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Yury Nikulin | Marko M. Mäkelä

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Yury Nikulin
University of Turku, Department of Mathematics
FI-20014 Turku, Finland
yurnik@utu.fi
Marko M. Mäkelä
University of Turku, Department of Mathematics
FI-20014 Turku, Finland
makela@utu.fi

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Abstract

A multicriteria boolean programming problem with linear cost functions in which initial coefficients of the cost functions are subject to perturbations is considered. For any optimal alternative, with respect to parameterized principle of optimality ”from Condorcet to Pareto”, appropriate measures of the quality are introduced. These measures correspond to the so-called stability and accuracy functions defined earlier for optimal solutions of a generic multicriteria combinatorial optimization problem with Pareto and lexicographic optimality principles. Various properties of such functions are studied and maximum norms of perturbations for which an optimal alternative preserves its optimality are calculated. To illustrate the way how the stability and accuracy functions can be used as efficient tools for post-optimal analysis, an application from the voting theory is considered.

Keywords: Condorcet optimality, Pareto set, stability and accuracy, parameterization, multicriteria optimization, voting principles

1 Introduction

The stability theory has its roots originating from the definition of a well-posed mathematical problem given by J. Hadamard in [8], who believed that mathematical models of physical phenomena should include, among others, the property of a solution to depend continuously on the data, in some reasonable topology. In optimization a question of stability of a problem arises in the case where the set of feasible solutions (alternatives) and/or the objective (cost) function depend on parameters. The presence of such parameters in optimization models is due to many reasons, for instance inaccuracy of initial data, non-adequacy of models to real processes, errors of numerical methods, errors of rounding off and other factors. Thus it appears to be important to allocate classes of problems in which small changes of the input data lead to small changes of the result. The problems with such properties are called stable. It is obvious that many optimization problems arising in practice cannot be correctly formulated, analyzed and solved without exploiting the results of the stability theory.

It is not very surprising that many researchers focus on analyzing various aspects of stability for large classes of optimization problems. For example, one can find a vast annotated bibliography for sensitivity and post-optimal analysis in integer programming and combinatorial optimization problems in [7].

The main object while studying stability of multicriteria optimization problems is usually a set of optimal (sometimes referred to as efficient) solutions or alternatives, i.e. the set of feasible solutions which satisfy a given optimality principle. In the case where the partial criteria of the problem have equal importance, the Pareto optimality principle (named after Vilfredo Pareto who proposed it in [14]), is more often used. Generally, a feasible solution is said to be Pareto optimal if there is no other feasible solution such that at least one its objective value is getting better without deteriorating any other objective values.

If we relax the demand of non-worsening objectives in such a way that worsening for some objective values is allowed but the number of objectives which values are allowed to be deteriorated is restricted above by the number of objectives with better values, then we get the concept of Condorcet optimality principle (named after marquis de Condorcet who proposed it in [4]).

It is clear that the set of optimal solutions defined by Condorcet optimality principle is a subset of the set of optimal solutions given by the Pareto optimality principles, i.e. Pareto optimality principle gives more freedom for solutions to become optimal compared to the Condorcet optimality principle.

The frequently used tool of stability theory and post-optimal analysis is so-called stability radius of some given optimal solution. In single objective optimization, it gives an upper bound on a subset of problem parameters for which this solution remains optimal (see [7] and bibliography therein). There exist already similar investigations in multiobjective case, where the stability radius defines extreme level of problem parameter perturbations preserving efficiency of the given solution. For example, in [6] one can find a large survey on sensitivity analysis of unconstrained vector integer linear programming, where the stability radius is a key object under investigation.

It is important to note that even in single objective case the stability radius does not provide us with any information about the quality of a given solution in the case when problem data are outside of the stability region. Some attempts to study quality of the problem solution in this case are connected with concepts of stability and accuracy functions, which were originally proposed in [10] and [11] for scalar combinatorial optimization problems. Later, the results were extended for the case of multicriteria combinatorial optimization problems with Pareto and lexicographic optimality principles in [12]. In [13], the similar questions of stability and accuracy were investigated under the framework of game theory, more precisely accuracy and stability functions for a coalition game with bans, linear payoffs, antagonistic strategies and parameterized principle of optimality ”from Nash to Pareto” were studied.

In this paper, we give an extension of the concepts of stability and accuracy functions under the parameterized optimality principle ”from Condorcet to Pareto”. The paper is organized as follows. In section 2, we formulate a general approach to post-optimal analysis using various quantitative measures. For a given solution we introduce an appropriate absolute error as a function representing deviation from optimality. Afterwards, we define the so called stability and accuracy radii as extreme norms of perturbations of problem parameters for which the stability and accuracy functions are equal to zero. Section 3 introduces two traditional optimality principles (Condocet and Pareto optimality) and a parameterization of these principles. In section 4, we consider a multicriteria Boolean linear programming problem, and specify some particular results about stability and accuracy functions valid for this particular problem. We give formulae to calculate values of both functions and corresponding radii. In section 5, an example from voting theory is considered to illustrate the way how the stability and accuracy functions can be used as efficient tools for post-optimal analysis. Final remarks and conclusions appear in section 6.

2 Postoptimal analysis under general framework

We consider a general multiobjective optimization problem with $m \geq 2$ cost functions representing the problem objectives. Let X be a finite set of feasible solutions or alternatives $x := \{x_1, ..., x_n\}^T$, where n denotes the problem size. To avoid trivial cases, we assume that $|X| \geq 2$, and $(0, 0, ..., 0)^T \notin X$.

For each solution $x \in X$, a vector of cost functions

$$f(C, x) := (f_1(C, x), ..., f_m(C, x))^T \longrightarrow \min_{x \in X} . \tag{1}$$

consists of individual cost functions $f_i(C, x)$, $i \in I_m := \{1, 2, ..., m\}$. Without loss of generality, we assume that $f_i(C,x)$ are minimized on the set of feasible solutions X for each $i \in I_m$. Here $C = [c_{ij}] \in \mathbf{R}_+^{m \times n}$, where $\mathbf{R}_+^{m \times n}$ is a set of $m \times n$ matrices (problem input data) with all elements being positive.

Contrary to the single objective case where the concept of optimal solution is unique, under multicriteria framework the concept of optimality may vary. The concept of optimality is usually based on binary relations reflecting preferability of one solutions over others. In its turn, any binary relation generates a principle of optimality (in other terminology, sometimes referred as a choice function).

To keep the general conceptual level of this section, we assume that some non-empty set of non-dominated (w.r.t. some binary relation $\prec$) solutions is searched for multiobjective optimization problem (1).

$$E^m(C) = \{x^* \in X \mid \ \nu(C,x^*) = \emptyset\},$$

where

$$\nu(C,x^*) := \{x \in X : \ x^* \prec x\}.$$

Notice that in single objective case $m = 1$, the set of non-dominated solutions transforms into the set of optimal solutions $E^1(C)$.

In postoptimal analysis, we assume some efficient solution was found for the problem with original input parameters, and we investigate the behavior of this solution under small changes (variations) of the input data. For these purposes some quantitative characteristics are used to express numerically how far the solution deviates from efficiency depending on a scale of variation of problem parameters.

Now assume that the set of feasible solutions X is fixed, but the matrix of input data C may vary or be estimated with errors. Moreover, we also assume that for some originally specified matrix $C^0 = [c_{ij}^0] \in \mathbf{R}_+^{m \times n}$ we know an efficient solution $x^* \in E^m(C^0)$.

The quality of the given solution $x^* \in E^1(C^0)$ in the problem with some matrix $C \in \mathbf{R}_+^{1 \times n}$ is evaluated based on the concept of **absolute error** $a^m(C,x^*)$, which in single objective case ($m = 1$) is defined as follows:

$$a^1(C,x^*) = f_1(C,x^*) - \min_{x \in X} f_1(C,x). \tag{2}$$

The absolute error contains essential information about how far the given solution deviates from being efficient in a situation where matrix C represents the problem input data. In a ”true” multiobjective case ($m \geq 2$), the expression for the absolute error $a^m(C,x^*)$ crucially depends on properties of the binary relation $\prec$ which is used to define $E^m(C)$. We specify an explicit form $a^m(C,x^*)$ in the next section.

Notice that sometimes, instead of the absolute error, its relative analogue can be used (see e.g. [10]). For example, the relative analog of $a^1(C,x^*)$ is defined in [11]. However, it may lead to practical limitations on a usage of the relative error, because of possible severe computational difficulties due to the presence of the division operator.

In the following we are interested, in fact, in the maximum value of the error $a^m(C, x^*)$ when the matrix C belongs to some specified set which describes possible absolute perturbations of the original matrix C^0. Two particular cases are considered: In the first case we are interested in **absolute perturbations** of the elements of matrix C^0 and the quality of a given solution $x^* \in E^m(C^0)$ is described by the so-called stability function. For a given $0 \leq \rho \leq ub$ the value of the stability function is equal to the maximum absolute error of a given situation under the assumption that none of elements of C^0 are increased or decreased by more than ρ. The parameter ub restricts admissible perturbations from above. Typically, its value is set up to the value of minimal element of the original matrix C^0.

In the second case we deal with **relative perturbations** of the elements of matrix C^0. This leads to the concept of the accuracy function. The value of the accuracy function for a given $\delta \in [0, 1)$ is equal to the maximum absolute error of the solution $x^* \in E^m(C^0)$ under the assumption that the elements of C^0 are perturbed by no more than $\delta \cdot 100\%$ of their original values.

The two types of perturbations, absolute and relative, reflect different types of input data uncertainty that may appear in the problem. While the absolute perturbations are usually specified by some global parameter which reflects admissible perturbation range valid for all elements, the relative perturbations incorporate discrepancy in element ranges, i.e. the range of actual admissible perturbations depends on the nominal element range.

For a given $\rho \in [0, q(C^0))$, where $q(C^0) = \min\{c^0_{ij} : \ i \in I_m, \ j \in I_n\}$, we consider a set of admissible perturbed matrices in the case of absolute perturbations:

$$\Omega_\rho(C^0) := \{C \in \mathbf{R}^{m\times n}_+ : \ |c_{ij} - c^0_{ij}| \leq \rho, \ i \in I_m, \ j \in I_n\}. \tag{3}$$

For $x^* \in E^m(C^0)$ and $\rho \in [0, q(C^0))$, the value of the **stability function** is defined as follows:

$$S(C^0, x^*, \rho) := \max_{C \in \Omega_\rho(C^0)} a^m(C, x^*).$$

In a similar way, for a given $\delta \in [0, 1)$, we consider a set of admissible perturbed matrices in the case of relative perturbations:

$$\Theta_\delta(C^0) := \{C \in \mathbf{R}^{m\times n}_+ : \ |c_{ij} - c^0_{ij}| \leq \delta \cdot c^0_{ij}, \ i \in I_m, \ j \in I_n\}. \tag{4}$$

For $x^* \in E^m(C^0)$ and $\delta \in [0, 1)$, the value of the **accuracy function** is defined as follows:

$$A(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} a^m(C, x^*).$$

It is easy to check that $S(C^0, x^*, \rho) \geq 0$ for any $\rho \in [0, q(C^0))$ as well as $A(C^0, x^*, \delta) \geq 0$ for each $\delta \in [0, 1)$.

Observe that if we compare two optimal solutions from the point of view of their robustness on data perturbations or inaccuracy, then smaller value of the stability or accuracy function for a given norm of data perturbation is more preferable. Thus, both defined functions may be used to evaluate the quality of solutions, which are optimal in the original problem.

Sometimes, it is of special interest to know the extreme values of ρ and δ, for which $S(C^0, x^*, \rho) = 0$ and $A(C^0, x^*, \delta) = 0$, respectively, because these values determine maximum norms of perturbations which preserve the property of a given solution to be efficient. These values are analogous to the so-called stability and accuracy radii introduced earlier for single/multiple objective combinatorial optimization problems (see e.g. [6]). Formally, the stability radius $R^S(C^0, x^*)$ and the accuracy radius $R^A(C^0, x^*)$ are defined in the following way:

$$R^S(C^0, x^*) := \sup \{\rho \in [0, q(C^0)) : \ S(C^0, x^*, \rho) = 0\},$$

$$R^A(C^0, x^*) := \sup \{\delta \in [0, 1) : \ A(C^0, x^*, \delta) = 0\}.$$

If these radii are equal to zero, then this means that there exist arbitrary small perturbations of the original matrix C^0 such that the initially efficient solution x^* loses its efficiency in the perturbed problem. Otherwise, x^* remains efficient for any problem with matrix $C \in \Omega_\rho(C^0), \ \rho < R^S(C^0, x^*)$ or $C \in \Theta_\delta(C^0), \ \delta < R^A(C^0, x^*)$.

In the next section, we first formulate two traditional multiobjective optimality principles (Condocet and Pareto optimality). Afterwards, we introduce a natural parameterization of these principles to provide the decision maker with more flexible tool of expressing the compromise between conflicting objectives.

3 Parameterizing ”from Condorcet to Pareto”

As it has been already mentioned in introduction, in voting theory two main rules - majority and unanimity - are commonly in use. Application of these two rules as binary relations of preference between two solutions, which has to be compared pairwise, lead to the definitions of Condorcet and Pareto optimality principles, respectively.

For any $x, \ x' \in X$ and $C \in \mathbf{R}_+^{m \times n}$, we put

$$m^+(C, x, x') := |\{i \in I_m : \ f_i(C, x) > f_i(C, x')\}|;$$

$$m^-(C, x, x') := |\{i \in I_m : \ f_i(C, x) < f_i(C, x')\}|;$$

$$m^0(C, x, x') := |\{i \in I_m : \ f_i(C, x) = f_i(C, x')\}|.$$

Obviously,

$$m^+(C, x, x') + m^-(C, x, x') + m^0(C, x, x') = m. \tag{5}$$

The binary relations $x \prec x'$ of a strict preference between two feasible solutions x and x' (x' is preferred to x) are given according to the formulae:
- Condorcet (majority) domination relation $x \prec_\mu x'$:

$$m^+(C,x,x') > m^-(C,x,x'); \tag{6}$$

- Pareto (unanimity) domination relation $x \prec_\pi x'$:

$$m^+(C,x,x') > (m-1) \cdot m^-(C,x,x'). \tag{7}$$

Note, that

$$m^+(C,x,x') > \zeta \cdot m^-(C,x,x'),$$

for all $\zeta \geq m-1$, also defines the Pareto domination relation $x \prec_\pi x'$, however $m-1$ in (7) is the smallest integer value of ζ which may guarantee $x \prec_\pi x'$.

A solution $x^* \in X$ is called **Condorcet optimal** if

$$\mu(C,x^*) = \emptyset,$$

where

$$\mu(C,x^*) := \{x \in X :\ x^* \prec_\mu x\}.$$

We will refer to the set of all Condorcet optimal solutions as the **Condorcet set** and denote it by $M^m(C)$. Respectively, a solution $x^* \in X$ is called **Pareto optimal** if

$$\pi(C,x^*) = \emptyset,$$

where

$$\pi(C,x^*) := \{x \in X :\ x^* \prec_\pi x\}.$$

We will refer to the set of all Pareto optimal solutions as the **Pareto set** and denote it by $P^m(C)$. Notice that $P^m(C)$ is always non-empty, since the set of feasible solutions is finite (see e.g. [5]).

The Condorcet principle of optimality realizes the well-known procedure of decision-making by the majority of votes. It is easy to understand that the binary relation $\prec_\mu$ is not always transitive, not even for $m = 3$. This is known as the well-known Condorcet paradox of voting [4], which was comprehensively analyzed by Kenneth Arrow [1] based on the axiomatic approach to the mechanism of collective decision-making [2].

In order to give a decision maker more flexibility in defining optimality one can smooth the difference between Pareto and Condorcet optimality principles by introducing an integer parameter s and defining the binary preference relation as follows [2]:
- s**-domination relation** $x \prec_s x'$:

$$m^+(C,x,x') > s \cdot m^-(C,x,x'),\ s \in I_{m-1}. \tag{8}$$

The meaning of (8) in words can be formulated as follows: x' is preferred to x if the number of objectives where x has worse objective values (larger than corresponding objective values of x') is strictly bigger than s times number of objectives where x has better objective values. In other words, in order x not to be dominated by x' w.r.t. $\prec_s$, the number of objectives for x with worse objective values should not exceed s times the number of objectives with better values. Obviously for $s = 1$, $\prec_s$ transforms into the Condorcet domination relation $\prec_\mu$, and for $s = m - 1$, $\prec_s$ becomes the Pareto domination relation $\prec_\pi$.

A solution $x^* \in X$ is called s-**optimal** if

$$\nu_s(C, x^*) = \emptyset,$$

where

$$\nu_s(C, x^*) := \{x \in X : \ x^* \prec_s x\}.$$

We will refer to the set of all s-optimal solutions as the **s-optimal set** and denote it by $N_s^m(C)$. It is clear that $N_1^m(C) = M^m(C)$ and $N_{m-1}^m(C) = P^m(C)$. Clearly, the set $N_s^m(C)$ can be defined also by the following equivalent form:

$$N_s^m(C) := \{x \in X : \ m^+(C, x, x') \leq s \cdot m^-(C, x, x') \quad \forall\, x' \in X\},\ s \in I_{m-1}.$$

Evidently, for any $C \in \mathbf{R}_+^{m\times n}$ and $m \geq 2$, we have

$$M^m(C) = N_1^m(C) \subseteq N_2^m(C) \subseteq ... \subseteq N_{m-1}^m(C) = P^m(C) \neq \emptyset$$

with $M^2(C) = P^2(C)$.

The parameter s controls the ratio between the number of objectives with ”better” and ”worse” objective values in any pairwise comparison of a pair of solutions. The s-domination binary relation gives more freedom to the decision maker, since e.g. voting schemes can be represented by using this relation e.g. as in the following example.

Example 1. *Assume we have the following situation:*
$\diamond$ *members of a parliament (MPs) voting for a bill;*
$\diamond$ *C is a matrix of preferences, C_i defines individual preferences of i-th MP*
$\diamond$ *the number of objectives m represents the number of MPs.*
$\diamond$ *selection between two bills x and x^*;*
$\diamond$ *voting is without compromise, no option ”abstain” is available, i.e. $m^0(C, x, x') = 0$;*
$\diamond$ $m^+(C, x, x^*)$ *- amount of MPs voting ”for x^*”, i.e. voting ”against x”;*
$\diamond$ $m^-(C, x, x^*)$ *- amount of MPs voting ”against x^*”, i.e. voting ”for x”.*
The situation, when the simple majority of votes (a half plus one vote) is needed to pass the bill x^*, can be described by means of 1-domination binary relation: $x \prec_1 x^*$, i.e. the Condorcet optimality principle holds. If we assume that one bill is preferred over the other if $2/3$ of the total amount of MPs plus one vote for

that bill, then a situation when the bill x^* passes can be described by means of 2-domination binary relation: $x \prec_2 x^*$. In other words the parameterized optimality principle is in use. Thus by means of introducing integer parameter s, a decision maker could define the so-called super majority or a qualified majority principles which guarantee for a proposal to gain a specified level or type of support which exceeds a simple majority in order to have effect. In some cases, for example, parliamentary procedure requires that any action that may alter the rights of the minority or constitutional regulations has a super majority requirement (e.g. two-thirds, three-fourth, four-fifth majority etc). In case if super majority ratio results in fractional number of votes the last is rounded off to the minimal larger integer, otherwise one voice must be added to guarantee that the majority is clear in terms of votes. This ends the example.

In what follows we will assume that the one or several s-optimal solutions have been detected, and concentrate on analyzing some quality measures of these solutions with respect to small perturbations of the original matrix of coefficients.

4 A Boolean linear programming problem

Now we would like to present the problem specific results, assuming that individual cost functions are defined as linear functions, i.e.:

$$f_i(C, x) := C_i x.$$

Here C_i is i-th row of matrix $C = [c_{ij}] \in \mathbf{R}_+^{m \times n}$.

Assume also that the decision variables are binary, i.e. that $X \subseteq 2^{\{0,1\}^n} \backslash \emptyset,\ |X| \geq 2,$ is a finite set of feasible solutions. We call the problem of finding $N_s^m(C)$, defined in the previous section, an m**-criteria Boolean linear programming problem**.

For any two fixed solutions $x^* \in N_s^m(C)$ and $x \in X$, put the deviation measure

$$\gamma_i(C, x^*, x) := f_i(C, x^*) - f_i(C, x) = C_i(x^* - x), \ \ i \in I_m.$$

Let us arrange all numbers $\gamma_i(C, x^*, x)$ in non-decreasing order p :

$$\gamma_{p_1}(C, x^*, x) \leq \gamma_{p_2}(C, x^*, x) \leq ... \leq \gamma_{p_m}(C, x^*, x). \tag{9}$$

W.l.o.g. we may assume that all inequalities in (9) are strict. When coefficients of objective functions change, then initially s-optimal solution may lose its optimality. We will evaluate the quality of this solution from the point of view of its robustness on possible data perturbations. Namely, for a given matrix $C \in \mathbf{R}_+^{m \times n}$ and $x^* \in N_s^m(C^0)$, we introduce the so-called **absolute error** (see also the previous section) of this solution:

$$a^m(C, x^*) := \max_{x \in X} \gamma_{p_k}(C, x^*, x) = \max_{x \in X} \{f_{p_k}(C, x^*) - f_{p_k}(C, x)\} = \max_{x \in X} \{C_{p_k}(x^* - x)\}, \tag{10}$$

where $k = \lceil \frac{m-1}{s+1} \rceil$ is the least integer no less than $\frac{m-1}{s+1}$. Note that $a^m(C, x^*) \geq 0$ for all $C \in \mathbf{R}_+^{m \times n}$ and $x^* \in N_s^m(C^0)$. The k-th element in ordering p serves as major indicator of s-optimality, which helps to determine whether x^* remains s-optimal for matrix C or not.

Obviously if $s = m - 1$, then $k = 1$ and (10) transforms into

$$a^{m-1}(C, x^*) = \gamma_{p_1}(C, x^*, x),$$

where p is objective ordering according to (9), i.e.

$$a^{m-1}(C, x^*) = \max_{x \in X} \min_{i \in I_m} \{f_i(C, x^*) - f_i(C, x)\}, \tag{11}$$

whose relative analog was previously known in [12]. In the scalar case, i.e. for $m = 1$, the Pareto set transforms into the set of optimal solutions. Therefore the absolute error $a^1(C, x^*)$ converts into (2).

The use of relative error is evidently advantageous to the use of simple absolute error, since the deviation from the optimal solution is measured taking into account cost function ranges. However if the objectives are normalized (i.e. their ranges are already counted) the usage of absolute error is computationally more reasonable.

In the scalar case the equality $a^1(C, x^*) = 0$ gives necessary and sufficient conditions that $x^* \in N_s^1(C)$. But in the multicriteria case the situation is a bit different.

Lemma 1 *If $x^* \in N_s^m(C)$, then $a^m(C, x^*) = 0$ for any $C \in \mathbf{R}_+^{m \times n}$.*

Indeed, for arbitrary $C \in \mathbf{R}_+^{m \times n}$ and $x^* \in N_s^m(C)$, we have $a^m(C, x^*) \geq 0$. If $a^m(C, x^*) > 0$, then $C_{p_k}x^* - C_{p_k}x > 0$. Due to ordering (9), we obtain $C_{p_i}x^* - C_{p_i}x > 0$ for all $i \in I_m \backslash I_k$, i.e. $m^+(C, x^*, x) > s \cdot m^-(C, x^*, x)$. The last implies that $x^* \notin N_s^m(C)$.

The equality $a^m(C, x^*) = 0$ formulates in general only necessary condition for x^* to be s-optimal, i.e. $a^m(C, x^*) = 0$ does not guarantee that $x^* \in N_s^m(C)$. Indeed, consider the following example.

Example 2. *Let $m = 4$, $n = 2$, $s = 2$, and*

$$C^1 = \begin{pmatrix} 2 & 1 \\ 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Assume also that $X = \{x^1, x^2\}$, $x^1 = (1, 0)^T$, $x^2 = (0, 1)^T$. Then

$$f(C^1, x^1) = (2, 2, 2, 1)^T, \ f(C^1, x^2) = (1, 2, 2, 2)^T,$$

$$m^+(C^1, x^1, x^2) = 1,\ m^0(C^1, x^1, x^2) = 2,\ m^-(C^1, x^1, x^2) = 1,$$

i.e. $N_2^4(C^1) = \{x^1, x^2\}$*. Moreover, for* $x^2 \in N_2^4(C^1)$ *we calculate*

$$\gamma_1(C^1, x^2, x^1) = -1,\ \gamma_2(C^1, x^2, x^1) = 0,\ \gamma_3(C^1, x^2, x^1) = 0,\ \gamma_4(C^1, x^2, x^1) = 1,$$

$p = (1, 2, 3, 4)$*,* $k = 1$*,* $p_1 = 1$*, and* $a^4(C^1, x^2) = 0$*.*

If we consider matrix

$$C^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix},$$

then

$$f(C^2, x^1) = (2, 1, 1, 1)^T,\ f(C^2, x^2) = (1, 2, 2, 2)^T,$$

$$m^+(C^2, x^1, x^2) = 1,\ m^0(C^2, x^1, x^2) = 0,\ m^-(C^2, x^1, x^2) = 3,$$

i.e. $N_2^4(C^2) = \{x^1\}$*. Then for* $x^2 \notin N_2^4(C^2)$ *we calculate*

$$\gamma_1(C^2, x^2, x^1) = -1,\ \gamma_2(C^2, x^2, x^1) = 1,\ \gamma_3(C^2, x^2, x^1) = 1,\ \gamma_4(C^2, x^2, x^1) = 1,$$

$p = (1, 2, 3, 4)$*,* $k = 1$*,* $p_1 = 1$*, and* $a^4(C^2, x^2) = 0$*. This ends the example.*

But later we will show, that if the equality $a(C, x^*) = 0$ is valid for any matrix in some open neighborhood of C^0, i.e. there is $\phi > 0$ such that $a(C, x^*) = 0$ for any C, $\| C - C^0 \| < \phi$, where $\| \cdot \|$ denotes a norm in $\mathbf{R}^{m \times n}$, then this equality provides also a sufficient condition for the solution x^* to belong $N_s^m(C^0)$.

Now assume again that the set of feasible solutions X is fixed, but the matrix of input data C may vary or be estimated with errors. Moreover, we also assume that for some originally specified matrix $C^0 = [c_{ij}^0] \in \mathbf{R}_+^{m \times n}$ we know an s-optimal solution x^* which is an element of the set of s-optimal solutions $N_s^m(C^0)$.

Let

$$\Omega'_\rho(C^0) := \{C \in \mathbf{R}_+^{m \times n} :\ |c_{ij} - c_{ij}^0| < \rho,\ i \in I_m,\ j \in I_n\},$$

$$\Theta'_\delta(C^0) := \{C \in \mathbf{R}_+^{m \times n} :\ |c_{ij} - c_{ij}^0| < \delta \cdot c_{ij}^0,\ i \in I_m,\ j \in I_m\}.$$

Note that $\mathrm{cl}(\Omega'_\rho(C^0)) = \Omega_\rho(C^0)$ as well as $\mathrm{cl}(\Theta'_\rho(C^0)) = \Theta_\rho(C^0)$, where $\Omega_\rho(C^0)$ and $\Theta_\rho(C^0)$ defined in (3) and (4), respectively.

Proposition 1 *For* $x^* \in N_s^m(C^0)$*,* $s \in I_{m-1}$*, and* $\rho \in [0, q(C^0))$*, we have* $x^* \in N_s^m(C)$ *for any* $C \in \Omega'_\rho(C^0)$ *if and only if* $S(C^0, x^*, \rho) = 0$*.*

Proof. *Necessity.* Let $\rho \in [0, q(C^0))$. If $x^* \in N_s^m(C)$, $s \in I_{m-1}$ for some $C \in \Omega'_\rho(C^0)$, then directly from the definition of $N_s^m(C)$, we have that

$$m^+(C, x^*, x) \leq s \cdot m^-(C, x^*, x)\ \ \forall x \in X$$

holds.

Using (5) we deduce

$$m \leq (s+1)\cdot m^-(C,x^*,x) + m^0(C,x^*,x) \ \ \forall x \in X,$$

then

$$m-1 < (s+1)\cdot \Big(m^-(C,x^*,x) + m^0(C,x^*,x)\Big) \ \ \forall x \in X,$$

and finally

$$\frac{m-1}{s+1} < m^-(C,x^*,x) + m^0(C,x^*,x)$$

i.e.

$$f_{p_k}(C,x^*) - f_{p_k}(C,x) \leq 0 \ \ \forall x \in X,$$

where $k = \lceil \frac{m-1}{s+1} \rceil$. Then $a^m(C,x^*) = 0$ for $C \in \Omega'_\rho(C^0)$. Due to the arbitrary choice of matrix C, what has been proven above is valid for any $C \in \Omega'_\rho(C^0)$.

Consider now the case $C \in \Omega_\rho(C^0)\backslash\Omega'_\rho(C^0)$. Let us show that in this case, even if x^* loses s-optimality for such matrix C, the absolute error $a^m(C,x^*)$ is still equal to 0. Indeed, - due to continuity of objectives as linear functions - we get

$$f_{p_k}(C,x^*) - f_{p_k}(C,x) = 0 \ \ \forall x \in X.$$

Thus, $a^m(C,x^*) = 0$ for $C \in \Omega_\rho(C^0)\backslash\Omega'_\rho(C^0)$. Again, due to the arbitrary choice of matrix C, what has been proven above is valid for any $C \in \Omega_\rho(C^0)\backslash\Omega'_\rho(C^0)$. Combining the two cases, we get $a^m(C,x^*) = 0$ for any $C \in \Omega_\rho(C^0)$. The last means $S(C^0,x^*,\rho) = 0$.

Sufficiency. To prove that for $\rho \in [0, q(C^0))$, $S(C^0,x^*,\rho) = 0$ implies that $x^* \in N^m_s(C)$ for any $C \in \Omega'_\rho(C^0)$, suppose that $S(C^0,x^*,\rho) = 0$, but there exists a matrix $C' \in \Omega'_\rho(C^0)$, such that $x^* \notin N^m_s(C')$. We will show that such assumption must lead to a contradiction. Indeed, $x^* \notin N^m_s(C')$ means that there exists $\hat{x} \in X$ such that

$$m^+(C',x^*,\hat{x}) > s\cdot m^-(C',x^*,\hat{x}),$$

i.e.

$$f_{p_k}(C',x^*) - f_{p_k}(C',\hat{x}) \geq 0.$$

Consider matrix $\tilde{C}' \in \mathbf{R}^{m\times n}_+$ with elements

$$\tilde{c}'_{ij} = \begin{cases} c'_{ij} - \phi \text{ if } \ i = p_k, \ x^*_j = 0, \\ c'_{ij} + \phi \text{ if } \ i = p_k, \ x^*_j = 1, \\ c'_{ij} \text{ otherwise,} \end{cases} \tag{12}$$

where $\phi > 0$ is taken small enough to satisfy $\tilde{C}' \in \Omega'_\rho(C^0)$ and not violate the ordering p_k. Now it is easy to see that

$$f_{p_k}(\tilde{C}',x^*) - f_{p_k}(\tilde{C}',\hat{x}) > 0,$$

i.e. $a(\tilde{C}', x^*) > 0$, that implies $S(C^0, x^*, \rho) > 0$. Thus we have a contradiction which completes the proof.

Proposition 2 *For $x^* \in N_s^m(C^0)$, $s \in I_{m-1}$, and $\delta \in [0,1)$, we have $x^* \in N_s^m(C)$ for any $C \in \Theta'_\delta(C^0)$ if and only if $A(C^0, x^*, \delta) = 0$.*

The proof of proposition 2 is analogous to the proof of proposition 1.

Thus, this positive value of the absolute error may be treated as a measure of inefficiency of x^*.

For any $z \in \mathbf{R}^m$, we denote two norms: **linear norm** $\| z \|_1$ and **Chebyshev norm** $\| z \|_\infty$.

The following statements are true for any vectors $z, z' \in \{0,1\}^n,\ c \in \mathbf{R}^n$:

$$|\langle c, z\rangle| \leq \| c \|_\infty \cdot \| z \|_1, \tag{13}$$

$$\| z - z' \|_1 = \| z \|_1 + \| z' \|_1 - 2\langle z, z'\rangle, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of two vectors. Note, that the left-hand side of equality (14) is the Hamming distance between Boolean vectors z and z'. It is easy to prove equality (14) using induction (on the number n) [9].

For any two $x, x^* \in X$ denote

$$\Delta(x^*, x) := \| x - x^* \|_1 = \| x \|_1 + \| x^* \|_1 - 2\langle x, x^*\rangle.$$

The following theorem gives a formula for calculating value of the stability function.

Theorem 1 *For $x^* \in N_s^m(C^0)$ and $\rho \in [0, q(C^0))$, the stability function can be expressed by the formula:*

$$S(C^0, x^*, \rho) = \max_{x \in X} \{C^0_{p^0_k}(x^* - x) + \rho\Delta(x^*, x)\}, \tag{15}$$

where $p^0 = (p^0_1, ..., p^0_m)$ is objective ordering according to (9) specified for each $x \in X$ and original matrix C^0.

Proof. Let $\Gamma(C^0, x^*, \rho)$ be the right-hand side of (15). We yield

$$S(C^0, x^*, \rho) = \max_{C \in \Omega_\rho(C^0)} a(C, x^*) = \max_{C \in \Omega_\rho(C^0)} \max_{x \in X} \{C_{p_k}(x^* - x)\} =$$

$$\max_{x \in X} \max_{C \in \Omega_\rho(C^0)} \{C_{p_k}(x^* - x)\}.$$

Note that the reordering of the two maximums is possible since X is finite and $\Omega_\rho(C^0)$ is compact.

For any fixed $x \in X$, the maximum $C_{p_k}(x^* - x)$ over $C \in \Omega_\rho(C^0)$ is attained when

$$c^*_{ij} = \begin{cases} c^0_{ij} - \rho \text{ if } x^*_j = 0,\ i \geq p^0_k, \\ c^0_{ij} + \rho \text{ if } x^*_j = 1,\ i \geq p^0_k, \\ c^0_{ij} \text{ otherwise.} \end{cases} \tag{16}$$

Obviously, $C^* \in \Omega_\rho(C^0)$. Then, taking into account that $0 \le \rho < q(C^0)$, we continue

$$\max_{x \in X} \{C^*_{p_k}(x^* - x)\} = \max_{x \in X} \{C^0_{p^0_k}(x^* - x) + \rho\Delta(x^*, x)\} = \Gamma(C^0, x^*, \rho).$$

This completes the proof.

As a corollary from theorem 1, we get the following results concerning Pareto optimality (c.f. [12]):

Corollary 1 *The stability function of $x^* \in P^m(C^0)$ can be expressed by the formula*

$$S(C^0, x^*, \rho) = \max_{x \in X} \min_{i \in I_m} \{C^0_i(x^* - x) + \rho\Delta(x^*, x)\}. \tag{17}$$

To formulate the further results we need the following definitions. For any two solutions $x^* \in N^m_s(C)$ and $x \in X,\ x \neq x^*$ put

$$q_i(C, x^*, x) := f_i(C, x) - f_i(C, x^*) = C_i(x - x^*),\ \ i \in I_m.$$

Let us arrange all numbers $q_i(C, x^*, x)$ in non-increasing order r:

$$q_{r_1}(C, x^*, x) \ge q_{r_2}(C, x^*, x) \ge ... \ge q_{r_m}(C, x^*, x). \tag{18}$$

Note that ordering (18) is identical to ordering (9), i.e. $(p_1, ..., p_m) = (r_1, ..., r_m)$. Recall, that w.l.o.g. we assumed that all inequalities in (9) are strict, and hence all inequalities in (18) are also strict.

To prove the some further statements we will need the following fact

Lemma 2 *Let $p = (p_1, p_2, ..., p_m)$ be the objective ordering specified by (9) and $r = (r_1, r_2, ..., r_m)$ be the objective ordering specified by (18), both are specified for the original matrix C^0. Assume also that $x^* \in N^m_s(C^0),\ x \in X$ and $\rho > 0$. Then the inequality*

$$\max_{x \in X} \{C^0_{p^0_k}(x^* - x) + \rho\Delta(x^*, x)\} > 0$$

is valid if and only if

$$\rho > \min_{x \in X \setminus \{x^*\}} \frac{C^0_{r^0_k}(x - x^*)}{\Delta(x^*, x)}$$

holds.

Proof. *Necessity*. Suppose that

$$\max_{x \in X} \{C^0_{p^0_k}(x^* - x) + \rho\Delta(x^*, x)\} > 0$$

is true. We will prove the necessity by contradiction. Assume that

$$\rho \le \min_{x \in X \setminus \{x^*\}} \frac{C^0_{r^0_k}(x - x^*)}{\Delta(x^*, x)},$$

i.e.

$$\rho \le \frac{C^0_{r^0_k}(x-x^*)}{\Delta(x^*,x)} \le \frac{C^0_{r^0_{k-1}}(x-x^*)}{\Delta(x^*,x)} \le ... \le \frac{C^0_{r^0_1}(x-x^*)}{\Delta(x^*,x)} \quad \forall x \in X\backslash\{x^*\},$$

i.e. due to $\Delta(x^*,x) > 0$ for any $x \in X\backslash\{x^*\}$, we obtain

$$0 \ge C^0_{r^0_k}(x^*-x)+\rho\Delta(x^*,x) \ge C^0_{r^0_{k-1}}(x^*-x)+\rho\Delta(x^*,x) \ge ... \ge C^0_{r^0_1}(x^*-x)+\rho\Delta(x^*,x)$$

for every $x \in X\backslash\{x^*\}$.

Now recall that orderings r^0 and p^0 are identical, and for ordering p^0 it is true that

$$C^0_{p^0_1}(x^*-x) \le ... \le C^0_{p^0_k}(x^*-x) \le ... \le C^0_{p^0_m}(x^*-x),$$

i.e. since $\rho > 0$ and $\Delta(x^*,x) > 0$ for any $x \in X\backslash\{x^*\}$, we get

$$C^0_{p^0_1}(x^*-x)+\rho\Delta(x^*,x) \le ... \le C^0_{p^0_k}(x^*-x)+\rho\Delta(x^*,x) \le ... \le C^0_{p^0_m}(x^*-x)+\rho\Delta(x^*,x)$$

for every $x \in X\backslash\{x^*\}$.

Since $\rho \in [0, q(C^0))$, we get

$$\max_{x\in X} C^0_{p^0_k}(x^*-x)+\rho\Delta(x^*,x) = 0.$$

This contradiction ends the first part of the proof.

Sufficiency. Now we assume that

$$\rho > \min_{x\in X\backslash\{x^*\}} \frac{C^0_{p^0_k}(x-x^*)}{\Delta(x^*,x)}.$$

We will prove the sufficiency by contradiction again. Suppose that

$$\max_{x\in X}\{C^0_{r^0_k}(x^*-x)+\rho\Delta(x^*,x)\} \le 0,$$

i.e.

$$C^0_{r^0_1}(x^*-x)+\rho\Delta(x^*,x) \le ... \le C^0_{r^0_k}(x^*-x)+\rho\Delta(x^*,x) \le 0, \quad \forall x \in X\backslash\{x^*\}.$$

Since $\rho \in [0, q(C^0))$, we get

$$C^0_{r^0_k}(x^*-x)+\rho\Delta(x^*,x) \le 0, \quad \forall x \in X\backslash\{x^*\}.$$

and, hence, because $\Delta(x^*,x) > 0$ $\forall x \in X\backslash\{x^*\}$, and orderings r^0 and p^0 are identical, we derive

$$\rho \le \frac{C^0_{p^0_i}(x-x^*)}{\Delta(x^*,x)}, \; i \in I_k \;\; \forall x \in X\backslash\{x^*\}.$$

The last implies that

$$\rho \le \min_{x\in X\backslash\{x^*\}} \frac{C^0_{p^0_k}(x-x^*)}{\Delta(x^*,x)}.$$

The obtained contradiction ends the proof of the second part and either completes the entire proof of the lemma.

Theorem 2 *For $x^* \in N_s^m(C^0)$, the stability radius can be expressed by the formula:*

$$R^S(C^0, x^*) = \min\left\{q(C^0), \min_{x \in X \setminus \{x^*\}} \frac{C^0_{r^0_k}(x - x^*)}{\Delta(x^*, x)}\right\}, \tag{19}$$

where $k = \lceil \frac{m-1}{s+1} \rceil$ is the least integer no less than $\frac{m-1}{s+1}$ and $(r_1^0, r_2^0, ..., r_m^0)$ is objective ordering according to (18).

Proof. If $\rho = 0$, then $S(C^0, x^*, 0) = 0$. Now consider non-trivial case $\rho > 0$. Assume $S(C^0, x^*, \rho) > 0$. Using formula (15) specified by theorem 1, we derive that $S(C^0, x^*, \rho) > 0$ if and only if

$$\max_{x \in X} \frac{C^0_{p^0_k}(x^* - x) + \rho\Delta(x^*, x)}{C^0_{p^0_k} x - \rho \parallel x \parallel_1} > 0,$$

where $p = (p_1^0, p_2^0, ..., p_m^0)$ is objective ordering specified by (9) for the original matrix C^0.

Due to lemma 2, the last inequality holds if and only if

$$\rho > \tilde{\rho} := \min_{x \in X \setminus \{x^*\}} \frac{C^0_{r^0_k}(x - x^*)}{\Delta(x^*, x)}.$$

Thus, if $\tilde{\rho} \leq q(C^0)$, then we get that $S(C^0, x^*, \rho) = 0$ on interval $[0, \tilde{\rho})$. Otherwise the stability function is equal to zero on $[0, q(C^0))$. This ends the proof.

As corollaries from theorem 2, we get the following well-known results (c.f. [2], [12]):

Corollary 2 *The stability radius of $x^* \in P^m(C^0)$ can be expressed by the formula*

$$R^S(C^0, x^*) = \min\left\{q(C^0), \min_{x \in X \setminus \{x^*\}} \max_{i \in I_m} \frac{C^0_i(x - x^*)}{\Delta(x^*, x)}\right\}.$$

Corollary 3 *The stability radius of $x^* \in M^m(C^0)$ can be expressed by the formula*

$$R^S(C^0, x^*) = \min\left\{q(C^0), \min_{x \in X \setminus \{x^*\}} \frac{C^0_{p^0_q}(x - x^*)}{\Delta(x^*, x)}\right\},$$

where $q = \lfloor \frac{m+1}{2} \rfloor$ is an integer part of $\frac{m+1}{2}$ and $p^0 = (p_1^0, ..., p_m^0)$ is objective ordering according to (9).

The following theorems, which give formulae for calculating values of the accuracy function and accuracy radius, can be proven by analogy with the proofs of theorem 1 and 2.

Theorem 3 *For $x^* \in N_s^m(C^0)$ and $\delta \in [0,1)$, the accuracy function can be expressed by the formula:*

$$A(C^0, x^*, \delta) = \max_{x \in X} C^0_{p_k^0}(x^* - x) + \delta \sum_{j \in I_n} c^0_{p_k^0 j}|x_j^* - x_j|.$$

Theorem 4 *For $x^* \in N_s^m(C^0)$ and $\delta \in [0,1)$, the accuracy radius can be expressed by the formula:*

$$R^A(C^0, x^*, \delta) = \min\Big\{1, \min_{x \in X \setminus \{x^*\}} \frac{C^0_{r_k^0}(x - x^*)}{\sum\limits_{j \in I_n} c^0_{r_k j}|x_j - x_j^*|}\Big\}.$$

5 Example

Consider the following example. Assume we have a group of people $X = \{x^1, x^2, x^3\}$ belonging to some community or organization. Assume also that there is a decision committee consisting of $m = 5$ persons making an assessment of activity within the group members and deciding whether to continue or stop membership of every person in the group based on $n = 3$ evaluation criteria. For every person $x^h = (x_1^h, x_2^h, x_3^h)^T,\ h \in I_3$, $x_j^h = 1$ if x_j^h does not satisfy j-th evaluation criterion, and $x_j^h = 0$ otherwise. Let $x^1 = (0,0,1)^T$, $x^2 = (0,1,0)^T$ and $x^3 = (1,0,0)^T$. Each i-th member of the decision committee ($i \in I_m$) has own preferences according to the importance of evaluation criteria, and hence own penalty costs c_{ij}^0 if some person does not meet j-th evaluation criterion. Then $C^0 \in \mathbf{R}_+^{m \times n}$ defines the penalty cost matrix:

$$C^0 = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 3 & 4 \\ 3 & 2 & 3 \\ 3 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

Thus, the penalty costs received by every group member from the decision committee are the following:

$$f(C^0, x^1) = (6,4,3,5,6)^T,\ f(C^0, x^2) = (2,3,2,3,4)^T,\ f(C^0, x^3) = (1,3,3,3,2)^T.$$

The assessment is going according to the following rules. Every member of the decision committee will independently evaluate activity of all persons in the organization by specifying some penalty costs. The less satisfactory performance is demonstrated by the persons the higher penalty cost may be given to them. As a result of the entire assessment, a penalty cost vector is associated to each person. Each component of the vector represents the penalty cost received by this person from the corresponding member of the decision committee. Any person

Table 1: The objective ordering p for different pairs x^* and x.

x^*	x	p
x^2	x^1	$(1,4,5,2,3)$
x^2	x^3	$(3,4,2,1,5)$
x^3	x^1	$(1,5,4,2,3)$
x^3	x^2	$(5,1,2,4,3)$

can be excluded from the organization if there is another person in the organization with better (in some sense) performance. Otherwise, the person will keep the membership within the organization. The pairwise comparison and establishing domination relations between the penalty vectors are used to decide whose performance was better. One penalty vector will dominate the other one in sense of s-domination relations (8), where $s = 1, 2, 3, 4$. The person with non-dominated penalty vector will keep the membership otherwise not.

If the penalty cost matrix is subject to uncertainty which may be the case when the members of the decision group are not absolutely sure about exact values of penalty costs that should be given, then there is a risk for some people to loose their membership if some small changes in penalty costs are done. We preliminary calculate

$$\Delta(x^1, x^2) = \Delta(x^2, x^3) = \Delta(x^1, x^3) = 2.$$

When we compare different solutions we get various objective orderings p specified in Table 1.

To estimate this risk we evaluate the value of stability functions according to (15).

$s = 1$

$$N_1^5(C^0) = M_1^5(C^0) = \{x^3\},\ k = 2,$$

$$S(C^0, x^3, \rho) = \max\Big\{0, -1 + 2\rho, -4 + 2\rho\Big\},\ \rho \in [0, 1);$$

$s = 2$

$$N_2^5(C^0) = \{x^2,\ x^3\},\ k = 2,$$

$$S(C^0, x^2, \rho) = \max\Big\{0, 2\rho, -2 + 2\rho\Big\},\ \rho \in [0, 1);$$

$$S(C^0, x^3, \rho) = \max\Big\{0, -1 + 2\rho, -4 + 2\rho\Big\},\ \rho \in [0, 1);$$

$s = 3$

$$N_3^5(C^0) = \{x^2,\ x^3\},\ k = 1,$$

$$S(C^0, x^2, \rho) = \max\Big\{0, -1 + 2\rho, -4 + 2\rho\Big\},\ \rho \in [0, 1);$$

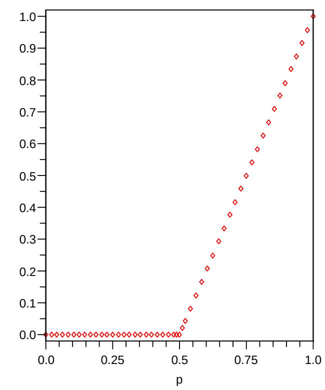

Figure 1: $s = 1$: Stability function $S(C^0, x^3, \rho)$ for $\rho \in [0, 1)$.

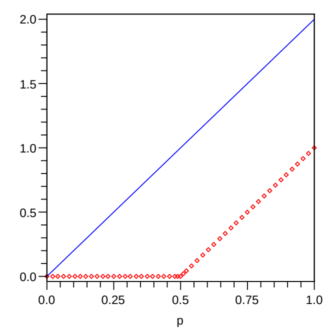

Figure 2: $s = 2$: Stability functions $S(C^0, x^2, \rho)$ and $S(C^0, x^3, \rho)$ for $\rho \in [0, 1)$.

$$S(C^0, x^3, \rho) = \max\Big\{0, -2 + 2\rho, -5 + 2\rho\Big\},\ \rho \in [0, 1);$$

$s = 4$

$$N_4^5(C^0) = N_4^5(C^0) = \{x^2,\ x^3\},\ k = 1,$$

$$S(C^0, x^2, \rho) = \max\Big\{0, -1 + 2\rho, -4 + 2\rho\Big\},\ \rho \in [0, 1);$$

$$S(C^0, x^3, \rho) = \max\Big\{0, -2 + 2\rho, -5 + 2\rho\Big\},\ \rho \in [0, 1).$$

Graphics for $S(C^0, x^2, \rho)$ (continuous line) and $S(C^0, x^3, \rho)$ (dotted line) are depicted in Figures 1 – 3. Now let us make a short analysis of the results.

$s = 1$ There is only one 1-optimal solution x^3. Its stability function is depicted on Figure 1. This solution is stable for $0 \leq \rho \leq 0.5$.

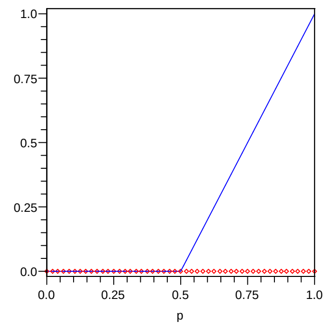

Figure 3: $s = 3, 4$: Stability functions $S(C^0, x^2, \rho)$ and $S(C^0, x^3, \rho)$ for $\rho \in [0, 1)$.

$s = 2$ As one can see from the graphics (see Figure 2), x^2 is non-stable for $0 < \rho \leq 1$, and x^3 is stable for $0 \leq \rho \leq 0.5$. Moreover, for $0 \leq \rho < 1$, $S(C^0, x^2, \rho) \geq S(C^0, x^3, \rho)$. So, we may conclude that x^3 is more preferable under the possible uncertainty of penalty cost matrix C^0.

$s = 3, 4$ As one can see from the graphics (see Figure 3), x^2 is stable for $0 \leq \rho \leq 0.5$, and x^3 is stable for $0 \leq \rho < 1$. Moreover, for $0 \leq \rho < 1$, $S(C^0, x^2, \rho) \geq S(C^0, x^3, \rho)$. So, we may conclude that x^3 is more preferable under the possible uncertainty of penalty cost matrix C^0.

6 Conclusions

The example in previous section suggests that small changes or inaccuracies in estimating objective function coefficients may have significant influence on the set of s-optimal solutions. Moreover, some solutions being initially optimal, cannot be considered as stable, because very small changes of input data destroy their properties of being optimal.

The simplest measure of the stability of the optimum is its stability or accuracy radius. But frequently, this measure is not sufficient to rank the solutions, among multiple optimal solutions, which so often occur in multicriteria optimization. Therefore, calculating stability radii only may be not sufficient to make a conclusion about solution stability, so it is necessary to calculate some complementary measures reflecting more information about solution behavior under uncertainty.

The accuracy and stability functions describe the quality of s-optimal solutions in the problem with uncertain coefficients of objective functions. The definitions of these functions are directly connected with given optimality principle. Most common optimality principles in voting theory, as Pareto and Condorcet optimality principles, may not fully cover all of the decision maker preferences. Sometimes, introducing a parameterized version of optimality principles may reflect the desirable preference specific much better.

As possible continuation of the research within this topic, it would be interesting to study whether the approach presented in this paper may be extended for the case of non-linear objective functions and non-boolean decision variables. Finding efficient strategies to compute stability and accuracy functions for the problems with larger dimension may be also promising to study. Analytical expressions obtained in the present paper imply full enumeration of the set of feasible solutions whose cardinality may depend exponentially on n. So, these formulae do not lead to the efficient (polynomial) way of calculating the values of stability and accuracy functions. The main focus could be given to calculating reasonable lower and upper bounds as well as getting good quality approximation by means of ad-hoc heuristics. One more research possibility is to study the properties of stability and accuracy functions for answering questions ”is this function concave?”; ”is the number of function segments polynomial with respect to n?”; ”is this function piece-wise linear?” etc. Having this questions answered positively, one can exploit these facts to design an efficient computational procedure similar to [3].

References

[1] **K. J. Arrow**, *Social choice and individual values*, John Wiley, New York, 1963.

[2] **S. Bukhtoyarov, V. Emelichev and Y. Stepanishina**, (2006). ”Stability of Discrete Vector Problems with the Parametric Principle of Optimality”. *Cybernetics and systems analysis 39*, 604 – 614.

[3] **N. Chakravarti, A. Wagelmans,** ”Calculation of stability radius for combinatorial optimization problems”, *Operations Research Letters 23*, 1 – 7.

[4] **M. de Condorcet**, *Essai sur l’application de l’analyse la probabilit des décisions rendues la pluralité des voix (Essay on the Application of Analysis to the Probability of Majority Decisions)*, Paris, 1785.

[5] **M. Ehrgott**, *Multicriteria optimization*, Springer, Berlin, 2000.

[6] **V. Emelichev, E. Girlich, Y. Nikulin and D. Podkopaev**, (2002). ”Stability and regularization of vector problems of integer linear programming”. *Optimization 51*, 645 – 676.

[7] **H. Greenberg**, (1998). ”An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization.” In D. Woodruff (ed.) Advances in computational and stochastic optimization, Logic programming and heuristic search, 97 – 148. Dordrecht: Kluwer Academic Publishers.

[8] **J. Hadamard**, *Lecons sur la propagation des ondes et les equations de l'hydrodynamique (Lectures on the propagation of waves and the equations of hydrodynamic)*, Hermann, Paris, 1903.

[9] **R. Hamming**, *Coding and information theory*, Prentice Hall, 1980.

[10] **M. Libura**, (1999). "On accuracy of solution for combinatorial optimization problems with perturbed coefficients of the objective function". *Annals of Operations Research 86*, 53 – 62.

[11] **M. Libura**, (2000). "Quality of solutions for perturbed combinatorial optimization problems". *Control and Cybernetics 29*, 199 – 219.

[12] **M. Libura and Y. Nikulin**, (2006). "Stability and accuracy functions in multicriteria linear combinatorial optimization problems". *Annals of Operations Research 147*, 255 – 267.

[13] **Y. Nikulin**, (2008). "Stability and accuracy functions in a coalition game with bans, linear payoffs and antagonistic strategies". (to appear in *Annals of Operations Research*)

[14] **V. Pareto**, *Manuel d'ecoonomie politique (Manual of political economy)*, Qiard, Paris, 1909.

Turku Centre *for* Computer Science

Lemminkäisenkatu 14 A, 20520 Turku, Finland | www.tucs.fi

University of Turku
- Department of Information Technology
- Department of Mathematics

Åbo Akademi University
- Department of Computer Science
- Institute for Advanced Management Systems Research

Turku School of Economics and Business Administration
- Institute of Information Systems Sciences

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