Structured Derivations Supporting High-School Mathematics

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ABSTRACT

At high school and lower level education, using exact formalism in definitions and proofs is usually avoided on mathematics courses. Many students consider mathematical proofs as tricks and, indeed, as they develop fear of the whole subject, rigorous mathematical reasoning remains an everlasting secret to most of them. In this study, we examine if the use of a special paradigm and notation called structured derivation can improve high-school students' understanding of mathematical reasoning and problem solving. The method is theoretically well founded and more rigorous and exact than the traditional methods used in high-school mathematics, and it contributes not only to the preciseness of expression but also to more systematic and straightforward presentation of mathematical reasoning. Furthermore, this method allows mathematical expressions and derivations to be considered on various abstraction levels according to teacher's desires and students' skills. The use of structured derivations in mathematics teaching has been examined and tested at a Finnish High School in Turku on the compulsory courses in advanced mathematics since autumn 2001. The study is done with two groups at the same entry level using the new method in teaching the test group and traditional methods with the control group. We are also interested in comparing advantages and disadvantages of the method with groups at different entry levels. The project is still in progress but the preliminary results are very encouraging, suggesting that this method can improve the performance of high school students and, moreover, deepen their learning on mathematical thinking.

Key words: high-school mathematics, structured derivation

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1. INTRODUCTION

According to the current high school level curriculum issued by the Finnish National Board of Education in 1994, the aims of mathematics studies are, among other things, that students learn to understand and use mathematical language, follow the presentation of mathematical information, argue about mathematics, read mathematical text and appreciate the exactness and clarity in presentations. Furthermore the aims are that the students are able to handle information in the way characteristic to mathematics, get used to make assumptions, examine their correctness, write exact justifications and learn to evaluate the validity of the justifications and generalization of the results (Opetushallitus, 1994).

The new curriculum, which will be taken into use in the year 2005, emphasizes besides the above the evaluation of students' skills focusing to basic mathematic skills to the selection of appropriate method and to the justification of exact conclusions. Learning to see the mathematical information as a logical structure has been added to the teaching aims (Opetushallitus, 2003).

Good success in mathematics does not necessarily imply the understanding of high-level mathematical notions and operations. Several students learn to use ad-hoc methods for solving different kinds of exercises without understanding the real meaning of the procedure they have applied. (Artigue, 1991; Tall, 1991) Students and also many teachers prefer shallow strategies leading to fast development of mechanic mathematic skills but overlook entirely deeper examination of mathematical structures and abstract notions (Repo, 1996).

The students consider the exact and logical justification of reasoning needed especially in proofs and problem solving exercises difficult. Often students study the exercise only by following some previously shown solution without designing their own justified chains of reasoning. A solution typically involves a number of interrelated formulae or expressions, but the relationships are not made explicit.

In advanced mathematics, it is very important for the student to take the step from general explanations to formal proofs. The student should understand the role and the significance of proofs and their structure. The proofs should be direct and straightforward, and structured in such a way that it becomes clear what is going on at any given time (Tall, 1991). According to Hanna (1991), the formalism should not be seen as a side issue but as an important tool for clarification, validation and understanding. When there is a need for justification and when this need can be met with an appropriate degree of rigour, learning will be greatly enhanced.

Thus it seems obvious, that we need new kind of approaches to encourage and facilitate students to do rigorous reasoning and use exact notation also in those fields of mathematics they consider difficult, such as mathematical proving and problem solving. Our aim is to get the students to be explicit about what strategies and rules are used, which order, and why.

A proof of a theorem should provide evidence for belief in the validity of the theorem consisting of facts and explanations of how the facts interact. A good presentation of a proof should clearly explain the facts and how they are combined. It should also make the proof appear so obvious that the reader can see how it was developed, can explain it to others, and perhaps can prove other similar theorems himself. Mathematics and rigorous thinking can be taught by first teaching the design of proofs using a formal logic (Gries & Schneider, 1995).

In practise, logical notation is used very seldom in high school mathematics. According to the new curriculum, logic, quantifiers, and the training of proving principles come along only on the optional advanced courses following the compulsory courses. In the compulsory courses, however, it could be better to use logic as a tool rather than an object of study. In this way, using logic could be supported already before the optional courses according to the aims of The Finnish National Board of Education. Structured derivation, which is an extension on the calculational proof style written with logical notation, could be used in high-school mathematics to facilitate the students to write the justifications of the solutions. In the next chapters we shall first consider what the structured derivation is about. Then we shall describe our experiment in using them in the compulsory courses on high-school mathematics, and finally present our results and conclusions.

2. METHOD

2.1. Structured derivations

Dijkstra and Scholten (1990) introduce the calculational proof format in their book *Predicate Calculus and Program Semantics*. They begin by making the observation that a great many proofs can be described as a series of transformations. Inspired by the clarity and the readability of the format, calculational paradigm for manipulating mathematical expressions emerged. It has been attributed to W. Feijen, and described in detail by van Gasteren (1990). According to the paradigm, mathematical expressions are transformed step by step from the initial expression to a solution. Each new version of the expression is written on a new line and between the two lines is written a symbol denoting the relationship between the expressions together with a justification for the validity of the step. Gries and Schneider (1993) have used the calculational proof style in their book *A Logical Approach to Discrete Math*, and their experiences in teaching this way have been positive (Gries & Schneider, 1995).

Structured derivations are based on the calculational proof paradigm and they are developed originally for the formal refinement of computer programs and reasoning about their correctness (Back & von Wright, 1998). Based on their work with program specifications, Back and von Wright have developed further the idea of structured derivations. They extended the method with the possibility of connecting subderivations to long and complicated solutions. This method can be used in high-school mathematics as a way of writing solutions to typical problems (Back, Grundy & von Wright, 1996; Back & von Wright, 1999). Furthermore, when solutions are very long, subderivations can be hidden and replaced with a link giving more detailed view of the partial solution.

2.2. Expressions and proofs

The first high school course in advanced mathematics (Function and Equations 1) contains basic algebra, calculus, and some function theory. A major part of the course content has been taught in the comprehensive school and so, according to our experiences, the students are eager to learn something new. Therefore, the beginning of this course is a very suitable time to teach the foundations of logic needed for using structured derivations.

Following the proposed method, solutions to typical expression reduction problems are written so that the new version of an expression is written on a new line and between the lines is written a semi-formal justification for the step. With subderivations, auxiliary details can be added to the solution in an organized way. The idea of subderivations is especially suitable for presentation in hypertext format with a web browser, because the hierarchical nature of structured derivations can be shown or hidden dynamically at will. Thus, problem solving is seen as a rather formal and incremental transformation with well-defined steps from the initial problem description to a final irreducible statement yielding the solution. With this, the solving of a problem forms a logical whole, where every step of the reasoning can be shown true and justified. The method has a natural and solid relation to proofs and mathematical problem solving as well as the use of abstractions. In fact, using this method, problem solving can be seen as proving and the solution as a proof.

Concise and exact expression has been prevalent in mathematics already during decades. This frugality of presentation culminates in the matriculation examination, where the lack of students' argumentation skills has repeatedly been a major concern. Attaching a short verbal justification to each transformation step, it becomes easier for students to understand and write rigorous solutions themselves.

Verbal description and commentary of a mathematical concept is an important part of the construction process, where the student has to consider the characteristics on the concept, and to reflect and analyse her mathematical thinking (Joutsenlahti, 2003). This commentary can be shortened by omitting basic facts as they become familiar and well enough learned.

In the following examples we see what kind of things have been taught to the students. The students are taught the logical equivalence (Example 1.), the set of truth-values (Example 2.), logical connectives and how to simplify logical expressions (rules of de Morgan) (Example 3.).

(a) $3x - 4 = 8 \quad | + 4$

$3x = 12 \quad | : 3$

$x = 4$

(b) $3x - 4 = 8$

$\equiv$ {add 4 to both sides}

$3x = 12$

$\equiv$ {divide both sides by 3}

$x = 4$

Example 1. Solution of a linear equation written in the traditional way (a) and in the calculational style (b)

(a) $3x - 4 = 3x$

$\equiv$ {subtract $3x$ from both sides}

$-4 = 0$

$\equiv$ {unequal numbers}

F

(b) $3x = x + 2(x - 1) + 2$

$\equiv$ {simplify right-hand side}

$3x = 3x$

$\equiv$ {equality is reflexive}

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Example 2. Two examples of logical truth and falsity

Subderivations are presented precisely where they are needed. We can do so by visually subordinating them through indentation and by marking their beginning with the symbol '•' and end with '…'.

$(x\text{-}1)/(x^2\text{-}1)$ is defined

$\equiv$ {definedness of rational expressions}

$x^2 - 1 \neq 0$

$\equiv$ {switch to logical notation}

$\neg(x^2\text{-}1 = 0)$

$\equiv$ {solve $x^2\text{-}1 = 0$}

• $x^2\text{-}1 = 0$

$\equiv$ {factorization rule}

$(x - 1)(x + 1) = 0$

$\equiv$ {rule for zero product}

$x = -1 \vee x = 1$

… $\neg(x = -1 \vee x = 1)$

$\equiv$ {de Morgan rules}

$\neg(x = -1) \wedge \neg(x = 1)$

$\equiv$ {change notation}

$x \neq -1 \wedge x \neq 1$

Example 3. Using connectives and simplifying the expression. For what values of x *is the expression* $(x\text{-}1)/(x^2\text{-}1)$ *defined?*

2.3. Teaching experiment

The method has been studied and tested in Kupittaa High School, in Turku since August 2001 (Kavander et al 2001). This autumn the third group of students began studying mathematics using the method. Every year before the studies, all the new students who have chosen advanced mathematics, answer to an attitude enquiry. They also do an exam which tells us their mathematical skills and after that they are divided into three groups: the test group, the control group and the group, which does not participate the research. To the test groups we have

Table1. Test and control groups

Group	N	Group	N
Test group 01	25	Control group 01	24
Test group 02	28	Control group 02	23
Test group 03	21	Control group 03	26
Total.	74		73

selected students interested in computer science, because we believe that the structured derivation method will turn out useful later when studying discrete mathematics and computer science. We have tried to select students to the test and control groups so that they are as much as possible on the same entry level according to the comprehensive school leaving certificate and the mathematical skills tested in our exam. Groups are named with the number of the year they began their high-school studies. (Table 1)

The method of structured derivations has been used when teaching new basic theory both on the board and with the computer. Subderivations consist of detailed solution on the part of the problem. When teacher is using computer and browser, subderivations can be hidden or shown with a link depending on how detailed solution students want to see for instance when checking their homework.

As an example of applying the method on the courses in detail, solving an inequality, is first shown in fully expanded form (a) as written on the board and then in the form (b) with hidden subderivations.

In the beginning of the study, autumn 2001, on the first two courses textbook was not in use but the teacher made the primary study material while the control group used normal mathematics course book. Both groups use the same exercise book related to the course book. The importance of a "real" printed textbook appeared in the students' feedback and since that they have had it in use besides the teacher's material. However, the teaching goes ahead with structured derivation method and not according to the textbook.

(a) $|x-3| \geq 2x$

≡ {property of absolute values}

$(x-3 \geq 2x \lor x-3 \leq -2x)$

≡ {solve the inequalities}

- $x-3 \geq 2x$

  ≡ {add 3 to both sides of the inequality}

  $x \geq 2x+3$

  ≡ {subtract $2x$ from both sides of the inequality}

  $-x \geq 3$

  ≡ {multiply both sides with -1}

  $x \leq -3$

- $x-3 \leq -2x$

  ≡ {add 3 to both sides of the inequality}

  $x \leq -2x+3$

  ≡ {add $2x$ to both sides of the inequality}

  $3x \leq +3$

  ≡ {divide both sides with 3}

  $x \leq 1$

… $x \leq -3 \lor x \leq 1$

≡ {simplify}

$x \leq 1$

Example 4(a). Solving the inequality $|x-3| \geq 2x$.

(b) $| x\text{-}3 | \geq 2x$

$\equiv$ {property of absolute values}

$(x\text{-}3 \geq 2x \vee x\text{-}3 \leq \text{-}2x)$

$\equiv$ {solve the inequalities}

… $x \leq \text{-}3 \vee x \leq 1$

$\equiv$ {simplify}

$x \leq 1$

Example 4(b). Solving the inequality | x-*3* | ≥ 2x.

3. RESULTS

The results of the test groups have been compared with those of the control groups using common course, repeating, and raising examinations after every course. Grading principles have been exactly the same during the study (Table 2., Figure1. and Figure2.). Some of the students in test groups have not wanted to use logic and structured derivations when solving the problems in exams. The results of these students have been compared with those in the test group using the method and with those who are taught with the traditional way.

Table 2. Means of the grades in mathematics in comprehensive school leaving certificates and means of grades of the first mathematics course at high school level and the differences between test and control groups

	comprehensive school			high-school		
beginning year	test group	control group	difference	test group	control group	difference
2001	9.67	8.95	0.72	8.68	6.96	1.72
2002	9.32	8.52	0.80	8.43	6.04	2.39
2003	8.86	8.31	0.55	7.48	6.12	1.36
all	9.30	8.57	0.73	8.24	6.37	1.87

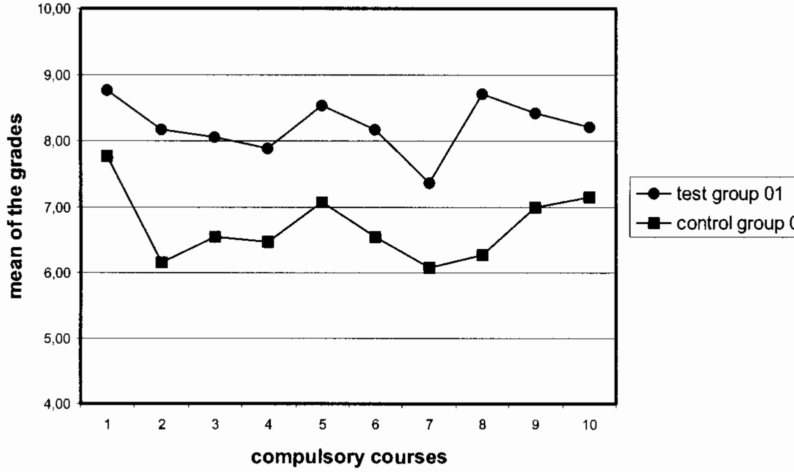

Figure 1. Means of course grades of all ten compulsory courses in advanced mathematics in test group (N=25) and in control group (N=24) containing students having started high-school studies year 2001

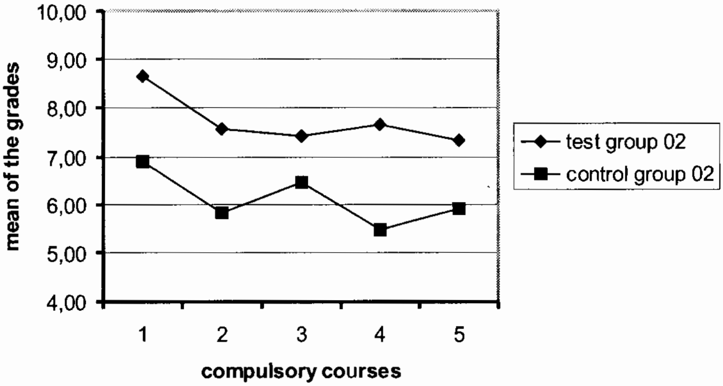

Figure 2. Means of course grades of the first five compulsory courses in advanced mathematics in test group (N=28) and in control group (N=23) containing students having started high-school studies year 2002

4. DISCUSSIONS AND CONCLUSIONS

Since the study began autumn 2001, the test groups have had the same teacher all the time. The control groups had two years the same teacher, however different from the test group. Now the control groups have a new teacher.

The use of the method on the geometry course has been quite low, because combining pictures to structured derivations does not yet work satisfactorily. Proving is taught for the first time on geometry course and the using of the method is very suitable for the proofs related to congruence theorems (Back et al, 2002). According to our experience, however, students have negative attitude towards problems where proving skills are required. In the exam, they typically don't solve this kind of problems as they have a chance to choose eight problems out of ten. So the results of using the method are fine on the first two courses while on the geometry course, the effect was negligible

In our future plans is analysing the differences in the students' skills in programming. Most of the students from the test and the control groups have had or will have the same programming courses. Then the structure and logic of the programs written by the students could be examined in more details.

According to Dijkstra (2002):

> As time went by, we accepted as challenge to avoid pulling rabbits out of the magician's hat. There is a mathematical style, in which proofs are presented as strings of unmotivated tricks that miraculously do the job, but we found greater intellectual satisfaction in showing how each next step in the argument, if not actually forced, is at least something sweetly reasonable to try. Another reason for avoiding rabbits as much as possible was that we did not want to teach proofs; we wanted to teach proof design.

Finally we note that new ideas in teaching are slow to catch on. People don't like changing their habits – especially if it requires them to change their own way of thinking. However, we believe that structured derivations can improve high-school students' understanding of mathematical reasoning and problem solving and, moreover, deepen their learning on mathematical thinking and diminish their fear of mathematics.

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