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Abstract

A new type of syntactic monoid and semigroup of tree languages is introduced. For each $n \geq 1$, we define for any tree language T its n-ary syntactic monoid $M^n(T)$ and its n-ary syntactic semigroup $S^n(T)$ as quotients of the monoid or semigroup, respectively, formed by certain new generalized contexts. For $n = 1$ these contexts are just the ordinary contexts (or 'special trees') and $M^1(T)$ is the syntactic monoid introduced by W. Thomas (1982,1984). Several properties of these monoids and semigroups are proved. For example, it is shown that $M^n(T)$ and $S^n(T)$ are isomorphic to certain monoids and semigroups associated with the minimal tree recognizer of T. Using these syntactic monoids or semigroups, we can associate with any variety of finite monoids or semigroups, respectively, a variety of tree languages. Although there are varieties of tree languages that cannot be obtained this way, we prove that the definite tree languages can be characterized by the syntactic semigroups $S^2(T)$, which is not possible using the classical syntactic monoids or semigroups.

Keywords: tree languages, universal algebras, semigroups, syntactic monoids

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1 Introduction

The classification theory of regular (string) languages based on syntactic monoids and syntactic semigroups has been a great success ever since M.P. Schützenberger [21] characterized the star-free languages using syntactic monoids, and S. Eilenberg's [5] variety theory has served well as a general framework for such studies (cf. [5, 16, 2] for expositions of these matters). Hence, it is quite natural to try something similar for tree languages. In [28] W. Thomas introduced syntactic monoids of tree languages and characterized the aperiodic tree languages in terms of them. Many fundamental properties of these monoids were presented by K. Salomaa in his Master's Thesis [20], and later an essentially equivalent notion for languages of binary trees was studied by M. Nivat and A. Podelski [13, 17]. By a natural modification, one can also define the syntactic semigroups of tree languages.

Let **M** be a variety of finite monoids (or semigroups), i.e., a class of finite monoids closed under submonoids, homomorphic images and finite direct products. As noted in [25], it follows from results of [20] that the tree languages whose syntactic monoid belongs to **M**, form a variety of tree languages, that is to say, it is closed under certain operations, and if the ranked alphabet is not fixed, they form a generalized variety of tree languages (cf. [25]). On the other hand, T. Wilke [29] proved that the variety of frontier-testable (i.e., reverse definite) tree languages cannot be defined this way by syntactic monoids or semigroups, and V. Piirainen [15] shows that this is the case also for the piecewise testable tree languages (cf. [25] for a further example). Finally, S. Salehi [18, 19] characterized the varieties of tree languages that can be defined by syntactic monoids (or semigroups). His result confirms the impression that the defining power of syntactic monoids or semigroups of tree languages is limited compared with that of syntactic algebras (cf. [1, 23, 24, 25, 26]. In particular, it shows that definite tree languages cannot be defined by them (contrary to what has been claimed in the literature).

In this paper we propose a family of new syntactic monoids and semigroups of tree languages: for any tree language T, we define for each $n \geq 1$, its *n-ary syntactic monoid* $M^n(T)$ and its *n-ary syntactic semigroup* $S^n(T)$. If $T \subseteq T_\Sigma(X)$ is a ΣX-tree language, where Σ is a ranked alphabet and X is a leaf alphabet, then $M^n(T)$ is the quotient monoid of the monoid of (what we call) *ΣXn-contexts* with respect to the *n-ary syntactic monoid congruence* of T. A ΣXn-context is an n-tuple $\langle u_1, \ldots, u_n \rangle$ of terms $u_1, \ldots, u_n \in T_\Sigma(X \cup \{\xi_1, \ldots, \xi_n\})$ with variables such that each one of the variables $\xi_1, \ldots, \xi_n$ appears exactly once in some component

u_i. For $n = 1$, we get the usual contexts, but for values $n > 1$ the new contexts may represent more informative horizontal cross-sections of trees. Similarly, the n-ary syntactic semigroup $S^n(T)$ is a quotient of the semigroup of the *proper* ΣXn-contexts in which no component is just a variable. We shall also introduce and study *n-ary translation monoids* of algebras and *n-ary transformation monoids* of tree recognizers, and show that for any regular tree language T, the n-ary syntactic monoid $M^n(T)$ is isomorphic to the n-ary transformation monoid of the minimal recognizer of T as well as to the n-ary translation monoid of the underlying algebra. Similar facts hold for the semigroups $S^n(T)$. For any $n \geq 1$, each variety $\mathbf{M}$ of finite monoids (or semigroups) the tree languages T such that $M^n(T) \in \mathbf{M}$ (or $S^n(T) \in \mathbf{M}$, resp.) form a variety of tree languages in the sense of [24, 26], for example.

This paper is mainly a study of the general properties of our new syntactic monoids and semigroups, as well as of the related translation and transformation monoids and semigroups, but we shall demonstrate the potential of the new notions by showing that the variety of *definite tree languages* can be characterized by 2-ary syntactic semigroups. On the other hand, we show that not all varieties of tree languages are definable by our monoids or semigroups.

The paper is organized as follows. In Section 2 we recall and introduce some general notions. In Section 3 we consider terms with variables and n-tuples of such terms and some operations on them. In Section 4 we define ΣXn-contexts and present some basic properties of the monoids and semigroups formed by them. The following section deals with n-ary translations of a given Σ-algebra and the monoids and semigroups formed by them. In particular, we show how the n-ary translation monoids or semigroups of subalgebras, homomorphic images and direct products of any given algebras relate to the n-ary translation monoids or semigroups of the original algebras. In Section 6, we consider the monoids and semigroups of state transformations induced in a ΣX-tree recognizer by ΣXn-contexts and proper ΣXn-contexts, respectively. For a connected ΣX-recognizer, these are shown to be isomorphic to the n-ary translation monoids and semigroups of the underlying Σ-algebra, and that they are finite if and only if the recognized tree language is regular.

Our n-ary syntactic monoid and semigroup congruences of tree languages are introduced in Section 7. Moreover, we prove that the syntactic congruences of the Boolean combinations, quotients or homomorphic pre-images of some given tree languages, relate to the syntactic congruences of the original tree languages similarly as in the cases of other syntactic congruences. In Section 8 we define the n-ary syntactic monoid $M^n(T)$ and the n-ary syntactic semigroup $S^n(T)$ of a tree language T in the natural way. It is shown that they are isomorphic, respectively, to the n-ary transformation monoid and semigroup of the minimal ΣX-recognizer of T. This implies that $M^n(T)$ and $S^n(T)$ are finite for every $n \geq 1$, if and only if the tree language T is recognizable. The n-ary syntactic monoids and semigroups of the Boolean combinations, quotients or homomorphic pre-images of any given tree languages relate again to the syntactic monoids or semigroups of the original tree languages as one would expect. We also show that the n-ary syntactic monoids and semigroups of any tree language T form ascending chains $M^1(T) \sqsubseteq M^2(T) \sqsubseteq \ldots$ and $S^1(T) \preceq S^2(T) \preceq \ldots$, where $M \sqsubseteq M'$ means that M is isomorphic to a submonoid of M' and $S \preceq S'$ means that S is a homomorphic image of a subsemigroup of S'.

In Section 9, we recall the notion of a *variety of Σ-tree languages* (a Σ-VTL, for short) [24, 26] and show then that for any $n \geq 1$ and any variety $\mathbf{M}$ of finite monoids, the tree languages T such that $M^n(T) \in \mathbf{M}$ form a Σ-VTL $\mathcal{V}^n_{\mathbf{M}}$. Similarly, any variety $\mathbf{S}$ of finite semigroups defines a Σ-VTL $\mathcal{V}^n_{\mathbf{S}}$ by the condition $S^n(T) \in \mathbf{S}$, but we show that not every Σ-VTL is obtained this way from a variety of finite monoids or a variety of finite semigroups. On the other hand, in Section 10 we prove that for any $k \geq 0$, there is a variety $\mathbf{D}_k$ of finite semigroups such that $\mathcal{V}^2_{\mathbf{D}_K}$ is the Σ-VTL of the k-definite Σ-tree languages, and then that the family of all the definite Σ-tree languages can be characterized as the Σ-VTL $\mathcal{V}^2_{\mathbf{D}}$ for a certain variety $\mathbf{D}$ of finite semigroups. In Section 11 we make some concluding remarks and note a few further questions to be considered.

2 Preliminaries

We shall often write $A := B$ to indicate that some object A is defined to be B. For any integer $n \geq 0$, let $[n]$ denote the set $\{1, \ldots, n\}$. For any relation $\rho \subseteq A \times B$, the fact that $(a, b) \in \rho$ for some $a \in A$ and $b \in B$, will usually be expressed by writing $a \, \rho \, b$. For any $a \in A$, let $a\rho := \{b \mid a\rho b\}$. In case of an equivalence relation, we write $[a]_\rho$, or just $[a]$, for $a\rho$. Moreover, for any $A' \subseteq A$, we denote by $A'\rho$ the set of all $b \in B$ such that $a\rho b$ for some $a \in A'$. The *converse* of ρ is the relation $\rho^{-1} := \{(b, a) \mid a\rho b\}$ ($\subseteq B \times A$). The *domain* and the *range* of ρ are $B\rho^{-1}$ and $A\rho$, respectively. The *composition* of two relations $\rho \subseteq A \times B$ and $\rho' \subseteq B \times C$ is the relation

$$\rho \circ \rho' := \{(a, c) \mid a \in A, c \in C, (\exists b \in B)\, a\rho b \text{ and } b\rho' c\}.$$

The *diagonal relation* $\{(a,a) \mid a \in A\}$ of a set A is denoted by Δ_A. A mapping $\varphi : A \to B$ may also be viewed as a relation ($\subseteq A \times B$), and $a\varphi$ ($a \in A$) denotes either the image $\varphi(a)$ of a or the set formed by it. Especially homomorphisms will be written this way as right operators that are also composed from left to right, i.e., the composition of $\varphi : A \to B$ and $\psi : B \to C$ is written as $\varphi\psi$.

In what follows, Σ is always a *ranked alphabet*, i.e., a finite set of symbols each of which has a given nonnegative integer arity. For any $m \geq 0$, the set of m-ary symbols in Σ is denoted by Σ_m. We suppose that $\Sigma_m \cap \Sigma_n = \emptyset$ for $m \neq n$. In addition to ranked alphabets, we use ordinary finite alphabets $X, Y, \ldots$ that we call *leaf alphabets*. These are assumed to be disjoint from the ranked alphabets. If $\Sigma_0 \neq \emptyset$, we may allow also the empty leaf alphabet.

The set $T_\Sigma(X)$ of Σ-*terms over* X is the smallest set T such that $X \cup \Sigma_0 \subseteq T$, and $f(t_1, \ldots, t_m) \in T$ whenever $m > 0$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T$. Such terms are regarded in the usual way as representations of labelled trees, and we call them ΣX-*trees*. Subsets of $T_\Sigma(X)$ are called ΣX-*tree languages*. We may also speak simply about *trees* and *tree languages* without specifying the alphabets.

Let ξ be a special symbol that appears neither in Σ nor in any of the leaf alphabets. A $\Sigma(X \cup \{\xi\})$-tree in which ξ appears exactly once, is called a ΣX-*context*. The set of all ΣX-contexts is denoted by $C_\Sigma(X)$. The elements of $C_\Sigma^+(X) := C_\Sigma(X) \setminus \{\xi\}$ are called *proper* ΣX-*contexts*. If $p, q \in C_\Sigma(X)$, then $p \cdot q = q(p)$ is the ΣX-context obtained from q by replacing the ξ in it with p. Similarly, if $t \in T_\Sigma(X)$ and $p \in C_\Sigma(X)$, then $t \cdot p = p(t)$ is the ΣX-tree obtained when the ξ in p is replaced with t. The ξ-*depth* $\mathrm{d}_\xi(p)$, i.e., the distance of the ξ-labelled leaf from the root, of a ΣX-context $p \in C_\Sigma(X)$ is defined as follows:

(1) $\mathrm{d}_\xi(\xi) = 0$;

(2) $\mathrm{d}_\xi(p) = \mathrm{d}_\xi(q) + 1$ for any $p = f(t_1, \ldots, t_{i-1}, q, t_{i+1}, \ldots, t_m)$ where $m > 0$, $f \in \Sigma_m$, $i \in [m]$, $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_m \in T_\Sigma(X)$ and $q \in C_\Sigma(X)$.

The ranked alphabet Σ is also used as a set of operation symbols, and a Σ-*algebra* $\mathcal{A}$ consists of a nonempty set A of elements and a Σ-indexed family of operations $(f^\mathcal{A} \mid f \in \Sigma)$ on A such that if $f \in \Sigma_m$ is a m-ary symbol, then $f^\mathcal{A} : A^m \to A$ is an m-ary operation on A. In particular, any nullary symbol $c \in \Sigma_0$ fixes a constant in A that we write as $c^\mathcal{A}$ (rather than $c^\mathcal{A}()$). We write simply $\mathcal{A} = (A, \Sigma)$ without any symbol for the assignment $f \mapsto f^\mathcal{A}$. Subalgebras, homomorphisms, direct products of such algebras are defined as usual (cf. [3] or [4], for example). If there is an isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, $\mathcal{A} \cong \mathcal{B}$ in symbols, and if there is an epimorphism $\varphi : \mathcal{A} \to \mathcal{B}$, then $\mathcal{B}$ is an *(epimorphic) image* of $\mathcal{A}$, $\mathcal{A} \twoheadleftarrow \mathcal{B}$ in symbols. A monomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is also called an *embedding*. Such an embedding exists exactly in case $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{B}$, and we express this situation by writing $\mathcal{A} \sqsubseteq \mathcal{B}$. Furthermore, $\mathcal{B}$ is said to *cover* $\mathcal{A}$ if $\mathcal{A}$ is an image of some subalgebra of $\mathcal{B}$. This we express by writing $\mathcal{A} \preceq \mathcal{B}$. Clearly, $\preceq$ generalizes both the subalgebra relation $\sqsubseteq$ and the epimorphic image relation $\twoheadleftarrow$.

A mapping $p : A \to A$ is called an *elementary translation* of $\mathcal{A} = (A, \Sigma)$ if there exist an $m > 0$, an $f \in \Sigma_m$, an $i \in [k]$, and elements $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in C$ such that $p(a) = f^{\mathcal{A}}(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m)$ for every $a \in A$. Let $\mathrm{ETr}(\mathcal{A})$ denote the set of elementary translations of $\mathcal{A}$. The set $\mathrm{Tr}(\mathcal{A})$ of all *translations* of $\mathcal{A}$ is defined as the smallest set of unary operations on A that contains the identity map $1_A : A \to A, a \mapsto a$, and all the elementary translations, and is closed under composition. It is well known (cf. [3, 4], for example) that any congruence of an algebra $\mathcal{A}$ is invariant with respect to every translation of $\mathcal{A}$, and that an equivalence on A is a congruence on $\mathcal{A}$ if it is invariant with respect to every elementary translation of $\mathcal{A}$.

The following lemma (cf. [24]) will be needed several times.

Lemma 2.1 *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism between two Σ-algebras $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$. For every translation $p \in \mathrm{Tr}(\mathcal{A})$ of $\mathcal{A}$ there is a translation $p_\varphi \in \mathrm{Tr}(\mathcal{B})$ of $\mathcal{B}$ such that $p(a)\varphi = p_\varphi(a\varphi)$ for every $a \in A$. If φ is surjective, then there exists for every $q \in \mathrm{Tr}(\mathcal{B})$ a $p \in \mathrm{Tr}(\mathcal{A})$ such that $q = p_\varphi$.* $\square$

If $T_\Sigma(X) \neq \emptyset$, i.e., if $\Sigma_0 \cup X \neq \emptyset$, then the ΣX-trees form the ΣX*-term algebra* $\mathcal{T}_\Sigma(X) = (T_\Sigma(X), \Sigma)$, where $c^{\mathcal{T}_\Sigma(X)} = c$ for any $c \in \Sigma_0$, and $f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ for all $m > 0$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T_\Sigma(X)$. The ΣX-term algebra $\mathcal{T}_\Sigma(X)$ is *freely generated* by X over the class of all Σ-algebras, that is to say, it is generated by X and any mapping $\alpha : X \to A$ of X into any Σ-algebra $\mathcal{A} = (A, \Sigma)$ has a unique extension to a homomorphism $\widehat{\alpha} : \mathcal{T}_\Sigma(X) \to \mathcal{A}$. There is a bijective correspondence between the translations of the term algebra $\mathcal{T}_\Sigma(X)$ and the ΣX-contexts: for any $p \in \mathrm{Tr}(\mathcal{T}_\Sigma(X))$, there is a unique $q \in C_\Sigma(X)$ such that $p(t) = q(t)$ for every $t \in T_\Sigma(X)$, and conversely every ΣX-context defines a translation of $\mathcal{T}_\Sigma(X)$.

Let A be a nonempty set and let $n \geq 1$. If $\varphi_1 : A^n \to A, \ldots, \varphi_n : A^n \to A$ are any n-ary operations on A, then the n-tuple $\langle \varphi_1, \ldots, \varphi_n \rangle$ defines an operation

$$\varphi : A^n \to A^n,\ \mathbf{a} \mapsto \langle \varphi_1(\mathbf{a}), \ldots, \varphi_n(\mathbf{a}) \rangle \qquad (\mathbf{a} \in A^n).$$

We write $\varphi = \langle\varphi_1, \ldots, \varphi_n\rangle$ and call such mappings $A^n \to A^n$ the *n-operations* on A. Let $TM^n(A)$ denote the set of all n-operations on A. The *composition* of two n-operations $\varphi = \langle\varphi_1, \ldots, \varphi_n\rangle$ and $\psi = \langle\psi_1, \ldots, \psi_n\rangle$ on A is the mapping

$$\varphi \circ \psi : A^n \to A^n,\ \mathbf{a} \mapsto \psi(\varphi(\mathbf{a})),$$

i.e., $(\varphi \circ \psi)(\mathbf{a}) = \langle\psi_1(\varphi(\mathbf{a})), \ldots, \psi_n(\varphi_n(\mathbf{a}))\rangle$ for every $\mathbf{a} \in A^n$. Obviously, $\varphi \circ \psi \in TM^n(A)$. Moreover, it is clear that $\varphi \circ (\psi \circ \eta) = (\varphi \circ \psi) \circ \eta$ for all $\varphi, \psi, \eta \in TM^n(A)$, and that $1_{A^n} \circ \varphi = \varphi \circ 1_{A^n} = \varphi$ for every $\varphi \in TM^n(A)$. Moreover, $1_{A^n} = \langle\pi_1, \ldots, \pi_n\rangle$, where $\pi_i : A^n \to A$, $\mathbf{a} \mapsto a_i$, is the i^{th} n-ary projection ($i \in [n]$). This means that $(TM^n(A), \circ, 1_{A^n})$ is a monoid. Note, however, that $TM^n(A)$ is not the full transformation monoid of A^n because its elements consist of n independent n-ary operations on A.

3 Terms with variables

In this section we consider terms with variables and various related notions and some technical facts about them.

We arbitrarily fix a ranked alphabet Σ and a leaf alphabet X. Let $\Xi := \{\xi_1, \xi_2, \ldots\}$ be a countably infinite set of symbols that are treated as *variables* and do not appear in Σ or X. For any $n \geq 0$, let $\Xi_n := \{\xi_1, \ldots, \xi_n\}$. The elements of $T_\Sigma(X \cup \Xi_n)$ we call *n-ary ΣX-terms*, and the elements of $T_\Sigma(X \cup \Xi) = \bigcup_{n\geq 0} T_\Sigma(X \cup \Xi_n)$ are called *ΣX-terms with variables*.

For a ΣX-term with variables $t \in T_\Sigma(X \cup \Xi)$, let $\mathrm{nv}(t)$ be the number of occurrences of variables $\xi_i \in \Xi$ in t. Furthermore, we define $\mathrm{root}(t)$ (the label of the *root* of t), the *height* $\mathrm{hg}(t)$ and the *depth* $\mathrm{dp}(t)$ (the minimal distance of a Ξ-labelled leaf from the root) as follows:

(1) $\mathrm{root}(\xi_i) = \xi_i$ and $\mathrm{hg}(\xi_i) = \mathrm{dp}(\xi_i) = 0$ for $\xi_i \in \Xi$,

(2) $\mathrm{root}(t) = t$, $\mathrm{hg}(t) = 0$ and $\mathrm{dp}(t) = \infty$ for $t \in X \cup \Sigma_0$, and

(3) if $t = f(t_1, \ldots, t_m)$, then $\mathrm{root}(t) = f$, $\mathrm{hg}(t) = \max\{\mathrm{hg}(t_1), \ldots, \mathrm{hg}(t_m)\} + 1$ and $\mathrm{dp}(t) = 1 + \min\{\mathrm{dp}(t_1), \ldots, \mathrm{dp}(t_m)\}$.

Obviously, these notions are generalizations of the usual ones defined for ΣX-trees or ΣX-contexts (cf. [7, 8], for example).

Let $T_\Sigma(X \cup \Xi)^*$ be the set of all finite sequences of ΣX-terms with variables, including the empty sequence $\langle\rangle$, i.e., $T_\Sigma(X \cup \Xi)^* = \bigcup_{n\geq 0} T_\Sigma(X \cup \Xi)^n$, where for each

$n \geq 0$, $T_\Sigma(X \cup \Xi)^n$ is the set of all n-tuples $\langle u_1, \ldots, u_n \rangle$ of ΣX-terms with variables. The concatenation $\langle u_1, \ldots, u_h, v_1, \ldots, v_i \rangle$ of any two such sequences $\mathbf{u} = \langle u_1, \ldots, u_h \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_i \rangle$ is denoted by $\mathbf{u} \oplus \mathbf{v}$.

Definition 3.1 For any $n, m, k \geq 0$, $v \in T_\Sigma(X \cup \Xi_m)$, $p \in C_\Sigma(X)$, $\mathbf{u} = \langle u_1, \ldots, u_m \rangle \in T_\Sigma(X \cup \Xi_n)^m$ and $\mathbf{v} = \langle v_1, \ldots, v_k \rangle \in T_\Sigma(X \cup \Xi_m)^k$, let

(1) $v \cdot p = p(v)$ be the m-ary ΣX-tree obtained when the ξ in p is replaced with t,

(2) $\mathbf{u} \cdot v = v(\mathbf{u})$ be the n-ary ΣX-term obtained by replacing each variable ξ_i in v with the corresponding u_i ($i \in [m]$), and let

(3) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v}(\mathbf{u}) := \langle \mathbf{u} \cdot v_1, \ldots, \mathbf{u} \cdot v_k \rangle$.

Furthermore, let $\mathbf{1}_n := \langle \xi_1, \ldots, \xi_n \rangle$. □

Clearly, $\mathbf{u} \cdot \mathbf{v} \in T_\Sigma(X \cup \Xi_n)^k$ for any $\mathbf{u} \in T_\Sigma(X \cup \Xi_n)^m$ and $\mathbf{v} \in T_\Sigma(X \cup \Xi_m)^k$ (as above). In particular, $\mathbf{t} \cdot \mathbf{v} \in T_\Sigma(X)^k$ for every m-tuple $\mathbf{t} = \langle t_1, \ldots, t_m \rangle \in T_\Sigma(X)^m$ of ΣX-trees. The following properties of these products are also quite obvious.

Lemma 3.2 *For any $k, l, m, n \geq 0$, and any $\mathbf{u} \in T_\Sigma(X \cup \Xi_n)^m$, $\mathbf{v} \in T_\Sigma(X \cup \Xi_m)^l$, $\mathbf{w} \in T_\Sigma(X \cup \Xi_l)^k$, $\mathbf{t} \in T_\Sigma(X)^n$, $u \in T_\Sigma(X \cup \Xi_m)$, $v \in T_\Sigma(X \cup \Xi_n)$, and $p \in C_\Sigma(X)$,*

(a) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

(b) $\mathbf{t} \cdot (\mathbf{u} \cdot \mathbf{v}) = (\mathbf{t} \cdot \mathbf{u}) \cdot \mathbf{v}$

(c) $\mathbf{1}_n \cdot \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} \cdot \mathbf{1}_m = \mathbf{u}$

(e) $(\mathbf{u} \cdot u) \cdot p = \mathbf{u} \cdot (u \cdot p)$

(f) $(\mathbf{t} \cdot v) \cdot p = \mathbf{t} \cdot (v \cdot p)$

(g) $\mathbf{t} \cdot (\mathbf{u} \cdot u) = (\mathbf{t} \cdot \mathbf{u}) \cdot u$.

□

4 Generalized contexts

In this section we introduce the generalized contexts that will be used for defining our new syntactic monoids and semigroups of tree languages.

Definition 4.1 For any $n \geq 0$, a ΣXn*-context* is an n-tuple $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ of n-ary ΣX-terms in which each of the variables $\xi_1, \ldots, \xi_n$ appears exactly once (in exactly one of the components u_i). A ΣXn-context $\mathbf{u}$ is *proper* if none of its components u_i is a variable (in Ξ_n). Let us denote the sets of ΣXn-contexts and proper ΣXn-contexts by $M_\Sigma^n(X)$ and $S_\Sigma^n(X)$, respectively.

Obviously, $S^n_\Sigma(X) \subseteq M^n_\Sigma(X) \subseteq T_\Sigma(X \cup \Xi_n)^n$ for every $n \geq 0$. Note that $M^1_\Sigma(X) = C_\Sigma(X)$ if we replace ξ_1 with ξ and identify any $\Sigma X1$-context $\langle u \rangle$ with the ΣX-context u. It is also immediately clear that

(a) if $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$, then $\mathbf{u} \cdot \mathbf{v} \in M^n_\Sigma(X)$,

(b) if $\mathbf{u}, \mathbf{v} \in S^n_\Sigma(X)$, then $\mathbf{u} \cdot \mathbf{v} \in S^n_\Sigma(X)$, and that

(c) $\mathbf{t} \cdot \mathbf{u} \in T_\Sigma(X)^n$ for any $\mathbf{t} \in T_\Sigma(X)^n$ and $\mathbf{u} \in M^n_\Sigma(X)$.

Moreover, from Lemma 3.2 it follows, that for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in M^n_\Sigma(X)$,

(d) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ and $\mathbf{1}_n \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{1}_n = \mathbf{u}$,

and hence $(M^n_\Sigma(X), \cdot\ , \mathbf{1}_n)$ is a monoid and $(S^n_\Sigma(X), \cdot)$ is a semigroup. As usual, they are denoted simply by $M^n_\Sigma(X)$ and $S^n_\Sigma(X)$, respectively. Note that $M^0_\Sigma(X) = S^0_\Sigma(X) = \{\langle\rangle\}$ is the trivial semigroup (and monoid). We shall mostly ignore this special case.

Lemma 4.2 *$M^n_\Sigma(X) \sqsubseteq M^{n+1}_\Sigma(X)$ and $S^n_\Sigma(X) \preceq S^{n+1}_\Sigma(X)$ for every $n \geq 0$.*

Proof. It is easy to see that $\varphi : M^n_\Sigma(X) \to M^{n+1}_\Sigma(X)$, $\mathbf{u} \mapsto \mathbf{u} \oplus \langle \xi_{n+1} \rangle$, is a monomorphism of monoids, and hence $M^n_\Sigma(X) \sqsubseteq M^{n+1}_\Sigma(X)$ holds. As to the second claim, it suffices to note that

$$S := \{\langle u_1, \ldots, u_n, u_{n+1} \rangle \in S^{n+1}_\Sigma(X) \mid \langle u_1, \ldots, u_n \rangle \in S^n_\Sigma(X)\}$$

is a subsemigroup of $S^{n+1}_\Sigma(X)$ and that

$$\psi : S \to S^n_\Sigma(X),\ \langle u_1, \ldots, u_n, u_{n+1} \rangle \mapsto \langle u_1, \ldots, u_n \rangle,$$

is an epimorphism. □

The following lemma is quite obvious.

Lemma 4.3 *If $X \subseteq Y$, then $M^n_\Sigma(X) \sqsubseteq M^n_\Sigma(Y)$ and $S^n_\Sigma(X) \sqsubseteq S^n_\Sigma(Y)$ for all $n \geq 1$.* □

5 n-ary translations of algebras

It is easy to see that the translations of a Σ-algebra $\mathcal{A} = (A, \Sigma)$ are defined in a natural way by ΣA-contexts. We shall now generalize this idea by introducing for each $n \geq 1$, the n-ary translations of an algebra defined by ΣAn-contexts.

Let us consider any Σ-algebra $\mathcal{A} = (A, \Sigma)$ and any $n \geq 1$. We associate with any n-ary ΣA-term $u \in T_\Sigma(A \cup \Xi_n)$ an n-ary operation $u^\mathcal{A} : A^n \to A$ on A as follows: for any $\mathbf{a} = \langle a_1, \ldots, a_n \rangle \in A^n$, let

(1) $\xi_i^\mathcal{A}(\mathbf{a}) = a_i$ for every $i \in [n]$ (i.e., $\xi_i^\mathcal{A}$ is the i^{th} n-ary projection operation), $a^\mathcal{A}(\mathbf{a}) = a$ for every $a \in A$, and $c^\mathcal{A}(\mathbf{a}) = c^\mathcal{A}$ for every $c \in \Sigma_0$, and let

(2) $t^\mathcal{A}(\mathbf{a}) = f^\mathcal{A}(u_1^\mathcal{A}(\mathbf{a}), \ldots, u_m^\mathcal{A}(\mathbf{a}))$ if $t = f(u_1, \ldots, u_m)$ for some $m \geq 1$, $f \in \Sigma_m$ and $u_1, \ldots, u_m \in T_\Sigma(A \cup \Xi_n)$.

Hence, $a^\mathcal{A}$ and $c^\mathcal{A}$ are constant operations ($a \in A, c \in \Sigma_0$). Note that the n-ary ΣA-terms are actually the *n-ary polynomial symbols* of $\mathcal{A}$, and the operations $u^\mathcal{A}$ we just defined are, in fact, the *n-ary polynomial functions* of $\mathcal{A}$ (cf. [3, 12], for example).

Now, any n-tuple $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ of n-ary ΣA-terms, defines an n-operation

$$\mathbf{u}^\mathcal{A} : A^n \to A^n,\ \mathbf{a} \mapsto \langle u_1^\mathcal{A}(\mathbf{a}), \ldots, u_n^\mathcal{A}(\mathbf{a}) \rangle,$$

on A. The composition of any two such n-operations $\mathbf{u}^\mathbf{A}$ and $\mathbf{v}^\mathbf{A}$, where $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ are n-tuples of n-ary ΣA-terms, is the n-operation

$$\mathbf{u}^\mathbf{A} \circ \mathbf{v}^\mathbf{A} : A^n \to A^n,\ \mathbf{a} \mapsto \langle v_1^\mathbf{A}(\mathbf{u}^\mathbf{A}(\mathbf{a})), \ldots, v_n^\mathbf{A}(\mathbf{u}^\mathbf{A}(\mathbf{a})) \rangle.$$

It is easy to verify the following facts.

Lemma 5.1 *Let $\mathcal{A} = (A, \Sigma)$ be a Σ-algebra and let $n \geq 1$. For any n-tuples $\mathbf{u}$ and $\mathbf{v}$ of n-ary ΣA-terms, $\mathbf{u}^\mathcal{A} \circ \mathbf{v}^\mathcal{A} = (\mathbf{u} \cdot \mathbf{v})^\mathcal{A}$. Furthermore, $\mathbf{1}_n^\mathcal{A} = 1_{A^n}$.* $\square$

The lemma implies that the mappings $\mathbf{u}^\mathcal{A}$ form a submonoid of the monoid $TM^n(A)$ of n-operations on A. We shall be mainly interested in the following submonoid and subsemigroup of this monoid.

Definition 5.2 For any Σ-algebra $\mathcal{A} = (A, \Sigma)$ and any $n \geq 1$,

$$TM^n(\mathcal{A}) := \{\mathbf{u}^\mathcal{A} \mid \mathbf{u} \in M_\Sigma^n(A)\} \quad \text{and} \quad TS^n(\mathcal{A}) := \{\mathbf{u}^\mathcal{A} \mid \mathbf{u} \in S_\Sigma^n(A)\}$$

are called the *monoid of n-translations* and the *semigroup of proper n-translations* of $\mathcal{A}$, respectively. $\square$

Since $\mathbf{u} \cdot \mathbf{v} \in M_\Sigma^n(A)$ for any $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(A)$ and $\mathbf{1}_n \in M_\Sigma^n(A)$, it follows from Lemma 5.1 that $TM^n(\mathcal{A})$ indeed forms a monoid. Similarly, $TS^n(\mathcal{A})$ forms a semigroup because $\mathbf{u} \cdot \mathbf{v} \in S_\Sigma^n(A)$ for all $\mathbf{u}, \mathbf{v} \in S_\Sigma^n(A)$. Let us also note that the monoids $TM^n(\mathcal{A})$ are submonoids of the monoids considered by Sommerhalder in [22]. Moreover, Lemma 5.1 yields the following facts.

Corollary 5.3 *For any algebra $\mathcal{A} = (A, \Sigma)$ and any $n \geq 1$, the mappings*

$$M_\Sigma^n(A) \to TM^n(\mathcal{A}),\ \mathbf{u} \mapsto \mathbf{u}^\mathcal{A}, \quad \text{and} \quad S_\Sigma^n(A) \to TS^n(\mathcal{A}),\ \mathbf{u} \mapsto \mathbf{u}^\mathcal{A},$$

are epimorphisms of monoids and semigroups, respectively. □

Let us now establish some of the basic properties of these monoids and semigroups.

Proposition 5.4 *Let $\mathcal{A}$ be a Σ-algebra. For every $n \geq 1$,*

(a) $TM^n(\mathcal{A}) \sqsubseteq TM^{n+1}(\mathcal{A})$, *and*

(b) $TS^n(\mathcal{A}) \preceq TS^{n+1}(\mathcal{A})$.

Proof. To prove (a), it suffices to verify that

$$\varphi : TM^n(\mathcal{A}) \to TM^{n+1}(\mathcal{A}),\ \mathbf{u}^\mathcal{A} \mapsto (\mathbf{u} \oplus \langle \xi_{n+1} \rangle)^\mathcal{A},$$

is a well-defined monomorphism of monoids. To prove (b), we first note that

$$S := \{\langle u_1, \ldots, u_n, u_{n+1} \rangle^\mathcal{A} \in TS^{n+1}(\mathcal{A}) \mid \langle u_1, \ldots, u_n \rangle \in S_\Sigma^n(A)\}$$

is a subsemigroup of $TS^{n+1}(\mathcal{A})$. Then it suffices to verify that

$$\psi : S \to TS^n(\mathcal{A}),\ \langle u_1, \ldots, u_n, u_{n+1} \rangle^\mathcal{A} \mapsto \langle u_1, \ldots, u_n \rangle^\mathcal{A},$$

is an epimorphism. □

The following observation is easy to verify.

Lemma 5.5 *If $\mathcal{A} = (A, \Sigma)$ is a subalgebra of $\mathcal{B} = (B, \Sigma)$, then $\mathbf{u}^\mathcal{A} = \mathbf{u}^\mathcal{B}\restriction_A$ for any $n \geq 1$ and any n-tuple of n-ary ΣA-terms $\mathbf{u}$.* □

Proposition 5.6 *Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be Σ-algebras. If $\mathcal{A} \sqsubseteq \mathcal{B}$, then $TM^n(\mathcal{A}) \preceq TM^n(\mathcal{B})$ and $SM^n(\mathcal{A}) \preceq SM^n(\mathcal{B})$ for every $n \geq 1$.*

Proof. Let us prove the statement that concerns monoids; the other one has a similar proof. We may assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, and let us consider any $n \geq 1$. Then $M^n_\Sigma(A) \subseteq M^n_\Sigma(B)$ and clearly $M := \{\mathbf{u}^\mathcal{B} \mid \mathbf{u} \in M^n_\Sigma(A)\}$ is a submonoid of $TM^n(\mathcal{B})$. It is immediately clear that

$$\varphi : M \to TM^n(\mathcal{A}),\ \mathbf{u}^\mathcal{B} \mapsto \mathbf{u}^\mathcal{A},$$

is well-defined and surjective. Moreover, for any $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(A)$,

$$(\mathbf{u}^\mathcal{B} \circ \mathbf{v}^\mathcal{B})\varphi = (\mathbf{u} \cdot \mathbf{v})^\mathcal{B}\varphi = (\mathbf{u} \cdot \mathbf{v})^\mathcal{A} = \mathbf{u}^\mathcal{A} \circ \mathbf{v}^\mathcal{A} = \mathbf{u}^\mathcal{B}\varphi \circ \mathbf{v}^\mathcal{B}\varphi.$$

and $1_{B^n}\varphi = \mathbf{1}^\mathcal{B}_n\varphi = \mathbf{1}^\mathcal{A}_n = 1_{A^n}$. Hence φ is an epimorphism. $\square$

Proposition 5.7 *Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be any Σ-algebras. If $\mathcal{A} \twoheadleftarrow \mathcal{B}$, then $TM^n(\mathcal{A}) \twoheadleftarrow TM^n(\mathcal{B})$ and $SM^n(\mathcal{A}) \twoheadleftarrow SM^n(\mathcal{B})$ for every $n \geq 1$.*

Proof. Again, we consider the monoid case. Let $\varphi : \mathcal{B} \to \mathcal{A}$ be an epimorphism. It is easy to verify that it can be extended to an epimorphism $\overline{\varphi} : \mathcal{T}_\Sigma(B \cup \Xi_n) \to \mathcal{T}_\Sigma(A \cup \Xi_n)$ of term algebras by the following conditions:

(1) $b\overline{\varphi} = b\varphi$ for $b \in B$; $\xi_i\overline{\varphi} = \xi_i$ for $i \in [n]$; $c\overline{\varphi} = c$ for $c \in \Sigma_0$;

(2) $f(t_1, \ldots, t_n)\overline{\varphi} = f(t_1\overline{\varphi}, \ldots, t_m\overline{\varphi})$ for $m > 0$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T_\Sigma(B \cup \Xi_n)$.

If we write $\mathbf{b}\varphi := \langle b_1\varphi, \ldots, b_n\varphi\rangle$ for any $\mathbf{b} = \langle b_1, \ldots, b_n\rangle \in B^n$, then it is easy to show by induction on t that

$$t^\mathcal{B}(\mathbf{b})\varphi = (t\overline{\varphi})^\mathcal{A}(\mathbf{b}\varphi), \tag{1}$$

for all $t \in T_\Sigma(B \cup \Xi_n)$ and $\mathbf{b} \in B^n$. It is also easy to verify that

$$\psi : M^n_\Sigma(B) \to M^n_\Sigma(A),\ \langle u_1, \ldots, u_n\rangle \mapsto \langle u_1\overline{\varphi}, \ldots, u_n\overline{\varphi}\rangle,$$

is a monoid epimorphism, i.e., $\mathbf{u}\psi \in M^n_\Sigma(A)$ for every $\mathbf{u} \in M^n_\Sigma(B)$, $(\mathbf{u}\cdot\mathbf{v})\psi = \mathbf{u}\psi\cdot\mathbf{v}\psi$ for all $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(B)$, $\mathbf{1}_n\psi = \mathbf{1}_n$, and ψ is surjective. Now we are ready to show that

$$\eta : TM^n(\mathcal{B}) \to TM^n(\mathcal{A}),\ \mathbf{u}^\mathcal{B} \mapsto (\mathbf{u}\psi)^\mathcal{A},$$

yields the required epimorphism.

1. To show that η is well-defined, let $\mathbf{u} = \langle u_1, \ldots, u_n \rangle, \mathbf{v} = \langle v_1, \ldots, v_n \rangle \in M_\Sigma^n(B)$ be such that $\mathbf{u}^\mathcal{B} = \mathbf{v}^\mathcal{B}$, and consider any $\mathbf{a} = \langle a_1, \ldots, a_n \rangle \in A^n$. Since φ is surjective, $\mathbf{a} = \mathbf{b}\varphi$ for some $\mathbf{b} \in B^n$. Hence

$$\begin{aligned}
(\mathbf{u}^\mathcal{B}\eta)(\mathbf{a}) &= (\mathbf{u}\psi)^\mathcal{A}(\mathbf{b}\varphi) \\
&= \langle (u_1\overline{\varphi})^\mathcal{A}(\mathbf{b}\varphi), \ldots, (u_n\overline{\varphi})^\mathcal{A}(\mathbf{b}\varphi) \rangle \\
&= \langle u_1^\mathcal{B}(\mathbf{b})\varphi, \ldots, u_n^\mathcal{B}(\mathbf{b})\varphi \rangle \qquad \text{(by equation (1))} \\
&= \mathbf{u}^\mathcal{B}(\mathbf{b})\varphi \\
&= \mathbf{v}^\mathcal{B}(\mathbf{b})\varphi \;=\; \ldots \;=\; (\mathbf{v}^\mathcal{B}\eta)(\mathbf{a})
\end{aligned}$$

2. For all $\mathbf{u} = \langle u_1, \ldots, u_n \rangle, \mathbf{v} = \langle v_1, \ldots, v_n \rangle \in M_\Sigma^n(B)$,

$$\begin{aligned}
(\mathbf{u}^\mathcal{B} \circ \mathbf{v}^\mathcal{B})\eta &= (\mathbf{u} \cdot \mathbf{v})^\mathcal{B}\eta = ((\mathbf{u} \cdot \mathbf{v})\psi)^\mathcal{A} = ((\mathbf{u}\psi) \cdot (\mathbf{v}\psi))^\mathcal{A} \\
&= (\mathbf{u}\psi)^\mathcal{A} \circ (\mathbf{v}\psi)^\mathcal{A} = \mathbf{u}^\mathcal{B}\eta \circ \mathbf{v}^\mathcal{B}\eta.
\end{aligned}$$

3. $1_{B^n}\eta = \mathbf{1}_n^\mathcal{B}\eta = (\mathbf{1}_n\psi)^\mathcal{A} = \mathbf{1}_n^\mathcal{A} = 1_{A^n}$. □

Proposition 5.8 *For any Σ-algebras $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma))$ and every $n \geq 1$,*

$$TM^n(\mathcal{A} \times \mathcal{B}) \sqsubseteq TM^n(\mathcal{A}) \times TM^n(\mathcal{B}) \quad \text{and} \quad TS^n(\mathcal{A} \times \mathcal{B}) \sqsubseteq TS^n(\mathcal{A}) \times TS^n(\mathcal{B}).$$

Proof. We show just the first relation; a proof for the second one can be obtained by easy modifications. Let

$$\pi_A : T_\Sigma((A \times B) \cup \Xi_n) \to T_\Sigma(A \cup \Xi_n) \quad \text{and} \quad \pi_B : T_\Sigma((A \times B) \cup \Xi_n) \to T_\Sigma(B \cup \Xi_n)$$

be the epimorphisms such that $(a, b)\pi_A = a$ and $(a, b)\pi_B = b$ for every $(a, b) \in A \times B$, and $\xi_i\pi_A = \xi_i\pi_B = \xi_i$ for every $i \in [n]$. For any $t \in T_\Sigma((A \times B) \cup \Xi_n)$, the images $t\pi_A$ and $t\pi_B$ are obtained simply by replacing in t every appearance of a symbol $(a, b) \in A \times B$ by a and b, respectively. Hence, it is clear that for any $\mathbf{u} = \langle u_1, \ldots, u_n \rangle \in M_\Sigma^n(A \times B)$,

$$\mathbf{u}\psi_1 := \langle u_1\pi_A, \ldots, u_n\pi_A \rangle \in M_\Sigma^n(A) \quad \text{and} \quad \mathbf{u}\psi_2 := \langle u_1\pi_B, \ldots, u_n\pi_B \rangle \in M_\Sigma^n(B).$$

In fact, $\psi_1 : M_\Sigma^n(A \times B) \to M_\Sigma^n(A)$ and $\psi_2 : M_\Sigma^n(A \times B) \to M_\Sigma^n(B)$ are epimorphisms of monoids such that for any $\mathbf{u} \in M_\Sigma^n(A \times B)$, $\mathbf{u}\psi_1$ and $\mathbf{u}\psi_2$ are obtained from $\mathbf{u}$ by replacing everywhere all symbols $(a, b) \in A \times B$ by a and b, respectively. This also means that, for any $\mathbf{u} \in M_\Sigma^n(A \times B)$ and $\langle (a_1, b_1), \ldots, (a_n, b_n) \rangle \in (A \times B)^n$, we have

$$\mathbf{u}^{\mathcal{A}\times\mathcal{B}}((a_1, b_1), \ldots, (a_n, b_n)) = ((\mathbf{u}\psi_1)^\mathcal{A}(a_1, \ldots, a_n), (\mathbf{u}\psi_2)^\mathcal{B}(b_1, \ldots, b_n)). \tag{2}$$

Now, we can conclude the proof by showing that

$$\varphi : TM^n(\mathcal{A}\times\mathcal{B}) \to TM^n(\mathcal{A})\times TM^n(\mathcal{B}),\ \mathbf{u}^{\mathcal{A}\times\mathcal{B}} \mapsto ((\mathbf{u}\psi_1)^{\mathcal{A}}, (\mathbf{u}\psi_2)^{\mathcal{B}}),$$

is a monomorphism.

1. For any $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(A\times B)$,

   $$\mathbf{u}^{\mathcal{A}\times\mathcal{B}}\varphi = \mathbf{v}^{\mathcal{A}\times\mathcal{B}}\varphi \ \Leftrightarrow\ (\mathbf{u}\psi_1)^{\mathcal{A}} = (\mathbf{v}\psi_1)^{\mathcal{A}},\ (\mathbf{u}\psi_2)^{\mathcal{B}} = (\mathbf{v}\psi_2)^{\mathcal{B}} \ \Leftrightarrow\ \mathbf{u}^{\mathcal{A}\times\mathcal{B}} = \mathbf{v}^{\mathcal{A}\times\mathcal{B}},$$

   where we made use of equation (2). This shows that φ is well-defined and injective.

2. For any $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(A\times B)$,

   $$\begin{aligned}(\mathbf{u}^{\mathcal{A}\times\mathcal{B}} \circ \mathbf{v}^{\mathcal{A}\times\mathcal{B}})\varphi &= (\mathbf{u}\cdot\mathbf{v})^{\mathcal{A}\times\mathcal{B}}\varphi \\ &= (((\mathbf{u}\cdot\mathbf{v})\psi_1)^{\mathcal{A}}, ((\mathbf{u}\cdot\mathbf{v})\psi_2)^{\mathcal{B}}) \\ &= ((\mathbf{u}\psi_1\cdot\mathbf{v}\psi_1)^{\mathcal{A}}, (\mathbf{u}\psi_2\cdot\mathbf{v}\psi_2)^{\mathcal{B}}) \\ &= ((\mathbf{u}\psi_1)^{\mathcal{A}}\circ(\mathbf{v}\psi_1)^{\mathcal{A}}, (\mathbf{u}\psi_2)^{\mathcal{B}}\circ(\mathbf{v}\psi_2)^{\mathcal{B}}) \\ &= ((\mathbf{u}\psi_1)^{\mathcal{A}}, (\mathbf{u}\psi_2)^{\mathcal{B}})\circ((\mathbf{v}\psi_1)^{\mathcal{A}}, (\mathbf{v}\psi_2)^{\mathcal{B}}) \\ &= \mathbf{u}^{\mathcal{A}\times\mathcal{B}}\varphi\circ\mathbf{v}^{\mathcal{A}\times\mathcal{B}}\varphi.\end{aligned}$$

3. Finally, $\mathbf{1}_n^{\mathcal{A}\times\mathcal{B}}\varphi = ((\mathbf{1}_n\psi_1)^{\mathcal{A}}, (\mathbf{1}_n\psi_2)^{\mathcal{B}}) = (\mathbf{1}_n^{\mathcal{A}}, \mathbf{1}_n^{\mathcal{B}})$. □

6 Transformation semigroups of tree recognizers

Let us recall that a (*deterministic bottom-up*) ΣX-*recognizer* $\mathbf{A} = (\mathcal{A}, \alpha, F)$ consists of a Σ-algebra $\mathcal{A} = (A, \Sigma)$, an *initial assignment* $\alpha : X \to A$, and a set $F \subseteq A$ of *final states*; A is the *state set*. The ΣX-tree language *recognized* by $\mathbf{A}$ is the set

$$T(\mathbf{A}) \ := \ \{t \in T_\Sigma(X) \mid t\widehat{\alpha} \in F\},$$

where $\widehat{\alpha} : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ is the homomorphic extension of $\alpha : X \to A$. The ΣX-recognizer $\mathbf{A}$ is *finite* if $\mathcal{A}$ is a finite algebra, i.e., the set of states A is finite. A ΣX-tree language is called *recognizable*, or *regular*, if it is recognized by some ΣX-recognizer. Let $\mathrm{Rec}_\Sigma(X)$ be the set of all recognizable ΣX-tree languages.

Let $\mathbf{A} = (\mathcal{A}, \alpha, F)$ be any ΣX-recognizer. The homomorphism $\widehat{\alpha} : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ can be extended in a natural way to the mapping

$$\widetilde{\alpha}_n : T_\Sigma(X)^n \to A^n,\ \langle t_1, \ldots, t_n\rangle \mapsto \langle t_1\widehat{\alpha}, \ldots, t_n\widehat{\alpha}\rangle.$$

Moreover, each n-ary ΣX-term $u \in T_\Sigma(X \cup \Xi_n)$ defines an operation $u^{\mathbf{A}} : A^n \to A$ as follows. For any $\mathbf{a} = \langle a_1, \ldots, a_n \rangle \in A^n$, let

(1) $x^{\mathbf{A}}(\mathbf{a}) = \alpha(x)$ for $x \in X$, $c^{\mathbf{A}}(\mathbf{a}) = c^{\mathcal{A}}$ for $c \in \Sigma_0$, $\xi_i^{\mathbf{A}}(\mathbf{a}) = a_i$ for $\xi_i \in \Xi_n$, and

(2) $u^{\mathbf{A}}(\mathbf{a}) = f^{\mathcal{A}}(u_1^{\mathbf{A}}(\mathbf{a}), \ldots, u_m^{\mathbf{A}}(\mathbf{a}))$ if $u = f(u_1, \ldots, u_m)$ for some $m > 0$, $f \in \Sigma_m$ and $u_1, \ldots, u_m \in T_\Sigma(X \cup \Xi_n)$.

Note that for any ΣX-term $t \in T_\Sigma(X)$, we get $t^{\mathbf{A}}(\mathbf{a}) = t\widehat{\alpha}$ for every $\mathbf{a} \in A^n$. Now, any n-tuple $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ of n-ary ΣX-terms defines in $\mathbf{A}$ an n-operation

$$\mathbf{u}^{\mathbf{A}} : A^n \to A^n,\ \mathbf{a} \mapsto \langle u_1^{\mathbf{A}}(\mathbf{a}), \ldots, u_n^{\mathbf{A}}(\mathbf{a}) \rangle.$$

The composition of any two such mappings $\mathbf{u}^{\mathbf{A}}$ and $\mathbf{v}^{\mathbf{A}}$, where $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, is the mapping $\mathbf{u}^{\mathbf{A}} \circ \mathbf{v}^{\mathbf{A}} : A^n \to A^n$ such that for every $\mathbf{a} \in A^n$,

$$(\mathbf{u}^{\mathbf{A}} \circ \mathbf{v}^{\mathbf{A}})(\mathbf{a}) = \langle v_1^{\mathbf{A}}(\mathbf{u}^{\mathbf{A}}(\mathbf{a})), \ldots, v_n^{\mathbf{A}}(\mathbf{u}^{\mathbf{A}}(\mathbf{a})) \rangle.$$

It is not hard to prove the following facts.

Lemma 6.1 *If $\mathbf{A} = (\mathcal{A}, \alpha, F)$ is a ΣX-recognizer and $n \geq 1$, then $\mathbf{u}^{\mathbf{A}} \circ \mathbf{v}^{\mathbf{A}} = (\mathbf{u} \cdot \mathbf{v})^{\mathbf{A}}$ for all $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$. Moreover, $\mathbf{1}_n^{\mathbf{A}} = 1_{A^n}$.* □

We shall focus on the n-operations $\mathbf{u}^{\mathbf{A}}$ defined by ΣXn-contexts. Thus, let $TM^n(\mathbf{A}) := \{\mathbf{u}^{\mathbf{A}} \mid \mathbf{u} \in M_\Sigma^n(X)\}$ and let $TS^n(\mathbf{A}) := \{\mathbf{u}^{\mathbf{A}} \mid \mathbf{u} \in S_\Sigma^n(X)\}$. We call the elements of $TM^n(\mathbf{A})$ the *n-transformations* of $\mathbf{A}$. Similarly, the elements of $TS^n(\mathbf{A})$ are the *proper n-transformations*. It follows from Lemma 6.1 that we get

(1) the *monoid of n-transformations* $(TM^n(\mathbf{A}), \cdot, \mathbf{1}_n^{\mathbf{A}})$, and

(2) the *semigroup of proper n-transformations* $(TS^n(\mathbf{A}), \cdot)$.

Of course, $TS^n(\mathbf{A})$ is a subsemigroup of $TM^n(\mathbf{A})$. Moreover, we can note that $TM^1(\mathbf{A})$ is the monoid $\mathrm{MT}(\mathbf{A}) = \{p^{\mathbf{A}} \mid p \in C_\Sigma(X)\}$ considered by Salomaa [20], and $TS^1(\mathbf{A})$ is the corresponding semigroup.

Let us recall (cf. [7, 8], for example) that a state $a \in A$ of a ΣX-recognizer $\mathbf{A} = (\mathcal{A}, \alpha, F)$ is *reachable* if $a = t\widehat{\alpha}$ for some $t \in T_\Sigma(X)$. If all the states of $\mathbf{A}$ are reachable, then $\mathbf{A}$ is said to be *connected*.

Proposition 6.2 *If $\mathbf{A} = (\mathcal{A}, \alpha, F)$ is a connected ΣX-recognizer, then $TM^n(\mathbf{A}) = TM^n(\mathcal{A})$ and $TS^n(\mathbf{A}) = TS^n(\mathcal{A})$ for every $n \geq 1$.*

Proof. To prove the inclusion $TM^n(\mathcal{A}) \subseteq TM^n(\mathbf{A})$, we begin by defining a mapping $\psi : T_\Sigma(A \cup \Xi_n) \to T_\Sigma(X \cup \Xi_n)$ as follows. Since $\mathbf{A}$ is connected, we may fix for each $a \in A$ a ΣX-tree $t_a \in T_\Sigma(X)$ such that $t_a\widehat{\alpha} = a$. Now, ψ is defined as follows:

(1) $a\psi = t_a$ for $a \in A$; $\xi_i\psi = \xi_i$ for $i \in [n]$; $c\psi = c$ for $c \in \Sigma_0$;

(2) $f(t_1, \ldots, t_m)\psi = f(t_1\psi, \ldots, t_m\psi)$ for all $m \geq 1$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T_\Sigma(X)$.

It is clear that ψ is a homomorphism of term algebras, and we may verify by induction on t that $t^{\mathcal{A}} = (t\psi)^{\mathbf{A}}$ for every $t \in T_\Sigma(A \cup \Xi_n)$. Moreover, if we extend ψ to n-tuples of n-ary ΣA-terms by setting $\mathbf{u}\psi = \langle u_1\psi, \ldots, u_n\psi\rangle$ for every $\mathbf{u} = \langle u_1, \ldots, u_n\rangle \in T_\Sigma(A \cup \Xi_n)^n$, then it is clear that $\mathbf{u}\psi \in M^n_\Sigma(X)$ for every $\mathbf{u} \in M^n_\Sigma(A)$. Hence, $\mathbf{u}^{\mathbf{A}} = (\mathbf{u}\psi)^{\mathcal{A}} \in TM^n(\mathbf{A})$ for every $\mathbf{u} \in M^n_\Sigma(A)$.

For the converse inclusion, we define $\varphi : T_\Sigma(X \cup \Xi_n) \to T_\Sigma(A \cup \Xi_n)$ as follows:

(1) $x\varphi = \alpha(x)$ for $x \in X$; $\xi_i\varphi = \xi_i$ for $i \in [n]$; $c\varphi = c$ for every $c \in \Sigma_0$;

(2) $f(t_1, \ldots, t_m)\varphi = f(t_1\varphi, \ldots, t_m\varphi)$ for all $m \geq 1$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T_\Sigma(X)$.

Obviously, φ is a homomorphism of term algebras, and it is easy to see that $t^{\mathbf{A}} = (t\varphi)^{\mathcal{A}}$ for every $t \in T_\Sigma(X \cup \Xi_n)$. Moreover, it is clear that $\mathbf{u}\varphi := \langle u_1\varphi, \ldots, u_n\varphi\rangle \in M^n_\Sigma(A)$ for every $\mathbf{u} = \langle u_1, \ldots, u_n\rangle \in M^n_\Sigma(X)$. Hence, $\mathbf{u}^{\mathbf{A}} = (\mathbf{u}\varphi)^{\mathcal{A}} \in TM^n(\mathcal{A})$ for every $\mathbf{u} \in M^n_\Sigma(X)$.

For the semigroups, the proof is quite similar. □

The following lemma is also useful.

Lemma 6.3 *For any connected ΣX-recognizer $\mathbf{A} = (\mathcal{A}, \alpha, F)$, the following conditions are pairwise equivalent:*

(a) $\mathbf{A}$ *is finite;*

(b) $TM^n(\mathbf{A})$ *is finite for every* $n \geq 1$; (b') $TS^n(\mathbf{A})$ *is finite for every* $n \geq 1$;

(c) $TM^n(\mathbf{A})$ *is finite for some* $n \geq 1$; (c') $TS^n(\mathbf{A})$ *is finite for some* $n \geq 1$.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (b) $\Rightarrow$ (b$'$) $\Rightarrow$ (c$'$) and (c) $\Rightarrow$ (c$'$) are obvious, so it suffices to show that (c') implies (a).

Assume that $TS^n(\mathbf{A})$ is finite for some $n \geq 1$. Consider any state $a \in A$. Since $\mathbf{A}$ is connected, there exists a ΣX-tree t such that $t\widehat{\alpha} = a$. If t is a tree of height ≥ 1, we can construct a proper ΣXn-context $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ such that u_1 is obtained from t by replacing the label of any single leaf by ξ_1, and hence $\mathbf{u}^{\mathbf{A}}(\mathbf{a}) = \mathbf{b}$ for some $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, \ldots, b_n \rangle$, where $a_1 = c^{\mathcal{A}}$ or $a_1 = \alpha(x)$ depending on whether the replaced symbol in t was $c \in \Sigma_0$ or $x \in X$, and $b_1 = a$. Since the set $\{c^{\mathcal{A}} \mid c \in \Sigma_0\} \cup \{\alpha(x) \mid x \in X\}$ is finite, there are only finitely many states $a \in A$ for which there is no $t \in T_\Sigma(X)$ such that $t\widehat{\alpha} = a$ and $\mathrm{hg}(t) \geq 1$. This shows that if $\mathbf{A}$ is infinite, then so is $TS^n(\mathbf{A})$, and hence also the implication (c$'$) $\Rightarrow$ (a) holds. □

7 n-ary syntactic congruences of tree languages

We shall now consider the congruences to be used for defining the syntactic monoids and semigroups introduced and studied in the next section.

Again, Σ is a ranked alphabet, X a leaf alphabet, and $n \geq 1$. Moreover, unless stated otherwise, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the vectors (of ΣX-terms with variables) $\langle u_1, \ldots, u_n \rangle$, $\langle v_1, \ldots, v_n \rangle$ and $\langle w_1, \ldots, w_n \rangle$, respectively.

Definition 7.1 For any $n \geq 1$, the *n-ary syntactic monoid congruence* of a ΣX-tree language T is the relation μ_T^n on $M_\Sigma^n(X)$ defined as follows: for any $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$,

$$\mathbf{u}\, \mu_T^n\, \mathbf{v} \;\Leftrightarrow\; (\forall i \in [n])(\forall \mathbf{t} \in T_\Sigma(X)^n)(\forall p \in C_\Sigma(X))(\mathbf{t} \cdot u_i \cdot p \in T \Leftrightarrow \mathbf{t} \cdot v_i \cdot p \in T).$$

Similarly, the *n-ary syntactic semigroup congruence* of T is the relation σ_T^n on $S_\Sigma^n(X)$ defined by stating that

$$\mathbf{u}\, \sigma_T^n\, \mathbf{v} \;\Leftrightarrow\; (\forall i \in [n])(\forall \mathbf{t} \in T_\Sigma(X)^n)(\forall p \in C_\Sigma(X))(\mathbf{t} \cdot u_i \cdot p \in T \Leftrightarrow \mathbf{t} \cdot v_i \cdot p \in T),$$

for any $\mathbf{u}, \mathbf{v} \in S_\Sigma^n(X)$. □

In Proposition 7.3 we will show that μ_T^n and σ_T^n really are congruences, and in Lemma 7.2 an alternative description of them is given.

For any $n \geq 1$, let us call an n-ary ΣX-term an *n-ary ΣX-context* if it contains at least one variable and none of the variables $\xi_1, \ldots, \xi_n$ appears in it more than once. Let $C_\Sigma^n(X)$ denote the set these generalized contexts.

Lemma 7.2 *Let T be a ΣX-tree language and let $n \geq 1$. For any $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$,*

$$\mathbf{u}\, \mu_T^n\, \mathbf{v} \;\Leftrightarrow\; (\forall \mathbf{t} \in T_\Sigma(X)^n)(\forall r \in C_\Sigma^n(X))(\mathbf{t} \cdot \mathbf{u} \cdot r \in T \Leftrightarrow \mathbf{t} \cdot \mathbf{v} \cdot r \in T).$$

Similarly,

$$\mathbf{u}\,\sigma_T^n\,\mathbf{v} \;\Leftrightarrow\; (\forall \mathbf{t} \in T_\Sigma(X)^n)(\forall r \in C_\Sigma^n(X))(\mathbf{t}\cdot\mathbf{u}\cdot r \in T \Leftrightarrow \mathbf{t}\cdot\mathbf{v}\cdot r \in T),$$

for all $\mathbf{u}, \mathbf{v} \in S_\Sigma^n(X)$. □

Let us say that a congruence θ on the monoid $M_\Sigma^n(X)$ *saturates* a ΣX-tree language T if, for all $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$,

$$\mathbf{u}\,\theta\,\mathbf{v} \;\Rightarrow\; (\forall i \in [n])(\forall \mathbf{t} \in T_\Sigma(X)^n)(\forall p \in C_\Sigma(X))(\mathbf{t}\cdot u_i \cdot p \in T \;\Leftrightarrow\; \mathbf{t}\cdot v_i \cdot p \in T).$$

The notion is defined for congruences on $S_\Sigma^n(X)$ exactly the same way. The following proposition expresses an important property of our syntactic congruences.

Proposition 7.3 *For any ΣX-tree language T and any $n \geq 1$, μ_T^n is the greatest congruence on the monoid $M_\Sigma^n(X)$ that saturates T. Similarly, σ_T^n is the greatest congruence on $S_\Sigma^n(X)$ that saturates T.*

Proof. We consider just μ_T^n; for σ_T^n the proof is almost the same. It is clear that μ_T^n is an equivalence on $M_\Sigma^n(X)$. Let us assume that $\mathbf{u}\,\mu_T^n\,\mathbf{v}$ and consider any $\mathbf{w} \in M_\Sigma^n(X)$. For any $\mathbf{t} \in T_\Sigma(X)^n$, $p \in C_\Sigma(X)$, and $i \in [n]$,

$$\mathbf{t}\cdot(\mathbf{u}\cdot w_i)\cdot p \in T \;\Leftrightarrow\; \mathbf{t}\cdot\mathbf{u}\cdot(w_i\cdot p) \in T \;\Leftrightarrow\; \mathbf{t}\cdot\mathbf{v}\cdot(w_i\cdot p) \in T \;\Leftrightarrow\; \mathbf{t}\cdot(\mathbf{v}\cdot w_i)\cdot p \in T,$$

where we used Lemma 3.2, the fact that $w_i \cdot p \in C_\Sigma^n(X)$ and Lemma 7.2, and hence $\mathbf{u}\cdot\mathbf{w}\;\mu_T^n\;\mathbf{v}\cdot\mathbf{w}$. Similarly, we get $\mathbf{w}\cdot\mathbf{u}\;\mu_T^n\;\mathbf{w}\cdot\mathbf{v}$ because

$$\mathbf{t}\cdot(\mathbf{w}\cdot u_i)\cdot p \in T \;\Leftrightarrow\; (\mathbf{t}\cdot\mathbf{w})\cdot u_i\cdot p \in T \;\Leftrightarrow\; (\mathbf{t}\cdot\mathbf{w})\cdot v_i\cdot p \in T \;\Leftrightarrow\; \mathbf{t}\cdot(\mathbf{w}\cdot v_i)\cdot p \in T,$$

for all $i \in [n]$, $\mathbf{t} \in T_\Sigma(X)^n$ and $p \in C_\Sigma(X)$. Hence, μ_T^n is a congruence.

It is immediately clear by the definitions that μ_T^n saturates $M_\Sigma^n(X)$. Assume that θ is a congruence on $M_\Sigma^n(X)$ that saturates T, and let $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$ be such that $\mathbf{u}\,\theta\,\mathbf{v}$. Then, if $\mathbf{t}\cdot u_i\cdot p \in T$ for some $\mathbf{t} \in T_\Sigma(X)^n$, $p \in C_\Sigma^n(X)$ and $i \in [n]$, then also $\mathbf{t}\cdot v_i \cdot p \in T$, and conversely. This means that $\mathbf{u}\,\mu_T^n\,\mathbf{v}$, and hence $\theta \subseteq \mu_T^n$. □

In statement (d) of Proposition 7.5 we use the following extensions of a given homomorphism $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ of term algebras. For any $n \geq 1$, we first extend φ to a homomorphism $\varphi_n : \mathcal{T}_\Sigma(X \cup \Xi_n) \to \mathcal{T}_\Sigma(Y \cup \Xi_n)$ by setting $\xi_i\varphi_n = \xi_i$ for every $i \in [n]$. Next, we extend φ_n to

$$\overline{\varphi}_n : T_\Sigma(X \cup \Xi_n)^n \to T_\Sigma(Y \cup \Xi_n)^n,\ \langle u_1, \ldots, u_n\rangle \mapsto \langle u_1\varphi_n, \ldots, u_n\varphi_n\rangle.$$

The following observations have straightforward proofs.

Lemma 7.4 *For any homomorphism $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ and any $n \geq 1$, the following hold.*

(a) $(\mathbf{u} \cdot v)\varphi_n = \mathbf{u}\overline{\varphi}_n \cdot v\varphi_n$ *for all* $\mathbf{u} \in T_\Sigma(X \cup \Xi_n)^n$ *and* $v \in T_\Sigma(X \cup \Xi_n)$.

(b) *Restricted to* $M^n_\Sigma(X)$, $\overline{\varphi}_n$ *yields a monoid homomorphism* $\widehat{\varphi}_n : M^n_\Sigma(X) \to M^n_\Sigma(Y)$, *i.e., if we set* $\mathbf{u}\widehat{\varphi}_n := \mathbf{u}\overline{\varphi}_n \ (\in M^n_\Sigma(Y))$ *for every* $\mathbf{u} \in M^n_\Sigma(X)$, *then* $(\mathbf{u} \circ \mathbf{v})\widehat{\varphi}_n = \mathbf{u}\widehat{\varphi}_n \circ \mathbf{v}\widehat{\varphi}_n$ *for all* $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$, *and* $\mathbf{1}_n\widehat{\varphi}_n = \mathbf{1}_n$.

(c) *If* φ *is surjective, then so are* φ_n, $\overline{\varphi}_n$ *and* $\widehat{\varphi}_n$. $\square$

Proposition 7.5 *If* $T, U \subseteq T_\Sigma(X)$ *and* $V \subseteq T_\Sigma(Y)$ *for some leaf alphabets* X *and* Y, *then the following hold for all* $n \geq 1$:

(a) $\mu^n_{T_\Sigma(X)\setminus T} = \mu^n_T$.

(b) $\mu^n_T \cap \mu^n_U \subseteq \mu^n_{T\cap U}$ *and* $\mu^n_T \cap \mu^n_U \subseteq \mu^n_{T\cup U}$.

(c) $\mu^n_T \subseteq \mu^n_{p^{-1}(T)}$ *for every* $p \in C_\Sigma(X)$.

(d) $\widehat{\varphi}_n \circ \mu^n_V \circ \widehat{\varphi}_n^{-1} \subseteq \mu^n_{V\varphi^{-1}}$ *for every homomorphism* $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$, *and equality holds if* φ *is an epimorphism.*

All the corresponding statements hold for the syntactic semigroup congruences σ^n_T.

Proof. Again, we present the proofs for the monoid congruences only. Statement (a) is completely obvious. To prove (b), assume that $\mathbf{u}\, \mu^n_T \cap \mu^n_U\, \mathbf{v}$, for some $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$. Then both $\mathbf{u}\, \mu^n_T\, \mathbf{v}$ and $\mathbf{u}\, \mu^n_U\, \mathbf{v}$. This means that for any $\mathbf{t} \in T_\Sigma(X)^n$, $p \in C_\Sigma(X)$, and $i \in [n]$, we have

$$\mathbf{t} \cdot u_i \cdot p \in T \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot p \in T \ \text{ and } \ \mathbf{t} \cdot u_i \cdot p \in U \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot p \in U,$$

and hence

$$\mathbf{t} \cdot u_i \cdot p \in T \cap U \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot p \in T \cap U \ \text{ and } \ \mathbf{t} \cdot u_i \cdot p \in T \cup U \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot p \in T \cup U,$$

from which the inclusions $\mu^n_T \cap \mu^n_U \subseteq \mu^n_{T\cap U}$ and $\mu^n_T \cap \mu^n_U \subseteq \mu^n_{T\cup U}$ follow.

To prove (c), we just have to note that for any $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$,

$$\begin{aligned}
\mathbf{u}\, \mu^n_T\, \mathbf{v} \quad &\Leftrightarrow \quad (\forall i)(\forall \mathbf{t})(\forall q)(\mathbf{t} \cdot u_i \cdot q \in T \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot q \in T) \\
&\Rightarrow \quad (\forall i)(\forall \mathbf{t})(\forall q)(\mathbf{t} \cdot u_i \cdot (q \cdot p) \in T \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot (q \cdot p) \in T) \\
&\Leftrightarrow \quad (\forall i)(\forall \mathbf{t})(\forall q)((\mathbf{t} \cdot u_i \cdot q) \cdot p \in T \ \Leftrightarrow\ (\mathbf{t} \cdot v_i \cdot q) \cdot p \in T) \\
&\Leftrightarrow \quad (\forall i)(\forall \mathbf{t})(\forall q)(\mathbf{t} \cdot u_i \cdot q \in p^{-1}(T) \ \Leftrightarrow\ \mathbf{t} \cdot v_i \cdot q \in p^{-1}(T)) \\
&\Leftrightarrow \quad \mathbf{u}\, \mu_{p^{-1}(T)}\, \mathbf{v},
\end{aligned}$$

where i ranges over $[n]$, $\mathbf{t}$ over $T_\Sigma(X)^n$, and q over $C_\Sigma(X)$.

Finally, for any $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$,

$$
\begin{aligned}
\mathbf{u}\,\widehat{\varphi}_n \circ \mu^n_T \circ \widehat{\varphi}_n^{-1}\,\mathbf{v} \;&\Leftrightarrow\; \mathbf{u}\widehat{\varphi}_n\,\mu^n_T\,\mathbf{v}\widehat{\varphi}_n \\
&\Leftrightarrow\; (\forall i \in [n])(\forall \mathbf{t} \in T_\Sigma(Y)^n)(\forall p \in C_\Sigma(Y)) \\
&\qquad\qquad (\mathbf{t} \cdot (u_i\varphi_n) \cdot p \in T \Leftrightarrow \mathbf{t} \cdot (v_i\varphi_n) \cdot p \in T) \\
&\Rightarrow\; (\forall i \in [n])(\forall \mathbf{s} \in T_\Sigma(X)^n)(\forall q \in C_\Sigma(X)) \\
&\qquad\qquad (\mathbf{s}\overline{\varphi}_n \cdot (u_i\varphi_n) \cdot q_\varphi \in T \Leftrightarrow \mathbf{s}\overline{\varphi}_n \cdot (v_i\varphi_n) \cdot q_\varphi \in T) \\
&\Leftrightarrow\; (\forall i \in [n])(\forall \mathbf{s} \in T_\Sigma(X)^n)(\forall q \in C_\Sigma(X)) \\
&\qquad\qquad ((\mathbf{s} \cdot u_i \cdot q)\varphi \in T \Leftrightarrow (\mathbf{s} \cdot v_i \cdot q)\varphi \in T) \\
&\Leftrightarrow\; (\forall i \in [n])(\forall \mathbf{s} \in T_\Sigma(X)^n)(\forall q \in C_\Sigma(X)) \\
&\qquad\qquad (\mathbf{s} \cdot u_i \cdot q \in T\varphi^{-1} \Leftrightarrow \mathbf{s} \cdot v_i \cdot q \in T\varphi^{-1}) \\
&\Leftrightarrow\; \mathbf{u}\,\mu^n_{T\varphi^{-1}}\,\mathbf{v},
\end{aligned}
$$

where we used Lemmas 2.1 and 7.4. This implies the inclusion of (d). If φ is an epimorphism, then every $\mathbf{t} \in T_\Sigma(Y)^n$ is of the form $\mathbf{s}\overline{\varphi}_n$ for some $\mathbf{s} \in T_\Sigma(X)^n$ and every $p \in C_\Sigma(Y)$ is of the form q_φ for some $q \in C_\Sigma(X)$, and hence the only implication in the above derivation also becomes an equivalence and $\widehat{\varphi}_n\varphi \circ \mu^n_V \circ \widehat{\varphi}_n^{-1} = \mu_{V\varphi^{-1}}$ holds. $\square$

8 n-ary syntactic monoids and semigroups

In this section we associate with any tree language a sequence of syntactic monoids and a sequence of syntactic semigroups.

Definition 8.1 Let T be any ΣX-tree language. For the sake of simplicity, we denote the μ^n_T-class of a ΣXn-context $\mathbf{u}$ by $[\mathbf{u}]^n_T$. The *n-ary syntactic monoid* of T is the quotient monoid $M^n(T) := M^n_\Sigma(X)/\mu^n_T$, and the canonical homomorphism

$$\nu^n_T : M^n_\Sigma(X) \to M^n(T),\ \mathbf{u} \mapsto [\mathbf{u}]^n_T,$$

is called the *syntactic (monoid) homomorphism* of T. The *n-ary syntactic semigroup* of T is similarly defined as the quotient semigroup $S^n(T) := S^n_\Sigma(X)/\sigma^n_T$. When speaking about these semigroups, we let $[\mathbf{u}]^n_T$ denote the σ^n_T-class of $\mathbf{u} \in S^n_\Sigma(X)$, and ν^n_T is then the *syntactic (semigroup) homomorphism* $S^n_\Sigma(X) \to S^n(T)$, $\mathbf{u} \mapsto [\mathbf{u}]^n_T$. $\square$

The monoids $M^n(T)$ and semigroups $S^n(T)$ generalize the syntactic monoids $SM(T)$ and semigroups $SS(T)$ studied by Thomas [27, 28], Salomaa [20] and Salehi [18, 19]. In fact, $M^1(T) \cong SM(T)$ and $S^1(T) \cong SS(T)$.

As a preparation for the proof of Proposition 8.3 below, we note the following facts.

Lemma 8.2 *Let* $\mathbf{A} = (\mathcal{A}, \alpha, F)$ *be any* ΣX*-recognizer.*

(a) *If* $\mathbf{A}$ *is connected, then for every* $\mathbf{a} \in A^n$ *there exists an* n*-tuple of* ΣX*-trees* $\mathbf{t} \in T_\Sigma(X)^n$ *such that* $\mathbf{t}\widetilde{\alpha}_n = \mathbf{a}$.

(b) $\mathbf{u}^{\mathbf{A}}(\mathbf{t}\widetilde{\alpha}_n) = (\mathbf{t} \cdot \mathbf{u})\widetilde{\alpha}_n$ *for all* $\mathbf{t} \in T_\Sigma(X)^n$ *and* $\mathbf{u} \in M^n_\Sigma(X)$.

Let us also recall (cf. [7, 8], for example) that the *equivalence* of states of a ΣX-recognizer $\mathbf{A} = (\mathcal{A}, \alpha, F)$ can be defined by

$$a \sim_{\mathbf{A}} b \;\Leftrightarrow\; (\forall p \in C_\Sigma(X))(p^{\mathbf{A}}(a) \in F \Leftrightarrow p^{\mathbf{A}}(b) \in F) \qquad (a, b \in A),$$

and that $\mathbf{A}$ is *reduced* if $\sim_{\mathbf{A}} = \Delta_A$. A ΣX-recognizer is *minimal* if is both reduced and connected. Every regular ΣX-tree language has a minimal ΣX-recognizer and this is unique up to isomorphism.

Proposition 8.3 *If* $\mathbf{A}$ *is the minimal* ΣX*-recognizer of a* ΣX*-tree language* T*, then* $M^n(T) \cong TM^n(\mathbf{A})$ *and* $S^n(T) \cong TS^n(\mathbf{A})$.

Proof. Let $\mathbf{A} = (\mathcal{A}, \alpha, F)$ be the minimal ΣX-recognizer of a given ΣX-tree language T. To prove $M^n(T) \cong TM^n(\mathbf{A})$, it suffices to show that

$$\varphi : M^n_\Sigma(X) \to TM^n(\mathbf{A}),\; \mathbf{u} \mapsto \mathbf{u}^{\mathbf{A}},$$

is an epimorphism such that $\ker \varphi = \mu^n_T$. It is clear that φ is surjective, and it is a homomorphism since, by Lemma 6.1, $\mathbf{1}^{\mathbf{A}}_n = 1_{A^n}$ and $(\mathbf{u} \cdot \mathbf{v})^{\mathbf{A}} = \mathbf{u}^{\mathbf{A}} \circ \mathbf{v}^{\mathbf{A}}$ for all $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$. Furthermore, for any $\mathbf{u}, \mathbf{v} \in M^n_\Sigma(X)$,

$$\begin{aligned}
\mathbf{u}\varphi = \mathbf{v}\varphi \;&\Leftrightarrow\; (\forall \mathbf{a} \in A^n)(\mathbf{u}^{\mathbf{A}}(\mathbf{a}) = \mathbf{v}^{\mathbf{A}}(\mathbf{a})) \\
&\Leftrightarrow\; (\forall \mathbf{t} \in T_\Sigma(X)^n)(\mathbf{u}^{\mathbf{A}}(\mathbf{t}\widetilde{\alpha}_n) = \mathbf{v}^{\mathbf{A}}(\mathbf{t}\widetilde{\alpha}_n)) \\
&\Leftrightarrow\; (\forall \mathbf{t} \in T_\Sigma(X)^n)((\mathbf{t} \cdot \mathbf{u})\widetilde{\alpha}_n = (\mathbf{t} \cdot \mathbf{v})\widetilde{\alpha}_n) \\
&\Leftrightarrow\; (\forall i \in [n])(\forall \mathbf{t} \in T_\Sigma(X)^n)((\mathbf{t} \cdot u_i)\widehat{\alpha} = (\mathbf{t} \cdot v_i)\widehat{\alpha}) \\
&\Leftrightarrow\; (\forall i)(\forall \mathbf{t})(\forall p \in C_\Sigma(X))(p^{\mathbf{A}}((\mathbf{t} \cdot u_i)\widehat{\alpha}) \in F \leftrightarrow p^{\mathbf{A}}((\mathbf{t} \cdot v_i)\widehat{\alpha}) \in F) \\
&\Leftrightarrow\; (\forall i)(\forall \mathbf{t})(\forall p)((\mathbf{t} \cdot u_i \cdot p)\widehat{\alpha} \in F \leftrightarrow (\mathbf{t} \cdot u_i \cdot p)\widehat{\alpha} \in F) \\
&\Leftrightarrow\; (\forall i)(\forall \mathbf{t})(\forall p)(\mathbf{t} \cdot u_i \cdot p \in T \leftrightarrow \mathbf{t} \cdot u_i \cdot p \in T) \\
&\Leftrightarrow\; \mathbf{u}\, \mu^n_T\, \mathbf{v},
\end{aligned}$$

where we used Lemma 8.2 and the fact that $\mathbf{A}$ is reduced. Hence, $\ker\varphi = \mu_T^n$, and therefore $TM^n(\mathbf{A}) \cong M_\Sigma^n(X)/\ker\varphi = M^n(T)$. The second isomorphism can be proved the same way. □

Let us recall that the *syntactic algebra* $\mathrm{SA}(T)$ of a ΣX-tree language T is the quotient algebra $\mathcal{T}_\Sigma(X)/\theta_T$, where θ_T is the *syntactic congruence* of T defined by

$$s\,\theta_T\,t \;\Leftrightarrow\; (\forall p \in C_\Sigma(X))(p(s) \in T \;\leftrightarrow\; p(t) \in T) \quad (s, t \in T_\Sigma(X)).$$

It is well known that (cf. [24, 26], for example) that the underlying Σ-algebra of the minimal ΣX-recognizer of a regular ΣX-tree language T is isomorphic to the syntactic algebra $\mathrm{SA}(T)$ of T. Hence, Propositions 6.2 and 8.3 together yield the following facts.

Corollary 8.4 *For any $T \in \mathrm{Rec}_\Sigma(X)$ and any $n \geq 1$, $M^n(T) \cong TM^n(\mathrm{SA}(T))$ and $S^n(T) \cong TS^n(\mathrm{SA}(T))$.*

Proposition 8.3 and Lemma 6.3 together yield the following result.

Corollary 8.5 *For any $T \subseteq T_\Sigma(X)$, the following conditions are equivalent:*

(a) $T \in \mathrm{Rec}_\Sigma(X)$;

(b) $M^n(T)$ *is finite for every* $n \geq 1$; (b') $S^n(T)$ *is finite for every* $n \geq 1$;

(c) $M^n(T)$ *is finite for some* $n \geq 1$; (c') $S^n(T)$ *is finite for some* $n \geq 1$.

Next we present the monoid and semigroup counterpart of Proposition 7.5.

Proposition 8.6 *Let $T, U \subseteq T_\Sigma(X)$ and $V \subseteq T_\Sigma(Y)$ for any leaf alphabets X and Y. Then the following hold for all $n \geq 1$:*

(a) $M^n(T_\Sigma(X) \setminus T) = M^n(T)$.

(b) $M^n(T \cap U) \preceq M^n(T) \times M^n(U)$ *and* $M^n(T \cup U) \preceq M^n(T) \times M^n(U)$.

(c) $M^n(p^{-1}(T)) \twoheadleftarrow M^n(T)$ *for every* $p \in C_\Sigma(X)$.

(d) $M^n(V\varphi^{-1}) \preceq M^n(V)$ *for every homomorphism* $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$, *and if* φ *is an epimorphism, then* $M^n(V\varphi^{-1}) \cong M^n(V)$.

All the corresponding statements hold for n-ary syntactic semigroups.

Proof. Statement (a) follows directly from statement (a) of Proposition 7.5. Also (b) and (c) follow from their counterparts in Proposition 7.5 by general algebra. As to (c), this is immediately clear because $\mu_T^n \subseteq \mu_{p^{-1}(T)}^n$ implies

$$M^n(p^{-1}(T)) = M_\Sigma^n(X)/\mu_{p^{-1}(T)}^n \twoheadleftarrow M_\Sigma^n(X)/\mu_T^n = M^n(T).$$

To get (b) one may use the slightly less obvious general fact that if θ_1, θ_2 and ρ are congruences of an algebra $\mathcal{A}$ such that $\theta_1 \cap \theta_2 \subseteq \rho$, then $\mathcal{A}/\rho \preceq \mathcal{A}/\theta_1 \times \mathcal{A}/\theta_2$. In the cases at hand, it suffices to note that $M := \{([\mathbf{u}]_T^n, [\mathbf{u}]_U^n) \mid \mathbf{u} \in M_\Sigma^n(X)\}$ is a submonoid of $M^n(T) \times M^n(U)$ and that

$$\psi_\cap : M \to M^n(T \cap U),\ ([\mathbf{u}]_T^n, [\mathbf{u}]_U^n) \mapsto [\mathbf{u}]_{T\cap U}^n,$$

and

$$\psi_\cup : M \to M^n(T \cup U),\ ([\mathbf{u}]_T^n, [\mathbf{u}]_U^n) \mapsto [\mathbf{u}]_{T\cup U}^n,$$

are epimorphisms which are well-defined by Proposition 7.5.

To prove (d), let us first assume that $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ is an epimorphism. Then

$$\psi : M^n(V\varphi^{-1}) \to M^n(V),\ [\mathbf{u}]_{V\varphi^{-1}}^n \mapsto [\mathbf{u}\widehat{\varphi}_n]_V^n,$$

where $\widehat{\varphi}_n : M_\Sigma^n(X) \to M_\Sigma^n(Y)$ is the homomorphism of Lemma 7.4, gives the required isomorphism as we shall show.

1. ψ is well-defined and injective: for any $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$,

$$\begin{aligned} [\mathbf{u}]_{V\varphi^{-1}}^n\psi = [\mathbf{v}]_{V\varphi^{-1}}^n\psi &\Leftrightarrow \mathbf{u}\widehat{\varphi}_n\, \mu_V^n\, \mathbf{v}\widehat{\varphi}_n \\ &\Leftrightarrow \mathbf{u}\, \mu_{V\varphi^{-1}}\, \mathbf{v} \quad \text{(Proposition 7.5 (d))} \\ &\Leftrightarrow [\mathbf{u}]_{V\varphi^{-1}}^n = [\mathbf{v}]_{V\varphi^{-1}}^n. \end{aligned}$$

2. ψ is surjective because φ is surjective.

3. ψ is a homomorphism: for any $\mathbf{u}, \mathbf{v} \in M_\Sigma^n(X)$,

$$\begin{aligned} ([\mathbf{u}]_{V\varphi^{-1}}^n \cdot [\mathbf{v}]_{V\varphi^{-1}}^n)\psi &= [\mathbf{u}\cdot\mathbf{v}]_{V\varphi^{-1}}\psi = [(\mathbf{u}\cdot\mathbf{v})\widehat{\varphi}_n]_V^n \\ &= [\mathbf{u}\widehat{\varphi}_n \cdot \mathbf{v}\widehat{\varphi}_n]_V^n = [\mathbf{u}\widehat{\varphi}_n]_V^n \cdot [\mathbf{v}\widehat{\varphi}_n]_V^n \\ &= [\mathbf{u}]_{V\varphi^{-1}}^n \cdot [\mathbf{v}]_{V\varphi^{-1}}^n, \end{aligned}$$

   and, moreover, $[\mathbf{1}_n]_{V\varphi^{-1}}^n\psi = [\mathbf{1}_n]_V^n$.

Consider now a general homomorphism $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$. Let $M := M^n_\Sigma(X)\widehat{\varphi}_n$, and let $\theta := \mu^n_V \restriction M$ be the restriction of μ^n_V to M. Let us define $\psi : M/\theta \to M^n(V\varphi^{-1})$ as follows. For any class $[\mathbf{v}]_\theta$ in M/θ, we may choose a $\mathbf{u} \in M^n_\Sigma(X)$ such that $\mathbf{u}\widehat{\varphi}_n = \mathbf{v}$. Then we set $[\mathbf{v}]_\theta\psi = [\mathbf{u}]^n_{V\varphi^{-1}}$. Since $M/\theta \sqsubseteq M^n(V)$, it now suffices to prove that ψ is a well-defined epimorphism.

1. To show that ψ is well-defined, we note that for any $\mathbf{u}, \mathbf{u}' \in M^n_\Sigma(X)$,

$$[\mathbf{u}\widehat{\varphi}_n]_\theta = [\mathbf{u}'\widehat{\varphi}_n]_\theta \;\Rightarrow\; \mathbf{u}\widehat{\varphi}_n \mu^n_V \mathbf{u}'\widehat{\varphi}_n \;\Rightarrow\; \mathbf{u}\widehat{\varphi}_n \circ \mu^n_V \circ \widehat{\varphi}_n^{-1} \mathbf{u}' \;\Rightarrow\; [\mathbf{u}]^n_{V\varphi^{-1}} = [\mathbf{u}']^n_{V\varphi^{-1}}.$$

2. Clearly, ψ is surjective.

3. Assume now that $\mathbf{u}\widehat{\varphi}_n = \mathbf{v}$ and $\mathbf{u}'\widehat{\varphi}_n = \mathbf{v}'$ for some $\mathbf{u}, \mathbf{u}' \in M^n_\Sigma(X)$ and $\mathbf{v}, \mathbf{v}' \in M$. Then $(\mathbf{u} \cdot \mathbf{u}')\widehat{\varphi}_n = \mathbf{v} \cdot \mathbf{v}'$, and hence

$$([\mathbf{v}]_\theta \cdot [\mathbf{v}']_\theta)\psi = [\mathbf{v} \cdot \mathbf{v}']_\theta\psi = [\mathbf{u} \cdot \mathbf{u}']^n_{V\varphi^{-1}} = [\mathbf{u}]^n_{V\varphi^{-1}} \cdot [\mathbf{u}']^n_{V\varphi^{-1}} = [\mathbf{v}]_\theta\psi \cdot [\mathbf{v}']_\theta\psi.$$

4. Clearly, $[\mathbf{1}_n]_\theta\psi = [\mathbf{1}_n]^n_{V\varphi^{-1}}$.

For semigroups, the proofs are quite analogous. □

Proposition 8.7 *For any $T \in \mathrm{Rec}_\Sigma(X)$ and every $n \geq 1$, $M^n(T) \sqsubseteq M^{n+1}(T)$ and $S^n(T) \preceq S^{n+1}(T)$.*

Proof. Both assertions follow directly from Corollary 8.4 and Proposition 5.4:

$$M^n(T) \cong TM^n(\mathrm{SA}(T)) \sqsubseteq TM^{n+1}(\mathrm{SA}(T)) \cong M^{n+1}(T),$$

and similarly $S^n(T) \cong TS^n(\mathrm{SA}(T)) \preceq TS^{n+1}(\mathrm{SA}(T)) \cong MS^{n+1}(T)$. □

9 Varieties of tree languages and n-ary syntactic monoids and semigroups

We shall now show that each variety of finite semigroups (monoids) defines via n-ary syntactic semigroups (monoids) a variety of tree languages, and we shall present some properties of these varieties of tree languages. First we recall some basic notions and facts from the theory of varieties of tree languages following [24, 26].

Let Σ be a ranked alphabet. A *family of (regular) Σ-tree languages* is a mapping $\mathcal{V}$ that assigns to every leaf alphabet X a set $\mathcal{V}(X)$ of (regular) ΣX-tree languages.

We write such a family as $\mathcal{V} = \{\mathcal{V}(X)\}_X$. For any two families of Σ-tree languages $\mathcal{U}$ and $\mathcal{V}$, let us set $\mathcal{U} \subseteq \mathcal{V}$ iff $\mathcal{U}(X) \subseteq \mathcal{V}(X)$ for every X. The unions and intersections of families of Σ-tree languages are defined by similar componentwise conditions.

A *variety of Σ-tree languages* (a Σ-VTL for short) is family of regular Σ-tree languages $\mathcal{V} = \{\mathcal{V}(X)\}_X$ such that for all leaf alphabets X and Y,

(1) $\mathcal{V}(X)$ is a Boolean subalgebra of $\mathrm{Rec}_\Sigma(X)$,

(2) $T \in \mathcal{V}(X)$ implies $p^{-1}(T) \in \mathcal{V}(X)$ for any $p \in C_\Sigma(X)$, and

(3) $T \in \mathcal{V}(Y)$ implies $T\varphi^{-1} \in \mathcal{V}(X)$ for any homomorphism $\varphi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$.

Recall also that a *variety of finite Σ-algebras* (a Σ-VFA for short) is a class of finite Σ-algebras closed under the formation of subalgebras, homomorphic images and finite direct products. There is a bijective correspondence between the Σ-VTLs and the Σ-VFAs: for any Σ-VFA $\mathbf{K}$, the corresponding Σ-VTL $\mathbf{K}^t = \{\mathbf{K}^t(X)\}_X$ is defined by the condition that a ΣX-tree language T is in $\mathbf{K}^t(X)$ iff its syntactic algebra $\mathrm{SA}(T)$ is in $\mathbf{K}$. We shall show that in a similar way a Σ-VTL can be associated with any variety of finite semigroups or monoids and any $n \geq 1$ via our n-ary syntactic semigroups or monoids, respectively. Of course, a *variety of finite semigroups* (VFS) is a class of finite semigroups that contains all subsemigroups, homomorphic images and finite direct products of its members, and a *variety of finite monoids* (VFM) is defined similarly.

Definition 9.1 For any class $\mathbf{M}$ of finite monoids and any $n \geq 1$, let $\mathcal{V}^n_{\mathbf{M}} = \{\mathcal{V}^n_{\mathbf{M}}(X)\}_X$ be the family of Σ-tree languages such that for any X,

$$\mathcal{V}^n_{\mathbf{M}}(X) := \{T \subseteq T_\Sigma(X) \mid M^n(T) \in \mathbf{M}\}.$$

Similarly, for any class of finite semigroups $\mathbf{S}$ and any $n \geq 1$, let $\mathcal{V}^n_{\mathbf{S}} = \{\mathcal{V}^n_{\mathbf{S}}(X)\}_X$, where $\mathcal{V}^n_{\mathbf{S}}(X) := \{T \subseteq T_\Sigma(X) \mid S^n(T) \in \mathbf{S}\}$ for each X. $\square$

Corollary 8.5 and Proposition 8.6 immediately yield the following basic facts.

Proposition 9.2 *For any VFM $\mathbf{M}$ and every $n \geq 1$, $\mathcal{V}^n_{\mathbf{M}}$ is a Σ-VTL. Similarly, $\mathcal{V}^n_{\mathbf{S}}$ is a Σ-VTL for every VFS $\mathbf{S}$ and every $n \geq 1$.* $\square$

As shown by the following example, not every Σ-VTL is of the form $\mathcal{V}^n_{\mathbf{M}}$ for some VFM $\mathbf{M}$ and some $n \geq 1$, or of the form $\mathcal{V}^n_{\mathbf{S}}$ for some VFS $\mathbf{S}$ and some $n \geq 1$.

$f^{\mathcal{A}_1}$	0	1	2
0	1	1	2
1	2	2	2
2	2	2	2

$g^{\mathcal{A}_1}$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

Figure 1: $f^{\mathcal{A}_1}$ and $g^{\mathcal{A}_1}$

$f^{\mathcal{A}_2}$	0	1	2
0	1	2	2
1	2	2	2
2	2	2	2

$g^{\mathcal{A}_2}$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

Figure 2: $f^{\mathcal{A}_2}$ and $g^{\mathcal{A}_2}$

Example 9.3 Let us consider the alphabets $\Sigma = \{f/2, g/2\}$ and $X = \{x\}$. Furthermore, let $\mathbf{K}$ be the Σ-VFA consisting of all finite Σ-algebras in which the f-operation is commutative, i.e., that satisfy the identity $f(\xi_1, \xi_2) \approx f(\xi_2, \xi_1)$, and let $\mathcal{V} := \mathbf{K}^t$ be the corresponding Σ-VTL. We shall show that there is no VFM $\mathbf{M}$ and no $n \geq 1$ such that $\mathcal{V} = \mathcal{V}^n_{\mathbf{M}}$. To do this, we consider the ΣX-tree languages

$$T_1 \;:=\; \{p(f(f(s_1, s_2), s_3)) \mid p \in C_\Sigma(X),\ s_1, s_2, s_3 \in T_\Sigma(X)\}$$

and

$$T_2 \;:=\; T_1 \cup \{p(f(s_1, f(s_2, s_3))) \mid p \in C_\Sigma(X),\ s_1, s_2, s_3 \in T_\Sigma(X)\}.$$

It is easy to see that $\mathbf{A}_1 = (\mathcal{A}_1, \alpha, F)$ and $\mathbf{A}_2 = (\mathcal{A}_2, \alpha, F)$ are the minimal ΣX-recognizers of T_1 and T_2, respectively, when

(1) $\mathcal{A}_1 = (A, \Sigma)$, $A = \{0, 1, 2\}$, $\alpha(x) = 0$, $F = \{2\}$, and $f^{\mathcal{A}_1}$ and $g^{\mathcal{A}_1}$ are defined by the tables in Figure 1, and

(2) $\mathcal{A}_2 = (A, \Sigma)$, $A = \{0, 1, 2\}$, $\alpha(x) = 0$, $F = \{2\}$, and $f^{\mathcal{A}_2}$ and $g^{\mathcal{A}_2}$ are defined by the tables in Figure 2.

Clearly, $f^{\mathcal{A}_2}$ is commutative while $f^{\mathcal{A}_1}$ is not, that is to say, $\mathcal{A}_2 \in \mathbf{K}$ but $\mathcal{A}_1 \notin \mathbf{K}$. Since $\mathrm{SA}(T_1) \cong \mathcal{A}_1$ and $\mathrm{SA}(T_2) \cong \mathcal{A}_2$ (cf. [24] or [26], for example), this means that $T_1 \notin \mathcal{V}(X)$ while $T_2 \in \mathcal{V}(X)$. To prove that there is no VFM $\mathbf{M}$ such that $\mathcal{V} = \mathcal{V}^n_{\mathbf{M}}$ for some $n \geq 1$, it now suffices to show that $M^n(T_1) \cong M^n(T_2)$ for every $n \geq 1$, and by Proposition 8.3 this can be done by showing that $TM^n(\mathbf{A}_1) \cong TM^n(\mathbf{A}_2)$ for every $n \geq 1$. In fact, we prove that $TM^n(\mathbf{A}_1) = TM^n(\mathbf{A}_2)$ for every $n \geq 1$.

Lemma For any $n \geq 1$ and any n-ary ΣX-term u, there exists an n-ary ΣX-term $\dot{u}$ such that each variable ξ_i ($i \in [n]$) appears exactly the same number of times in $\dot{u}$ as it appears in u, and $u^{\mathbf{A}_1} = \dot{u}^{\mathbf{A}_2}$. Conversely, for any $v \in T_\Sigma(X \cup \Xi_n)$, there is a $\widehat{v} \in T_\Sigma(X \cup \Xi_n)$ such that each variable ξ_i ($i \in [n]$) appears exactly the same number of times in $\widehat{v}$ as it appears in v, and $v^{\mathbf{A}_2} = \widehat{v}^{\mathbf{A}_1}$.

Proof. We can verify the first claim by tree induction on u.

1. Obviously, we can let $\dot{x} := x$ and $\dot{\xi_i} := \xi_i$ for every $i \in [n]$.

2. Suppose that $\dot{u_1}$ and $\dot{u_2}$ have been defined as required.If $u = g(u_1, u_2)$, we may choose simply $\dot{u} := g(\dot{u_1}, \dot{u_2})$ because $g^{\mathcal{A}_1} = g^{\mathcal{A}_2}$. If $u = f(u_1, u_2)$, we set $\dot{u} := f(\dot{u_1}, g(\dot{u_2}, x))$. It is clear that this $\dot{u}$ satisfies the variable conditions. To show that $u^{\mathbf{A}_1} = \dot{u}^{\mathbf{A}_2}$, it suffices to verify that $f^{\mathcal{A}_1}(a, b) = f^{\mathcal{A}_2}(a, g^{\mathcal{A}_2}(b, 0))$ for all $a, b \in A$. It is clear that both sides of this equality assume the value 2 whenever $a \in \{1, 2\}$ or $b = 2$, and in the remaining two cases both sides equal 1.

The second claim is proved similarly by induction on v, but for $v = f(v_1, v_2)$ we choose $\widehat{v} := f(\widehat{v_1}, f(\widehat{v_2}, x))$. $\square$

The equality $TM^n(\mathbf{A}_1) = TM^n(\mathbf{A}_2)$ follows immediately from the Lemma: for any $\mathbf{u} = \langle u_1, \ldots, u_n \rangle \in M^n_\Sigma(X)$, we have $\mathbf{u}^{\mathbf{A}_1} = \dot{\mathbf{u}}^{\mathbf{A}_2} \in TM^n(\mathbf{A}_2)$ when $\dot{\mathbf{u}} := \langle \dot{u}_1, \ldots, \dot{u}_n \rangle$, and similarly, for any $\mathbf{v} = \langle v_1, \ldots, v_n \rangle \in M^n_\Sigma(X)$, we have $\mathbf{v}^{\mathbf{A}_2} = \widehat{\mathbf{v}}^{\mathbf{A}_1} \in TM^n(\mathbf{A}_1)$ when $\widehat{\mathbf{v}} := \langle \widehat{v}_1, \ldots, \widehat{v}_n \rangle$.

It is clear that if $\mathbf{u} \in S^n_\Sigma(X)$ or $\mathbf{v} \in S^n_\Sigma(X)$, then also $\dot{\mathbf{u}}$ or $\widehat{\mathbf{v}}$, respectively, is a proper n-ary context, and hence we can also conclude that $\mathcal{V} = \mathcal{V}^n_{\mathbf{S}}$ for no VFS $\mathbf{S}$ and no $n \geq 1$. $\square$

In [18, 19] Salehi characterized the varieties of tree languages definable by the syntactic monoids of Thomas [27, 28] or the corresponding semigroups. Because $M^1(T) \cong SM(T)$ for every tree language T, these varieties are exactly the ones definable by our 1-ary syntactic monoids, and the same applies to syntactic semigroups. On the other hand, in the next section we shall see that the variety of definite tree languages can be defined by our syntactic semigroups although it is known that this is not possible using ordinary syntactic semigroups or monoids. Hence, we may conclude that the variety-defining power of our syntactic semigroups lies properly between that of syntactic algebras and that of ordinary syntactic semigroups.

Let us also note the following facts.

Proposition 9.4 *For any VFM* $\mathbf{M}$ *and any* $n \geq 1$, $\mathcal{V}_{\mathbf{M}}^{n} \supseteq \mathcal{V}_{\mathbf{M}}^{n+1}$. *Similarly,* $\mathcal{V}_{\mathbf{S}}^{n} \supseteq \mathcal{V}_{\mathbf{S}}^{n+1}$ *for any VFS* $\mathbf{S}$ *and any* $n \geq 1$.

Proof. If $T \in \mathcal{V}_{\mathbf{M}}^{n+1}(X)$ for some X, then by definition $M^{n+1}(T) \in \mathbf{M}$. On the other hand, $M^{n}(T) \sqsubseteq M^{n+1}(T)$ by Proposition 8.7. Since $\mathbf{M}$ is a VFM, this means that $M^{n}(T) \in \mathbf{M}$ and hence $T \in \mathcal{V}_{\mathbf{M}}^{n}(X)$. The inclusion $\mathcal{V}_{\mathbf{S}}^{n} \supseteq \mathcal{V}_{\mathbf{S}}^{n+1}$ has a similar proof. $\square$

By the next two examples we show that for any $n \geq 1$, the inclusions $\mathcal{V}_{\mathbf{M}}^{n} \supseteq \mathcal{V}_{\mathbf{M}}^{n+1}$ and $\mathcal{V}_{\mathbf{S}}^{n} \supseteq \mathcal{V}_{\mathbf{S}}^{n+1}$ may be proper.

Example 9.5 Let us consider any given $n \geq 1$, and let $\mathbf{M}$ be the VFM defined by the identity $\xi^{n^2} \approx \xi^{(n+n!)^2}$. Furthermore, let $\Sigma = \{f/1, g/1\}$ and $X = \{x\}$. For any $s \in T_\Sigma(X \cup \Xi_n)$, let $f^0(s) = s$ and $f^{i+1}(s) = f(f^i(s))$ for $i \geq 0$. We shall show that the ΣX-tree language

$$T_n := \{f^m(s) \mid m \geq n, s \in T_\Sigma(X)\}$$

belongs to $\mathcal{V}_{\mathbf{M}}^{n}(X) \setminus \mathcal{V}_{\mathbf{M}}^{n+1}(X)$. It is easy to see that the minimal ΣX-recognizer $\mathbf{A} = (\mathcal{A}, \alpha, F)$ of T_n can be defined as follows:

(1) $\mathcal{A} = (A, \Sigma)$ is the Σ-algebra such that $A = \{0, 1, \ldots, n\}$, $f^{\mathcal{A}}(a) = a + 1$ for $a \in \{0, \ldots, n-1\}$, $f^{\mathcal{A}}(n) = n$, and $g^{\mathcal{A}}(a) = 0$ for every $a \in A$;

(2) $\alpha(x) = 0$ and $F = \{n\}$.

By Proposition 8.3, it suffices to show that $TM^n(\mathbf{A}) \in \mathbf{M}$ while $TM^{n+1}(\mathbf{A}) \notin \mathbf{M}$.

To prove that $TM^n(\mathbf{A}) \in \mathbf{M}$, we consider any $\mathbf{u} \in M_\Sigma^n(X)$. If $\mathbf{u}^{n^2} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{u}^{(n+n!)^2} = \langle w_1, \ldots, w_n \rangle$, then the claim is that $v_i^{\mathbf{A}} = w_i^{\mathbf{A}}$ for every $i \in [n]$. Let us fix $i \in [n]$ arbitrarily.

Since $\Sigma = \Sigma_1$, we have $\mathbf{u} = \langle q_1(\xi_{\sigma(1)}), \ldots, q_n(\xi_{\sigma(n)}) \rangle$, for some $q_1, \cdots, q_n \in C_\Sigma(X)$ and some permutation σ of $[n]$. Then

$$v_i = q_i(q_{\sigma(i)}(q_{\sigma^2(i)}(\ldots(q_{\sigma^{n^2-1}(i)}(\xi_{\sigma^{n^2}(i)})\ldots))))$$

and

$$w_i = q_i(q_{\sigma(i)}(q_{\sigma^2(i)}(\ldots(q_{\sigma^{n^2-1}(i)}(\ldots(q_{\sigma^{(n+n!)^2-1}(i)}(\xi_{\sigma^{(n+n!)^2}(i)})\ldots)\ldots))))).$$

Assume that the cycle $(i\,\sigma(i)\,\ldots\,\sigma^{m-1}(i))$ of σ in which i appears is of length m. Since $n^2 \equiv_m (n+n!)^2$, we have $\sigma^{n^2}(i) = \sigma^{(n+n!)^2}(i)$ and hence $v_i = p(\xi_l)$ and $w_i = q(\xi_l)$ for some $p, q \in C_\Sigma(X)$ and $l \in [n]$. Let

$$p_1(\xi_1') := q_i(\xi_{\sigma(i)}),\ p_2(\xi_2') := q_i(q_{\sigma(i)}(\xi_{\sigma^2(i)})), \ldots,\ p_m(\xi_m') := q_i(q_{\sigma(i)}(\ldots q_{\sigma^{m-1}(i)}(\xi_i)\ldots))$$

be the n-ary ΣX-contexts appearing in the above representations of v_i and w_i. There are now three possibilities to consider.

1. For every $k \in [m]$, $p_k(\xi_k') = f^h(\xi_k')$ for some $h \geq 0$, and $h \geq 1$ for at least one $k \in [m]$. Then $v_i = f^n(p'(\xi_l))$ and $w_i = f^n(q'(\xi_l))$ for some $p', q' \in C_\Sigma(X)$, and hence $v_i^{\mathbf{A}}(\mathbf{a}) = n = w_i^{\mathbf{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^n$.

2. The symbol g appears in at least one p_k. Let $k \in [m]$ be the least index for which this is the case. Then

$$p_1(\xi_1') = f^{n_1}(\xi_1'),\ \ldots\ , p_{k-1}(\xi_{k-1}') = f^{n_{k-1}}(\xi_{k-1}),\ p_k(\xi_k') = f^{n_k}(g(r(\xi_k'))),$$

   for some $n_1, \ldots, n_{k-1}, n_k \geq 0$ and $r \in C_\Sigma(X)$. If $h := n_1 + \ldots + n_{k-1} + n_k \geq n$, then we have again $v_i = f^n(p'(\xi_l))$ and $w_i = f^n(q'(\xi_l))$ for some $p', q' \in C_\Sigma(X)$ as in the first case. If $h < n$, then $v_i^{\mathbf{A}}(\mathbf{a}) = h = w_i^{\mathbf{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^n$.

3. Finally, if $p_k(\xi_k') = \xi_k'$ for every $k \in [m]$, then $v_i = w_i = \xi_l$, and hence $v_i^{\mathbf{A}}(\mathbf{a}) = a_l = w_i^{\mathbf{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^n$.

It remains to prove that $TM^{n+1}(\mathbf{A}) \notin \mathbf{M}$. Let us consider the $\Sigma X(n+1)$-context $\mathbf{u} = \langle \xi_2, \xi_3, \ldots, \xi_{n+1}, f(\xi_1)\rangle$, and let $\mathbf{u}^{n^2} = \langle v_1, \ldots, v_{n+1}\rangle$ and $\mathbf{u}^{(n+n!)^2} = \langle w_1, \ldots, w_{n+1}\rangle$. Then it suffices to show that $w_1^{\mathbf{A}} \neq v_1^{\mathbf{A}}$. Let $\mathbf{a} := \langle 0, 0, \ldots, 0\rangle \in A^{n+1}$. It is easy to see that

$$\mathbf{v} = \langle f^{n-1}(\xi_2), f^n(\xi_3), \ldots, f^n(\xi_{n+1}), f^n(\xi_1)\rangle,$$

and hence $v_1^{\mathbf{A}}(\mathbf{a}) = n - 1$. On the other hand, it is clear that $w_1 = f^k(\xi_l)$ for some $k \geq n$ and $l \in [n+1]$, and hence $w_i^{\mathbf{A}}(\mathbf{a}) = n$. □

Example 9.6 In this example we show that the proper inclusion $\mathcal{V}_{\mathbf{S}}^n \supset \mathcal{V}_{\mathbf{S}}^{n+1}$ may hold for any $n \geq 1$. For this, consider any given $n \geq 1$, and let $\mathbf{S}$ be the VFS defined by the identity $\xi^n \approx \xi^{n+n!}$. Furthermore, let $\Sigma = \{f/1, g/1\}$ and $X = \{x\}$. We shall show that the ΣX-tree language

$$T := \{p(f(s)) \mid p \in C_\Sigma(X), s \in T_\Sigma(X)\}$$

belongs to $\mathcal{V}_{\mathbf{S}}^n(X) \setminus \mathcal{V}_{\mathbf{S}}^{n+1}(X)$. It is easy to see that the minimal ΣX-recognizer $\mathbf{A} = (\mathcal{A}, \alpha, F)$ of T can be defined as follows:

(1) $\mathcal{A} = (A, \Sigma)$ is the Σ-algebra such that $A = \{0, 1\}$, and $f^{\mathcal{A}}(a) = 1$ and $g^{\mathcal{A}}(a) = a$ for every $a \in A$;

(2) $\alpha(x) = 0$ and $F = \{1\}$.

By Proposition 8.3, it suffices to show that $TS^n(\mathbf{A}) \in \mathbf{S}$ while $TS^{n+1}(\mathbf{A}) \notin \mathbf{S}$.

To prove that $TS^n(\mathbf{A}) \in \mathbf{S}$, take any $\mathbf{u} \in M^n_\Sigma(X)$. If $\mathbf{u}^n = \langle v_1, \ldots, v_n\rangle$ and $\mathbf{u}^{n+n!} = \langle w_1, \ldots, w_n\rangle$, then the claim is that $v_i^{\mathbf{A}} = w_i^{\mathbf{A}}$ for every $i \in [n]$. Let us consider any $i \in [n]$.

Since $\Sigma = \Sigma_1$, we have $\mathbf{u} = \langle q_1(\xi_{\sigma(1)}), \ldots, q_n(\xi_{\sigma(n)})\rangle$, for some proper contexts $q_1, \cdots, q_n \in C^+_\Sigma(X)$ and some permutation σ of $[n]$. Then

$$v_i = q_i(q_{\sigma(i)}(q_{\sigma^2(i)}(\ldots(q_{\sigma^{n-1}(i)}(\xi_{\sigma^n(i)})\ldots))))$$

and

$$w_i = q_i(q_{\sigma(i)}(q_{\sigma^2(i)}(\ldots(q_{\sigma^{n-1}(i)}(\ldots(q_{\sigma^{n+n!-1}(i)}(\xi_{\sigma^{n+n!}(i)})\ldots)\ldots))))).$$

Assume that the cycle $(i\,\sigma(i)\,\ldots\,\sigma^{m-1}(i))$ of σ in which i appears is of length m. Since $n \equiv_m n + n!$, we have $\sigma^n(i) = \sigma^{n+n!}(i)$ and hence $v_i = p(\xi_l)$ and $w_i = q(\xi_l)$ for some $p, q \in C_\Sigma(X)$ and $l \in [n]$. Let

$$p_1(\xi'_1) := q_i(\xi_{\sigma(i)}),\ p_2(\xi'_2) := q_i(q_{\sigma(i)}(\xi_{\sigma^2(i)})), \ldots,\ p_m(\xi'_m) := q_i(q_{\sigma(i)}(\ldots q_{\sigma^{m-1}(i)}(\xi_i)\ldots))$$

be the n-ary ΣX-contexts appearing in the above representations of v_i and w_i. There are now two possibilities to consider.

1. For every $k \in [m]$, $p_k(\xi'_k) = g^h(\xi'_k)$ for some $h \geq 1$. Then $v_i = g^{h_v}(\xi_l)$ and $w_i = g^{h_w}(\xi_l)$ for some $h_v \geq n$ and $h_w \geq n + n!$, and hence $v_i^{\mathbf{A}}(\mathbf{a}) = 0 = w_i^{\mathbf{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^n$.

2. For some $k \in [m]$, $p_k(\xi'_k) = p(f(q(\xi'_k)))$ for some $p, q \in C_\Sigma(X)$. Then $v_i = p(f(q(\xi'_k)))$ for some $p, q \in C_\Sigma(X)$ and $w_i = p'(f(q'(\xi'_k)))$ for some $p', q' \in C_\Sigma(X)$, and hence $v_i^{\mathbf{A}}(\mathbf{a}) = 1 = w_i^{\mathbf{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^n$.

To prove that $TS^{n+1}(\mathbf{A}) \notin \mathbf{S}$, we consider the $\Sigma X(n + 1)$-context

$$\mathbf{u} = \langle g(\xi_2), g(\xi_3), \ldots, g(\xi_{n+1}), f(\xi_1)\rangle.$$

If $\mathbf{u}^n = \langle v_1, \ldots, v_{n+1}\rangle$ and $\mathbf{u}^{n+n!} = \langle w_1, \ldots, w_{n+1}\rangle$, then it suffices to show that $w_1^{\mathbf{A}} \neq v_1^{\mathbf{A}}$. Let $\mathbf{a} := \langle 0, 0, \ldots, 0\rangle \in A^{n+1}$. It is easy to see that

$$\mathbf{v} = \langle g^n(\xi_{n+1}), g^{n-1}(f(\xi_1)), \ldots, g(f(g^{n-2}(\xi_{n-1}))), f(g^{n-1}(\xi_n))\rangle,$$

and hence $v_1^{\mathbf{A}}(\mathbf{a}) = 0$. On the other hand, it is clear that $w_1 = p(f(q(\xi_1)))$ for some $p, q \in C_\Sigma(X)$, and hence $w_i^{\mathbf{A}}(\mathbf{a}) = 1$. $\square$

10 Definite tree languages

Let us recall that a string language L is *definite* if there is a $k \geq 0$ such that whether a word of length $\geq k$ is in L depends on its suffix of length k only (cf. [11, 14]). Similarly, a tree language T is said to be definite if the membership of a tree t in T can be decided by looking at the root segment of t of some given height k. Definite tree languages were first studied by Heuter [9, 10]. In [24, 25] the variety properties of the definite tree languages were noted, and in [6] Ésik considers definite tree automata.

Definition 10.1 For any $k \geq 0$, the *k-root* $\mathrm{rt}_k(t)$ of a ΣX-tree t is defined as follows:

(1) $\mathrm{rt}_0(t) = \varepsilon$ for every $t \in T_\Sigma(X)$; here ε represents the "empty root segment" that gives no information about the tree.

(2) $\mathrm{rt}_1(t) = \mathrm{root}(t)$ for every $t \in T_\Sigma(X)$.

(3) Let $k \geq 2$. If $\mathrm{hg}(t) < k$, then $\mathrm{rt}_k(t) = t$. If $\mathrm{hg}(t) \geq k$ and $t = f(t_1, \ldots, t_m)$, then $\mathrm{rt}_k(t) = f(\mathrm{rt}_{k-1}(t_1), \ldots, \mathrm{rt}_{k-1}(t_m))$.

For any $k \geq 0$, a tree language $T \subseteq T_\Sigma(X)$ is called *k-definite* if, for any $s, t \in T_\Sigma(X)$, if $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$, then $s \in T$ iff $t \in T$. A tree language is *definite* if it is k-definite for some $k \geq 0$. The set of k-definite ΣX-tree languages is denoted by $\mathrm{Def}_\Sigma(X, k)$ and the set of all definite ΣX-tree languages by $\mathrm{Def}_\Sigma(X)$. Furthermore, let $\mathrm{Def}_\Sigma = \{\mathrm{Def}_\Sigma(X)\}_X$ and $\mathrm{Def}_\Sigma(k) = \{\mathrm{Def}_\Sigma(X, k)\}_X$ denote the families of definite and k-definite Σ-tree languages, respectively. □

We shall now present a couple of notions and auxiliary results to be used in the proof of the main result of this section.

If $r \in C^n_\Sigma(X)$ is an n-ary ΣX-context in which the variable ξ_i appears for some $i \in [n]$, then the ξ_i*-depth* $\mathrm{d}_{\xi_i}(r)$ of r is the distance of the ξ_i-labelled leaf from the root, i.e.,

(1) $\mathrm{d}_{\xi_i}(\xi_i) = 0$, and

(2) if $r = f(r_1, \ldots, r_m)$, where ξ_i appears in r_j, then $\mathrm{d}_{\xi_i}(r) = \mathrm{d}_{\xi_i}(r_j) + 1$.

Lemma 10.2 *Let $s, t \in T_\Sigma(X)$ and $k \geq 0$. If $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$, then either $s = t$ or then there exist an $n \geq 1$, an n-ary ΣX-context $r \in C^n_\Sigma(X)$ in which all of the variables $\xi_1, \ldots, \xi_n$ appear, and $\mathbf{s} = \langle s_1, \ldots, s_n \rangle, \mathbf{t} = \langle t_1, \ldots, t_n \rangle \in T_\Sigma(X)^n$ such that $s = \mathbf{s} \cdot r$, $t = \mathbf{t} \cdot r$, and $\mathrm{d}_{\xi_i}(r) = k$ for every $i \in [n]$.*

Proof. If $\mathrm{rt}_k(s) = s$ or $\mathrm{rt}_k(t) = t$, i.e., if $\mathrm{hg}(s) < k$ or $\mathrm{hg}(t) < k$, then $s = t$. Hence we assume that $\mathrm{hg}(s), \mathrm{hg}(t) \geq k$. We can now proceed by induction on $k \geq 0$.

(1) If $k = 0$, we can set $n = 1$, $r = \xi_1$, $\mathbf{s} = \langle s \rangle$ and $\mathbf{t} = \langle t \rangle$.

(2) If $k = 1$, then $\mathrm{root}(s) = \mathrm{root}(t) = f$ for some $f \in \Sigma_m$ where $m \geq 1$. Hence, $s = f(s_1, \ldots, s_m)$ and $t = f(t_1, \ldots, t_m)$ for some $s_1, \ldots, s_m, t_1, \ldots, t_m \in T_\Sigma(X)$, and we may choose $r = f(\xi_1, \ldots, \xi_m)$, $\mathbf{s} = \langle s_1, \ldots, s_m \rangle$ and $\mathbf{t} = \langle t_1, \ldots, t_m \rangle$.

(3) Let $k \geq 2$ and assume that the assertion holds for all lesser values of k. Since we assumed that $\mathrm{hg}(s), \mathrm{hg}(t) \geq k$, we have $s = f(s_1, \ldots, s_m)$ and $t = f(t_1, \ldots, t_m)$ for some $m \geq 1$, $f \in \Sigma_m$, and $s_1, \ldots, s_m, t_1, \ldots, t_m \in T_\Sigma(X)$, and moreover $\mathrm{rt}_{k-1}(s_i) = \mathrm{rt}_{k-1}(t_i)$ for every $i \in [m]$. For each $i \in [m]$, either $s_i = \mathrm{rt}_{k-1}(s_i) = \mathrm{rt}_{k-1}(t_i) = t_i$, or $\mathrm{hg}(s_i), \mathrm{hg}(t_i) \geq k-1$. In the former case, we set $n_i := 0$ and $r_i := s_i (= t_i)$. Otherwise, there exist $n_i \geq 1$, $r_i \in C_\Sigma^{n_i}(X)$ and $\mathbf{s}_i, \mathbf{t}_i \in T_\Sigma(X)^{n_i}$ such that $s_i = \mathbf{s}_i \cdot r_i$, $t_i = \mathbf{t}_i \cdot r_i$, and for every $j \in [n_i]$, the variable ξ_j appears in r_i and $\mathrm{d}_{\xi_j}(r_i) = k-1$. The required representations for s and t can be defined as follows. Firstly, let $n := n_1 + \ldots + n_m$, and secondly, let $r := f(r_1, r_2', \ldots, r_m') (\in C_\Sigma^n(X))$, where r_2' is obtained from r_2 by incrementing the indices of the variables by n_1, r_3' is obtained from r_3 by incrementing the indices of the variables by $n_1 + n_2$ etc. Finally, let $\mathbf{s} := \mathbf{s}_1 \oplus \ldots \oplus \mathbf{s}_m$ and $\mathbf{t} := \mathbf{t}_1 \oplus \ldots \oplus \mathbf{t}_m$. □

We shall also need the following obvious fact.

Lemma 10.3 *Let $p \in C_\Sigma(X)$ be any ΣX-context. If $\mathrm{d}_\xi(p) = n$, where $n \geq 0$, then there are n ΣX-contexts $p_1, \ldots, p_n \in C_\Sigma(X)$, each of ξ-depth 1, such that $p = p_1 \cdot p_2 \cdot \ldots \cdot p_n$.*

Proof. The lemma can be proved by induction on $n \geq 0$. Note that if $n = 0$, then $p = \xi$ and the empty product $p_1 \cdot p_2 \cdot \ldots \cdot p_n$ is also interpreted as ξ. □

We shall need also the following property of the semigroups $S_\Sigma^n(X)$.

Lemma 10.4 *Let $\mathbf{u}_1, \ldots, \mathbf{u}_k \in S_\Sigma^n(X)$ for some $n, k \geq 1$. If $\mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k = \langle w_1, \ldots, w_n \rangle$, then for each $i \in [n]$, either*

(1) *$w_i \in T_\Sigma(X)$, or*

(2) *$w_i \in T_\Sigma(X \cup \Xi_n) \setminus T_\Sigma(X)$ and $\mathrm{hg}(w_i), \mathrm{dp}(w_i) \geq k$.*

Proof. Let us fix $n \geq 1$ arbitrarily and proceed by induction on k.

For $k = 1$, we have $\langle w_1, \ldots, w_n \rangle = \mathbf{u}_1 \in S^n_\Sigma(X)$. From the definition of $S^n_\Sigma(X)$, it follows that $w_i \in T_\Sigma(X)$, or $\mathrm{hg}(w_i), \mathrm{dp}(w_i) \geq 1$ for every $i \in [n]$.

Consider any $k > 1$ and assume that the assertion holds for every smaller value of k. If $\mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_{k-1} = \mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{u}_k = \langle u_1, \ldots, u_n \rangle$, then $\langle w_1, \ldots, w_n \rangle = \mathbf{v} \cdot \mathbf{u}_k$ where $w_i = \mathbf{v} \cdot u_i$ for each $i \in [n]$. If $u_i \in T_\Sigma(X)$, then $w_i = u_i \in T_\Sigma(X)$. Assume then that $u_i \in T_\Sigma(X \cup \Xi_n) \setminus T_\Sigma(X)$. If $v_j \in T_\Sigma(X)$ for every variable ξ_j ($j \in [n]$) appearing in u_i, then $w_i = \mathbf{v} \cdot u_i$ is also in $T_\Sigma(X)$. Otherwise, there is at least one variable ξ_j in u_i such that $v_j \in T_\Sigma(X \cup \Xi_n) \setminus T_\Sigma(X)$. By the inductive assumption, $\mathrm{hg}(v_j) \geq k - 1$. Since $u_i \neq \xi_j$, this means that $\mathrm{hg}(w_i) \geq k$. Furthermore,

$$\mathrm{dp}(w_i) \geq \mathrm{dp}(u_i) + \min(\mathrm{dp}(v_1), \ldots, \mathrm{dp}(v_n)) \geq 1 + (k-1) = k,$$

since $\mathrm{dp}(u_i) \geq 1$ and, by the inductive assumption, $\mathrm{dp}(v_i) \geq k - 1$ for every $i \in [n]$. $\square$

Let us call a ranked alphabet Σ *proper* if $\Sigma_m \neq \emptyset$ for some $m \geq 2$. Note that if $\Sigma_0 = \emptyset$, we excluded the empty leaf alphabet, and hence properness of Σ guarantees that for any $n \geq 0$, there always is a ΣX-tree with variables in which there are exactly n leaves labeled with a variable. Following Eilenberg [5], we denote by $\mathbf{D}_k$ the VFS of the finite semigroups satisfying the identity $\mathbf{v} \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k \approx \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k$ ($k \geq 0$). Note that we included the value $k = 0$ for which $\mathbf{D}_k$ consists of the trivial semigroups only.

Proposition 10.5 *If Σ is a proper ranked alphabet, then* $\mathrm{Def}_\Sigma(k) = \mathcal{V}^2_{\mathbf{D}_k}$ *for every* $k \geq 0$.

Proof. Let T be a regular ΣX-tree language T for some leaf alphabet X. The proposition claims that T is k-definite if and only if

$$S^2(T) \models \mathbf{v} \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k \approx \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k.$$

Let $\mathbf{A} = (\mathcal{A}, \alpha, F)$ be the minimal ΣX-recognizer of T. Since $S^2(T) \cong TS^2(\mathbf{A})$, it suffices to show that T is k-definite if and only if $(\mathbf{v} \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}} = (\mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}}$ for all $\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_k \in S^2_\Sigma(X)$.

Let us first assume that $T \in \mathrm{Def}_\Sigma(X, k)$ and consider any $\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_k \in S^2_\Sigma(X)$. Let $\mathbf{w} = \langle w_1, w_2 \rangle := \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k$ and $\mathbf{z} = \langle z_1, z_2 \rangle := \mathbf{v} \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k$. We should now prove that $w_1^{\mathbf{A}}(\mathbf{a}) = z_1^{\mathbf{A}}(\mathbf{a})$ and $w_2^{\mathbf{A}}(\mathbf{a}) = z_2^{\mathbf{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^2$. Let $i = 1$ or $i = 2$ and consider any $\mathbf{a} \in A^2$. With Lemma 10.4 in mind, we distinguish two cases.

1. If $w_i \in T_\Sigma(X)$, then $z_i = \mathbf{v} \cdot w_i = w_i$ and $w_i^{\mathbf{A}}(\mathbf{a}) = z_i^{\mathbf{A}}(\mathbf{a})$ trivially holds.

2. If $w_i \notin T_\Sigma(X)$, then $\mathrm{hg}(w_i), \mathrm{dp}(w_i) \geq k$. By Lemma 8.2, there is a $\mathbf{t} \in T_\Sigma(X)^2$ such that $\mathbf{t}\widehat{\alpha}_2 = \mathbf{a}$. Then $w_i^{\mathbf{A}}(\mathbf{a}) = (\mathbf{t} \cdot w_i)\widehat{\alpha}$ and $z_i^{\mathbf{A}}(\mathbf{a}) = (\mathbf{t} \cdot \mathbf{v} \cdot w_i)\widehat{\alpha}$. Since $\mathrm{dp}(w_i) \geq k$, we have $\mathrm{rt}_k(\mathbf{t} \cdot w_i \cdot p) = \mathrm{rt}_k(\mathbf{t} \cdot \mathbf{v} \cdot w_i \cdot p)$ for every $p \in C_\Sigma(X)$. Because $T = T(\mathbf{A})$ is k-definite, this means that

$$(\forall p \in C_\Sigma(X))\big(p^{\mathbf{A}}((\mathbf{t} \cdot w_i)\widehat{\alpha}) \in F \ \leftrightarrow\ p^{\mathbf{A}}((\mathbf{t} \cdot \mathbf{v} \cdot w_i)\widehat{\alpha}) \in F\big),$$

   i.e., that $(\mathbf{t} \cdot w_i)\widehat{\alpha} \sim_{\mathbf{A}} (\mathbf{t} \cdot \mathbf{v} \cdot w_i)\widehat{\alpha}$. As $\mathbf{A}$ is reduced, we get $w_i^{\mathbf{A}}(\mathbf{a}) = (\mathbf{t} \cdot w_i)\widehat{\alpha} = (\mathbf{t} \cdot \mathbf{v} \cdot w_i)\widehat{\alpha} = z_i^{\mathbf{A}}(\mathbf{a})$.

Assume now that $(\mathbf{v} \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}} = (\mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}}$ for all $\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_k \in S_\Sigma^2(X)$ and consider any ΣX-trees s and t such that $\mathrm{rt}_k(s) = \mathrm{rt}_k(t)$. To prove that T is k-definite, it suffices to show that $s\widehat{\alpha} = t\widehat{\alpha}$.

If $\mathrm{hg}(s) < k$ or $\mathrm{hg}(t) < k$, then $s = \mathrm{rt}_k(s) = \mathrm{rt}_k(t) = t$. Hence we may assume that $\mathrm{hg}(s), \mathrm{hg}(t) \geq k$ and that $s \neq t$. By Lemma 10.2, there exist an $n \geq 1$, an n-ary ΣX-context $r \in C_\Sigma^n(X)$ in which each variable $\xi_1, \ldots, \xi_n$ appears and $\mathrm{d}_{\xi_i}(r) = k$ for every $i \in [n]$, and two n-tuples of ΣX-trees $\mathbf{s} = \langle s_1, \ldots, s_n\rangle, \mathbf{t} = \langle t_1, \ldots, t_n\rangle \in T_\Sigma(X)^n$ such that $s = \mathbf{s} \cdot r$, $t = \mathbf{t} \cdot r$. For each $i \in [n]$, let

$$p_i \ := \ \langle t_1, \ldots, t_{i-1}, \xi, s_{i+1}, \ldots, s_n\rangle \cdot r$$

be the ΣX-context obtained from r when the variables $\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots, \xi_n$ are replaced by $t_1, \ldots, t_{i-1}, \xi, s_{i+1}, \ldots, s_n$, respectively. Then

$$s = p_1(s_1), p_1(t_1) = p_2(s_2), p_2(t_2) = p_3(s_3), \ldots, p_{n-1}(t_{n-1}) = p_n(s_n), p_n(t_n) = t.$$

Hence, it suffices to show that $p_i(s_i)\widehat{\alpha} = p_i(t_i)\widehat{\alpha}$ for every $i \in [n]$.

Let us consider any $i \in [n]$. By Lemma 10.3, we may write $p_i = q_1 \cdot \ldots \cdot q_k$ for some SX-contexts $q_1, \ldots, q_k$ (of ξ-depth 1). Since Σ is proper, we may fix a tree $w \in T_\Sigma(X \cup \{\xi_2\}) \setminus \{\xi_2\}$ in which ξ_2 appears exactly once and a binary ΣX-context $z \in C_\Sigma^2(X)$ in which ξ_1 and ξ_2 both appear. For each $j \in [k]$, let q_j' be the unary ΣX-context obtained from q_j by replacing ξ with ξ_1. Then

$$\mathbf{u}_1 := \langle q_1', w\rangle, \ldots, \mathbf{u}_k := \langle q_k', w\rangle, \mathbf{v} := \langle s_i, z\rangle, \mathbf{v}' := \langle t_i, z\rangle \in S_\Sigma^2(X),$$

and hence

$$(\mathbf{v} \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}} \ = \ (\mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}} \ = \ (\mathbf{v}' \cdot \mathbf{u}_1 \cdot \ldots \cdot \mathbf{u}_k)^{\mathbf{A}}$$

by our assumption about $\mathbf{A}$. But the first components of $\mathbf{v}\cdot\mathbf{u}_1\cdot\ldots\cdot\mathbf{u}_k$ and $\mathbf{v}'\cdot\mathbf{u}_1\cdot\ldots\cdot\mathbf{u}_k$ are easily seen to be the ΣX-trees $s_i \cdot q_1' \cdot \ldots \cdot q_k' = p_i(s_i)$ and $t_i \cdot q_1' \cdot \ldots \cdot q_k' = p_i(t_i)$, respectively, and hence $p_i(s_i)\widehat{\alpha} = p_i(t_i)\widehat{\alpha}$. $\square$

Let us recall that the VFS $\mathbf{D}_k$ corresponds in Eilenberg's variety theory to the +-variety of k-definite (string) languages (cf. [5], pp. 214–216), and that the union of the chain $\mathbf{D}_0 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \ldots$ is the VFS $\mathbf{D}$ that corresponds to the +-variety of all definite languages. Since $\mathrm{Def}_\Sigma(0) \subset \mathrm{Def}_\Sigma(1) \subset \mathrm{Def}_\Sigma(2) \subset \ldots$ and $\mathrm{Def}_\Sigma = \bigcup_{k\geq 0} \mathrm{Def}_\Sigma(k)$, Proposition 10.5 yields a the following corresponding fact.

Corollary 10.6 $\mathrm{Def}_\Sigma = \mathcal{V}^2_{\mathbf{D}}$ *for any proper ranked alphabet* Σ. $\square$

Finally, let us note that in Proposition 10.5 and Corollary 10.6 we could write $\mathrm{Def}_\Sigma(k) = \mathcal{V}^n_{\mathbf{D}_k}$ and $\mathrm{Def}_\Sigma = \mathcal{V}^n_{\mathbf{D}}$, respectively, for any $n \geq 2$. Actually, the only modification required in the proof of Proposition 10.5 concerns the definition of the $\Sigma X2$-contexts $\mathbf{u}_1, \ldots, \mathbf{u}_k$, $\mathbf{v}$ and $\mathbf{v}'$ that should be made ΣXn-contexts. If $n \geq 3$, it suffices that $\Sigma_m \neq \emptyset$ for some $m \geq 1$.

11 Concluding remarks

We have established several basic properties of the new syntactic monoids $M^n(T)$ and semigroups $S^n(T)$ of tree languages introduced in this paper. In particular, we have shown that $M^n(T)$ and $S^n(T)$ are isomorphic, respectively, to the n-ary transformation monoid and semigroup of the minimal recognizer of T, as well as to the n-ary translation monoid and semigroup of the syntactic algebra of T. We have also shown how any variety of finite monoids or semigroups yields a variety of tree languages via these syntactic monoids or semigroups, respectively. Hereby, it turned out that the variety-defining power of our monoids or semigroups lies properly between that of ordinary syntactic monoids or semigroups and that of syntactic algebras. However, a great number of natural questions remain open. Let us note some of them.

In spite of Examples 9.5 and 9.6, we don't have examples of proper infinite hierarchies $\mathcal{V}^1_{\mathbf{M}} \supset \mathcal{V}^2_{\mathbf{M}} \supset \mathcal{V}^3_{\mathbf{M}} \supset \ldots$ or $\mathcal{V}^1_{\mathbf{S}} \supset \mathcal{V}^2_{\mathbf{S}} \supset \mathcal{V}^3_{\mathbf{S}} \supset \ldots$. An even more interesting question concerns the hierarchies of the classes of varieties of tree languages definable by our syntactic monoids or semigroups. To state this question more precisely, let $VTL_\Sigma(Mon, n)$ denote the class of Σ-VTLs $\mathcal{V}$ such that $\mathcal{V} = \mathcal{V}^n_{\mathbf{M}}$ for some VFM $\mathbf{M}$. Now, the conjecture is that $VTL_\Sigma(Mon, n) \subset VTL_\Sigma(Mon, n+1)$ for every

$n \geq 1$ and any non-trivial ranked alphabet Σ. Similarly, it seems plausible that $VLT_\Sigma(Sg, n) \subset VTL_\Sigma(Sg, n+1)$ holds for the corresponding classes $VLT_\Sigma(Sg, n)$ of Σ-VTLs definable by our syntactic semigroups. Another major problem is to find a characterization of the varieties of the form $\mathcal{V}^n_{\mathbf{M}}$ or $\mathcal{V}^n_{\mathbf{S}}$, similar to the one given by Salehi [18, 19] for the varieties of tree languages definable by the classical syntactic monoids or semigroups of Thomas [27, 28]. Such a result would naturally also give valuable guidance in the search for further characterizations of varieties of tree languages in terms of our monoids or semigroups. It seems natural to compute the monoids $M^n(T)$ and semigroups $S^n(T)$ as translation monoids or semigroups, respectively, of the syntactic algebra of T, but since they are likely to be quite big even in simple cases, we would need efficient ways to extract the crucial information about them without too much computation. However, it seems that this can be done only by utilizing the special properties of each variety at hand. Finally, let us note that since monoids and semigroups are not associated with any particular ranked alphabet, it would be natural to consider *generalized varieties of tree languages* (cf. [25]) associated with a VFM or a VFS via our syntactic monoids or semigroups, respectively.

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