Possibility versus probability: falling shadows versus falling integrals

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Abstract

In 2001 Carlsson and Fullér introduced the possibilistic mean value, variance and covariance of fuzzy numbers. In 2003 Fullér and Majlender introduced the notations of crisp weighted possibilistic mean value, variance and covariance of fuzzy numbers, which are consistent with the extension principle. Summarizing our results, in this paper we will consider fuzzy numbers from a normative point of view and will illustrate the concepts of possibilistic covariance and correlation by several examples.

1 Probability

In probability theory, the dependency between two random variables can be characterized through their joint probability density function. Namely, if X and Y are two random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then the density function, $f_{X,Y}(x,y)$, of their joint random variable (X,Y), should satisfy the following properties

$$\int_{\mathbb{R}} f_{X,Y}(x,t)dt = f_X(x), \int_{\mathbb{R}} f_{X,Y}(t,y)dt = f_Y(y),$$

for all $x, y \in \mathbb{R}$. $f_X(x)$ and $f_Y(y)$ are called the the marginal probability density functions of random variable (X,Y). X and Y are said to be independent if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, holds for all x, y.

The covariance between two random variables X and Y is defined as

$$\text{Cov}(X,Y) = \int_{\mathbb{R}^2} xy f_{X,Y}(x,y)dxdy - \int_{\mathbb{R}} x f_X(x)dx \int_{\mathbb{R}} y f_Y(y)dy,$$

and if X and Y are independent then $\text{Cov}(X,Y) = 0$. Let X and Y be random variables with finite variances $\text{Var}(X)$ and $\text{Var}(Y)$. Then the correlation coefficient between X and Y is defined by

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}},$$

and it is clear that $-1 \le \rho(X,Y) \le 1$.

2 Possibility

A fuzzy set A in $\mathbb{R}$ is said to be a fuzzy number if it is normal, fuzzy convex and has an upper semi-continuous membership function of bounded support. The family of all fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy set A in $\mathbb{R}^m$ is defined by $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \ge \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = \text{cl}\{x \in \mathbb{R}^m : A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$. If $A \in \mathcal{F}$ is a fuzzy number then $[A]^\gamma$ is a convex and compact subset of $\mathbb{R}$ for all $\gamma \in [0,1]$.

Fuzzy numbers can be considered as possibility distributions [6]. A fuzzy set B in $\mathbb{R}^m$ is said to be a joint possibility distribution of fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, m$, if it satisfies the relationship

$$\max_{x_j \in \mathbb{R},\, j \ne i} B(x_1, \dots, x_m) = A_i(x_i),$$

for all $x_i \in \mathbb{R}$, $i = 1, \dots, m$. Furthermore, A_i is called the i-th marginal possibility distribution of B,

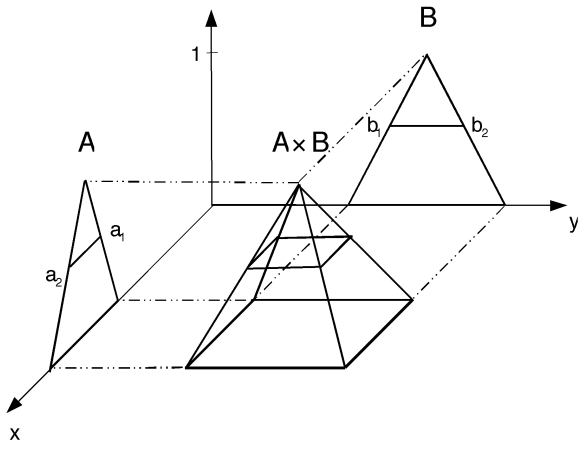

Figure 1: Non-interactive possibility distributions.

and the projection of B on the i-th axis is A_i for $i = 1, \dots, m$.

Definition 2.1 *Fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, m$, are said to be non-interactive if their joint possibility distribution, B, is given by*

$$B(x_1, \dots, x_m) = \min\{A_1(x_1), \dots, A_m(x_m)\},$$

or, equivalently, $[B]^\gamma = [A_1]^\gamma \times \cdots \times [A_m]^\gamma$, for all $x_1, \dots, x_m \in \mathbb{R}$ and $\gamma \in [0, 1]$.

It is clear that in this case any change in the membership function of A does not effect the second marginal possibility distribution and vice versa. On the other hand, A and B are said to be interactive if they can not take their values independently of each other [3].

Note 2.1 *Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'.*

3 Central values

Let B be a joint possibility distribution in $\mathbb{R}^n$, let $\gamma \in [0, 1]$ and let $g\colon \mathbb{R}^n \to \mathbb{R}$ be an integrable function. It is well-known from analysis that the average value of function g on $[B]^\gamma$ can be computed by

$$\mathcal{C}_{[B]^\gamma}(g) = \frac{1}{\int_{[B]^\gamma} dx} \int_{[B]^\gamma} g(x)dx$$

We will call $\mathcal{C}$ as the central value operator [5].

If $g\colon \mathbb{R} \to \mathbb{R}$ is an integrable function and $A \in \mathcal{F}$ then the average value of function g on $[A]^\gamma$ is defined by

$$\mathcal{C}_{[A]^\gamma}(g) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} g(x)dx.$$

Especially, if $g(x) = x$, for all $x \in \mathbb{R}$ is the identity function ($g = \mathrm{id}$) and $A \in \mathcal{F}$ is a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ then the average value of the identity function on $[A]^\gamma$ is computed by

$$\mathcal{C}_{[A]^\gamma}(\mathrm{id}) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx = \frac{a_1(\gamma) + a_2(\gamma)}{2},$$

which remains valid in the limit case $a_2(\gamma) - a_1(\gamma) = 0$ for some γ. Because $\mathcal{C}_{[A]^\gamma}(\mathrm{id})$ is nothing else, but the center of $[A]^\gamma$ we will use the shorter notation $C([A]^\gamma)$ for $\mathcal{C}_{[A]^\gamma}(\mathrm{id})$.

Definition 3.1 *[4] A function $f\colon [0, 1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition*

$$\int_0^1 f(\gamma)d\gamma = 1.$$

Different weighting functions can give different (case-dependent) importances to γ-levels sets of fuzzy numbers.

We can use the principle of central values to introduce the notion of expected value of functions on fuzzy sets. Let $g\colon \mathbb{R} \to \mathbb{R}$ be an integrable function and let $A \in \mathcal{F}$.

Definition 3.2 *[5] The expected value of function g on A with respect to a weighting function f is defined by*

$$E_f(g; A) = \int_0^1 \mathcal{C}_{[A]^\gamma}(g) f(\gamma) d\gamma$$

$$= \int_0^1 \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} g(x) dx f(\gamma) d\gamma.$$

Especially, if g is the identity function then we get

$$E_f(\mathrm{id}; A) = E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

which is the$f$-weighted possibilistic expected value value of A introduced in [5].

Let us denote $\mathcal{R}_{[A]^\gamma}(\mathrm{id},\mathrm{id})$ the average value of function $g(x) = (x - \mathcal{C}([A]^\gamma))^2$ on the γ-level set of an individual fuzzy number A. That is,

$$\mathcal{R}_{[A]^\gamma}(\mathrm{id},\mathrm{id}) = \frac{1}{\int_{[A]^\gamma} dx}\int_{[A]^\gamma} x^2 dx - \left(\frac{1}{\int_{[A]^\gamma} dx}\int_{[A]^\gamma} x dx\right)^2.$$

Definition 3.3 *[5] The variance of A is defined as the expected value of function $g(x) = (x - \mathcal{C}([A]^\gamma))^2$ on A. That is,*

$$\mathrm{Var}_f(A) = E_f(g; A) = \int_0^1 \mathcal{R}_{[A]^\gamma}(\mathrm{id},\mathrm{id}) f(\gamma) d\gamma.$$

After some calculations we get,

$$\mathrm{Var}_f(A) = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

Definition 3.4 *[5] Let C be a joint possibility distribution with marginal possibility distributions $A, B \in \mathcal{F}$, and let $\gamma \in [0,1]$. The measure of interactivity between the γ-level sets of A and B is defined by*

$$\begin{aligned}&\mathcal{R}_{[C]^\gamma}(\pi_x, \pi_y)\\ &= \mathcal{C}_{[C]^\gamma}\big((\pi_x - \mathcal{C}_{[C]^\gamma}(\pi_x))(\pi_y - \mathcal{C}_{[C]^\gamma}(\pi_y))\big)\\ &= \mathcal{C}_{[C]^\gamma}(\pi_x\pi_y) - \mathcal{C}_{[C]^\gamma}(\pi_x)\cdot\mathcal{C}_{[C]^\gamma}(\pi_y)\end{aligned}$$

The interactivity relation computes the average value of the interactivity function

$$g(x,y) = (x - \mathcal{C}_{[C]^\gamma}(\pi_x))(y - \mathcal{C}_{[C]^\gamma}(\pi_y)),$$

on $[C]^\gamma$.

Based on the notion of central values we introduced a novel definition of covariance, that agrees with the principle of 'falling shadows'.

Definition 3.5 *[5] Let C be a joint possibility distribution in $\mathbb{R}^2$. Let $A, B \in \mathcal{F}$ denote its marginal possibility distributions. The covariance of A and B with respect to a weighting function f (and with respect to their joint possibility distributioin C) is defined by*

$$\mathrm{Cov}_f(A,B) = \int_0^1 \mathcal{R}_{[C]^\gamma}(\pi_x,\pi_y) f(\gamma) d\gamma = \int_0^1 \left[\mathcal{C}_{[C]^\gamma}(\pi_x\pi_y) - \mathcal{C}_{[C]^\gamma}(\pi_x)\cdot\mathcal{C}_{[C]^\gamma}(\pi_y)\right] f(\gamma) d\gamma.$$

The covariance between marginal distributions A and B of a joint possibility distribution C is nothing else but the expected value of their interactivity function on C (with respect to a weighting function f). Furthermore, the covariance has been interpreted as a measure of interactivity between marginal distributions [5].

Theorem 3.1 *[5] If $A, B \in \mathcal{F}$ are non-interactive then* $\mathrm{Cov}_f(A,B) = 0$ *for any weighting function f.*

However, zero correlation does not always imply non-interactivity (see Fig. 2).

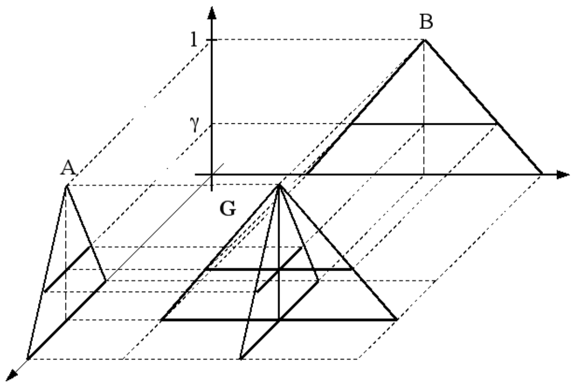

Figure 2: $\rho_f(A,B) = 0$ for interactive fuzzy numbers.

Theorem 3.2 *[2] Let $A, B \in \mathcal{F}$ be fuzzy numbers (with $\mathrm{Var}_f(A) \neq 0$ and $\mathrm{Var}_f(B) \neq 0$) with joint possibility distribution C. Then, the correlation coefficient between A and B, defined by*

$$\rho_f(A,B) = \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}.$$

satisfies the property $-1 \leq \rho_f(A,B) \leq 1$ for any weighting function f.

Let us consider some interesting cases. In [2] we proved that if A and B are non-interactive, then $\rho_f(A,B) = 0$. Consider now the case when shadows of the joint possibility distribution move in tandem (Fig. 3): if $A(u) \geq \gamma$ for some $u \in \mathbb{R}$ then there exists a unique $v \in \mathbb{R}$ that B can take, furthermore, if u is moved to the left (right) then the corresponding value (that B can take) will also move to the left (right). It can be shown [2] that in this case $\rho_f(A,B) = 1$.

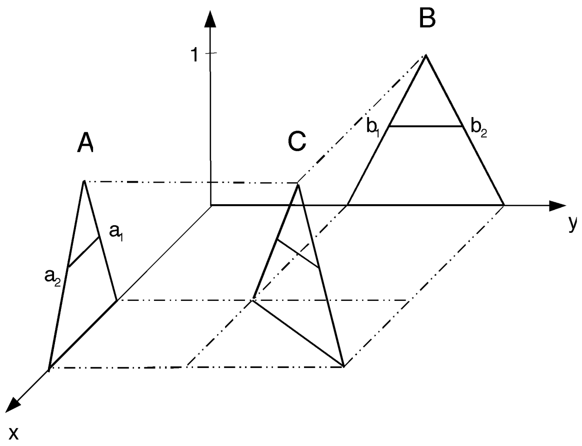

Figure 3: $\rho_f(A, B) = 1$.

Consider now the case when the shadows of the joint possibility distribution move in the opposite direction (Fig. 4): if $A(u) \geq \gamma$ for some $u \in \mathbb{R}$ then there exists a unique $v \in \mathbb{R}$ that B can take, furthermore, if u is moved to the left (right) then the corresponding value (that B can take) will move to the right (left). We have shown [2] that $\rho_f = -1$.

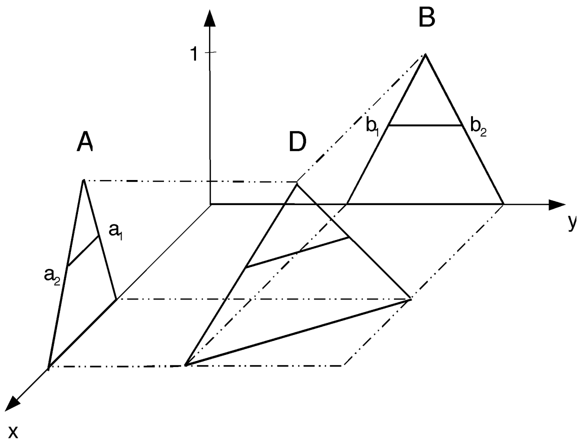

Figure 4: $\rho_f(A, B) = -1$.

Consider now the case depicted in Fig. 5. Since,

$$\mathrm{Cov}_f(A, B) = -\frac{1}{36}\int_0^1 (1-\gamma)^2 f(\gamma) d\gamma,$$

$$\mathrm{Var}_f(A) = \frac{1}{12}\int_0^1 (1-\gamma)^2 f(\gamma) d\gamma,$$

therefore $\rho_f(A, B) = -1/3$ for any weighting function f.

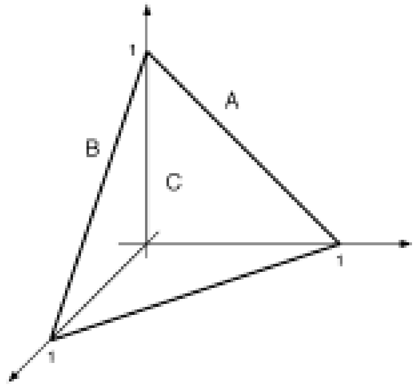

Figure 5: $\rho_f(A, B) = -1/3$.

4 Summary

In this paper we have summarized our results on interactive fuzzy numbers.

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