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# Preservation of revealed preference and congruence indicators by similarity

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**Abstract**

The revealed preference axioms $WAFRP$, $SAFRP$ and the congruence axioms $WFCA$, $SFCA$ are conditions which appear in the study of the rationality of fuzzy choices. The indicators of revealed preference $WAFRP(C)$, $SAFRP(C)$ and the indicators of congruence $WFCA(C)$, $SFCA(C)$ express the degree to which a fuzzy choice function $C$ verifies each of these conditions.

In this paper these indicators are related to the similarity of fuzzy choice functions. The main result establishes the way the indicators $WAFRP(C)$, $SAFRP(C)$, $WFCA(C)$, $SFCA(C)$ are preserved by the similarity.



**Keywords:** fuzzy choice function, revealed preference, similarity relation

# 1 Introduction

The revealed preference theory is one of the paradigms in actual social choice theory.Revealed preference is a concept introduced by Samuelson in 1938 in order to describe the rationality of a consumer's behaviour in terms of a preference relation associated with a demand function.In revealed preference theory first choice are given, then preference are defined by choices.

An abstract form of this theme has been developed by Arrow [1], Sen [15], [16], Suzumura [17] in the framework of choice functions theory. The core of the abstract revealed preference theory is the connection between choice functions and preference relations. Every choice function determines some preference relations on the set $X$ of alternatives. Conversely, some preference relations on $X$ induce choice functions. The exist two ways of defining the rationality of a choice function w.r.t. a preference relation $Q$ on $X$: the choice of the $Q$-greatest alternatives and the choice of the $Q$–maximal alternatives [17]. The rational behaviour of a choice function is expressed by various conditions, as the axioms of revealed preference $WARP$ and $SARP$([1], [4], the congruence axioms $WCA$ and $SCA$ [13], [15], [16]), consistency properties [15] ,etc.

On the other hand, there exist situations in social and economic domains where the preferences and choices are fuzzy. Some situations concern fuzzy preferences and exact choices [4] and some others concern fuzzy preferences and fuzzy choices [2], [3], [12], [6], [7], [8], [9], [10].

A very general notion of fuzzy choice function was defined in [3], [6], [7], [8] and [12]. In [2], [3] the domain of a choice function is made by crisp sets and the codomain is made from fuzzy sets. In papers [6], [7], [8], [12] both the domain and the codomain of a fuzzy choice function is made by fuzzy subsets of the universe of alternatives.

In [6] we have been developed a theory of revealed preference for fuzzy choice functions. The rationality of fuzzy choice functions was studied by means of the fuzzy revealed preference axioms $WAFRP$, $SAFRP$ and the fuzzy congruence axioms $WFCA$, $SFCA$. $WAFRP$ , $SAFRP$ are fuzzy generalizations of $WARP$ , $SARP$ and $WFCA$, $SFCA$ are fuzzy generalizations of $WCA$,$SCA$.

The study of properties of fuzzy phenomena necessitates often an approach different from the crisp case. Instead of checking whether such a property $P$ is true or false, it is more interesting to measure "the degree to which $P$ is true".

To exemplify, let us consider the axiom $WAFRP$. Then, instead of checking whether a fuzzy choice function $C$ verifies or not $WAFRP$, we need an indicator which should express "the degree to which $C$ verifies $WAFRP$".

In [7] there have been introduced the indicators of revealed preference $WAFRP(C)$, $SAFRP(C)$ corresponding to the axioms $WAFRP$, $SAFRP$ and the indicators of congruence $WFCA(C)$, $SFCA(C)$ corresponding to the axioms $WFCA$, $SFCA$.

The similarity of fuzzy sets is a concept introduced by Zadeh in 1971 [18]. In

[9] we defined the similarity of fuzzy choice functions and we investigated the its relationship with the similarity of the associated revealed preference relations.

In this paper we connect the similarity and the indicators $WAFRP(C)$, $SAFRP(C)$, $WFCA(C)$ and $SFCA(C)$. Our main theorem is an answer to a problem of the following type : if the choices $C_1$, $C_2$ of two agents $Agent_1$, $Agent_2$ are similar, then to what extent is the rationality of $C_1$ (expressed by $WAFRP(C_1)$, $SAFRP(C_1)$, $WFCA(C_1)$ or $SFCA(C_1)$ closer to the rationality of $C_2$?

As a corollary of the main theorem, we prove that if the degree of similarity of $C_1$ and $C_2$ is a real number $\alpha$ then $|WAFRP(C_1) - WAFRP(C_2) \leq 1 - \alpha$, $|SAFRP(C_1) - SAFRP(C_2) \leq 1 - \alpha$, $|WFCA(C_1) - WFCA(C_2) \leq 1 - \alpha$, $|SFCA(C_1) - SFCA(C_2) \leq 1 - \alpha$ . In other words, the more similar the choices $C_1$, $C_2$ are, the closer the $WAFRP$–rationality of $C_1$ is to the $WAFRP$–rationality of $C_2$, etc.

# 2 Preliminaries

In this section we recall some notions and basic results on the minimum operator and on fuzzy binary relations. The basic references are [4, 5, 11, 19].

For any $\{a_i\}_{i\in I} \subseteq [0,1]$ we shall denote $\bigvee\limits_{i\in I} a_i = \sup\{a_i | i \in I\}$ and $\bigwedge\limits_{i\in I} a_i = \inf\{a_i | i \in I\}$. In particular, for any $a, b \in [0,1]$, $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$.

We consider the residuum operation associated with the minimum operator $\wedge$:
$a \to b = \bigvee\{c | a \wedge c \leq b\} = a \to b = \begin{cases} 1 & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$

The negation operation $\to$ associated with the minimum operator $\wedge$ is defined by $\neg a = a \to 0$ for any $a \in [0,1]$.

The following lemmas contain the main properties of these operations:

**Lemma 2.1** *[4, 5, 11] For any $a, b, c \in [0,1]$ the following properties are true:*

*(1)* $a \leq b \to c$*;*

*(2)* $a \wedge (a \to b) = a \wedge b$*;*

*(3)* $a \leq b$ *iff* $a \to b = 1$*;*

*(4)* $a \to 1 = 1$*;*

*(5)* $1 \to a = a$*;*

*(6) If* $a \leq b$ *then* $b \to c \leq a \to c$ *and* $c \to a \leq c \to b$*;*

*(7)* $a \to (b \to c) = (a \wedge b) \to c = b \to (a \to c)$*;*

*(8)* $a \leq \neg b$ *iff* $a \wedge b = 0$.

**Lemma 2.2** *[4, 5, 11] Let* $\{a_i\} \subseteq [0,1]$ *and* $b \in [0,1]$. *Then*

*(1)* $\bigvee_{i\in I} a_i \wedge b = \bigvee_{i\in I} (a_i \wedge b)$*;*

*(2)* $(\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I} (a_i \to b)$*;*

*(3)* $b \to (\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} (b \to a_i)$.

Another operation on $[0,1]$ is the biresiduum $\leftrightarrow$ defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Let $X$ be a non–empty set (*the universe*). A *fuzzy subset* of $X$ is a function $A : X \to [0,1]$; if $x \in X$ then $A(x)$ is called the *degree of membership* of $x$. We denote by $\mathcal{P}(X)$ the powerset of $X$ and by $\mathcal{F}(X)$ the family of fuzzy subsets of $X$. For $U \subseteq X$ we denote by $\chi_U$ the characteristic function of $U$:

$$\chi_U = \begin{cases} 1 & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U \end{cases}$$

If we identify $U$ with its characteristic function $\chi_U$ then $\mathcal{P}(X)$ is identified with $\{0,1\}^X \subseteq \mathcal{F}(X)$. For any $A, B \in \mathcal{F}(X)$ we write $A \subseteq B$ if $A(x) \leq B(x)$ for all $x \in X$.

A *fuzzy relation* $R$ on $X$ is a function $R : X^2 \to [0,1]$.

If $A, B \in \mathcal{F}(X)$ then we denote

$$I(A,B) = \bigwedge_{x\in X} (A(x) \to B(x)) = \bigwedge_{x\in X} \{B(x) | x \in X, A(x) > B(x)\};$$

$$E(A,B) = \bigwedge_{x\in X} (A(x) \leftrightarrow B(x)) = \bigwedge_{x\in X} \{A(x) \wedge B(x) | x \in X, A(x) \neq B(x)\};$$

$I(A,B)$ is called the *subsethood degree* of $A$ in $B$ and $E(A,B)$ the *degree of equality* of $A$ and $B$. Intuitively $I(A,B)$ expresses the truth value of the statement "$A$ is included in $B$."

# 3 Fuzzy revealed preference

In this section we shall recall some basic notions of fuzzy revealed preference theory [6], [7]. We shall present the notion of fuzzy choice function in its general form [6], [7], [12] then we shall define the main fuzzy preference relations associated with a fuzzy choice function. On this basis, the axioms of revealed preference $WAFRP$, $SAFRP$ and the axioms of congruence $WFCA$, $SFCA$ will be formulated. Then we shall define the indicators of revealed preference $WAFRP(C)$, $SAFRP(C)$ and the indicators of congruence $WFCA(C)$, $SFCA(C)$ associated with a fuzzy choice function.

The framework in which we shall develop the fuzzy revealed preference theory is given by the concept of fuzzy choice space. A fuzzy choice space is a pair $(X, \mathcal{B})$ where $X$ is a non–empty universe of alternatives and $\mathcal{B}$ is a nonempty family of non–empty fuzzy subsets of $X$. The members of $\mathcal{B}$ are called available fuzzy sets; in interpretation, they represent vague attributes or vague criteria on alternatives. If $x \in X$ is an alternative and $S \in \mathcal{B}$ then the real number $S(x)$ is called the availability degree of $x$ with respect to $S$.

A fuzzy choice function [6] on a fuzzy choice space $(X, \mathcal{B})$ is a function $C : \mathcal{B} \to \mathcal{F}(X)$ such that for any $S \in \mathcal{B}$, $C(S)$ is a non–empty subset of $X$ and $C(S) \subseteq S$. If $x$ is an alternative and $S$ represents a criterion then $C(S)(x)$ measures the potentiality of $x$ of being chosen with respect to criterion $S$.

This definition of fuzzy choice functions contains that of Banerjee [2]. The domain of a fuzzy choice function in the sense of Banerjee consists of crisp subsets and the range of a fuzzy choice function consists of fuzzy sets of alternatives.

Let $C$ be a fuzzy choice function on $(X, \mathcal{B})$. We define the fuzzy preference relations $R_C$, $\tilde{P}_C$ on $X$ by

$$R_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y))$$

$$\tilde{P}_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) \wedge S(y) \wedge \neg C(S)(y))$$

for any $x, y \in X$. At the same time, we shall consider the fuzzy preference relations $W_C = T(R_C)$ and $P^*{}_C = T(\tilde{P}_C)$. Straight from the definition it follows that $\tilde{P}_C \subseteq R_C$, therefore $P^*{}_C \subseteq W_C$.

Particularizing the form of $R_C$ and $\tilde{P}_C$ for crisp choice functions we find the preference relations (denoted also by $R_C$ and $\tilde{P}_C$) studied by Samuelson [14], Arrow [1], Sen [15], [16], Suzumura [17]. A great part of classic theory on revealed preference theory concentrates on the revealed preference axioms $WARP$, $SARP$ and congruence axioms $WCA$, $SCA$. In [6] the fuzzy versions of these axioms have been introduced:

$WAFRP$ (Weak Axiom of Fuzzy Revealed Preference)
$\tilde{P}(x, y) \leq \neg R(y, x)$ for all $x, y \in X$;

$SAFRP$ (Strong Axiom of Fuzzy Revealed Preference)
$P^*(x, y) \leq \neg R(y, x)$ for all $x, y \in X$.

$WFCA$ (Weak Fuzzy Congruence Axiom)
For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds
$R(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.

$SFCA$ (Strong Fuzzy Congruence Axiom)
For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds
$W(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.

These statements are conditions which describe a rational behaviour of the choice function. Nevertheless there exist few situations when the vague choices verify such properties. In the study of the imprecise phenomena, it is more appropriate to evaluate the degree to which a property takes place. This point of view leads, in this case, to introducing the indicators $WAFRP(C)$, $SAFRP(C)$, $WFCA(C)$ and $SFCA(C)$ in order to measure the degree to which a fuzzy choice function $C$ verifies $WAFRP$, $SAFRP$, $WFCA$ and $SFCA$. The definition of these indicators starts exactly from the formulation in natural language of these axioms.

**Definition 3.1** *For a fuzzy choice function $C$ on $(X, \mathcal{B})$ we define the following indicators of the axioms $WAFRP$, $SAFRP$, $WFCA$ and $SFCA$:*

*(i)* $WAFRP(C) = \bigwedge\limits_{x,y \in X} [\tilde{P}_C(x,y) \rightarrow \neg R_C(y,x)]$;
*(ii)* $SAFRP(C) = \bigwedge\limits_{x,y \in X} [P^*_C(x,y) \rightarrow \neg R_C(y,x)]$;
*(iii)* $WFCA(C) = \bigwedge\limits_{x,y \in X} \bigwedge\limits_{S \in \mathcal{B}} [S(x) \wedge C(S)(y) \wedge R_C(x,y) \rightarrow C(S)(x)]$;
*(iv)* $SFCA(C) = \bigwedge\limits_{x,y \in X} \bigwedge\limits_{S \in \mathcal{B}} [S(x) \wedge C(S)(y) \wedge W_C(x,y) \rightarrow C(S)(x)]$.

**Remark 3.2** *For a choice function $C$ the following equivalences hold:*

$WAFRP(C) = 1$ *iff $C$ verifies $WAFRP$;*
$SAFRP(C) = 1$ *iff $C$ verifies $SAFRP$;*
$WFCA(C) = 1$ *iff $C$ verifies $WFCA$;*
$SFCA(C) = 1$ *iff $C$ verifies $SFCA$.*

For example, the indicator $WAFRP(C)$ indicates the degree to which the choice function $C$ verifies $WAFRP$. Similar interpretations can be given to the other three indicators. By these indicators we have information on each fuzzy choice function with respect to satisfying the corresponding axiom. Each indicator produces a criterion for comparison of two fuzzy choice functions. If $C_1$, $C_2$ are two fuzzy choice functions on $(X, \mathcal{B})$ and $WAFRP(C_1) \leq WAFRP(C_2)$ then $C_2$ is better than $C_1$ with respect to $WAFRP$. By such criteria one obtains hierarchies of families fuzzy choice functions.

# 4 Preservation properties of fuzzy choice functions

In this section two fundamental concepts of this paper are connected: the similarity and the indicators of revealed preference and congruence of fuzzy choice functions. The main theorem establishes the way the similarity preserves these four indicators, by obtaining this way an answer to Problem $B$ from the previous section.

Similarity is a notion related to imprecise phenomena. The identical behaviour of two vague entities is rare. Much more often we can find the situation when two fuzzy entities have a behaviour which makes them much closer to one another; therefore the similarity is a more operative concept.

The similarity of two fuzzy sets has been defined by Zadeh [18] as an extension of the notion of equivalence relation. A fuzzy relation $Q$ on a set $X$ is called an equivalent relation if it is reflexive, symmetric and transitive. If $x, y \in X$ then the real number $Q(x, y)$ is the degree of similarity of $x$ and $y$.

Let $X$ be a universe of alternatives. If $Q_1$, $Q_2$ are two preference relations on $X$ and $E(Q_1, Q_2) = \bigwedge_{x,y \in X} (Q_1(x, y) \leftrightarrow Q_2(x, y))$, then according to [9], the assignment $(Q_1, Q_2) \mapsto E(Q_1, Q_2)$ is a similarity relation on the set of fuzzy preference relations on $X$. In interpretation, $Q_1$, $Q_2$ can be considered as preferences of two agents; then the real number $E(Q_1, Q_2)$ appreciates how similar those preferences are.

Suppose the two agents have vague choices represented by fuzzy choice functions $C_1$, $C_2$. In this case we also need a notion which should measure how similar the choices of the two agents are.

If $C_1$, $C_2$ are two fuzzy choice functions on $(X, \mathcal{B})$ then according to [9], their degree of similarity is defined by

$$E(C_1, C_2) = \bigwedge_{S \in \mathcal{B}} \bigwedge_{x \in X} (C_1(S)(x) \leftrightarrow C_2(S)(x))$$

In [9], we proved that the assignment $(C_1, C_2) \mapsto E(C_1, C_2)$ is a similarity relation on the set of fuzzy preference relations on $(X, \mathcal{B})$. In [9] we also have studied the way the similarity of fuzzy choice functions is connected with the similarity of the fuzzy revealed preference relations associated with them.

Let $\alpha \in [0, 1]$. We shall say that the fuzzy choice functions $C_1$, $C_2$ are $\alpha$–similar if $E(C_1, C_2) \geq \alpha$, i. e. if for $C_1$ and $C_2$ a similarity with a degree greater or equal than $\alpha$ is ensured. In this case we shall write $C_1 =_\alpha C_2$.

Concerning the similarity of vague choices of two agents $Agent_1$ and $Agent_2$ we can formulate a generic problem and the problems $A$ and $B$ which derive from it.

**A generic problem**

Suppose the choices of two agents are made according to a criterion $Crit$. If the choices of the two agents are similar and the choices of $Agent_1$ verify criterion $Crit$ then do the choices of $Agent_2$ still verify $Crit$ ?

Criterion $Crit$ may be one of the axioms of revealed preferences $WAFRP$, $SAFRP$ or one of the axioms of congruence $WFCA$, $SFCA$. Suppose e. g. that $Crit$ is $WAFRP$. Then the above problem becomes:

**Problem $A$**

Suppose that $C_1$, $C_2$ are fuzzy choice functions corresponding to the agents $Agent_1$ and $Agent_2$. If $C_1$, $C_2$ are $\alpha$–similar and $C_1$ verifies $WAFRP$ then does $C_2$ verify $WAFRP$?

In most of the cases there is no positive answer to this question. Therefore we consider the following version of Problem $A$:

**Problem** $B$ If $C_1$, $C_2$ are $\alpha$–similar and $C_1$ verifies $WAFRP$, then which is the degree to which $C_2$ verifies $WAFRP$?

The answer to Problem $B$ comes down to evaluating the indicator $WAFRP(C_2)$ under the conditions that $C_1$ and $C_2$ are $\alpha$–similar and $WAFRP(C_1) = 1$. This observation leads to the idea of comparing $WAFRP(C_2)$ with the degree of similarity $E(C_1, C_2)$ and with $WAFRP(C_1)$.

Of course the previous discussion for the axiom $WAFRP$ can be extended to the axioms $SAFRP$, $WFCA$ and $SFCA$.

**Lemma 4.1** *[7] Let $C$, $C'$ be two fuzzy choice functions on $(X, \mathcal{B})$. Then for any $S \in \mathcal{B}$ and $x \in X$ we have*

*(i) $E(C, C') \wedge C(S)(x) \leq C'(S)(x)$;*

*(ii) $E(C, C') \wedge \neg C(S)(x) \leq \neg C'(S)(x)$.*

**Lemma 4.2** *[7] Let $C$ and $C'$ be two fuzzy choice functions on $(X, \mathcal{B})$and $x, y \in X$. Then*

*(i) $E(C, C') \wedge R_C(x, y) \leq R_{C'}(x, y)$;*

*(ii) $E(C, C') \wedge \neg R_C(x, y) \leq \neg R_{C'}(x, y)$.*

**Lemma 4.3** *[7] If $C$ and $C'$ are two fuzzy choice functions on $(X, \mathcal{B})$ and $x, y \in X$ then $E(C, C') \wedge \tilde{P}_C(x, y) \leq \tilde{P}_{C'}(x, y)$.*

The following theorem connects the four indicators $WAFRP(C)$, $SAFRP(C)$, $WFCA(C)$, $SFCA(C)$ and the similarity of fuzzy choice functions.

**Theorem 4.4** *If $C$, $C'$ are two fuzzy choice functions on $(X, \mathcal{B})$, then*

*(i) $WAFRP(C) \wedge E(C, C') \leq WAFRP(C')$;*

*(ii) $SAFRP(C) \wedge E(C, C') \leq SAFRP(C')$;*

*(iii) $WFCA(C) \wedge E(C, C') \leq WFCA(C')$;*

*(iv) $SFCA(C) \wedge E(C, C') \leq SFCA(C')$.*

Proof. (i) We have to prove that

$$E(C, C') \wedge WAFRP(C) \leq \bigwedge_{x,y \in X} [\tilde{P}_{C'}(x, y) \rightarrow \neg R_{C'}(y, x)]$$

which is equivalent with verifying for each $x, y \in X$ the following inequality

(a1) $E(C, C') \wedge WAFRP(C) \leq \tilde{P}_{C'}(x, y) \rightarrow \neg R_{C'}(y, x)$.

Let $x, y \in X$. By Lemma 2.1 (1), inequality (a1) is equivalent with

(b1) $E(C, C') \wedge WAFRP(C) \wedge \tilde{P}_{C'}(x, y) \leq \neg R_{C'}(y, x)$.

By Lemma 2.2 (1), the left hand side member of (b1) is computed:

$$E(C, C') \wedge WAFRP(C) \wedge \tilde{P}_{C'}(x, y) =$$

$$= E(C, C') \wedge WAFRP(C) \wedge \bigvee_{S \in \mathcal{B}} [C'(S)(x) \wedge S(y) \wedge \neg C'(S)(y)] =$$

$= \bigvee_{S\in\mathcal{B}} [E(C,C') \wedge WAFRP(C) \wedge C'(S)(x) \wedge S(y) \wedge \neg C'(S)(y)]$.

Then to prove (b1) means to verify that for each $S \in \mathcal{B}$ the following inequality takes place:

(c1) $E(C,C') \wedge WAFRP(C) \wedge C'(S)(x) \wedge S(y) \wedge \neg C'(S)(y) \leq \neg R_{C'}(y,x)$.

Let $S \in \mathcal{B}$. By Lemma 4.1 we have

$E(C,C') \wedge C'(S)(x) \leq C(S)(x)$ and

$E(C,C') \wedge \neg C'(S)(y) \leq \neg C(S)(y)$.

From these two inequalities and by Lemmas 2.1 (2) and 2.2 (2) it follows

$E(C,C') \wedge WAFRP(C) \wedge C'(S)(x) \wedge S(y) \wedge \neg C'(S)(y) \leq$

$\leq E(C,C') \wedge WAFRP(C) \wedge C(S)(x) \wedge S(y) \wedge \neg C(S)(y) =$

$= E(C,C') \wedge C(S)(x) \wedge S(y) \wedge \neg C(S)(y) \wedge [[\bigvee_{T\in\mathcal{B}} (C(T)(x) \wedge T(y) \wedge \neg C(T)(y))] \rightarrow \neg R_C(y,x) =$

$= E(C,C') \wedge C(S)(x) \wedge S(y) \wedge \neg C(S)(y) \wedge \bigwedge_{T\in\mathcal{B}} [(C(T)(x) \wedge T(y) \wedge \neg C(T)(y)) \rightarrow \neg R_C(y,x)] \leq$

$\leq E(C,C') \wedge C(S)(x) \wedge S(y) \wedge \neg C(S)(y) \wedge [(C(S)(x) \wedge S(y) \wedge \neg C(S)(y)) \rightarrow \neg R_C(y,x)] =$

$= E(C,C') \wedge C(S)(x) \wedge S(y) \wedge \neg C(S)(y) \wedge \neg R_C(y,x) \leq$

$\leq E(C,C') \wedge \neg R_C(y,x) \leq \neg R_{C'}(y,x)$.

With this the inequality (c1) is proved and also the proof of (i).

(ii) According to the definition of $SAFRP(C')$, we have to prove

$E(C,C') \wedge SAFRP(C) \leq \bigwedge_{x,y\in X} [P^*_{C'}(x,y) \rightarrow \neg R_{C'}(y,x)]$.

It suffices for any $x, y \in X$ to establish the inequality

(a2) $E(C,C') \wedge SAFRP(C) \leq P^*_{C'}(x,y) \rightarrow \neg R_{C'}(y,x)$.

By Lemma 2.1 (1), (a2) is equivalent with

(b2) $E(C,C') \wedge SAFRP(C) \wedge P^*_{C'}(x,y) \leq \neg R_{C'}(y,x)$.

The left hand side member of (b2) has the form:

$E(C,C') \wedge SAFRP(C) \wedge P^*_{C'}(x,y) =$

$E(C,C') \wedge SAFRP(C) \wedge [\tilde{P}_{C'}(x,y) \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n\in X} (\tilde{P}_{C'}(x,t_1) \wedge \ldots \wedge \tilde{P}_{C'}(t_n,y))] =$

$= [E(C,C') \wedge SAFRP(C) \wedge \tilde{P}_{C'}(x,y)] \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n\in X} [E(C,C') \wedge SAFRP(C) \wedge \tilde{P}_{C'}(x,t_1) \wedge \ldots \wedge \tilde{P}_{C'}(t_n,y)]$.

Then (b2) is equivalent with the conjunction of the following two assertions:

(c2) $E(C,C') \wedge SAFRP(C) \wedge \tilde{P}_{C'}(x,y) \leq \neg R_{C'}(y,x)$

(d2) For any $n \geq 1$ and $t_1, \ldots, t_n \in X$,

$E(C,C') \wedge SAFRP(C) \wedge \tilde{P}_{C'}(x,t_1) \wedge \ldots \wedge \tilde{P}_{C'}(t_n,y) \leq \neg R_{C'}(y,x)$.

We will check only (d2). Let $n \geq 1$ and $t_1, \ldots, t_n \in X$. By applying Lemmas 4.3 and 2.1 (2) we obtain:

$E(C,C') \wedge SAFRP(C) \wedge \tilde{P}_{C'}(x,t_1) \wedge \ldots \wedge \tilde{P}_{C'}(t_n,y) =$

$= SAFRP(C) \wedge [E(C,C') \wedge P_{C'}(x,t_1)] \wedge \ldots \wedge [E(C,C') \wedge P_{C'}(t_n,y)] \leq$

$\leq SAFRP(C) \wedge \tilde{P}_C(x,t_1) \wedge \ldots \wedge \tilde{P}_C(t_n,y) \leq$

$\leq \tilde{P}_C(x,t_1) \wedge \ldots \wedge \tilde{P}_C(t_n,y) \wedge [(P_C(x,y) \wedge \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n \in X} (\tilde{P}_C(x,t_1) \wedge \ldots \wedge \tilde{P}_C(t_n,y)) \to \neg R_C(x,y)]$

$\leq \tilde{P}_C(x,t_1) \wedge \ldots \wedge \tilde{P}_C(t_n,y) \wedge [(\tilde{P}_C(x,t_1) \wedge \ldots \wedge \tilde{P}_C(t_n,y) \to \neg R_C(y,x)] =$

$= \tilde{P}_C(x,t_1) \wedge \ldots \wedge \tilde{P}_C(t_n,y) \wedge \neg R_C(y,x) \leq \neg R_C(y,x)$.

By applying Lemma 4.2 it follows

$E(C,C') \wedge SAFRP(C') \wedge \tilde{P}_{C'}(x,t_1) \wedge \ldots \wedge \tilde{P}_{C'}(t_n,y) \leq E(C,C') \wedge \neg R_C(y,x) \leq \neg R_{C'}(y,x)$

since (d2) has been proved.

(iii) Let $S \in \mathcal{B}$ and $x, y \in X$. We will establish the inequality:

(a3) $WFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge R_{C'}(x,y) \leq C'(S)(x)$.

According to Lemma 4.3 and Lemma 4.4 we have

$WFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge R_{C'}(x,y) =$

$= WFCA(C) \wedge E(C,C') \wedge S(x) \wedge [E(C,C') \wedge C'(S)(y)] \wedge [E(C,C') \wedge R_{C'}(x,y)] \leq$

$WFCA(C) \wedge E(C,C') \wedge S(x) \wedge C(S)(y) \wedge R_C(x,y) =$

$= E(C,C') \wedge S(x) \wedge C(S)(y) \wedge R_C(x,y) \wedge \bigwedge_{u,v \in X} \bigwedge_{T \in \mathcal{B}} [(T(u) \wedge C(T)(v) \to R_C(u,v)) \to C(T)(u)] \leq$

$\leq E(C,C') \wedge S(x) \wedge C(S)(y) \wedge R_C(x,y) \wedge [(S(x) \wedge C(S)(y) \wedge R_C(x,y)) \to C(S)(x)] =$

$= E(C,C') \wedge S(x) \wedge C(S)(y) \wedge R_C(x,y) \wedge C(S)(x) \leq$

$\leq E(C,C') \wedge C(S)(x) \leq C'(S)(x)$.

From (a3) one obtains:

$WFCA(C) \wedge E(C,C') \leq (S(x) \wedge C'(S)(y) \wedge R_{C'}(x,y)) \to C'(S)(x)$

for any $x, y \in X$ and $S \in \mathcal{B}$. Therefore

$WFCA(C) \wedge E(C,C') \leq \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} [(S(x) \wedge C'(S)(y) \wedge R_{C'}(x,y)) \to C'(S)(x)] =$

$= WFCA(C')$.

(iv) To prove $SFCA(C) \wedge E(C,C') \leq SFCA(C')$ is equivalent with establishing the inequality

(a4) $SFCA(C) \wedge E(C,C') \leq (S(x) \wedge C'(S)(y) \wedge W_{C'}(x,y)) \to C'(S)(x)$

for any $x, y \in X$ and $S \in \mathcal{B}$.

Then let $x, y \in X$ and $S \in \mathcal{B}$. (a4) is equivalent with

(b4) $SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge W_{C'}(x,y) \leq C'(S)(x)$.

The we compute the left hand side member of (b4):

$SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge W_{C'}(x,y) =$

$= SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge [R_{C'}(x,y) \wedge \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n \in X} (R_{C'}(x,t_1) \wedge \ldots \wedge R_{C'}(t_n,y))] =$

$= [SFCA(C)\wedge E(C,C')\wedge S(x)\wedge C'(S)(y)\wedge R_{C'}(x,y)]\vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n\in X} [SFCA(C)\wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge R_{C'}(x,t_1) \wedge \ldots \wedge R_{C'}(t_n,y)]$.

Then (b4) is equivalent with the conjunction of the following two assertions:

(c4) $SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge R_{C'}(x,y) \leq C'(S)(x)$

(d4) For all $n \geq 1$ and $t_1,\ldots,t_n \in X$,

$SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge R_{C'}(x,t_1) \wedge \ldots \wedge R_{C'}(t_n,y) \leq C'(S)(x)$.

We prove only (d4). Let $n \geq 1$ and $t_1,\ldots t_n \in X$. We notice that

$R_C(x,t_1) \wedge E(C,C') \wedge S(x) \wedge C'(S)(y) \wedge R_{C'}(x,t_1) \wedge \ldots \wedge R_{C'}(t_n,y) =$

$SFCA(C)\wedge E(C,C')\wedge S(x)\wedge[E(C,C')\wedge C'(S)(y)]\wedge[E(C,C')\wedge R_{C'}(x,t_1)]\wedge \ldots \wedge [E(C,C') \wedge R_{C'}(t_n,y)] \leq$

$\leq SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C(S)(y) \wedge R_C(x,t_1) \wedge \ldots \wedge R_C(t_n,y) \leq$

$\leq SFCA(C) \wedge E(C,C') \wedge S(x) \wedge C(S)(y) \wedge W_C(x,y) =$

$= E(C,C') \wedge S(x) \wedge C(S)(y) \wedge W_C(x,y) \wedge \bigwedge_{u,v\in X} \bigwedge_{T\in\mathcal{B}} [(S(u) \wedge C(T)(v) \wedge W_C(u,v)) \rightarrow C(S)(u)] \leq$

$\leq E(C,C') \wedge S(x) \wedge C(S)(y) \wedge W_C(x,y) \wedge C(S)(x) \leq$

$\leq E(C,C') \wedge C(S)(x) \leq C'(S)(y)$.

With this (d4) has been verified. The proof of the theorem has finished.

∎

**Definition 4.5** *Let $0 \leq \delta \leq 1$. For the fuzzy choice functions $C$, $C'$ we introduce the following notions:*

*(i) $C\delta_{WAFRP}C'$ iff $\rho(WAFRP(C), WAFRP(C')) \geq \delta$;*

*(ii) $C\delta_{SAFRP}C'$ iff $\rho(SAFRP(C), SAFRP(C')) \geq \delta$;*

*(iii) $C\delta_{WFCA}C'$ iff $\rho(WFCA(C), WFCA(C')) \geq \delta$;*

*(iv) $C\delta_{SFCA}C'$ iff $\rho(SFCA(C), SFCA(C')) \geq \delta$.*

If $C\delta_{WAFRP}C'$ then we say that the fuzzy choice functions $C$, $C'$ are $WAFRP$–equivalent with the degree of confidence $\delta$. In the other three situations a similar terminology is introduced.

**Remark 4.6** *The concept $C_{WAFRP}C'$ evaluates how close the behaviour of the fuzzy choice functions $C$ and $C'$ is with respect to the verification of the axiom $WAFRP$.*

**Corollary 4.7** *Let $0 \leq \delta \leq 1$ and $C$, $C'$ be two fuzzy choice functions.*

*If $C =_\delta C'$ then $C\delta_{WAFRP}C'$, $C\delta_{SAFRP}C'$, $C\delta_{WFCA}C'$, $C\delta_{SFCA}C'$.*

Proof. By Theorem 4.4 we have

$E(C,C') \leq WAFRP(C) \rightarrow WAFRP(C')$;

$E(C,C') \leq WAFRP(C') \rightarrow WAFRP(C)$;

hence
$E(C, C') \leq WAFRP(C) \leftrightarrow WAFRP(C') = \rho(WAFRP(C), WAFRP(C'))$.
Then $E(C, C') \geq \delta$ implies $\rho(WAFRP(C), WAFRP(C')) \geq \delta$.
The other cases result analogously. ∎

# 5 Concluding Remarks

The axioms of revealed preference and congruence are conditions which ensure a rational behaviour of choice functions (in particular, of consumers) (see [1], [14], [15], [16], [17]).

When we study the rationality of fuzzy choices it is necessary to find appropriate fuzzy versions of these properties.

There are two ways in which we can transfer the axioms of revealed preference and congruence from the context of classic choice functions to the context of fuzzy choice functions.

(a) to express these properties by assertions of fuzzy logic ([2], [6], [8]).

(b) the introduction of some indicators which should measure the conditions of revealed preference and congruence ([7], [10]).

The second modality has the following advantages compared to the first one:

(i) these indicators are associated with any fuzzy choice functions;

(ii) by having a numerical expression, these indicators allow for classifications and hierarchies of fuzzy choice functions from the point of view of a property of rationality.

Similarity is specific to fuzzy phenomena (in particular to fuzzy choices). The degree of similarity of two fuzzy choice functions gives an appreciation on the closeness of these ones.

This paper linked this concept to the properties of revealed preference and congruence, by establishing the way the indicators $WAFRP(C)$, $SAFRP(C)$, $WFCA(C)$ and $SFCA(C)$ are preserved by the similarity of the fuzzy choice functions. More precisely, by the comparison of the indicators, there is evaluated the way two ”similar” choice functions have a close behaviour from the point of view of their rationality.

The definitions and the results of the paper can be formulated in the context of the theory of fuzzy sets associated to a left continuous t–norm , but the proofs are done in the paper only for the minimum t–norm. To obtain the results in a more general context remains an open problem.

On the other hand, it would be interested to study, in problems with concrete data, the way in which by applying the evaluations by these indicators we can reach the most rational choices.

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