Fair Consistency Evaluation in Fuzzy Preference Relations and in AHP

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Abstract. In this paper we first highlight what we consider a drawback of most consistency indices for pairwise comparison matrices. In our opinion they do not take into account whether the elicited judgements are close to indifference or, on the contrary, they express strong preferences in comparing the alternatives, with the result that the latter case is unfairly penalized. We then introduce a consistency preserving transformation. By means of this transformation we define an equivalence relation in the set of pairwise comparison matrices and propose a new method for a more fair evaluation of the consistency. Finally, we extend the new method to fuzzy preference relations.

Keywords: Pairwise Comparison Matrices, Consistency Indices, Fuzzy Preference Relations.

Introduction

Consistency of judgements is considered an important issue in decision making problems and the achievement of a satisfactory consistency level is viewed as a desirable property. In [5], for example, an aggregation process for group decision making problems is proposed where the more consistent the preferences of the various decision makers, the more importance is given to those preferences. Therefore, it is crucial that the consistency evaluation is carried out in a fair way. In this paper we argue that almost all known consistency indices suffer from the same drawback: they unfairly penalize the pairwise comparison matrices with strong preferences and favor, on the contrary, those with preferences close to indifference, even if they are contradictory. The paper is organized as follows. In section 1 we introduce the problem of pairwise comparison with some necessary notation. In section 2 we consider a consistency preserving transformation of a Pairwise Comparison Matrix, *PCM* in the following, and we describe what, in our opinion, is a drawback of almost all known consistency indices. In order to overcome the previously described drawback, we propose in section 3 a new method for evaluating the consistency of a *PCM*. Finally, in section 4, we extend our approach to the fuzzy preference relations framework by means of a simple transformation function.

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1 Pairwise Comparisons and Consistency Indices

Let $\Lambda = \{A_1, A_2, ..., A_n\}$ be a set of n alternatives and let the judgements of a decision maker be expressed by pairwise comparisons of the alternatives. We assume that the reader is familiar with Saaty's approach [14], where the decision maker's judgement a_{ij} estimates the ratio w_i/w_j of the preference intensities of alternative A_i over alternative A_j and the 1– 9 ratio scale is used. A *PCM* $A = [a_{ij}]$ is then positive and reciprocal, $a_{ji} = 1/a_{ij}, \quad \forall i, j$. We denote by Ω_n the set of the $n \times n$ pairwise comparison matrices.

A *PCM* $A = [a_{ij}]$ is called consistent if and only if

$$a_{ij} = a_{ih}a_{hj} \quad \forall i, j, h. \tag{1}$$

If (1) is not satisfied, it is a relevant issue to evaluate, by means of a consistency index, how 'far' is $A = [a_{ij}]$ from full consistency condition (1). In his seminal paper on the Analytical Hierarchy Process [13], T. Saaty proposes the consistency index *CI*,

$$CI(A) = \frac{\lambda_{max} - n}{n - 1}, \tag{2}$$

where λ_{max} is the maximum eigenvalue of A. By means of (2), Saaty defines the Consistency Ratio $CR(A) = CI(A)/RI$, where the Random Index RI is the mean value of the CI of random $n \times n$ *PCM*s. After that, several other authors proposed different consistency indices [1] [3] [5] [8] [12] [15]. For brevity, we do not describe here the various proposals and refer the interested reader to the cited papers.

2 A Consistency Preserving Transformation

Assuming that a *PCM* A is consistent, i.e. that (1) is satisfied, a transformation $f(\cdot)$ preserves consistency property (1) if and only if

$$f(a_{ij}) = f(a_{ih})f(a_{hj}). \tag{3}$$

From (1) and (3) the following Cauchy's functional equation is obtained,

$$f(a_{ih}a_{hj}) = f(a_{ih})f(a_{hj}). \tag{4}$$

Excluding the trivial solution $f(a_{ij}) = 0$, for positive arguments and assuming continuity [2], the general solution of (4) is

$$f(a_{ij}) = (a_{ij})^k, \quad k \in \Re. \tag{5}$$

As a consequence, if a *PCM* $A = [a_{ij}]$ is consistent, then it is consistent also every *PCM* $A' = [a'_{ij}]$ obtained from A by means of (5), for every real value of k,

$$a'_{ij} = f(a_{ij}) = (a_{ij})^k. \tag{6}$$

The general result stated above clearly requires that the scale $\frac{1}{9}, \ldots, 9$ originally proposed by Saaty is extended to the whole set of positive real numbers. If it is required that the entries of the *PCM* remain in the interval $[\frac{1}{9}, 9]$, it is sufficient to conveniently bound the value of k in (5).

Transformation (5) is used in [9] in order to rescale in Saaty's interval $[\frac{1}{9}, 9]$ a consistent *PCM* with some entries not belonging to this interval. Note that (5) also preserves properties weaker than (1) introduced and studied in [4] [6].

At this point we can better explain why we consider most of the known consistency indices to be unfair. The first argument is the following. A *PCM* with all entries equal to one is evaluated as fully consistent by all the known consistency indices. Note that this is the case where the decision maker expresses indifference in every pairwise comparison between alternatives. A small change in the values of the entries causes a small change in the value of the consistency index, except for the index proposed in [3]. Therefore, this means that the *PCM* remains very close to consistency, and this happens whatever are the changes, even if cycles arise stressing contradictory preferences. The following example should clarify this fact. Let $A \in \Omega_n,\ A_{modif} \in \Omega_n$, with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} , \quad A_{modif} = \begin{pmatrix} 1 & \frac{5}{4} & \frac{4}{5} \\ \frac{4}{5} & 1 & \frac{5}{4} \\ \frac{5}{4} & \frac{4}{5} & 1 \end{pmatrix} .$$

Matrix A_{modif} contains a cycle: $A_1 \succ A_2 \succ A_3 \succ A_1$ and therefore it is contradictory. Nevertheless, since all the judgements are still close to indifference, its consistency index (2) is quite good: $CI = \frac{1}{40}$. To summarize: in our opinion, not all *PCM*s with entries close to one should be considered as satisfactorily consistent.

The second argument is the following. Given that transformation (6) preserves consistency for a consistent *PCM*, it is justified to assign the same consistency level to both *PCM*s A and A' (non necessarily consistent) if A' is obtained by applying (6) to A. In the following section we will formalize this idea.

3 A New Proposal for Consistency Evaluation

In the previous section we have argued that an unbiased method for consistency evaluation should take into account only the mutual coherence of the judgements and thus should be, in a suitable way, independent from the size of the entries of the corresponding *PCM*. Starting from consistency preserving transformation (6), let us define an equivalence relation in Ω_n.

Definition 1 (Consistency–Equivalence for *PCM*). *Let $A \in \Omega_n$ and $B \in \Omega_n$, $A = [a_{ij}]$, $B = [b_{ij}]$. A is said consistency–equivalent to B, $A \sim B$, If and only if* $\exists k \neq 0$ *s.t.* $a_{ij} = b_{ij}^k \ \forall i, j$.

Proposition 1. *Consistency–Equivalence $\sim$ is an equivalence relation.*

Proof

1. *Reflexivity.* Clearly, with $k = 1$, it is $A \sim A$.
2. *Symmetry.* From $a_{ij} = b_{ij}^k$, it is $b_{ij} = a_{ij}^{\frac{1}{k}}$; then $A \sim B \Rightarrow B \sim A$.
3. *Transitivity.* Let $C \in \Omega_n$, $C = [c_{ij}]$. If $a_{ij} = b_{ij}^k$ and $b_{ij} = c_{ij}^h$, then $a_{ij} = c_{ij}^{kh}$. Therefore $(A \sim B \text{ and } B \sim C) \Rightarrow A \sim C$. □

As a consequence of Proposition 1, set Ω_n is partitioned by $\sim$ into equivalence classes. Let $\Omega_n/\sim$ be the quotient set. To overcame the above mentioned drawback, we propose to consider equivalent, from the point of view of consistency, all the *PCM*s in the same equivalence class $\Theta \in \Omega_n/\sim$. Therefore, we assign to all the *PCM*s $A \in \Theta$ the same numerical value to quantify their consistency. We denote this value by $C(A)$ and we will call it *inconsistency level* of A. Since all the *PCM*s in Θ share the same inconsistency level, we can denote it by C_Θ. Using this notation, it is $A \in \Theta \Rightarrow C(A) = C_\Theta$.

We propose to define C_Θ in the following way. We chose a particular *PCM* $\hat{A} \in \Theta$ as representing the whole equivalence class, we compute its Consistency Ratio $CR(\hat{A})$ and we assign to all the matrices in Θ the inconsistency level

$$C_\Theta = CR(\hat{A}). \tag{7}$$

The choice of $\hat{A}$ can be made on the basis of different criteria. One possibility is to define $\hat{A}$ as the (unique) *PCM* in Θ with maximum entry equal to 9, i.e. the maximum feasible value in Saaty's scale. A second possibility is to define $\hat{A}$ as the (unique) *PCM* in Θ with 'average size entries'. More precisely, referring to 1– 9 Saaty's scale, which has mean value 5, it is possible to define $\hat{A}$ as the *PCM* having the mean value of the $\frac{n(n-1)}{2}$ comparisons with $a_{ij} \geq 1$ equal to 5.

The previous definition of C_Θ doesn't apply to the equivalence class containing the single matrix with all the entries equal to one. Clearly, the inconsistency level of this particular *PCM* is defined to be equal to zero.

Let us now apply (5), for $k \in]0, 10]$, to the entries of the matrix A_{modif} introduced in section 2. For every obtained matrix A_{modif}^k we compute Saaty's Consistency Index (2). Results are shown in Fig.1, where it is clear that the CI increases with respect to k.

According to our proposal, on the contrary, since all the matrices A_{modif}^k are in the same equivalence class, they share the same inconsistency level (7).

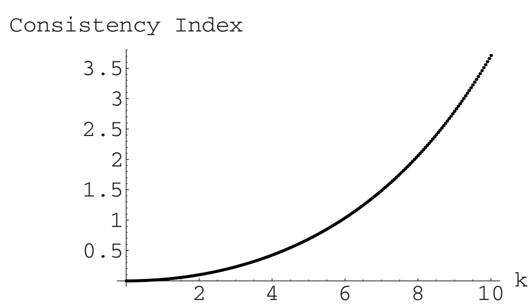

Fig. 1. Consistency Index of A_{modif}^k

Note that, instead of Saaty's Consistency Ratio, any other known consistency index can be used in (7), since what we propose is not a new index, but a new way of evaluating consistency, based on equivalence relation $\sim$. Analogously, any other ratio scale different from Saaty's one can be used. It is sufficient to substitute, in what described above, the number 9 with the maximum value of the considered scale. To our knowledge, the consistency index $RC(A)$ proposed in [3] by Barzilai is the only invariant one with respect to (5). Note that in [3] unbounded scales are used. In future research we will go deeper into the comparison between our proposal and Barzilai's index.

4 Extension to Fuzzy Preference Relations

The results reported in the previous sections can be easily extended to Fuzzy Preference Relations, *FPR* in the following, [10] [11] defined on the same set of alternatives $\Lambda = \{A_1, A_2, ..., A_n\}$. In fact, each *PCM* $A = [a_{ij}]$ can be transformed into a matrix $R = [r_{ij}]$ associated with a *FPR* and vice versa using the following function $g : [\frac{1}{9}, 9] \to [0, 1]$, introduced in [7],

$$r_{ij} = g(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}). \tag{8}$$

Function (8) transforms the a_{ij} values into the r_{ij} values in such a way that all the relevant properties of $A = [a_{ij}]$ are transformed into the corresponding properties for $R = [r_{ij}]$. In particular, multiplicative reciprocity $a_{ji} = 1/a_{ij}$ is transformed into additive reciprocity $r_{ij} + r_{ji} = 1$ [16] and multiplicative consistency (1) is transformed into additive consistency [16],

$$(r_{ih} - 0.5) + (r_{hj} - 0.5) = (r_{ij} - 0.5), \quad i, j, h = 1, ..., n \ . \tag{9}$$

As a consequence, every result obtained for *PCM*s can immediately be reformulated into the corresponding one for *FPR*s and vice versa.

Thus, let $R = [r_{ij}]$ be a consistent *FPR*. The Consistency preserving transformation for R can be obtained from (6) through (8), and it is

$$r'_{ij} - 0.5 = k(r_{ij} - 0.5). \tag{10}$$

Clearly, the same result could be obtained following a method analogous to that described in section 2 for *PCM*s. In this case, instead of (4), the following Cauchy's functional equation should have been solved,

$$f((r_{ih} - 0.5) + (r_{hj} - 0.5)) = f(r_{ih} - 0.5) + f(r_{hj} - 0.5), \tag{11}$$

obtaining again (10) as a solution.

The extension to *FPR*s of the results exposed in section 3 is briefly presented in the following, having denoted by Ψ_n the set of the n–dimensional *FPR*s.

Definition 2 (Consistency–Equivalence for *FPR*). *Let $R \in \Psi_n$ and $S \in \Psi_n$, $R = [r_{ij}]$, $S = [s_{ij}]$. R is said consistency–equivalent to S, $R \sim S$, If and only if $\exists k \neq 0$ s.t. $r_{ij} - 0.5 = k(s_{ij} - 0.5)$ $\forall i, j$.*

Proposition 2. *Consistency–Equivalence for FPR $\sim$ is an equivalence relation.*

We skip the proof of Proposition 2 for brevity, as it is similar to the proof of Proposition 1.

The set Ψ_n is partitioned by $\sim$ in equivalence classes, $\Psi_n/\sim$ being the quotient set. All the FPRs in the same equivalence class Φ share the same inconsistency level C_Φ.

As observed for (5), the general validity of the results presented above requires the use of an open scale, as it is assumed in [3]. Nevertheless, if it is required that the entries of R remain in the interval $[0, 1]$, it is sufficient to conveniently bound in (10) the value of k.

In [9] a related problem is addressed: the function

$$f(x) = \frac{1}{1+2a} \cdot x + \frac{a}{1+2a} \tag{12}$$

has been used to rescale into the interval $[0, 1]$ a consistent FPR with entries in the interval $[-a, 1+a]$. Note that function (12) is a special case of (10), obtained for $k = \frac{1}{2a+1}$.

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