}(C) \wedge G(S, \bar{R}) \leq$

$(I([x, y], S) \wedge [x, y](y) \wedge C(S)(y)) \wedge [(I([x, y], S) \wedge [x, y](y) \wedge C(S)(y)) \to E([x, y] \cap C(S), C([x, y]))] \leq$

$\leq E([x, y] \cap C(S), C([x, y]))$.

Since $G(S, \bar{R})(x) \leq C([x, y])(x)$ we obtain

$\mathcal{A}(C) \wedge G(S, \bar{R})(x) \leq C([x, y], x) \wedge E([x, y] \cap C(S), C([x, y])) \leq$

$\leq C([x, y])(x) \wedge [[x, y](x) \wedge C(S)(x) \to C([x, y])(x)] =$

$= \bar{R}(x, y) \wedge [C(S)(x) \to \bar{R}(x, y)] \leq$

$\leq \bar{R}(x, y) \wedge [\bar{R}(x, y) \to C(S)(x)] = \bar{R}(x, y) \wedge C(S)(x) \leq C(S)(x)$

hence $\mathcal{A}(C) \leq G(S, \bar{R})(x) \to C(S)(x)$ and (i') is proved.

The inequalities (h') and (i') hold for all $S \in \mathcal{B}$ and $x \in X$ hence

$\mathcal{A}(C) \leq \bigwedge_{S \in \mathcal{B}} \bigwedge_{x \in X} [C(S)(x) \leftrightarrow G(S, \bar{R})(x)] = E(C, G(., \bar{R}))$.

The inequality (g') was proved. Since $\bar{R}$ is reflexive and strongly total, we have $Ref(\bar{R}) = ST(\bar{R}) = 1$. Thus, by Proposition 4.2 and (g') we get

$\mathcal{A}(C) \leq E(C, G(., \bar{R})) \wedge Trans(\bar{R}) =$

$= E(C, G(., \bar{R})) \wedge Trans(\bar{R}) \wedge Ref(\bar{R}) \wedge ST(\bar{R}) \leq$

$\leq \bigvee_{Q \in \mathcal{R}} [E(C, G(., Q)) \wedge Trans(Q) \wedge Ref(Q) \wedge ST(Q)]$.

The proof of the theorem is finished.

∎

Remark 4.5 *By Theorem 4.4 the following assertions are equivalent:*
- *C satisfies FAA;*
- *$E(C, G(., Q)) = Trans(Q) = Ref(Q) = ST(Q) = 1$ for some $Q \in \mathcal{R}$;*
- *C is full rational.*

We have obtained Theorem 3.6 as a corollary of Theorem 4.4.

Let us denote by W the transitive closure of the fuzzy revealed preference relation R (see [6] or [10]).

In [12] there have been introduced the following congruence axioms for fuzzy choice functions:

$WFCA$ (Weak Fuzzy Congruence Axiom)

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For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$R(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.

$SFCA$ (Strong Fuzzy Congruence Axiom)

For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$W(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.

$WFCA$, $SFCA$ are fuzzy versions of the classical congruence axioms WCA, SCA [24], [27], [30], [31].

In [16] the following indicators of the axioms $WFCA$ and $SFCA$ have been defined:

Definition 4.6 *[16] For a fuzzy choice function C on a fuzzy choice space $(X, \mathcal{B})$ let us define the indicators:*

(i) $WFCA(C) = \bigwedge\limits_{x, y \in X} \bigwedge\limits_{S \in \mathcal{B}} [(S(x) \wedge C(S)(y) \wedge R(x, y)) \to C(S)(x)]$

(ii) $SFCA(C) = \bigwedge\limits_{x, y \in X} \bigwedge\limits_{S \in \mathcal{B}} [(S(x) \wedge C(S)(y) \wedge W(x, y)) \to C(S)(x)]$.

Remark 4.7 *The following assertions are equivalent:*
(i) $WFCA(C) = 1$ *iff C verifies the axiom $WFCA$;*
(ii) $SFCA(C) = 1$ *iff C verifies the axiom $SFCA$.*

The indicator $WFCA(C)$ (resp. $SFCA(C)$) expresses the degree to which the fuzzy choice function C satisfies the axiom $WFCA$ (resp. $SFCA$).

We recall the following theorem from [16], p. 214:

Theorem 4.8 *If C is a fuzzy choice function then*
$WFCA(C) = SFCA(C) = E(C, G(., \bar{R})) \wedge Trans(R)$
$= E(C, G(., \bar{R})) \wedge Trans(\bar{R})$.

By combining Theorems 4.4 and 4.8 it follows:

Theorem 4.9 *If C is a fuzzy choice function then*
$\mathcal{A}(C) = WFCA(C) = SFCA(C) = E(C, G(., R)) \wedge Trans(R)$,
$= E(C, G(., \bar{R})) \wedge Trans(\bar{R}))$.

Remark 4.10 *The fuzzy Arrow–Sen theorem established in [11], [16] is a particular case of Theorem 4.9. In case when C is a crisp choice function, one obtains an important part of the classic Arrow–Sen theorem ([1], [27], [31]).*

5   Concluding Remarks

The treatment of rationality in fuzzy revealed preference theory emphasizes the connection between the fuzzy choice functions and the fuzzy preference relations which rationalize them. This interconnection is realized by conditions on fuzzy choice functions (axioms of revealed preference and congruence, consistency properties, etc.) Such condition related to the fuzzy choice functions is the

Fuzzy Arrow Axiom (FAA), a generalization in fuzzy context of the well known axiom of Arrow [1].

The main theorem of [14] shows the equivalence between FAA and the full rationality of the fuzzy choice function. The notion of full rationality combines two types of properties:

(a) properties of fuzzy choice functions: rationality;

(b) properties of fuzzy preference relations: reflexivity, transitivity, strong completeness.

In this paper the relation between FAA and the full rationality has been analyzed from another perspective:

• instead of the Arrow Axiom, one took the Arrow index $\mathcal{A}(C)$ of a fuzzy choice function, indicator which expresses the degree to which C verifies FAA;

• one considered a numerical measure of the full rationality, expressed by the degree of similarity of fuzzy choice functions and by the indicators of reflexivity, transitivity and strong completeness of the fuzzy preference relation.

The equivalence between FAA and the full rationality is here transformed into the equality between $\mathcal{A}(C)$ and a real number which expresses the degree of full rationality.

The main theorem of the paper (Theorem 4.4) consists of the equality of three numerical indicators, and the result of [14] is one if its particular cases. By combining Theorem 4.4 with a result of [16], p. 214 on the congruence indicators $WFCA(C)$ and $SFCA(C)$, one obtains their equality with the Arrow index, and an evaluation of their value in terms of the similarity and the transitivity indicator.

The result of this paper strengthens the one in [14] for the following two reasons:

(i) it consists of a numerical relation valid for any fuzzy choice function;

(ii) it provides a method of ranking the fuzzy choice functions with respect to their rationality.

It remains to investigate to which extent the theory of this paper can be developed without hypotheses H_1 and H_2. At the same time, an open problem is to investigate the validity of these results in the more general context offered by a left–continuous t–norm or for other particular t–norms.

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