Fuzzy linear programs with optimal tolerance levels

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Abstract

It is usually supposed that tolerance levels are determined by the decision maker *a priori* in a fuzzy linear program (FLP). In this paper we shall suppose that the decision maker does not care about the particular values of tolerance levels, but he wishes to minimize their weighted sum. This is a new statement of FLP, because here the tolerance levels are also treated as variables.

1 Preliminaries

A fuzzy set $\tilde{a}$ of the real line $\mathbb{R}$ is defined by its membership function (denoted also by $\tilde{a}$) $\tilde{a}\colon \mathbb{R} \to [0,1]$. A fuzzy number $\tilde{a}$ is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. A fuzzy set of the real line given by the membership function

$$\tilde{a}(t) = \begin{cases} 1 - \dfrac{|a-t|}{\alpha} & \text{if } |a-t| \leq \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ will be called a symmetrical triangular fuzzy number with center $a \in \mathbb{R}$ and width 2α and we shall refer to it by the pair (a, α).
Let $\tilde{a} = (a, \alpha)$ and $\tilde{b} = (b, \alpha')$ be two fuzzy numbers of symmetric triangular form, $\lambda \in \mathbb{R}$. Then it is easily verified from Zadeh's extension principle that we have

$$\tilde{a} + \tilde{b} = (a + b, \alpha + \alpha'), \quad \tilde{a} - \tilde{b} = (a - b, \alpha + \alpha'), \quad \lambda\tilde{a} = (\lambda a, |\lambda|\alpha). \tag{1}$$

Fuzzy numbers can also be considered as possibility distributions. If $\tilde{a} \in \mathcal{F}$ is a fuzzy number and $x \in \mathbb{R}$ a real number then $\tilde{a}(x)$ can be interpreted as the degree of possibility of the statement "x is $\tilde{a}$".
Let $\tilde{a}, \tilde{b} \in \mathcal{F}$ be fuzzy numbers. The degree of possibility that the proposition "$\tilde{a}$ is less than or equal to B" is true denoted by Pos$[\tilde{a} \leq \tilde{b}]$ and defined by the extension principle as [2]

$$\text{Pos}[\tilde{a} \leq \tilde{b}] = \sup_{x \leq y} \min\{\tilde{a}(x), \tilde{b}(y)\}, \tag{2}$$

This formula uses once more Zadeh's extension principle. Let $\tilde{a} = (a, \alpha)$ and $b = (b, \alpha')$ be two fuzzy numbers of symmetric triangular form. Then it can be easily checked that

$$\text{Pos}[\tilde{a} \leq \tilde{b}] = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - \dfrac{a - b}{\alpha + \alpha'} & \text{if } b < a \leq b + \alpha + \alpha', \\ 0 & \text{if } a > b + \alpha + \alpha'. \end{cases} \tag{3}$$

2 Fuzzy Linear Programs

Consider the following fuzzy linear programming problem

$$\langle \tilde{c}, x \rangle \to \min; \text{ subject to } \tilde{A}x \leq \tilde{b}, \ x \in \mathbb{R}^n. \tag{4}$$

where $\langle \tilde{c}, x \rangle = \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n$, denotes the weighted sum of fuzzy number coefficients $\tilde{c}_j$; $\langle \tilde{a}_i, x \rangle = \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n$ denotes the weighted sum of fuzzy number coefficients $\tilde{a}_{ij}$; $\tilde{b}$ is a vector of fuzzy numbers $\tilde{b}_i$ $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $\leq$ is understood in possibilistic sense. Supposing that the decision maker has a (fuzzy) aspiration level for the objective function, represented by a fuzzy number $\tilde{b}_0$, problem (4) can be stated as

$$\langle \tilde{c}, x \rangle \leq \tilde{b}_0; \text{ subject to } \tilde{A}x \leq \tilde{b}, \ x \in \mathbb{R}^n. \tag{5}$$

In the sequal we shall suppose that all of the fuzzy number coefficients in (4) are chosen from the class of symmetric triangular fuzzy numbers.
Consider now (5) with fuzzy number coefficients $\tilde{c}_j = (c_j, \alpha)$, $\tilde{b}_i = (b_i, d_i)$ and $\tilde{a}_{ij} = (a_{ij}, \alpha)$ of symmetric triangular form, $i = 0, 1, \ldots, m$, $j = 1, \ldots, n$, and rewrite it in the form: find $x \in \mathbb{R}^n$ such that,

$$(\langle c, x \rangle, \alpha\|x\|_1) \leq (b_0, d_0); \text{ subject to } \{(\langle a_i, x \rangle, \alpha\|x\|_1) \leq (b_i, d_i), i = 1, \ldots, m\}, \tag{6}$$

where $\|x\|_1 = |x_1| + \cdots + |x_n|$, $\langle c, x \rangle = c_1 x_1 + \cdots + c_n x_n$ and $\langle a_i, x \rangle = a_{i1} x_1 + \cdots + a_{in} x_n$; and in this case we shall call (6) a *flexible linear programming* problem and interpret it as a fuzzy extension of the crisp linear inequality problem: find $x \in \mathbb{R}^n$ such that,

$$\langle c, x \rangle \leq b_0; \text{ subject to } Ax \leq b, \tag{7}$$

and $d_i > 0$ is interpreted as the level of maximal admissible violation for the i-th crisp constraint $\langle a_i, x\rangle \leq b_i$, $i = 1, \dots, m$; d_0 is the level of maximal admissible violation for the crisp constraint $\langle c, x\rangle \leq b_0$; and b_0 is interpreted as the decision maker's crisp aspiration level for the crisp objective function $\langle c, x\rangle$.
Then the degree of possibility, denoted by $\mu_i(x)$, that x satisfies the i-th constraint in (6) is computed as in [3]

$$\mu_i(x) = \begin{cases} 1 & \text{if } \langle a_i, x\rangle \leq b_i \\ 1 - \dfrac{\langle a_i, x\rangle - b_i}{\alpha\|x\|_1 + d_i} & \text{if } b_i < \langle a_i, x\rangle \leq b_i + d_i \\ 0 & \text{if } \langle a_i, x\rangle > b_i + d_i, \end{cases}$$

for $i = 1, 2, \dots, m$, and the degree to which x satisfies the decision maker's goal is computed as

$$\mu_0(x) = \begin{cases} 1 & \text{if } \langle c, x\rangle \leq b_0 \\ 1 - \dfrac{\langle c, x\rangle - b_0}{\alpha\|x\|_1 + d_0} & \text{if } b_i < \langle c, x\rangle \leq b_0 + d_0 \\ 0 & \text{if } \langle c, x\rangle > b_0 + d_0. \end{cases}$$

The fuzzy solution to problem (6) is defined by Bellman and Zadeh's principle [1] as

$$D(x) = \min\{\mu_0(x), \mu_1(x), \dots, \mu_m(x)\},\ x \in \mathbb{R}^n,$$

and an optimal solution, $x^* \in \mathbb{R}^n$, is determined from the relationship

$$\lambda^* = D(x^*) = \max\{D(x) \,|\, x \in \mathbb{R}^n\}. \tag{8}$$

where λ^* is called the degree of consistency of (5). It is easy to see that problem (8) leads to the following nonlinear mathematical programming problem,

$$\begin{aligned} &\lambda \to \max \\ &1 - \frac{\langle c, x\rangle - b_0}{\alpha\|x\|_1 + d_0} \geq \lambda, \\ &1 - \frac{\langle a_i, x\rangle - b_i}{\alpha\|x\|_1 + d_i} \geq \lambda,\ i = 1, \dots, m, \\ &\lambda \in [0, 1], x \in \mathbb{R}^n \end{aligned}$$

In the extremal case $\alpha = 0$ (but $d_i > 0$), the problem of finding an optimal solution to (6) from equation (8) leads to the following linear programming problem,

$$\begin{aligned} &\lambda \to \max \\ &1 - \frac{\langle c, x\rangle - b_0}{d_0} \geq \lambda, \\ &1 - \frac{\langle a_i, x\rangle - b_i}{d_i} \geq \lambda,\ i = 1, \dots, m, \\ &\lambda \in [0, 1], x \in \mathbb{R}^n, \end{aligned}$$

which was introduced in [12]. Sensitivity analysis in FLP problems (with $\alpha = 0$) was first considered in [6], where a functional relationship between changes of parameters of the right-hand side and those of the optimal value of the primal objective function was derived for almost all conceivable cases. Tanaka [11] formulated an FLP problem with symmetrical triangular fuzzy numbe coefficients and discussed the value of information via sensitivity analysis. Stable embeddings of linear equality and inequality systems into fuzzified systems were discussed in [10].

3 FLP with a restricted overall flexibility level

Suppose that decision maker does not care about the particular values of d_i, but he wishes to reduce the overall level of violation, defined by $d_0 + d_1 + \cdots + d_m$, as much as possible, and the membership function of his *soft overall flexibility* is given by

$$\mu_{m+1}(d_0 + d_1 + \cdots + d_m) = \begin{cases} 1 & \text{if } \sum_{i=0}^{m} d_i \leq T \\ 1 - \dfrac{\sum_{i=0}^{m} d_i - T}{t} & \text{if } T < \sum_{i=0}^{m} d_i \leq T + t \\ 0 & \text{otherwise,} \end{cases}$$

where T is called the decision maker's crisp overall flexibility level, and $t > 0$ denotes his tolerance level for exceeding T. Therefore, T is nothing else but the cumulated violation of crisp inequalities in (7). Then the fuzzy decision problem (5) under soft overall flexibility constraint can be formulated as [5]

$$\begin{aligned} &\langle \tilde{c}, x \rangle \leq \tilde{b}_0 \\ &\tilde{A}x \leq \tilde{b}, \\ &d_0 + \cdots + d_m \leq (T, t),\ x \in \mathbb{R}^n, \end{aligned} \tag{9}$$

and its fuzzy solution is then defined by

$$D(x, d_0, d_1, \ldots, d_m) = \min\{\mu_0(x), \mu_1(x), \ldots, \mu_m(x), \mu_{m+1}(d_0 + d_1 + \cdots + d_m)\}, , \tag{10}$$

where $x \in \mathbb{R}^n$ and $d_0 > 0, \ldots, d_m > 0$. furthermore, an optimal solution to (9) can be obtained by solving the following nonlinear mathematical programming problem,

$$\begin{aligned} &\lambda \to \max \\ &1 - \frac{\langle c, x \rangle - b_0}{\alpha \|x\|_1 + d_0} \geq \lambda, \\ &1 - \frac{\langle a_i, x \rangle - b_i}{\alpha \|x\|_1 + d_i} \geq \lambda,\ i = 1, \ldots, m, \\ &1 - \frac{d_0 + \cdots + d_m - T}{t} \geq \lambda \\ &\lambda \in [0, 1], d_0 > 0, \ldots, d_m > 0, x \in \mathbb{R}^n. \end{aligned} \tag{11}$$

In [5] it was proved that the fuzzy solution and degree of consistency of FLP problem (9) depends continuously on the degree of flexiblity, that is, small changes in the decision maker's overall flexibility level can cause only small deviations both in the fuzzy solution and in the degree of consistency.

4 FLP with optimal tolerance levels

Suppose now that the decision maker is not in the position to identify his overall flexibility level, but he is concerned about minimizing the weighted sum of all tolerance levels, where the i-th weight shows the importance of the i-th constraint, $i = 0, 1, \dots, m$. In this case the fuzzy solution of FLP

$$\langle \tilde{c}, x \rangle \leq \tilde{b}_0; \text{ subject to } \tilde{A}x \leq \tilde{b},\ x \in \mathbb{R}^n. \tag{12}$$

can be written in the form

$$D(x, d) = \min\{\mu_0(x, d), \mu_1(x, d), \dots, \mu_m(x, d)\},\ x \in \mathbb{R}^n,$$

where we used the notation $d = (d_0, d_1, \dots, d_m)$. Then an optimal solution, $x^* \in \mathbb{R}^n$ and $d_0^* > 0, \dots, d_m^* > 0$ is obtained by finding a good compromise solution to the following nonlinear non-smooth biobjective problem

$$\left\{ \max\{D(x, d), \min\{w_0 d_0 + w_1 d_1 + \cdots + w_m d_m\} \right\} \tag{13}$$

$$\text{subject to } d_0 > 0, \dots, d_m > 0, x \in \mathbb{R}^n$$

There are several approaches to nonlinear non-smooth problem (13) (see [7, 8, 9]). In this paper we suggest the use of a scalarizing function, where the weight of the first objective is Ω and the weight of the second objective is $1 - \Omega$. Then (13) turns into the following single-objective optimization problem

$$\Omega D(x, d) - (1 - \Omega)(w_0 d_0 + w_1 d_1 + \cdots + w_m d_m) \to \max \tag{14}$$

$$\text{subject to } d_0 > 0, \dots, d_m > 0, x \in \mathbb{R}^n$$

That is,

$$\Omega \min\{\mu_0(x, d), \mu_1(x, d), \dots, \mu_m(x, d)\} - (1 - \Omega)(w_0 d_0 + w_1 d_1 + \cdots + w_m d_m) \to \max$$

$$\text{subject to } d_0 > 0, \dots, d_m > 0, x \in \mathbb{R}^n.$$

5 Summary

In this paper we have considered a novel statement of flexible linear programming problems where the decision maker does not care about the particular values of tolerance levels, but he wishes to minimize their weighted sum. These types of problems may arise in portfolio selection problems where the tolerance levels can be expressed in monetary terms and the weighted sum of tolerances levels denote the amount of extra capital the investor might find in order to improve portfolio performance. Treating tolerance levels as variables, the dimension of the original problem (8) increases by $(m + 1)$ new variables. Furthermore, to find a solution to the resulting biobjective nonlinear non-smooth problem (14) generally requires the use of genetic optimization techniques, and could be tricky.

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sciences* 17(1970), B141-B154.

[2] D. Dubois and H. Prade, System of linear fuzzy constraints, *Fuzzy Sets and Systems* **3**(1980) 37-48.

[3] R. Fullér, On stability in fuzzy linear programming problems, *Fuzzy Sets and Systems*, 30(1989) 339-344.

[4] R. Fullér, On stability in possibilistic linear equality systems with Lipschitzian fuzzy numbers, *Fuzzy Sets and Systems*, 34(1990) 347-353.

[5] R. Fullér, M.Kovács and G.Schuster, Flexible linear programs with a restricted overall flexibility level, *Fuzzy Sets and Systems*, 127(2002) 177-183.

[6] H. Hamacher, H. Leberling and H. -J. Zimmermann, Sensitivity analysis in fuzzy linear programming, *Fuzzy Sets and Systems* 1(1978) 269-281.

[7] K.Miettinen and M.M.Mäkelä, Interactive Bundle-Based Method for Nondifferentiable Multiobjective Optimization: NIMBUS, *Optimization*, 34(1995) 231-246.

[8] K.Miettinen and M.M.Mäkelä, On Scalarizing Functions in Multiobjective Optimization, *OR Spectrum*, 24(2002) 193-213.

[9] K.Miettinen and M.M.Mäkelä, Characterizing Generalized Trade-Off Directions, *Mathematical Method of Operations Research*, 57(2003) 89-100.

[10] M.Kovács, Stable embeddings of linear equality and inequality systems into fuzzified systems, *Fuzzy Sets and Systems*, 45(1992) 305-312.

[11] H. Tanaka, H. Ichihashi and K. Asai, A value of information in FLP problems via sensitivity analysis, *Fuzzy Sets and Systems* 18(1986) 119-129.

[12] H.J. Zimmermann, Description and optimization of fuzzy systems, *Internat. J. General Systems* 2(1975) 209-215.

[13] H.J.Zimmermann, Fuzzy set theory and mathematical programming, in: A. Jones et al. eds., *Fuzzy Sets Theory and Applications* (D. Reidel Publishing Company, Dordrecht, 1986) 99-114.