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Turku Centre *for* Computer Science

TUCS Technical Report
No 931, February 2009

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Abstract

Various type of optimal solutions of multiobjective optimization problems can be characterized by means of different cones. We consider here five different optimality principles which are very common in multiobjective optimization: efficiency, weak and proper Pareto optimality, strong and lexicographic optimality. The five optimality concepts can be characterized with the help of different geometrical concepts. The usage of contingent cone, normal cone and cone of feasible directions is a natural choice in the case of convex optimization. In nonconvex case two additional types of cones are helpful - tangent cone and cone of local feasible directions. Provided the partial objectives are not necessarily convex, we derive necessary and sufficient geometrical optimality conditions for strongly efficient and lexicographically optimal solutions by using the above-mentioned cones. Combining new results with previously known ones about efficiency, weak and proper Pareto optimality, we derive two general schemes reflecting structural properties and interconnections of the five optimality principles.

Keywords: Multiple criteria, strong efficiency, lexicographic optimality, tangent cone, contingent cone, normal cone.

1 Introduction

The major goal in multiobjective optimization is to find a compromise between several conflicting objectives which is best-fit to the needs of a decision maker. This compromise is usually refereed to as an optimality principle. Various mathematical definitions of the optimality principle can be derived in several different ways depending on the needs of the solution approaches used. Moreover, sometimes the use of one definition may be advantageous to the other due to computational complexity reasons.

The usage of contingent cone, normal cone and cone of feasible directions is a natural choice in the case of convex optimization [16]. In nonconvex optimization two additional types of cones are proved to be helpful - tangent cone and cone of local feasible directions [2]. The guaranteed property of convexity of these cones assures that they can be efficiently used to overcome some difficulties which appear in nonconvex case. Provided the partial objectives are not necessarily convex, the new results concerning some structural properties of strongly efficient and lexicographically optimal solutions are obtained using the geometrical cone characterization approach. These results are combined with the results previously known for the sets of efficient, weakly and proper Pareto optimal solutions. As a result, we derive two general schemes reflecting structural properties and interconnections of five different optimality concepts: weakly and properly Pareto optimality, efficiency and strongly efficiency as well as lexicographic optimality. Notice that similar results were obtained earlier for convex case in [10].

A solution is Pareto optimal if improvement in some objectives can only be obtained at the expense of some other objective(s). This traditional concept is also known as efficiency, non-dominance or non-inferiority. It reflects the equilibrium situation for many problems in economics, sociology and game theory (see e.g. [1], [3], [11]). The set of weakly Pareto optimal solutions contains the Pareto optimal solutions together with solutions where all the objectives cannot be improved simultaneously.

On the other hand, Pareto optimal solutions can be divided into properly and improperly Pareto optimal ones. The set of improperly Pareto optimal solutions represents a set of efficient points with certain abnormal or irregular properties. To eliminate such anomalous efficient points, various concepts of Proper pareto optimality have been introduced in literature. Comprehensive analysis of various concepts of proper Pareto optimality can be found in e.g. [3], [11]. Henceforth we use only one of the concepts, which is according to Henig [5]. This concept uses a convex cone, which interior part must contain an inverse of standard ordering cone, to prohibit tradeoffs towards directions within the cone. It generalizes many well-known concepts of proper Pareto optimality and appears to be very useful in nonconvex optimization.

Strong efficiency is generally referred to the solutions which deliver optimality to each objective. Despite feasibility of such solutions is rare, they provide an important information on the lower bound for each objective in the Pareto optimal set. The most well-known application of the strong efficiency is the usage of the so-called ideal and utopian points as a reference point in compromise programming and other methods which main goal is to optimize distance measure from the reference point to the feasible set. Strong efficiency also plays a crucial role in many other multiobjective methods and algorithms (see e.g. [7]).

Lexicographic optimality principle is generally applied to the situation where objectives have no equal importance anymore but ordered according to their significance. A rigid arrangements of partial criteria with respect to importance is often used for a wide spectrum of important optimization problems, for example problems of stochastic programming, problems of axiomatic systems of utility theory and so on [3], [15]. Observe also that any scalar constrained optimization problem may be transformed to unconstrained bicriteria lexicographic problem by using as first criterion some exact penalty function for problem constraints, and an original objective function as a second constraint.

As it was already mentioned above, the five optimality concepts can be characterized with the help of different geometrical concepts. Sometimes, exploiting geometrical characterization may be advantageous to using straightforward definitions of optimality due to potential decrease of computational efforts needed. The choice of particular geometrical tool (which cone to use) first of all depends on what type of optimization problem - convex or nonconvex - we deal with. In this paper, we report about new results on characterization optimality for two well-known classes of optimality which are strong efficiency and lexicographic optimality. This will lead to a more global view at structural properties of five well-know optimality principles in nonconvex case. The results are summarized in two interconnected schemes. The results are compared to those obtained earlier for the convex optimization case.

In what follows, we introduce the problem formulation as well as some well-known results in Section 2. The new results concerning the set of strongly efficient solutions are given in Section 3. The lexicographic optimality is a subject of throughout research in Section 4. In section 5, we compare the similarity and difference of the results obtained in convex and nonconvex cases. The paper is concluded in Section 6.

2 Problem Formulation and Preliminaries

We consider general multiobjective optimization problems of the following form:

$$\min_{x\in S} \{f_1(x), f_2(x), \dots, f_k(x)\}, \tag{1}$$

with the continuous *objective functions* $f_i : \mathbf{R}^n \to \mathbf{R}$ for all $i \in I_k := \{1, \ldots, k\}$.

The *decision vector* x belongs to the nonempty *feasible set* $S \subset \mathbf{R}^n$. The image of the feasible set is denoted by $Z \subset \mathbf{R}^k$, i.e. $Z := f(S)$ and it is supposed not necessarily to be convex. Elements of Z are termed *objective vectors* and they are denoted by $z = f(x) = (f_1(x), f_2(x), \ldots, f_k(x))^T$. Additionally, we assume $f(B(x;\varepsilon))$ to be open for all $x \in S$ and $\varepsilon > 0$, where $B(x;\varepsilon)$ is *an open ball* with radius ε and center x.

The Problem (1) is said to be convex if the objective functions f_i for all $i \in I_k$ and the feasible set S are convex.

The sum of two sets A and E is defined by $A+E = \{a+e \mid a \in A,\ e \in E\}$. The interior, closure, convex hull and complement of a set A are denoted by int A cl A, conv A and A^C, respectively.

A set A is a *cone* if $\lambda x \in A$ whenever $x \in A$ and $\lambda > 0$. We denote the negative orthant of $\mathbf{R}^k$ by $\mathbf{R}^k_- = \{d \in \mathbf{R}^k \mid d_i \leq 0 \text{ for every } i \in I_k\}$. The positive orthant $\mathbf{R}^k_+$, the *standard ordering cone*, is defined respectively. Note, that $\mathbf{R}^k_-$ and $\mathbf{R}^k_+$ are closed convex cones.

In what follows, the notation $z < y$ for $z, y \in \mathbf{R}^k$ means that $z_i < y_i$ for every $i \in I_k$ and, correspondingly, $z \leq y$ stands for $z_i \leq y_i$ for every $i \in I_k$.

Simultaneous optimization of several objectives for multiobjective optimization problem is not a straightforward task. Contrary to the the single objective case, the concept of optimality is not unique in multiobjective cases.

Below we give traditional definitions of two well-known and most fundamental principles of optimality.

Weak Pareto optimality. An objective vector $z^* \in Z$ is *weakly Pareto optimal* if there does not exist another objective vector $z \in Z$ such that $z_i < z_i^*$ for all $i \in I_k$.

Pareto optimality or efficiency. An objective vector $z^* \in Z$ is *Pareto optimal* or *efficient* if there does not exist another objective vector $z \in Z$ such that $z_i \leq z_i^*$ for all $i \in I_k$ and $z_j < z_j^*$ for at least one index j.

Next we define the sets of globally weakly Pareto, Pareto and properly Pareto optimal solutions by using the opposite of the standard ordering cone. It is trivial to verify that the definitions of weak Pareto optimality and efficiency formulated above are equivalent to those following below.

Definition 1 *The globally weakly Pareto optimal set is*

$$GWP(Z) := \{z \in Z \mid (z + \text{int}\, \mathbf{R}^k_-) \cap Z = \emptyset\},$$

the globally Pareto optimal set is

$$GP(Z) := \{z \in Z \mid (z + \mathbf{R}^k_- \setminus \{0\}) \cap Z = \emptyset\},$$

and the globally properly Pareto optimal set is defined as

$$GPP(Z) := \{z \in Z \mid (z + C \setminus \{0\}) \cap Z = \emptyset\}$$

for some convex cone C such that $\mathbf{R}^k_- \setminus \{0\} \subset \text{int}\, C$.

The corresponding local concepts are defined in the following. Naturally, in a convex case, local and global concepts are equal.

Definition 2 *The locally weakly Pareto optimal set with $z = f(x) \in Z$ is given as*

$$LWP(Z) = \bigcup_{\delta>0} \Big\{ z \in Z \mid (z + int\, \mathbf{R}^k_-) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\},$$

the locally Pareto optimal set as

$$LP(Z) = \bigcup_{\delta>0} \Big\{ z \in Z \mid (z + \mathbf{R}^k_- \setminus \{0\}) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\},$$

and the locally properly Pareto optimal set as

$$LPP(Z) = \bigcup_{\delta>0} \Big\{ z \in Z \mid (z + C \setminus \{0\}) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\}$$

for some convex cone C such that $\mathbf{R}^k_- \setminus \{0\} \subset \text{int}\, C$.

Notice that the concept of proper Pareto optimality originates from the idea of prohibiting an unbounded trade-off between objectives but preserving the requirement of Pareto optimality. This limitation can be imposed either analytically or geometrically that will lead to slightly different concepts of proper Pareto optimality. We used the definition of global proper Pareto optimality given by Henig in [5], since his definition uses geometrical characterization with help of convex ordering cone. He also defined local proper Pareto optimality which differs from Definition 2. Our approach is motivated by analogy with the general treatment of local optimality [12].

Obviously we have the following relationships between the different grades of Pareto optimality – see Fig. 1.

Next we define several geometrical basic cones (see e.g. [16]).

Definition 3 *The contingent cone of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is defined as*

$$K_z(Z) := \{d \in \mathbf{R}^k \mid \textit{there exist } t_j \searrow 0 \textit{ and } d_j \to d \textit{ such that } z + t_j d_j \in Z\}.$$

The cone of globally feasible directions of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is denoted by

$$D_z(Z) := \{d \in \mathbf{R}^k \mid \textit{ there exists } t > 0 \textit{ such that } z + td \in Z\}.$$

$$
\begin{array}{ccccc}
GPP(Z) & \subset & GP(Z) & \subset & GWP(Z) \\
\cap & & \cap & & \cap \\
LPP(Z) & \subset & LP(Z) & \subset & LWP(Z)
\end{array}
$$

Figure 1: Collection of the relationships between local and global weak, proper Pareto optimality and efficiency.

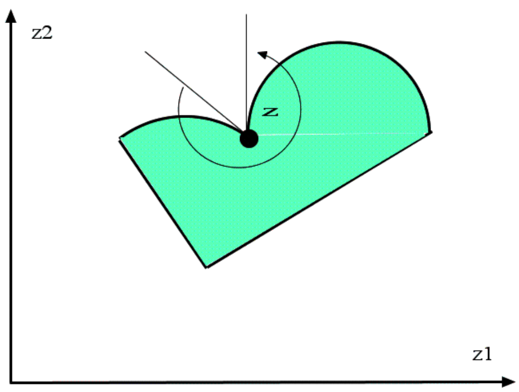

Figure 2: Nonconvex contingent cone $K_z(Z)$.

The definitions of contingent cones $K_z(Z)$ and cones of globally feasible directions $D_z(Z)$ are equally valid for both convex and nonconvex sets. Note, however, that the cone convexity, which holds for convex sets, is not guaranteed in nonconvex case (see Fig. 2).

In nonconvex case, the cone of feasible directions $D_z(Z)$ does not describe the shape of Z locally. Thus, we introduce a cone of locally feasible directions, which reflects the shape of Z locally.

Definition 4 *The cone of locally feasible directions of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is denoted by*

$$F_z(Z) = \{d \in \mathbf{R}^k \mid \text{ there exists } t > 0 \text{ such that } z + \tau d \in Z \text{ for all } \tau \in (0, t]\}.$$

The following definition provides local regularity condition for Z at $z \in Z$.

Definition 5 *The set Z is called locally regular at $z \in Z$ if $F_z(Z) = K_z(Z)$.*

Note that, under convexity assumption, for any $z \in Z$ we have cl $F_z(Z) = K_z(Z)$ (see e.g. [17]), i.e. local regularity defines a bit stronger requirement on a local structure of a set than the convexity assumption.

For nonconvex cases, Clarke [2] has defined a convex tangent cone in the following way.

Definition 6 *The tangent cone of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is given by the formula*

$$T_z(Z) = \{d \in \mathbf{R}^k \mid \textit{for all } t_j \searrow 0 \textit{ and } z_j \to z \textit{ with } z_j \in Z, \\ \textit{there exists } d_j \to d \textit{ with } z_j + t_j d_j \in Z\}.$$

The following basic relations can be derived from the definitions of the concepts used and from [4], [9], [17].

Lemma 1 *For the cones $K_z(Z)$, $D_z(Z)$, $T_z(Z)$ and $F_z(Z)$ we have the following*

a) *$K_z(Z)$ and $T_z(Z)$ are closed and $T_z(Z)$ is convex.*

b) $0 \in K_z(Z) \cap D_z(Z) \cap T_z(Z) \cap F_z(Z)$.

c) $Z \subset z + D_z(Z)$.

d) $\mathrm{cl}\, F_z(Z) \subset K_z(Z) \subset \mathrm{cl}\, D_z(Z)$.

e) $T_z(Z) \subset K_z(Z) \subset \mathrm{cl}\, D_z(Z)$.

f) *If Z is convex, then* $\mathrm{cl}\, F_z(Z) = T_z(Z) = K_z(Z) = \mathrm{cl}\, D_z(Z)$.

Let us point out once again that contingent cones can be nonconvex in which case their polar cones are irrelevant, in other words, $K_z(Z)^\circ = \{0\}$ independently of Z.

Even though contingent cones are generally nonconvex, their convexity is guaranteed under special circumstances.

Definition 7 *The set Z is called tangentially regular at $z \in Z$ if $T_z(Z) = K_z(Z)$.*

Trivially, we can see that e.g. convex sets are always tangentially regular.

The *normal cone* of Z at $z \in Z$ is the polar cone of the tangent cone, that is,

$$N_z(Z) := T_z(Z)^\circ = \{y \in \mathbf{R}^k \mid y^T d \leq 0 \text{ for all } d \in T_z(Z)\}.$$

Due to polarity and tangent cone convexity, the cone $N_z(Z)$ is always convex and contains zero.

The results related to the three optimality concepts (efficiency, weak and proper Pareto optimality) and different cones in case Z is nonconvex are collected in Fig. 3 (for details and proofs see [12]), where symbol $*$ denotes those cases for which tangent regularity must be held.

To make the global picture of properties depicted in Fig. 3 even more complete, we present here two more results related to global proper and local Pareto optimality, which were not mentioned in [12].

The first results specifies the necessary and sufficient condition for global proper Pareto optimality.

$$
\begin{array}{ccccccc}
 & & & & & & N_z(Z) \cap \text{int } \mathbf{R}^k_- \neq \emptyset \\
 & & & & & & \Updownarrow_* \\
 & & z \in GPP(Z) & \Rightarrow & z \in LPP(Z) & \Leftrightarrow & K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & \Leftrightarrow & z \in GP(Z) & \Rightarrow & z \in LP(Z) & \Rightarrow & F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset & \Leftrightarrow & z \in GWP(Z) & \Rightarrow & z \in LWP(Z) & \Rightarrow & K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset \\
 & & & & & & \Updownarrow^* \\
 & & & & & & N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset
\end{array}
$$

Figure 3: Collection of nonconvex results.

Theorem 1 *The vector $z \in GPP(Z)$ if and only if*

$$\text{cl } D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset.$$

Proof. *Necessity.* Assume $z \in GPP(Z)$. We will prove by contradiction. Suppose that cl $D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset$, i.e. there exists $d \in \text{cl } D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\}$. On the one hand, since $d \in \text{cl } D_z(Z)$, there exists $d_i \in D_z(Z)$ such that $d_i \to d$. This implies that for any i there exists $\lambda_i > 0$ such that $z + \lambda_i d_i \in Z$. On the other hand, since $d \in \mathbf{R}^k_- \setminus \{0\} \subset \text{int } C$ and $d_i \to d$, there exists $m > 0$ such that $d_i \in \text{int } C \subset C \setminus \{0\}$ for all $i \geq m$. Since C is a cone and $\lambda_i > 0$, we get $\lambda_i d_i \in C \setminus \{0\}$ for all $i \geq m$. The last implies $z + \lambda_i d_i \in z + C \setminus \{0\}$, and since $z + \lambda_i d_i \in Z$, we get $z + \lambda_i d_i \in (z + C \setminus \{0\}) \cap Z$, i.e. $(z + C \setminus \{0\}) \cap Z \neq \emptyset$. Thus $z \notin GPP(Z)$. The obtained contradiction proves the necessity.

Sufficiency. Assume that cl $D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$. By Theorem 2.1 in [6], there exists a convex cone C such that $\mathbf{R}^k_- \setminus \{0\} \subset \text{int } C$ and $(C \setminus \{0\}) \cap D_z(Z) = \emptyset$. By Lemma 1, we have $Z \subset z + D_z(Z)$. This means that $(z + C \setminus \{0\}) \cap Z = \emptyset$, and thus $z \in GPP(Z)$. This ends the proof.

As it was proved in [12], emptiness of $F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\}$ is a necessary condition of local Pareto optimality. In [12] (Fig. 3, p.243), a counterexample showing that the necessary condition above is not sufficient for local Pareto optimality was given. That counterexample contains some inaccuracy, so we present here another counterexample showing that sufficiency does not hold (see Fig. 4, where $F_z(Z) = \{z\}$ and $z \notin LP(Z)$). Nevertheless, under assumption of local regularity, sufficiency can be proved, so we have the following result.

Theorem 2 *If $F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$ and Z is locally regular at z, then $z \in LP(Z)$.*

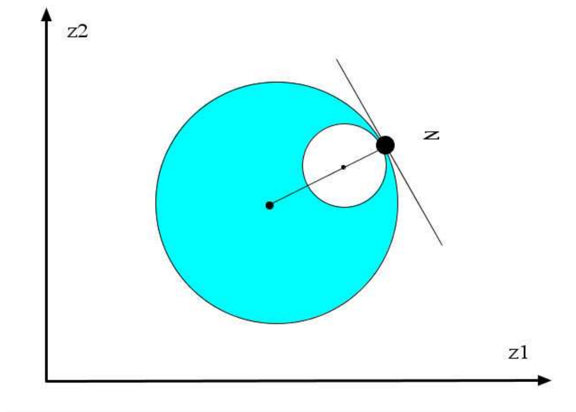

Figure 4: Counterexample for local Pareto optimality.

Proof. Let
$$F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset. \tag{2}$$
We will prove by contradiction. Assume that $z = f(x) \notin LP(Z)$. Then for any $\delta > 0$, there exists $d \in \mathbf{R}^k_- \setminus \{0\}$ such that $z + d \in Z \cap f(B(x;\delta))$. Due to objectives continuity, the last implies that there exist $t_j \searrow 0$, $d_j \in \mathbf{R}^k_-$ and $d^* \in \mathbf{R}^k_-$ such that $d_j \to d^*$ and $z + t_j d_j \in Z$, i.e. $d^* \in K_z(Z)$, and hence $d^* \in F_z(Z)$ due to local regularity. This contradicts (2).

In what follows we extend the results of [12] by introducing two other optimality principles - strong efficiency and lexicographic optimality. We present geometrical cone characterization for them and incorporate the obtained results into two interrelated schemes.

3 Strong Efficiency

Let us first define the concept of global strong optimality.

Definition 8 *The globally strongly efficient set is defined as*

$$GSE(Z) := \{z \in Z \mid (z + (\mathbf{R}^k_+)^C) \cap Z = \emptyset\}.$$

Globally strongly efficient solutions are sometimes called also *ideal solutions*. This is due to fact that

$$GSE(Z) = \bigcap_{i=1}^{k} \arg\min_{x \in S} f_i(x).$$

The corresponding local concept is defined in the following.

$$\begin{array}{ccccccc} GSE(Z) & \subset & GPP(Z) & \subset & GP(Z) & \subset & GWP(Z) \\ \cap & & \cap & & \cap & & \cap \\ LSE(Z) & \subset & LPP(Z) & \subset & LP(Z) & \subset & LWP(Z) \end{array}$$

Figure 5: Collection of the relationships between local and global strong, weak, proper Pareto optimality and efficiency.

Definition 9 *The locally strongly efficient set with $z = f(x)$ is defined as*

$$LSE(Z) := \bigcup_{\delta>0} \Big\{ z \in Z \mid (z + (\mathbf{R}^k_+)^C) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\}.$$

Clearly we have the following relationships between the different grades of optimality – see Fig. 5.

In this section we derive similar geometrical necessary and sufficient optimality conditions presented in previous section also for strongly efficient solutions.

Since the cone of feasible directions $D_z(Z)$ contains global information, it can be used to characterize global strong efficiency even in a nonconvex case. Thus, we have the following result, which is similar to the results obtained in [10] for convex case.

Note 1 *(c.f. [10]) The vector $z \in GSE(Z)$ if and only if*

$$D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z)$$

The second necessary and sufficient condition for global strong efficiency by means of contingent cones, which has been proved in [10] for convex case, is transformed into the following necessary condition for local strong efficiency in nonconvex case.

Theorem 3 *If $z \in LSE(Z)$, then*

$$K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z).$$

Proof. Let $z = f(x) \in LSE(Z)$. Then there exists $\delta > 0$ such that

$$(z + (\mathbf{R}^k_+)^C) \cap Z \cap f(B(x;\delta)) = \emptyset. \tag{3}$$

Let us suppose that $K_z(Z) \cap \mathbf{R}^k_+ \neq K_z(Z)$, i.e. there exists $d \in K_z(Z)$ such that $d \notin \mathbf{R}^k_+$. Then it follows from the definition of $K_z(Z)$ that there exists $t_j \searrow 0$ and $d_j \to d$ such that $z + t_j d_j \in Z$.

Since $d \notin \mathbf{R}^k_+$ and $d_j \to d$, there exists j_1 such that $d_j \notin \mathbf{R}^k_+$ for all $j \geq j_1$. Since $t_j > 0$, we have $t_j d_j \notin \mathbf{R}^k_+$ for all $j \geq j_1$, i.e. $t_j d_j \in (\mathbf{R}^k_+)^C$ for all $j \geq j_1$. On the other hand, since $t_j \searrow 0$ and $d_j \to d$ and the objective functions are continuous, there exists j_2 such that $z + t_j d_j \in f(B(x;\delta))$ for all $j \geq j_2$. Recall that $z + t_j d_j \in Z$. Let us define $m = \max\{j_1, j_2\}$. Then we have

$$z + t_m d_m \in (z + (\mathbf{R}^k_+)^C) \cap Z \cap f(B(x;\delta)),$$

which is contradiction to (3).

Theorem 4 *If $z \in LSE(Z)$, then*

$$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-.$$

Proof. The result follows directly from the result of Theorem 3 and observation that $T_z(Z) \subset K_z(Z)$.

Notice also that if Z is tangentially regular at $z \in Z$, then

$$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-$$

implies

$$K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z).$$

The results related to the four optimality concepts and different cones in case Z is nonconvex are collected in Table 1, where the tangent (local) regularity assumption is noted by $*$ ($**$).

4 Lexicographic Optimality

We start by giving a standard definition of the concept of lexicographic optimality (see e.g. [11]). An objective vector $z^* \in Z$ is *lexicographically optimal* if for any other objective vector $z \in Z$ one of the following two conditions holds:

1) $z = z^*$
2) $\exists\ i \in I_k :\ (z^*_i < z_i) \wedge (\forall j \in I_{i-1} :\ z^*_j = z_j)$, where $I_0 = \emptyset$.

Next we will give one more equivalent definition of the lexicographic optimality: an objective vector $z^* \in Z$ is lexicographically optimal if

$$\Big\{ z \in Z \mid z_i < z^*_i,\ i = \min\{j \in I_k \mid\ z_j \neq z^*_j\} \Big\} = \emptyset.$$

Note that the lexicographic optima may be obtained as a result of the solution of single objective (scalar) problems sequence

$$L^{(i)} = \min\{z_i \mid\ z \in L^{(i-1)}\},$$

where $i \in I_k$, $L^{(0)} = Z$, and z_i denotes i-th objective. Thus $L^{(k)}$ will constitute the set of lexicographically optimal solutions which we define below by using the complement of the lexicographic cone. It is simple to verify that all definitions are equivalent and referred to the following concept of lexicographic optimality.

Definition 10 *The globally lexicographically optimal set is*

$$GLO(Z) = \{z \in Z \mid (z + (C^k_{\text{lex}})^C) \cap Z = \emptyset\},$$

where the lexicographic cone is

$$C^k_{\text{lex}} := \{0\} \cup \{d \in \mathbf{R}^k \mid \exists\, i \in I_k \textit{ such that } d_i > 0 \textit{ and } d_j = 0\ \forall\, j < i\}.$$

Emphasize the following properties of the lexicographic cone [8]:

a) C^k_{lex} is pointed, i.e. $l(C^k_{\text{lex}}) = C^k_{\text{lex}} \cap -C^k_{\text{lex}} = \{0\}$;

b) C^k_{lex} is not correct, i.e. $\operatorname{cl} C^k_{\text{lex}} + C^k_{\text{lex}} \backslash l(C^k_{\text{lex}}) \not\subset C^k_{\text{lex}}$;

c) C^k_{lex} is not strictly supported, i.e. $C^k_{\text{lex}} \backslash l(C^k_{\text{lex}})$ is not contained in an open homogeneous half space.

Some more properties of C^k_{lex} can be easily verified:

d) C^k_{lex} is neither closed nor open;

e) $(C^k_{\text{lex}})^* := \{y \in \mathbf{R}^k \mid y^T d \geq 0 \text{ for all } d \in C^k_{\text{lex}}\} = \mathbf{R}_+$;

f) $(C^k_{\text{lex}})^\circ := \{y \in \mathbf{R}^k \mid y^T d \leq 0 \text{ for all } d \in C^k_{\text{lex}}\} = \mathbf{R}_-$.

Definition 11 *The locally lexicographically optimal set with $z = f(x)$ is*

$$LLO(Z) = \bigcup_{\delta > 0} \Big\{ z \in Z \mid (z + (C^k_{\text{lex}})^C) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\}.$$

It is evident that we have the following relationships between the different optimality principles – see Fig. 6.

However, nothing can be said in general case about the relation of $GLO(Z)$ and $GPP(Z)$ nor $LLO(Z)$ and $LPP(Z)$ (see [10]).

Now we will formulate the main results concerning lexicographic optimality characterization by means of different cones.

Again, as in previous section, the cone $D_z(Z)$, which contains global information about feasibility, can be directly used to characterize global lexicographic optimality even in a nonconvex case. Thus, we have the following result, which is similar to the results obtained in [10] for convex case.

$$\begin{array}{ccccccc} GSE(Z) & \subset & GLO(Z) & \subset & GP(Z) & \subset & GWP(Z) \\ \cap & & \cap & & \cap & & \cap \\ LSE(Z) & \subset & LLO(Z) & \subset & LP(Z) & \subset & LWP(Z) \end{array}$$

Figure 6: Collection of the relationships between local and global efficiency, strong efficiency, lexicographic and weak Pareto optimality.

Note 2 *(c.f. [10]) The vector $z \in GLO(Z)$ if and only if*

$$D_z(Z) \cap C^k_{\text{lex}} = D_z(Z)$$

The necessary condition for local lexicographic optimality in nonconvex case can be obtained if we replace the cone of globally feasible directions $D_z(Z)$ with the cone of locally feasible directions $F_z(Z)$, The condition will become also sufficient under assumption of local regularity, i.e. we get the following result.

Theorem 5 *If $z \in LLO(Z)$, then*

$$F_z(Z) \cap C^k_{\text{lex}} = F_z(Z).$$

Proof. Let $z = f(x) \in LLO(Z)$. Then there exists $\delta > 0$ such that

$$(z + (C^k_{\text{lex}})^C) \cap Z \cap f(B(x;\delta)) = \emptyset. \tag{4}$$

Let us suppose that $F_z(Z) \cap C^k_{\text{lex}} \neq F_z(Z)$. Then there exists $d \in F_z(Z)$ such that $d \notin C^k_{\text{lex}}$, i.e. $d \in (C^k_{\text{lex}})^C$. Then it follows from the definition of $F_z(Z)$ that there exists $t > 0$ such that $z + \tau d \in Z$ for all $\tau \in (0, t]$.

Since $\tau > 0$ and due to the definition of lexicographic cone, we have $\tau d \notin C^k_{\text{lex}}$, i.e. $\tau d \in (C^k_{\text{lex}})^C$. On the other hand, the continuity of the objective functions implies that there exists $\tilde{\tau} > 0$ such that $z + \tilde{\tau} d \in f(B(x;\delta))$. This contradicts (4) and ends the proof.

The results related to the four optimality concepts involving lexicographic optimality and different cones in case Z is nonconvex are collected in Table. 2, where the tangent (local) regularity assumption is noted by $*$ ($**$).

Sometimes, the lexicographic optimality principle is defined in more general way in order to reflect all possible objective orderings. This will lead to the so-called generalized lexicographic optimality concept which we define below.

Definition 12 *The global* generalized lexicographic set *$GGLO(Z)$ defined by all $k!$ permutations of objectives is:*

$$GGLO(Z) := \bigcup_{s \in S_k} GLO_s(Z),$$

where

$$GLO_s(Z) := \Big\{ z \in Z \mid \ \big(z = z^*\big) \vee$$

$$\big(\exists i \in I_k : \ (z^*_{s_i} < z_{s_i}) \wedge (\forall j \in I_{s_i - 1} : \ z^*_{s_j} = z_{s_j})\big)\Big\},$$

and S_k is a set of all $k!$ permutations of the numbers $1, 2, ..., k$.

The elements of the set $GLO_s(Z)$ are called lexicographic optima with respect to permutation s of objective order. Notice that $GLO_s(Z) = GLO(Z)$ if s is identity permutation, i.e. $s = (s_1, s_2, ..., s_k) = (1, 2, ..., k)$. The elements of the set $GGLO(Z)$ are called *global generalized lexicographic optima*. It is easy to see that any global generalized lexicographic optimum belongs to the Pareto set, i.e. the following chain of inclusions holds

$$GSE(Z) \subset GLO(Z) \subset GGLO(Z) \subset GP(Z) \subset GWP(Z).$$

The corresponding local concepts is defined in the following.

Definition 13 *The local generalized lexicographic optimum set is defined as*

$$LGSE(Z) := \bigcup_{\delta > 0} \Big\{ z \in Z \mid \bigcup_{s \in S_k} \Big(z + (C^k_{\text{lex}})^C \cap Z \cap f(B(x;\delta)) \Big)_s = \emptyset \Big\}$$

for some $\delta > 0$. Here $()_s$ means that $f(B(x;\delta))$ and C^k_{lex} are taken respectively for each $s \in S_k$.

Using Theorem 5 and Note 2, we obtain the following straightforward results.

Corollary 1 *Let $Z \subset \mathbf{R}^k$, then*

$$GGLO(Z) = \bigcup_{s \in S_k} \Big\{ z \in Z \mid \big(D_z(Z) \cap C^k_{\text{lex}} = D_z(Z) \big)_s \Big\},$$

where $()_s$ means that $D_z(Z)$ and C^k_{lex} are taken respectively for each $s \in S_k$.

Corollary 2 *Let $Z \subset \mathbf{R}^k$, then*

$$LGLO(Z) \subset \bigcup_{s \in S_k} \Big\{ z \in Z \mid \big(F_z(Z) \cap C^k_{\text{lex}} = F_z(Z) \big)_s \Big\}.$$

5 Similarity and difference between convex and nonconvex cases

Now we shortly analyze the similarity and difference between the results in two cases - convex and nonconvex. Here we would like to emphasize two tendencies about the ways how the results are modified while loosing convexity.

The first tendency is that some conditions, which are necessary and sufficient conditions for optimality in convex case, are transformed into necessary but not sufficient conditions for local optimality in nonconvex case for those principles where standard ordering cone is used in the definitions. For example, the condition $K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$, being a necessary and sufficient condition in convex case for weak Pareto optimality, becomes only necessary condition for local weak Pareto optimality. The counterexample in Fig. 7 shows that this condition is not sufficient for local weak Pareto optimality anymore. The same is also true with the condition $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$, which is necessary and sufficient for strong efficiency in convex case, is also necessary but not sufficient for local strong efficiency in nonconvex case. The counterexample illustrating this fact is presented in Fig. 8. The loss of sufficiency can be explained by the fact that the above-mentioned conditions use the contingent cone, which may have "bad" directions towards no feasibility.

In the case with proper Pareto optimality, replacement of the cone of globally feasible direction $D_z(Z)$ with the contingent cone $K_z(Z)$ is possible without loss of sufficiency due to the efficiency definition specific that uses a convex cone with inverse of the standard ordering cone as a part of interior. An appropriate choice of the convex cone makes possible to eliminate those "bad" directions, which appeared after replacement $D_z(Z)$ with $K_z(Z)$. The condition $\mathbf{R}^k_- \setminus \{0\} \subset \text{ int } C$, which was put in the definition of proper Pareto optimality, will guarantee that elimination is always possible, and this actually implies that the sufficiency always holds in the case of proper Pareto optimality.

The second tendency is that the usage of the cone of locally feasible directions $F_z(Z)$ instead of the cone of globally feasible directions $D_z(Z)$ allows to formulate only necessary conditions for local optimality in Pareto and lexicographic cases similar to the conditions which exploit $D_z(Z)$ in convex case. Sufficiency in general case is not guaranteed, but it can be achieved in Pareto case by imposing some regularity rules, which actually creates local convexity towards some directions but keep the remaining areas irregular, i.e. non-convex. Similar to the situation where the tangent regularity assumption is used to prove sufficient conditions operating with normal cones, the local regularity is used to prove the sufficient conditions which involves the usage of the cone of locally feasible directions. To investigate if the the assumptions of tangent regularity and local regularity could be weaken is an

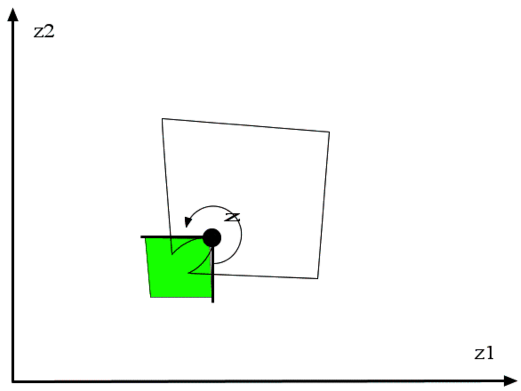

Figure 7: Counterexample for local weak Pareto optimality.

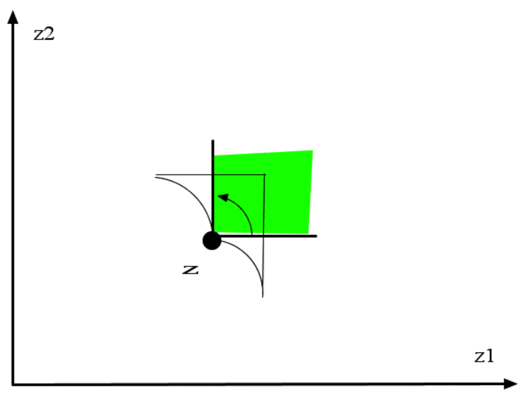

Figure 8: Counterexample for local strong efficiency.

interesting direction for continuation research in this area.

6 Concluding Remarks

Additionally to previously known cone characterizations of three optimality principles - efficiency, weakly and proper Pareto optimality, we have characterized two other optimality concepts - strongly efficiency and lexicographic optimality in terms of intersections of different cones in nonconvex case. The results were collected and summarized in two figures illustrating the interconnections between different optimality principles. The aim was to point out the differences and similarities between the five optimality principles as well as between convex and nonconvex cases. As a possible continuation of the research in the area, it seems to be interesting to check if some of the results of the paper can be specified more precisely for some classes of problems with non-convex functions but which are still possessing good properties similar to convexity (see e.g. [14], [18], [19]). For example, the class of invex functions could become the first promising candidate. The other potential direction is to consider various generalized optimality principles which are given by means of either some ordering cone or parameterization.

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$D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z)$	$\Leftrightarrow$	$z \in GSE(Z)$	$\Rightarrow$	$z \in LSE(Z)$	$\Rightarrow$	$K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$	$^* \Leftrightarrow$	$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow_*$
cl $D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$	$\Leftrightarrow$	$z \in GPP(Z)$	$\Rightarrow$	$z \in LPP(Z)$	$\Leftrightarrow$	$K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$	$^* \Leftrightarrow$	$N_z(Z) \cap \text{int } \mathbf{R}^k_- \neq \emptyset$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		
$D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$	$\Leftrightarrow$	$z \in GP(Z)$	$\Rightarrow$	$z \in LP(Z)$	$^{**} \Leftrightarrow$	$F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$		$\Downarrow_*$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		
$D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$	$\Leftrightarrow$	$z \in GWP(Z)$	$\Rightarrow$	$z \in LWP(Z)$	$\Rightarrow$	$K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$	$^* \Leftrightarrow$	$N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset$

Table 1: Collection of the relationships in nonconvex case with proper Pareto optimality.

$D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z)$	$\Leftrightarrow$	$z \in GSE(Z)$	$\Rightarrow$	$z \in LSE(Z)$	$\Rightarrow$	$K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$	$^* \Leftrightarrow$	$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		
$D_z(Z) \cap C^k_{\text{lex}} = D_z(Z)$	$\Leftrightarrow$	$z \in GLO(Z)$	$\Rightarrow$	$z \in LLO(Z)$	$\Rightarrow$	$F_z(Z) \cap C^k_{lex} = F_z(Z)$		
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow_*$
$D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$	$\Leftrightarrow$	$z \in GP(Z)$	$\Rightarrow$	$z \in LP(Z)$	$^{**} \Leftrightarrow$	$F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$		
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		
$D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$	$\Leftrightarrow$	$z \in GWP(Z)$	$\Rightarrow$	$z \in LWP(Z)$	$\Rightarrow$	$K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$	$^* \Leftrightarrow$	$N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset$

Table 2: Collection of the relationships in nonconvex case with lexicographic optimality.

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ISBN 978-952-12-2254-2
ISSN 1239-1891