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On generalized trade-off directions for basic optimality principles in convex and nonconvex multiobjective optimization

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Abstract

We consider a general multiobjective optimization problem with five basic optimality principles: efficiency, weak and proper Pareto optimality, strong efficiency and lexicographic optimality. We generalize the concept of trade-off directions defining them as some optimal surface of appropriate cones. In convex optimization, the contingent cone can be used for all optimality principles except lexicographic optimality where the cone of feasible directions is useful. In nonconvex case the contingent cone and the cone of locally feasible directions with lexicographic optimality are helpful. We derive necessary and sufficient geometrical optimality conditions in terms of corresponding trade-off directions for both convex and nonconvex cases. We analyze similarities and differences between the cases.

1 Introduction

The overall goal in multiobjective optimization is to find a compromise between several conflicting objectives which is best-fit to the needs of a decision maker. This compromise is usually refereed to as an optimality principle. Various mathematical definitions of the optimality principle can be derived in several different ways depending on the needs of the solution approaches used. Moreover, sometimes the use of one definition may be advantageous to the other due to computational complexity reasons.

The usage of trade-offs as a tool containing essential information about compromise have been suggested in series of papers (see e.g. [18] and [19]), where certain scalarizing functions were used to define the concept. Another approach, proposed in [6], [7], consists in generating solution satisfying some pre-specified bounds on trade-off information by means of a scalarizing function. In [5] for convex (including nondifferentiable) problems, the concept of trade-offs has been generalized into a cone of trade-off directions, which was defined as a Pareto optimal surface of a contingent (tangent) cone located at the point considered.

The usage of contingent and normal cones as well as the cone of feasible directions is a natural choice in the case of convex optimization [16]. In nonconvex optimization, the main difficulty arises due to the fact that the contingent cone as well as the cone of feasible directions may loose convexity. Two additional types of cones are proved to be helpful - tangent cone and cone of local feasible directions [2]. The guaranteed property of convexity of these cones assures that they can be used to overcome some difficulties which appear in nonconvex optimization. However in nonconvex case, tangent cones do not necessarily represent the shape of the set considered even locally and the relation to trade-off directions is lost. Therefore to define trade-off directions in nonconvex case, we must use nonconvex contingent cones as it was suggested originally in [8] for smooth problems and later generalized for not necessarily differentiable problems in [11].

The aim of this paper is to describe necessary and sufficient optimality conditions in terms of trade-off directions for both convex and nonconvex cases. The paper is organized as follows. In Section 2, we formulate a general multiobjective problem and introduce five basic optimality principles, which are the most common in multiobjective optimization. We give traditional definitions and geometrical ones via appropriate cones. For every optimality principle considered, we define generalized trade-off directions for convex and nonconvex cases in Section 3. Giving up convexity naturally means that we need local instead of global analysis. Section 4 presents the main results showing interrelation between optimal solutions and corresponding generalized trade-off directions. The results presented for convex and nonconvex cases and summarized in two schemes. The paper is concluded in Section 5,

where the differences and similarities between two cases are analyzed.

2 Basic optimality principles

We consider general multiobjective optimization problems of the following form:

$$\min_{x \in S} \{f_1(x), f_2(x), \ldots, f_k(x)\},$$

where the *objective functions* $f_i : \mathbf{R}^n \to \mathbf{R}$ for all $i \in I_k := \{1, \ldots, k\}$ are supposed to be continuous. The *decision vector* x belongs to the nonempty *feasible set* $S \subset \mathbf{R}^n$. The image of the feasible set is denoted by $Z \subset \mathbf{R}^k$, i.e. $Z := f(S)$. Elements of Z are termed *objective vectors* and they are denoted by $z = f(x) = (f_1(x), f_2(x), \ldots, f_k(x))^T$.

The sum of two sets A and E is defined by $A + E = \{a + e \mid a \in A,\ e \in E\}$. The interior, closure, convex hull and complement of a set A are denoted by int A, cl A, conv A and A^C, respectively.

A set A is a *cone* if $\lambda x \in A$ whenever $x \in A$ and $\lambda > 0$. We denote the positive orthant of $\mathbf{R}^k$ by $\mathbf{R}^k_+ = \{d \in \mathbf{R}^k \mid d_i \geq 0 \text{ for every } i \in I_k\}$. The positive orthant is also known as *standard ordering cone*. The negative orthant $\mathbf{R}^k_-$ is defined respectively. Note, that $\mathbf{R}^k_-$ and $\mathbf{R}^k_+$ are closed convex cones. Furthermore, an *open ball* with center x and radius δ is denoted by $B(x; \delta)$.

In what follows, the notation $z < y$ for $z, y \in \mathbf{R}^k$ means that $z_i < y_i$ for every $i \in I_k$ and, correspondingly, $z \leq y$ stands for $z_i \leq y_i$ for every $i \in I_k$.

Simultaneous optimization of several objectives for multiobjective optimization problem is not a straightforward task. Contrary to the the single objective case, the concept of optimality is not unique in multiobjective cases.

Below we give traditional definitions of five well-known and most fundamental principles of optimality (see e.g. [3], [9]).

Weak Pareto Optimality. An objective vector $z^* \in Z$ is *weakly Pareto optimal* if there does not exist another objective vector $z \in Z$ such that $z_i < z_i^*$ for all $i \in I_k$.

Pareto optimality or efficiency. An objective vector $z^* \in Z$ is *Pareto optimal* or *efficient* if there does not exist another objective vector $z \in Z$ such that $z_i \leq z_i^*$ for all $i \in I_k$ and $z_j < z_j^*$ for at least one index j.

Proper Pareto Optimality. An objective vector $z^* \in Z$ is *properly Pareto optimal* if there exists no objective vector $z \in Z$ such that $z \in C$ for some convex cone C, $\mathbf{R}^k_- \setminus \{0\} \subset$ int C, attached to z^*.

Strong Efficiency. An objective vector $z^* \in Z$ is *strongly Pareto optimal* if for all $i \in I_k$ there exists no objective vector $z \in Z$ such that $z_i < z_i^*$ or in other words $z^* \in Z$ optimizes all objectives z_i, $i \in I_k$.

Lexicographic Optimality. An objective vector $z^* \in Z$ is *lexicographically optimal* if for all other objective vector $z \in Z$ one of the following two conditions holds:
1) $z = z^*$
2) $\exists\, i \in I_k : \; (z_i^* < z_i) \wedge (\forall j \in I_{i-1} : \; z_j^* = z_j)$, where $I_0 = \emptyset$.

A solution is Pareto optimal if improvement in some objectives can only be obtained at the expense of some other objective(s) (see e.g. [3], [9]). The set of weakly Pareto optimal solutions contains the Pareto optimal solutions together with solutions where all the objectives cannot be improved simultaneously (see, e.g. [3], [9]). The set of improperly Pareto optimal solutions represents a set of efficient points with certain abnormal or irregular properties. Henceforth we use only one of the possible concepts of proper efficiency, which is according to Henig [4]. This concept uses a convex cone, which interior part must contain an inverse of standard ordering cone, to prohibit tradeoffs towards directions within the cone. Strong efficiency is generally referred to the solutions which deliver optimality to each objective. Despite feasibility of such solutions is rare, they provide an important information on the lower bound for each objective in the Pareto optimal set. Lexicographic optimality principle is generally applied to the situation where objectives have no equal importance anymore but ordered according to their significance.

Next we define the five sets of efficient solutions by using appropriate ordering cones. It is trivial to verify that the definitions of optimality and efficiency formulated above are equivalent to those following below.

Definition 1 *The weakly Pareto optimal set is*

$$WP(Z) := \{z \in Z \mid (z + \operatorname{int} \mathbf{R}_-^k) \cap Z = \emptyset\};$$

the Pareto optimal set is

$$PO(Z) := \{z \in Z \mid (z + \mathbf{R}_-^k \setminus \{0\}) \cap Z = \emptyset\};$$

the properly Pareto optimal set is defined as

$$PP(Z) := \{z \in Z \mid (z + C \setminus \{0\}) \cap Z = \emptyset\}$$

for some convex cone C such that $\mathbf{R}_-^k \setminus \{0\} \subset \operatorname{int} C$;
the strongly efficient set is

$$SE(Z) := \{z \in Z \mid (z + (\mathbf{R}_+^k)^C) \cap Z = \emptyset\};$$

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and the lexicographically optimal set is

$$LO(Z) = \{z \in Z \mid (z + (C_{\text{lex}}^k)^C) \cap Z = \emptyset\},$$

where the lexicographic cone is

$$C_{\text{lex}}^k := \{0\} \cup \{d \in \mathbf{R}^k \mid \exists i \in I_k \text{ such that } d_i > 0 \text{ and }$$

$$d_j = 0 \; \forall \, j < i\}.$$

Note that $SE(Z) \subset PP(Z) \subset PO(Z) \subset WP(Z)$ and $LO(Z) \subset PP(Z) \subset PO(Z) \subset WP(Z)$.

The corresponding local concepts are defined in the following. Naturally, in a convex case, local and global concepts are equal.

Definition 2 *The locally weakly Pareto optimal set with $z = f(x) \in Z$ is given as*

$$LWP(Z) = \bigcup_{\delta > 0} \Big\{ z \in Z \mid (z + \text{int } \mathbf{R}_-^k) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\};$$

the locally Pareto optimal set as

$$LPO(Z) = \bigcup_{\delta > 0} \Big\{ z \in Z \mid (z + \mathbf{R}_-^k \setminus \{0\}) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\};$$

the locally properly Pareto optimal set as

$$LPP(Z) = \bigcup_{\delta > 0} \Big\{ z \in Z \mid (z + C \setminus \{0\}) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\}$$

for some convex cone C such that $\mathbf{R}_-^k \setminus \{0\} \subset \text{int } C$;
the locally strongly efficient set with $z = f(x)$ is defined as

$$LSE(Z) := \bigcup_{\delta > 0} \Big\{ z \in Z \mid (z + (\mathbf{R}_+^k)^C) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\};$$

and the locally lexicographically optimal set with $z = f(x)$ is

$$LLO(Z) = \bigcup_{\delta > 0} \Big\{ z \in Z \mid (z + (C_{\text{lex}}^k)^C) \cap Z \cap f(B(x;\delta)) = \emptyset \Big\}.$$

Note that $LSE(Z) \subset LPP(Z) \subset LPO(Z) \subset LWP(Z)$ and $LLO(Z) \subset LPP(Z) \subset LPO(Z) \subset LWP(Z)$.

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3 Generalized trade-off directions

The concept of trade-offs in multiobjective optimization is a key point to define compromise between conflicting objectives. It can be used to describe solutions which linearly approximate the feasible region and which are mutually non-dominated with respect to the given optimality principle. The trade-off directions can be used in many algorithms which requires specifying directions which may lead fast to the solution that is most preferred by the decision maker (see e.g. [1], [9]).

Since the contingent cones linearly approximates the shape of the feasible set, equally well in both convex (global approximation) and nonconvex (local approximation) cases, it can be used to define the generalized trade-off directions. A (weakly) Pareto surface of the contingent cone serves for that purposes.

Definition 3 *The contingent cone of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is defined as*

$$K_z(Z) := \{d \in \mathbf{R}^k \mid$$

$$\text{there exist } t_j \searrow 0 \text{ and } d_j \to d \text{ such that } z + t_j d_j \in Z\}.$$

Definition 4 *The* cone of feasible directions *of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is denoted by*

$$D_z(Z) := \{d \in \mathbf{R}^k \mid$$

$$\text{there exists } t > 0 \text{ such that } z + td \in Z\}.$$

The following definition provides regularity condition for Z at $z \in Z$.

Definition 5 *The set Z is called regular at $z \in Z$ if $D_z(Z) = K_z(Z)$.*

In convex case, the sets of generalized trade-off directions can be defined as follows

Definition 6 *Let Z be convex. The set of generalized trade-off directions is defined as:*
- in case of weak Pareto optimality:

$$G_{WP}(Z) := WP(K_z(Z));$$

- in case of Pareto optimality (efficiency):

$$G_{PO}(Z) := PO(K_z(Z));$$

- in case of proper Pareto optimality:

$$G_{PP}(Z) := PO(K_z(Z));$$

- *in case of strong efficiency:*

$$G_{SE}(Z) := SE(K_z(Z)) = SE(D_z(Z));$$

- *in case of lexicographic optimality:*

$$G_{LO}(Z) := LO(D_z(Z)).$$

Note that $G_{PO}(Z) = G_{PP}(Z)$ by definition since Pareto optimality can be seen as a particular case of proper Pareto optimality with $C = \mathbf{R}^k_-$. It is also easy to see that in convex case $SE(K_z(Z)) = SE(D_z(Z))$ follows directly from the definitions and Lemma 1.

In nonconvex case, the cone of feasible directions $D_z(Z)$ does not describe the shape of Z locally. Thus, we introduce a cone of locally feasible directions, which reflects the shape of Z locally (see e.g. [12]).

Definition 7 *The cone of locally feasible directions of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is denoted by*

$$F_z(Z) = \{d \in \mathbf{R}^k \mid$$

there exists $t > 0$ such that $z + \tau d \in Z$ for all $\tau \in (0, t]\}.$

Notice that, since two solutions are considered to be mutually lexicographically non-dominated if they have the same objective vectors, we have to use the cone of feasible directions in the definition of the set of generalized trade-off directions in case with lexicographic optimality. Indeed, the set of generalized trade-off directions in case with local lexicographic optimality is either empty or one point 0 (origin of $F_z(Z)$) only, so it becomes indifferent if $D_z(Z)$ is closed or open, what is not true in cases with other types of local optimality.

The following definition provides local regularity condition for Z at $z \in Z$.

Definition 8 *The set Z is called locally regular at $z \in Z$ if $F_z(Z) = K_z(Z)$.*

For nonconvex cases, Clarke [2] has defined a convex tangent cone in the following way.

Definition 9 *The tangent cone of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is given by the formula*

$$T_z(Z) = \{d \in \mathbf{R}^k \mid$$

for all $t_j \searrow 0$ and $z_j \to z$ with $z_j \in Z$,

there exists $d_j \to d$ with $z_j + t_j d_j \in Z\}.$

The following basic relations can be derived from the definitions of the concepts used and from [12], [17].

$$\text{cl } D_z(Z) = K_z(Z) \quad \Leftarrow \qquad \text{Regularity}$$
$$\Uparrow$$
$$\text{Convexity} \qquad \Rightarrow \quad \text{Tangent Regularity}$$
$$\Downarrow$$
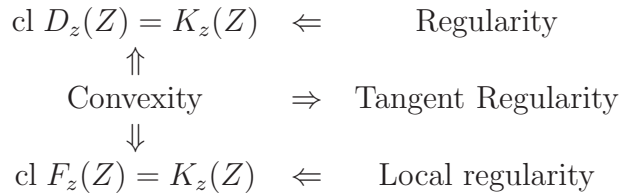$$\text{cl } F_z(Z) = K_z(Z) \quad \Leftarrow \qquad \text{Local regularity}$$

Figure 1: Interconnection between variuos types of regularity.

Lemma 1 *For the cones $K_z(Z)$, $D_z(Z)$, $T_z(Z)$ and $F_z(Z)$ we have the following*

a) $K_z(Z)$ and $T_z(Z)$ are closed and $T_z(Z)$ is convex.

b) $0 \in K_z(Z) \cap D_z(Z) \cap T_z(Z) \cap F_z(Z)$.

c) $Z \subset z + D_z(Z)$.

d) cl $F_z(Z) \subset K_z(Z) \subset$ cl $D_z(Z)$.

e) $T_z(Z) \subset K_z(Z)$.

f) *If Z is convex, then* cl $F_z(Z) = T_z(Z) = K_z(Z) =$ cl $D_z(Z)$.

Note that, under convexity assumption, for any $z \in Z$ we have cl $F_z(Z) = K_z(Z)$ (see e.g. [17]), i.e. local regularity defines a bit stronger requirement on a local structure of a set than the convexity assumption. At the same time local regularity does not necessarily imply cl $D_z(Z) = K_z(Z)$, the condition which is guaranteed under convexity assumption.

Let us point out once again that contingent cones can be nonconvex in which case their polar cones are irrelevant, in other words, $K_z(Z)^\circ = \{0\}$ independently of Z.

Even though contingent cones are generally nonconvex, their convexity is guaranteed under special circumstances.

Definition 10 *The set Z is called tangentially regular at $z \in Z$ if $T_z(Z) = K_z(Z)$.*

Trivially, we can see that e.g. convex sets are always tangentially regular.

Note that in order to formulate some of optimality conditions we use four different assumptions about structural properties of Z - convexity, tangent regularity, regularity and local regularity. In general all these are different and does not directly imply each other. The interconnection between the four regularity assumptions are presented in Figure 1.

In nonconvex case, the sets of generalized trade-off directions can be defined similar to Definition 6 for all optimality principles except the lexicographic one.

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Definition 11 *(cf. Definition 6) Let Z be nonconvex. The set of generalized trade-off directions is defined as:*
  - in case of local weak Pareto optimality:

$$G_{LWP}(Z) := WP(K_z(Z));$$

- in case of local Pareto optimality (local efficiency):

$$G_{LPO}(Z) := PO(K_z(Z));$$

- in case of local proper Pareto optimality:

$$G_{LPP}(Z) := PO(K_z(Z));$$

- in case of local strong efficiency:

$$G_{LSE}(Z) := SE(K_z(Z));$$

- in case of local lexicographic optimality:

$$G_{LLO}(Z) := LO(F_z(Z)).$$

Notice that, contrary to the convex case, $SE(K_z(Z))$ is not necessarily equal to $SE(D_z(Z))$ if Z is nonconvex. By analogy with convex case, we should use the cone of locally feasible directions (instead of contingent cone) in the definition of the set of generalized trade-off directions in case with local lexicographic optimality.

4 Main Results

4.1 Convex case

Here we formulate and prove the basic results concerning relations between optimality and corresponding set of generalized trade-off directions in convex case.

Theorem 1 *Let Z be convex. If $z \in WP(Z)$, then $G_{WP}(Z) \neq \emptyset$.*

This result directly follows from the result of theorem 6.

Theorem 2 *Let Z be convex. If $z \in PO(Z)$, then $G_{PO}(Z) \neq \emptyset$ under assumption that Z is regular.*

Proof. Assume $z \in PO(Z)$. Suppose that $G_{PO}(Z) = \emptyset$. Then $(d+\mathbf{R}^k_-\backslash\{0\})\cap K_z(Z) \neq \emptyset$ for all $d \in K_z(Z)$. Taking $d = 0$ $(0 \in K_z(Z))$, we get $(\mathbf{R}^k_-\backslash\{0\})\cap K_z(Z) \neq \emptyset$, and due to regularity assumption $(\mathbf{R}^k_-\backslash\{0\}) \cap D_z(Z) \neq \emptyset$. The last contradicts with the initial assumption that $z \in PO(Z)$. This ends the proof.

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Theorem 3 *Let Z be convex. The solution $z \in PP(Z)$ if and only if $G_{PP}(Z) \neq \emptyset$.*

This result directly follows from the result of theorem 8 and the fact that convex set is always tangentially regular.

Theorem 4 *Let Z be convex. The solution $z \in SE(Z)$ if and only if $G_{SE}(Z) \neq \emptyset$, or equivalently $G_{SE}(Z) = \{0\}$.*

Proof. First we show that $z \in SE(Z)$ if and only if $0 \in G_{SE}(Z)$. Indeed, (see [13])

$$z \in SE(Z) \Leftrightarrow K_z(Z) \cap \mathbf{R}_+^k = K_z(Z) \Leftrightarrow$$
$$(0 + (\mathbf{R}_+^k)^C) \cap K_z(Z) = \emptyset \Leftrightarrow 0 \in G_{SE}(Z).$$

Now it remains to show that if $y \in K_z(Z)$, $y \neq 0$, then $y \notin G_{SE}(Z)$. Indeed, if $y \in K_z(Z)$, $y \neq 0$, then $0 \in (y + (\mathbf{R}_+^k)^C) \cap K_z(Z)$, and then $y \notin G_{SE}(Z)$. This ends the proof.

Theorem 5 *Let Z be convex. The solution $z \in LO(Z)$ if and only if $G_{LO}(Z) \neq \emptyset$, or equivalently $G_{LO}(Z) = \{0\}$.*

Proof. First we show that $z \in LO(Z)$ if and only if $0 \in G_{LO}(Z)$. Indeed, (see [13])

$$z \in LO(Z) \Leftrightarrow D_z(Z) \cap (C_{lex}^k)^C = \emptyset \Leftrightarrow$$
$$(0 + (C_{lex}^k)^C) \cap D_z(Z) = \emptyset \Leftrightarrow 0 \in G_{LO}(Z).$$

Now it remains to show that if $d \in D_z(Z)$, $d \neq 0$, then $d \notin G_{LO}(Z)$. Indeed, if $d \in D_z(Z)$, $d \neq 0$, then $d \in C_{lex}^k$ and $-d \in (C_{lex}^k)^C$, i.e. $d + (-d) = 0 \in D_z(Z)$, and then $(d + (C_{lex}^k)^C) \cap D_z(Z) \neq \emptyset$. Thus $d \notin G_{LO}(Z)$. This ends the proof.

4.2 Nonconvex case

Here we formulate and prove the basic results concerning relations between optimality and corresponding set of generalized trade-off directions in nonconvex case.

Theorem 6 *[10] If $z \in LWP(Z)$, then $G_{LWP}(Z) \neq \emptyset$.*

Theorem 7 *If $z \in LPO(Z)$, then $G_{LPO}(Z) \neq \emptyset$ under assumption that Z is locally regular.*

Proof. Assume $z \in LPO(Z)$. Suppose that $G_{LPO}(Z) = \emptyset$. Then $(d + \mathbf{R}_-^k \setminus \{0\}) \cap K_z(Z) \cap f(B(x; \delta) \neq \emptyset$ for all $d \in K_z(Z)$. Taking $d = 0$ ($0 \in K_z(Z)$), we get $(\mathbf{R}_-^k \setminus \{0\}) \cap K_z(Z) \cap f(B(x; \delta) \neq \emptyset$, and due to local regularity assumption $(\mathbf{R}_-^k \setminus \{0\}) \cap F_z(Z) \cap f(B(x; \delta) \neq \emptyset$. The last contradicts with the initial assumption that $z \in LPO(Z)$. This ends the proof.

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Theorem 8 *[11] If $z \in LPP(Z)$, then $G_{LPP}(Z) \neq \emptyset$. The necessary condition above is also sufficient if Z is tangentially regular.*

Theorem 9 *If $z \in LSE(Z)$, then $G_{LSE}(Z) \neq \emptyset$, or equivalently $G_{LSE}(Z) = \{0\}$.*

Proof. Let $z \in LSE(Z)$. Then (see [14])

$$K_z(Z) \cap \mathbf{R}_+^k = K_z(Z).$$

Then it follows that

$$(0 + \mathbf{R}_-^k \backslash \{0\}) \cap K_z(Z) = \emptyset \Rightarrow 0 \in G_{LSE}(Z).$$

Suppose $y \in G_{LSE}(Z)$, $y \neq 0$, then $(y + \mathbf{R}_-^k \backslash \{0\}) \cap K_z(Z) = \emptyset$. If $y \in \mathbf{R}_+^k \backslash \{0\}$, then $0 \in (y + \mathbf{R}_-^k \backslash \{0\})$, and then $y \in (\mathbf{R}_+^k)^C$. It implies that $y \in (\mathbf{R}_+^k)^C \cap K_z(Z)$. The obtained contradiction ends the proof.

Theorem 10 *If $z \in LLO(Z)$, then $G_{LLO}(Z) \neq \emptyset$, or equivalently $G_{LLO}(Z) = \{0\}$.*

Proof. Let $z \in LLO(Z)$. Then (see [14])

$$F_z(Z) \cap C_{lex}^k = F_z(Z).$$

Suppose $G_{LLO}(Z) = \emptyset$. Then

$$(d + (C_{lex}^k)^C) \cap F_z(Z) \neq \emptyset \ \forall d \in F_z(Z).$$

Taking $d = 0$, we get $(C_{lex}^k)^C \cap F_z(Z) \neq \emptyset$ that contradicts $F_z(Z) \cap C_{lex}^k = F_z(Z)$. This contradiction ends the proof.

5 Conclusions

In that paper we introduced and characterized the concept of trade-off directions for five most common optimality principles in multiobjective optimization. We generally followed the approach, initially proposed by Henig and Buchanan [5] followed by Lee and Nakayama [8] as well as Miettinen and Mäkelä [10], [11], where trade-off directions are defined via some optimal surface of appropriate cones. We specified necessary and in some cases also sufficient conditions of optimality in terms of corresponding trade-off directions in both convex and nonconvex cases. The results obtained not only summarize and order already known facts about trade-off directions but also shed a new light on their structural properties, emphasizing on some fundamental similarities and differences existing in convex and nonconvex optimization. An interesting topic of further research is to investigate applicability of the proposed concepts in different multiobjective optimization algorithms. Some interactive methods [1] could be promising candidates for that purpose.

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