Structured Derivations: a Logic Based Approach to Teaching Mathematics

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Abstract

Being able to reason rigorously and comfortably in mathematics plays and essential role in computer science, particularly when working with formal methods. Unfortunately, the reasoning abilities of first year university students' are commonly rather poor due to lack of training in exact formalism and logic during prior education. In this paper we present *structured derivations*, a logic based approach to teaching mathematics, which promotes preciseness of expression and offers a systematic presentation of mathematical reasoning. The approach has been extensively evaluated at different levels of education with encouraging results, indicating that structured derivations provide many benefits both for students and teachers.

Keywords: Structured derivations, teaching mathematics, mathematics for formal methods

1 Introduction

Being able to reason rigorously and comfortably in mathematics is an essential prerequisite for studies in computer science (CS), especially when working with formal methods. Nevertheless, many CS students unfortunately show little understanding for and interest in mathematics in general and formal notation, logic and proofs in particular. For instance, Gries [10] notes that "students' reasoning abilities are poor, even after several math courses. Many students still fear math and notation, and the development of proofs remains a mystery to most." (p. 2) Almstrum [3] found that novice CS students experience more difficulty with the concepts of mathematical logic than with other CS concepts.

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This paper is electronically published in
Electronic Notes in Theoretical Computer Science
URL: www.elsevier.nl/locate/entcs

One reason for students' low level of skills in formal reasoning and proofs can be traced back to their prior education: exact formalism and proof are perceived as difficult and consequently avoided at e.g. high school level (e.g. [5,12,13,18]). For instance in Finland, high school students are offered the choice of two different mathematics syllabi, including a total of 21 courses (sixteen compulsory, five elective) [15]. Despite the large number of courses, proof and formal reasoning is only mentioned in the learning objectives for one of them, the elective course "Logic and Number Theory" in the advanced syllabus. This is also the only course that introduces logical notation and truth values. How could we expect first year university students to expose high levels of proficiency in topics such as logic, exact formalisms and constructing proofs, when their only prior chance to study these topics is in one (elective) course throughout their entire general education? Proofs should be considered a way of thinking that can be applied to any mathematical topic, instead of being viewed as a distinct topic [11,18].

In addition to the lack of training in formal reasoning and proofs, studies have indicated problems in the way proofs are approached and presented in education. For instance, Dreyfus [9] claims that students often receive mixed messages. As an example he notes that many mathematical textbooks offer intuitive explanations in one solution, use examples to clarify another, and give a rigorous proof for yet another. The differences between these justifications are however not explicated, but leave students with three different views of what *could* constitute a proof. As a result, students do not know what counts as an acceptable mathematical justification.

Moreover, students are likely to engage in activities that feel worth while and relevant for their studies as a whole. However, the prevalent curriculum strategy at CS departments is to divide courses "into areas of 'theory' and 'practice'... [which] causes both faculty and students to view the theory of computing as separate and distinct from the practice of computing." [2, p. 73] In order for mathematics to be considered useful by CS students, it should thus be presented in a way that clearly links it to the computing practice.

In this paper, we present *structured derivations* [4,6,7], a logic based approach to teaching mathematics, which we argue can be used to address all the aforementioned problems. Structured derivations promote preciseness of expression and offer a systematic and straightforward presentation of mathematical reasoning, without restricting the application area. Using structured derivations, logic becomes a tool for doing mathematics, rather than a object of mathematical study.

We begin with a brief description of the structured derivations approach to constructing proofs in Section 2. The approach has been extensively evaluated since 2001 and currently the evaluation involves five institutions at high school and university level. We summarize the high school experience in Section 3 and give a more detailed account of our experience from using the approach in a first year CS course in Section 4. We conclude with a discussion section including ideas for future work.

2 Structured Derivations

Structured derivations [4,6,7] is a further development of Dijkstra's *calculational proof style*, where we have added a mechanism for doing subderivations and for handling assumptions in proofs. With these extensions, structured derivations can be seen as an alternative notation for Gentzen like proofs in predicate calculus or higher order logic [6]. A structured derivation has the following general syntax:

$$\begin{array}{rl}
\text{derivation} ::= & \text{"}\bullet\text{"}[\text{ task " : "}]\ [\text{assumptionList}]\ [(\vdash \mid \Vdash)]\ [\text{proofSteps}] \\
\text{assumptionList} ::= & (\text{postulate} \mid \text{lemma})^{+} \\
\text{postulate} ::= & \text{"["}\ \text{identification}\ \text{"]"}\ \text{formula} \\
\text{lemma} ::= & \text{"("}\ \text{identification}\ \text{")"}\ \text{formula derivation} \\
\text{proofSteps} ::= & \text{term}\ (\text{basicStep} \mid \text{subderivation})^{+} \\
\text{basicStep} ::= & \text{relation}\ \text{"\{"}\ \text{motivation}\ \text{"\}"}\ \text{term} \\
\text{subderivation} ::= & \text{relation}\ \text{"\{"}\ \text{motivation}\ \text{"\}"}\ \text{derivation}^{+}\ \text{"}\ldots\text{"}\ \text{term}
\end{array}$$

Terminals are given inside quotes and nonterminals in roman font. The layout of a structured derivation is fixed. The general proof layout is as follows ("task" is a short informal explanation of what we want to do):

derivation:

```
•    task:
     assumptionList
⊢
     proofSteps
```

Below to the left, we show the layout for a derivation where all assumptions are postulates and where there are only basic proof steps. The middle box shows the layout of a lemma, where the proof of the lemma (the formula) is a derivation written directly after the formula but is indented one step to the right. On the right we show a proof step with a subderivation. A subderivation justifies the proof step and corresponds to the application of an inference rule in a Gentzen like proof system. One proof step may require one or more subderivations. The subderivations follow immediately after the motivation for the proof step and are indented one step.

simple derivation:

```
•     task:
[id]  formula
⋮
[id]  formula
⊢
      term
rel   {motivation}
      term
⋮
rel   {motivation}
      term
```

lemma with proof:

```
(id) formula
      •    task:
           assumptionList
      ⊢
           proofSteps
```

subderivation proof step:

```
      term
rel   {motivation}
      •    task:
           assumptionList
      ⊢
           proofSteps
...   term
```

This proof format fixes the overall structure and layout of a derivation (hence the name *structured derivations*) but it does not fix the syntax of basic entities such as task, formula, term, relation, motivation, or identification. Thus, we can use structured derivations for proofs on different domains, and with different levels of rigor and detail (from a completely intuitive argumentation to an axiomatic proof in a logical theory).

We illustrate structured derivations with two mathematical problems. The first one illustrates the use of logical rules in standard mathematical reasoning, while the second illustrates subderivations and the way in which we combine formal and informal reasoning in a structured derivation (the second problem is taken from the Finnish high school matriculation exam 2006).

Our first problem is to solve the equation $(x-1)(x^2+1)=0$. We have the following solution

---

- Solve the equation $(x-1)(x^2+1)=0$:

  $(x-1)(x^2+1)=0$

$\equiv$ {zero product rule: $ab=0 \equiv a=0 \vee b=0$}

  $x-1=0 \vee x^2+1=0$

$\equiv$ {add 1 to both sides in left disjunct and -1 to both sides in right disjunct}

  $x=1 \vee x^2=-1$

$\equiv$ {a square is always non-negative}

  $x=1 \vee F$

$\equiv$ {disjunction rule}

  $x=1$

The second problem is to determine the values of a for which the function $f(x)=-x^2+ax+a-3$ is always negative.

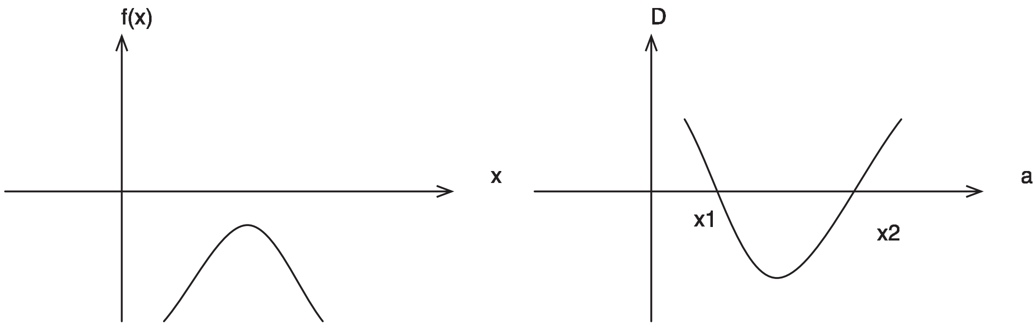

Figure 1. Downward and upward opening parabolas

The following structured derivation shows how we determine the value of a. Figure 1 illustrates the arguments used in the proof.

- Determine the values of a for which $-x^2+ax+a-3$ is always negative:

  $(\forall x \cdot -x^2+ax+a-3<0)$

$\equiv$ {the function is a parabola that opens downwards (the coefficient for x^2 is negative); such a function is always negative if it does not intersect the x-axis, i.e. has no roots (figure on the left)}

  $(\forall x \cdot -x^2+ax+a-3 \neq 0)$

$\equiv$ {this condition holds iff the discriminant D for the function is negative}
$D < 0$
$\equiv$ {substitute value of D}
- Determine the discriminant D:
  D
  $=$ {the discriminant for the equation $Ax^2 + Bx + C = 0$ is $B^2 - 4AC$}
  $a^2 - 4(-1)(a-3)$
  $=$ {simplify}
  $a^2 + 4a - 12$

$\ldots$ $a^2 + 4a - 12 < 0$
$\equiv$ {the function $a^2 + 4a - 12$ is a parabola that opens upwards (the coefficient for a^2 is positive); such a function is negative between its roots (figure on the right)}
- Solve the equation $a^2 + 4a - 12$:
  $a^2 + 4a - 12 = 0$
  $\equiv$ {square root formula}
  $a = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1}$
  $\equiv$ {simplify the expression}
  $a = 2 \vee a = -6$

$\ldots$ $-6 < a < 2$

This proves that

$$(\forall x \cdot -x^2 + ax + a - 3 < 0) \equiv -6 < a < 2$$

In other words, the function is always negative if and only if $-6 < a < 2$.

If we are using a computer supported tool with outlining features (like the TeXmacs plug-in mentioned below), we can choose to hide the two subderivations. Omitting the more detailed steps will give us a better view of the overall structure of the proof.

Traditional approaches to teaching and presenting mathematics contain much implicit information [9,14]. Using structured derivations, all steps in the derivation are explicitly motivated and the final product thus contains a documentation of the thinking the student was engaged in while completing the derivation. This facilitates reading and debugging both for students and teachers.

Moreover, as stated in the introduction, traditional approaches to teaching proofs leave students uncertain about what rigor is required for a particular proof in a certain situation [9]. Structured derivations provide a well-defined proof format, which gives students a concrete “model” for what constitutes a proof and which can guide them in how to carry out rigorous proofs in practice. A clear and familiar format functions as a mental support that gives students belief in their own skills to construct the proof. A defined format also lets students focus on the solution rather than spending time thinking about how to put their thoughts down on paper. Furthermore, our approach provides a structure that can be used to make the presentation of mathematics more consistent in textbooks and classrooms. Due to the well-defined syntax and simple structure, structured derivations are also well suited for presentation on the web.

3 Structured Derivations at High School Level

As stated in the introduction, Finnish students can graduate from high school without being exposed to logic or proof in their mathematics courses. This is alarming not only from a computing perspective, but from a science perspective in general. Thus, although our main concern as CS educators is to ensure that our students possess sufficient mathematical skills in order to be able to progress successfully in their studies, we feel that attention also should be put on mathematics education at lower levels. We summarize our experiences from introducing structured derivations at high school in this section, and describe our experiences from introducing the same method into a CS syllabus in the next section.

The two mathematics syllabi offered in Finnish high schools have different foci: the general syllabus focuses on developing the capabilities needed "to use mathematics in different situations in life and in further studies" [15, p. 119], whereas the advanced syllabus focuses on learning to "understand the nature of mathematical knowledge" [15, p. 122]. The advanced syllabus is practically the norm for students seeking admission to universities for further studies in, for instance, mathematics, CS, engineering, medicine and physics. Considering the need for mathematical maturity in these fields, the students would most certainly benefit from getting more training in formal reasoning and proof already at high school level (as mentioned before, these topics are currently only mentioned in one advanced elective course). This does, however, not necessarily imply that more specific courses on logic should be introduced, but rather that logic should be integrated in other courses [16].

In 2001, a longitudinal study was initiated in a high school in Turku, Finland [5,17]. The aim of the study was to investigate whether structured derivations could be used to integrate logic, proof and formal reasoning throughout high school mathematics education without the need for additional courses on logic. The research setting involved a test group and a control group, which were followed up during their entire high school period (three years). The students chose which group to belong to, but care was taken to ensure that the entry level of the students was as similar as possible in both groups. The groups had different teachers who taught the exact same material, only using different approaches; the test group teacher rewrote and taught all ten compulsory mathematics courses using structured derivations, whereas the control group teacher gave the courses in his usual presentation style. Moreover, the test group teacher spent a few hours at the beginning of the first course introducing basic notions of elementary logic and giving students formal and informal practice in working with logical connectives.

The results from the study were positive, indicating that logical notation and structured derivations can be successfully used in a high school setting. Students in the test group consistently outperformed the control group in all ten courses [5] as well as in the matriculation exam [17]. [5]

[5] Students take the matriculation exam at the end of their high school studies. The matriculations examination board approves of the use of structured derivations in the matriculation exam.

In addition to this longitudinal study, the approach has also been introduced in single courses at three other high schools. Clearly, this renders a completely different situation than when all courses are given using the same format. Despite the limited time available for the teacher to present the approach and for the students to get familiar with it and use it, the results from these courses have also been positive. Surveys and observations have shown that despite a somewhat negative initial reaction to the new strict format requiring additional writing, most students learned to use and appreciate the structured approach during one single course.

We are now in the process of developing more systematic teaching material to support the use of structured derivations in mathematics education. Back and von Wright have written "Mathematics with a Little Bit of Logic" [8], a text book that introduces the approach and that can also be used as a teachers manual. Moreover, two ordinary high school mathematics text books have been "translated" into structured derivations. In addition, all assignments in ten complete mathematics matriculation exams have been solved using structured derivations (altogether 150 solved problems). This collection of solutions is important not only as an example base, but also as a confirmation that the approach can in fact be applied on a wide variety of problems.

4 Structured Derivations for First Year CS Students

The need for practical skills in proving mathematical theorems becomes evident to our CS students already during their first year courses. A compulsory course on logic was introduced in the basic studies in the CS curriculum at our department already in the 1990s, but as it was rather theoretical, students did not see the connection to the real world and felt that the course did not give them any skills that could be useful in practice. The course was totally redesigned in fall 2006, when structured derivations were introduced to put more focus on enhancing students' logical reasoning and proof-writing abilities in practice.

The course was attended by 47 students and included 36 lectures (of 45 minutes), six exercise sessions (of 90 minutes) and a final exam. A pre- and postcourse survey as well as observations were used to evaluate the course and students' opinions about the approach. The idea of the course was to apply structured derivations to high school mathematics. We thought that applying the rigorous derivational format on familiar problems would make it easier for students to learn the methodology, as they would not have to learn any new mathematics at the same time. This, however, did not work as intended. Instead, the familiar domain hindered the students from seeing the purpose of the course, but thought that the course was just a repetition of high school mathematics, failing to understand the importance of the format used. The initial confusion was partly a result of miscommunication, as the teaching assistant did not enforce the use of the structured derivations format in the exercise sessions. This gave the students a message that was not consistent with the one they received during the lectures.

Resistance to relearn familiar material using a new format is understandable; why should one start using a new approach for doing something one has already been doing successfully in another way for 12 years. However, as the students realized that the topics covered in the course were indeed intended to be familiar, and that structured derivations was the new thing that they were supposed to learn, the resistance faded away. The results from the final exam were good (70% of the students passed, 30% with the highest grade) and the final feedback was in general positive. In the following, we list some of the positive and negative aspects brought up by students in the open questions of the post course survey.[6]

Students identified many of the same benefits of using structured derivations as was originally hypothesized when the approach was developed:

- *"When you write out everything, careless mistakes disappear"*
- *"Writing better mathematical derivations: same motivations as earlier, but more logically constructed"*
- *"Learning a systematic way of working"*
- *"I learned to think deeper on mathematical solutions"*
- *"Useful to practice problem solving, structure and divide problems"*
- *"The different proof strategies will be useful in math courses. But the logical motivations were also important to learn considering how to prove one's programs"*

Some students found writing the derivations a bit tedious, but nevertheless found the approach interesting and useful.

- *"They feel a bit unnecessary sometimes, but on the other hand you see much more clearly what you've meant when you look at the solution again later"*
- *"Structured derivations feels like unnecessary work, but the format does make the calculations clearer"*
- *"In many cases I feel that structured derivations is a way of complicating simple things. Sure, you should be able to motivate what you're doing, but there's no need to exaggerate. On a suitable level of abstraction, this is, however, an interesting way of thinking"*
- *"The derivations were important. Even if you don't like them, you may appreciate them more during further studies"*

Students also appreciated the structured derivations simply because it was a new format that appealed to them.

- *"The approach was pretty difficult and therefore interesting"*
- *"Interesting to learn how to write them and understand why you should write them"*
- *"The approach was interesting. I've always had problems proving things"*

The negative aspects brought up by the students were mainly related to the motivations and assumptions in proofs. One student also mentioned difficulties remembering the syntax of the format.

- *"Feels somewhat unnecessary to motivate everything"*
- *"I think it's difficult to write down the motivations as many of them are obvious"*
- *"Difficult to know when a derivation is correct. What can you assume and what can't you assume?"*
- *"Difficult to remember how to write them correctly"*

Finally, some students seemed to have a negative attitude towards mathematics in general.

- *"Derivations are always uninteresting"*
- *"I'm not interested in derivations and I don't think I'll need it in future CS studies"*

We were pleased to see that only a couple of students expressed negative opinions towards mathematics after the course. Most students stated that they appreciated the approach and the benefits it provides (clarifies solutions, facilitates debugging, makes relations explicit, enforces a systematic way of working, etc). As was found in the feedback on the high school courses, students initially felt frustrated with the extra writing needed in the form of motivations, but still found the format useful. We feel that the students' feedback indicates that the majority of them did no longer

[6] The quotations have been freely translated into English by one of the authors.

fear notation after this particular course. In our opinion, this is an encouraging finding.

Based on the experience from the first version of the course, some revisions were made before giving the course again starting in fall 2007. The main changes made were that instead of applying structured derivations on high school topics, the approach is now exemplified with problems in areas that the students are not familiar with from before: propositional and predicate logic, discrete mathematics, elementary algebra, lattice theory and boolean algebra. The focus is still on using the structured derivations framework and on emphasizing the need to develop skills in constructing mathematical proofs in practice. This course is going on as we write this, and the final evaluation is thus yet to be done. However, our preliminary observations indicate that students now "got the point" of the course from the very beginning, have used the structured derivations format in their own solutions and appreciate learning new mathematical topics.

5 Discussion

The experience from using structured derivations in education has been encouraging both at secondary and tertiary level. Although the results of introducing the approach in individual high school courses have been positive, we nevertheless believe that integrating logic as a tool in all mathematics courses is to be preferred. The effects of one single course are easily canceled out by the remaining courses not mentioning logic or proof at all.

Our experiences from teaching the structured derivations course as an introductory CS course has also been very encouraging, so much that structured derivations is now the standard approach used in the logic course. We are also planning a new course for first year university students in natural sciences and engineering. The course will be specifically designed as a bridging course between high school and university, focusing on improving students' proficiency in doing mathematical proofs with structured derivations.

We are presently working on tool support for making structured derivation proofs, both on a personal computer and on the web. We use TEXmacs [1], a wysiwyg LATEX editor, as the basic framework for writing mathematical documents. We have constructed a plug-in for TEXmacs that understands structured derivations and makes it easy to construct and browse derivations. In particular, TEXmacs now supports selective hiding and revealing of subderivations and lemma proofs. This has made it straightforward for both teachers and students to work with derivations and proofs in electronic format.

Moreover, we are also currently working on providing just-in-time on-the-spot assistance for students reading a mathematical proof. For instance, consider a step in a derivation that calls for solving an equation, like in the second example given in Section 2. In that example, the equation was solved in a subderivation. If all subderivations are initially hidden, then students who feel confident about how to solve an equation do not need to open the subderivation, while students who are uncertain can do that. Thus, one single example can be used for students at different skill levels. This feature also renders structured derivations suitable for self-study material, as examples can be made self-explanatory on different levels, providing the reader the choice of different levels of detail.

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