World Congress of International Fuzzy Systems Association 2011 and Asia Fuzzy Systems Society International Conference 2011, Surabaya-Bali, Indonesia, 21-25 June 2011, ISBN: 978-602-99359-0-5

A Quantitative Approach to Quasi Fuzzy Numbers

Christer Carlsson*
IAMSR, Åbo Akademi University
Joukahaisenkatu 3-5A
FIN-20520 Åbo, Finland
Email: christer.carlsson@abo.fi

Robert Fullér
IAMSR, Åbo Akademi University
Joukahaisenkatu 3-5A
FIN-20520 Åbo, Finland
Email: rfuller@abo.fi

József Mezei
Turku Centre for Computer Science
Joukahaisenkatu 3-5B
FIN-20520 Åbo, Finland
Email: jmezei@abo.fi

Abstract: In this paper we generalize the principles of possibilistic mean value, variance, covariance and correlation of fuzzy numbers to quasi fuzzy numbers. We will show some necessary and sufficient conditions for the existence of possibilistic mean value and variance for quasi fuzzy numbers. Considering the standard exponential probability distribution as a quasi fuzzy number we will compare the possibilistic and the probabilistic correlation coefficients.

I. Introduction

A *fuzzy number* A is a fuzzy set in $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by $\mathcal{F}$. A *quasi fuzzy number* A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions [2]

$$\lim_{t\to\infty} \mu_A(t) = 0, \quad \lim_{t\to-\infty} \mu_A(t) = 0.$$

A *quasi triangular* fuzzy number is a quasi fuzzy number with a unique maximizing point. Furthermore, we call $\mathcal{Q}$ the family of all quasi fuzzy numbers. Quasi fuzzy numbers can also be considered as possibility distributions [6]. A γ-level set of a fuzzy set A in $\mathbb{R}^m$ is defined by $[A]^\gamma = \{x \in \mathbb{R}^m : \mu_A(x) \geq \gamma\}$, if $\gamma > 0$ and $[A]^\gamma = \mathrm{cl}\{x \in \mathbb{R}^m : \mu_A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$.

If A is a fuzzy number, then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\gamma \in [0,1]$. If A is a quasi fuzzy number, then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for any $\gamma > 0$. Let us introduce the notations $a_1(\gamma) = \min[A]^\gamma$, $a_2(\gamma) = \max[A]^\gamma$ In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut, of A for any $\gamma \in [0,1]$. A fuzzy set C in $\mathbb{R}^2$ is said to be a joint possibility distribution of quasi fuzzy numbers $A, B \in \mathcal{Q}$, if it satisfies the relationships $\max\{x \mid \mu_C(x,y)\} = \mu_B(y)$, $\max\{y \mid \mu_C(x,y)\} = \mu_A(x)$, for all $x, y \in \mathbb{R}$. Furthermore, A and B are called the marginal possibility distributions of C. A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition

$$\int_0^1 f(\gamma)d\gamma = 1.$$

Different weighting functions can give different (case-dependent) importances to γ-levels sets of quasi fuzzy numbers. It is motivated in part by the desire to give less importance to the lower levels of fuzzy sets [11] (it is why f should be monotone increasing).

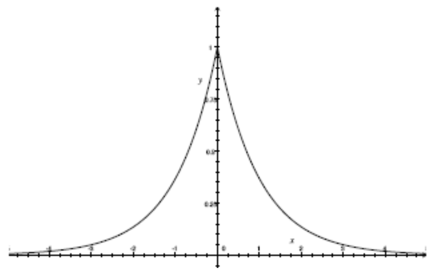

Fig. 1. A quasi triangular fuzzy number with membership function $e^{-|x|}$.

II. Possibilistic Mean Value, Variance, Covariance and Correlation of Quasi Fuzzy Numbers

The possibilistic mean (or expected value), variance and covariance can be defined from the measure of possibilistic interactivity (as shown in [3], [9], [10]) but for simplicity, we will present the concept of possibilistic mean value, variance, covariance in a pure probabilistic setting. Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and let U_γ denote a uniform probability distribution on $[A]^\gamma$, $\gamma \in [0,1]$. Recall that the probabilistic mean value of U_γ is equal to

$$M(U_\gamma) = \frac{a_1(\gamma) + a_2(\gamma)}{2},$$

and its probabilistic variance is computed by

$$\mathrm{var}(U_\gamma) = \frac{(a_2(\gamma) - a_1(\gamma))^2}{12}.$$

The f-weighted *possibilistic mean value* (or expected value) of $A \in \mathcal{F}$ is defined as [8]

$$E_f(A) = \int_0^1 E(U_\gamma)f(\gamma)d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)d\gamma,$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma \in [0,1]$. If $f(\gamma) = 1$ for all $\gamma \in [0,1]$ then we get

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma.$$

That is, $f(\gamma) \equiv 1$ the f-weighted possibilistic mean value coincides with the (i) generative expectation of fuzzy numbers introduced by Chanas and Nowakowski in ([4], page 47); (ii) middle-point-of-the-mean-interval defuzzication method proposed by Yager in ([15], page161). In this paper we will use the *natural* weighting function $f(\gamma) = 2\gamma$. In this case the possibilistic mean value is, denoted by $E(A)$, defined by,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \mathrm{d}\gamma \\ &= \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) \mathrm{d}\gamma, \end{aligned} \tag{1}$$

which the possibilistic mean value of A originally introduced by Carlsson and Fullér in 2001 [1]. We note here that from the equality

$$\begin{aligned} E(A) &= \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma \\ &= \frac{\displaystyle\int_0^1 \gamma \cdot \frac{a_1(\gamma) + a_2(\gamma)}{2}\, d\gamma}{\displaystyle\int_0^1 \gamma\, d\gamma}, \end{aligned}$$

it follows that $E(A)$ is nothing else but the level-weighted average of the arithmetic means of all γ-level sets, that is, the weight of the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$ is just γ.

Note 1. *There exist several other ways to define mean values of fuzzy numbers, e.g. Dubois and Prade [5] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Using evaluation measures, Yoshida et al [16] introduced a possibility mean, a necessity mean and a credibility mean of fuzzy numbers that are different from (1). Surveying the results in quantitative possibility theory, Dubois [7] showed that some notions (e.g. cumulative distributions, mean values) in statistics can naturally be interpreted in the language of possibility theory.*

Now we will extend the concept of possibilistic mean value to the family of quasi fuzzy numbers.

Definition II.1. *The f-weighted* possibilistic mean value *of $A \in \mathcal{Q}$ is defined as*

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma > 0$. The value of $E_f(A)$ does not depend on the boundedness of the support of A.

The possibilistic mean value is originally defined for fuzzy numbers (i.e. quasi fuzzy numbers with bounded support). If the support of a quasi fuzzy number A is unbounded then its possibilistic mean value might even not exist. However, for a symmetric quasi fuzzy number A we get $E_f(A) = a$, where a is the center of symmetry, for any weighting function f.

Now we will characterize the family of quasi fuzzy numbers for which it is possible to calculate the possibilistic mean value. First we show an example for a quasi triangular fuzzy number that does not have a mean value.

Example II.1. *Consider the following quasi triangular fuzzy number*

$$\mu_A(x) = \begin{cases} 0 & \textit{if } x \le 0 \\ \dfrac{1}{\sqrt{x+1}} & \textit{if } 0 \le x \end{cases}$$

In this case

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma^2} - 1,$$

and its possibilistic mean value can not be computed, since the following integral does not exist (not finite),

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2}\, 2\gamma \mathrm{d}\gamma \\ &= \int_0^1 \left(\frac{1}{\gamma^2} - 1\right)\gamma \mathrm{d}\gamma = \int_0^1 \left(\frac{1}{\gamma} - \gamma\right) \mathrm{d}\gamma. \end{aligned}$$

Note 2. *This example is very important: if the membership function of the quasi fuzzy number tends to zero slower than the function $1/\sqrt{x}$ then it is not possible to calculate the possibilistic mean value, (clearly, the value of the integral will be infinitive), otherwise the possibilistic mean value does exist.*

To show this, suppose that there exists $\varepsilon > 0$, such that the membership function of quasi fuzzy number A satisfies the property,

$$\mu_A(x) = O(x^{-\frac{1}{2}-\varepsilon})$$

if $x \to +\infty$. This means that there exists and $x_0 \in \mathbb{R}$ such that,

$$\mu_A(x) \le M x^{-\frac{1}{2}-\varepsilon},$$

if $x > x_0$ and where M is a positive real number. So the possibilistic mean value of A is bonded from above by

$$M^{\frac{1}{-\frac{1}{2}-\varepsilon}}$$

multiplied by the possibilistic mean value of a quasi fuzzy number with membership function $x^{-\frac{1}{2}-\varepsilon}$ plus an additional constant (because of the properties of a quasi fuzzy number we know that the interval $[0, x_0]$ accounts for a finite value in the integral).

Suppose that,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ x^{-\frac{1}{2}-\varepsilon} & \text{if } x \ge 1 \end{cases}$$

A similar reasoning holds for negative fuzzy numbers with membership function $(-x)^{-\frac{1}{2}-\varepsilon}$. Then we get,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \gamma^{\frac{1}{-\frac{1}{2}-\varepsilon}},$$

and since

$$\frac{\varepsilon - \frac{1}{2}}{\varepsilon + \frac{1}{2}} \neq 1,$$

we can calculate the possibilistic mean value of A as,

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \mathrm{d}\gamma = \int_0^1 \gamma^{-\frac{1}{\varepsilon+\frac{1}{2}}} \gamma \mathrm{d}\gamma$$
$$= \int_0^1 \gamma^{-\frac{\varepsilon-\frac{1}{2}}{\varepsilon+\frac{1}{2}}} \mathrm{d}\gamma = (\varepsilon + \frac{1}{2}) \left[\gamma^{\frac{1}{\varepsilon+\frac{1}{2}}}\right]_0^1 = \varepsilon + 1/2$$

Theorem II.1. *If A is a non-symmetric quasi fuzzy number then $E_f(A)$ exists if and only if there exist real numbers $\varepsilon, \delta > 0$, such that,*

$$\mu_A(x) = O\big(x^{-\frac{1}{2}-\varepsilon}\big),$$

if $x \to +\infty$ and

$$\mu_A(x) = O\big((-x)^{-\frac{1}{2}-\delta}\big),$$

if $x \to -\infty$.

Note 3. *If we consider other weighting functions, we need to require that $\mu_A(x) = O(x^{-1-\varepsilon})$, when $x \to +\infty$ (in the worst case, when $f(\gamma) = 1$, $\dfrac{1}{\gamma}$ is the critical growth rate.)*

Example II.2. *Consider the following quasi triangular fuzzy number,*

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{1}{x+1} & \text{if } 1 \le x \end{cases}$$

In this case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

and its possibilistic mean value is,

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \mathrm{d}\gamma = \int_0^1 \left(\frac{1}{\gamma} - 1\right) \gamma \mathrm{d}\gamma$$
$$= \int_0^1 (1 - \gamma) \mathrm{d}\gamma = 1/2.$$

This example is very important since the volume of A can not be normalized since $\int_0^\infty \mu_A(x)\mathrm{d}x$ does not exist. In other words, μ_A can not be considered as a density function of any random variable.

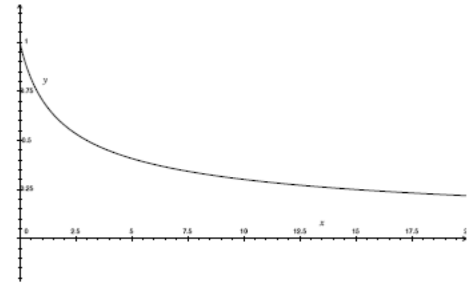

Fig. 2. Quasi triangular fuzzy number $1/(x+1), x \ge 0$.

The measure of f-weighted possibilistic variance of a fuzzy number A is the f-weighted average of the probabilistic variances of the respective uniform distributions on the level sets of A. That is, the f-weighted *possibilistic variance* of A is defined by [9]

$$\mathrm{Var}_f(A) = \int_0^1 \mathrm{var}(U_\gamma) f(\gamma) \mathrm{d}\gamma$$
$$= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) \mathrm{d}\gamma.$$

Now we will extend the concept of possibilistic variance to the family of quasi fuzzy numbers.

Definition II.2. *The measure of f-weighted possibilistic variance of a quasi fuzzy number A is the f-weighted average of the probabilistic variances of the respective uniform distributions on the level sets of A. That is, the f-weighted* possibilistic variance *of A is defined by*

$$\mathrm{Var}_f(A) = \int_0^1 \mathrm{var}(U_\gamma) f(\gamma) \mathrm{d}\gamma$$
$$= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) \mathrm{d}\gamma.$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma > 0$. The value of $\mathrm{Var}_f(A)$ does not depend on the boundedness of the support of A. If $f(\gamma) = 2\gamma$ then we simple write $\mathrm{Var}(A)$.

From the definition it follows that in this case we can not make any distinction between the symmetric and non-symmetric case. And it is also obvious, since in the definition we have the square of the $a_1(\gamma)$ and $a_2(\gamma)$ functions, that the decreasing rate of the membership function has to be the square of the mean value case. We can conclude:

Theorem II.2. *If A is a quasi fuzzy number then* $\mathrm{Var}(A)$ *exists if and only if there exist real numbers $\varepsilon, \delta > 0$, such*

$$\mu_A(x) = O(x^{-1-\varepsilon})$$

if $x \to +\infty$ and

$$\mu_A(x) = O((-x)^{-1-\delta}),$$

if $x \to -\infty$.

Note 4. *If we consider other weighting functions, we need to require that*

$$\mu_A(x) = O(x^{-2-\varepsilon}),$$

when $x \to +\infty$ *(in the worst case, when* $f(\gamma) = 1$, $\frac{1}{\sqrt{\gamma}}$ *is the critical growth rate.)*

Example II.3. *Consider again the quasi triangular fuzzy number,*

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{1}{x+1} & \text{if } 1 \le x \end{cases}$$

In this case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

and its possibilistic variance does not exist since

$$\int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma \mathrm{d}\gamma = \int_0^1 \frac{(1/\gamma - 1)^2}{12} 2\gamma \mathrm{d}\gamma = \infty.$$

In 2004 Fullér and Majlender [9] introduced a measure of possibilistic covariance between marginal distributions of a joint possibility distribution C as the expected value of the interactivity relation between the γ-level sets of its marginal distributions. In 2005 Carlsson, Fullér and Majlender [3] showed that the possibilistic covariance between fuzzy numbers A and B can be written as the weighted average of the probabilistic covariances between random variables with uniform joint distribution on the level sets of their joint possibility distribution C. The f-weighted *measure of possibilistic covariance* between $A, B \in \mathcal{F}$, (with respect to their joint distribution C), defined by [9], can be written as

$$\mathrm{Cov}_f(A, B) = \int_0^1 \mathrm{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$, and $\mathrm{cov}(X_\gamma, Y_\gamma)$ denotes their probabilistic covariance.

Now we will extend the concept of possibilistic covariance to the family of quasi fuzzy numbers.

Definition II.3. *The* f*-weighted* measure of possibilistic covariance *between* $A, B \in \mathcal{Q}$*, (with respect to their joint distribution* C*), is defined by,*

$$\mathrm{Cov}_f(A, B) = \int_0^1 \mathrm{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where X_γ *and* Y_γ *are random variables whose joint distribution is uniform on* $[C]^\gamma$ *for any* $\gamma > 0$.

It is easy to see that the possibilistic covariance is an absolute measure in the sense that it can take any value from the real line. To have a relative measure of interactivity between marginal distributions Fullér, Mezei and Várlaki introduced the normalized covariance in 2010 (see [10]). A normalized f-weighted index of interactivity of $A, B \in \mathcal{F}$ (with respect to their joint distribution C) is defined by

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) \mathrm{d}\gamma$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}}$$

and, where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$.

In other words, the (f-weighted) index of interactivity is nothing else, but the f-weighted average of the probabilistic correlation coefficients $\rho(X_\gamma, Y_\gamma)$ for all $\gamma \in [0, 1]$. It is clear that for any joint possibility distribution this correlation coefficient always takes its value from interval $[-1, 1]$, since $\rho(X_\gamma, Y_\gamma) \in [-1, 1]$ for any $\gamma \in [0, 1]$ and $\int_0^1 f(\gamma) d\gamma = 1$. Since $\rho_f(A, B)$ measures an average index of interactivity between the level sets of A and B, we may call this measure as the f-weighted possibilistic correlation coefficient.

Now we will extend the concept of possibilistic correlation to the family of quasi fuzzy numbers.

Definition II.4. *The* f*-weighted possibilistic correlation coefficient of* $A, B \in \mathcal{Q}$ *(with respect to their joint distribution* C*) is defined by*

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) \mathrm{d}\gamma$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}}$$

and, where X_γ *and* Y_γ *are random variables whose joint distribution is uniform on* $[C]^\gamma$ *for any* $\gamma > 0$.

III. Probability versus Possibility: The Case of Exponential Function

Now we will calculate the possibilistic mean value and variance of a quasi triangular fuzzy number defined by the membership function e^{-x}, $x \ge 0$, which can also be seen as a density function of a standard exponential random variable. In probability theory and statistics, the exponential distribution is a family of continuous probability distributions. It describes the time between events in a *Poisson process*, i.e. a process in which events occur continuously and independently at a constant average rate.

Consider the following quasi triangular fuzzy number

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x \ge 0 \end{cases}$$

From $\int_0^\infty \mu_A(x) \mathrm{d}x = 1$ it follows that μ_A can also be considered as the density function of a standard exponential random variable (with parameter one). It is well-known

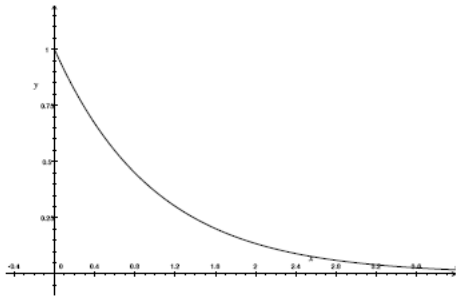

Fig. 3. Quasi triangular fuzzy number and density function of an exponential random variable with parameter one: e^{-x}, $x \geq 0$.

that the mean value and the variance of this probability distribution is equal to one. In the fuzzy case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = -\ln\gamma,$$

and its possibilistic mean value is

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} \, 2\gamma \mathrm{d}\gamma = \int_0^1 -(\ln\gamma)\gamma \mathrm{d}\gamma = \frac{1}{4},$$

and its possibilistic variance is,

$$\begin{aligned} \mathrm{Var}(A) &= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma \mathrm{d}\gamma \\ &= \int_0^1 \frac{(-\ln\gamma)^2}{6} \gamma \mathrm{d}\gamma = \frac{1}{24}. \end{aligned}$$

Let C be the joint possibility distribution, defined by the membership function,

$$\mu_C(x, y) = e^{-(x+y)}, \, x \geq 0, y \geq 0,$$

of quasi fuzzy numbers A and B with membership functions

$$\mu_A(x) = e^{-x}, \, x \geq 0, \quad \text{and} \quad \mu_B(y) = e^{-y}, \, y \geq 0.$$

In other words, the membership function of C is defined by a simple multiplication (by Larsen t-norm [13]) of the membership values of $\mu_A(x)$and $\mu_B(y)$, that is, $\mu_C(x, y) = \mu_A(x) \times \mu_B(y)$. The γ-cut of C can be computed by

$$[C]^\gamma = \{(x, y) \mid x + y \leq -\ln\gamma; \, x, y \geq 0\}.$$

Then

$$M(X_\gamma) = M(Y_\gamma) = -\frac{\ln\gamma}{3},$$

$$M(X_\gamma^2) = M(Y_\gamma^2) = \frac{(\ln\gamma)^2}{6},$$

and,

$$\begin{aligned} \mathrm{var}(X_\gamma) &= M(X_\gamma^2) - M(X_\gamma)^2 \\ &= \frac{(\ln\gamma)^2}{6} - \frac{(\ln\gamma)^2}{9} \\ &= \frac{(\ln\gamma)^2}{18}. \end{aligned}$$

Similarly we obtain,

$$\mathrm{var}(Y_\gamma) = \frac{(\ln\gamma)^2}{18}.$$

Furthermore,

$$M(X_\gamma Y_\gamma) = \frac{(\ln\gamma)^2}{12},$$

$$\mathrm{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(\ln\gamma)^2}{36},$$

we can calculate the probabilistic correlation by

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = -\frac{1}{2}.$$

That is, $\rho(X_\gamma, Y_\gamma) = -1/2$ for any $\gamma > 0$. Consequently, their possibilistic correlation coefficient is,

$$\rho_f(A, B) = -1/2$$

for any weighting function f.

On the other hand, in a probabilistic context, $\mu_C(x, y) = \mu_A(x) \times \mu_B(y) = e^{-(x+y)}$ can be also considered as the joint density function of independent exponential marginal probability distributions with parameter one. That is, in a probabilistic context, their (probabilistic) correlation coefficient is equal to zero.

Note 5. *The probabilistic correlation coefficient between two standard exponential marginal probability distributions can not go below* $(1 - \pi^2/6)$. *Really, the lower limit, denoted by* τ, *can be computed from,*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \big((1 - e^{-x} - e^{-y})^+ - (1 - e^{-x})(1 - e^{-y})\big) \mathrm{d}x\mathrm{d}y \\ &= -\int_0^\infty \int_0^\infty e^{-x}e^{-y} \mathrm{d}x\mathrm{d}y \\ &\quad + \iint\limits_{0<x,\, 0<y,\, 1<e^{-x}+e^{-y}} (1 - e^{-x} - e^{-y})^+ \mathrm{d}x\mathrm{d}y \\ &= -1 + \iint\limits_{0<x,\, 0<y,\, 1<e^{-x}+e^{-y}} (2e^{-x} - 1) \mathrm{d}x\mathrm{d}y = \tau \end{aligned}$$

using the substitutions $u = e^{-x}, v = e^{-y}$ *we get,*

$$\begin{aligned}
\tau &= -1 + \iint\limits_{u<1,\, v<1,\, u+v>1} \left(\frac{2}{u} - \frac{1}{uv}\right) \mathrm{d}u\mathrm{d}v \\
&= -1 + \int_0^1 \frac{1}{u} \int_0^1 \left(2 - \frac{1}{v}\right) \mathrm{d}v\mathrm{d}u \\
&= 1 + \int_0^1 \frac{2u + \log(1-u)}{u} \mathrm{d}u \\
&= -1 + \int_0^1 \frac{\log(1-u)}{u} \mathrm{d}u \\
&= \int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k}\, \mathrm{d}u \\
&= 1 - \sum_{k=1}^{\infty} \frac{1}{k^2} \\
&= 1 - \frac{\pi^2}{6}.
\end{aligned}$$

In the case of possibility distributions there is no known lower limit [12].

If the joint possibility distribution C is given by the minimum operator (Mamdani t-norm [14]),

$$\mu_C(x, y) = \min\{\mu_A(x), \mu_B(y)\} = \min\{e^{-x}, e^{-y}\},$$

$x \geq 0$, $y \geq 0$, then A and B are non-interactive marginal possibility distributions and, therefore, their possibilistic correlation coefficient equal to zero.

IV. Summary

We have generalized the principles of possibilistic mean value, variance, covariance and correlation of fuzzy numbers to a more general class of fuzzy quantities: to quasi fuzzy numbers. We have shown some necessary and sufficient conditions for the existence of possibilistic mean value and variance for quasi fuzzy numbers.

V. Acknowledgments

We are greatly indebted to Prof. Tamás Móri of Department of Probability Theory and Statistics, Eötvös Loránd University, Budapest, for his long-term help with probability distributions.

References

[1] C. Carlsson, R. Fullér, On possibilistic mean value and variance of fuzzy numbers, *Fuzzy Sets and Systems*, 122(2001) 315–326. doi: 10.1016/S0165-0114(00)00043-9

[2] C. Carlsson and R. Fullér, *Fuzzy Reasoning in Decision Making and Optimization*, Studies in Fuzziness and Soft Computing Series, Vol. 82, Springer-Verlag, 2002.

[3] C. Carlsson, R. Fullér and P. Majlender, On possibilistic correlation, *Fuzzy Sets and Systems*, 155(2005) 425-445. doi: 10.1016/j.fss.2005.04.014

[4] S. Chanas and M. Nowakowski, Single value simulation of fuzzy variable, Fuzzy Sets and Systems 25 (1988) 43–57. doi:10.1016/0165-0114(88)90098-X

[5] D. Dubois and H. Prade, The mean value of a fuzzy number, Fuzzy Sets and Systems, 24(1987) 279–300. doi: 10.1016/0165-0114(87)90028-5

[6] D. Dubois and H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.

[7] D. Dubois, Possibility theory and statistical reasoning, Computational Statistics & Data Analysis 51 (2006) 47–69. doi: 10.1016/j.csda.2006.04.015

[8] R. Fullér and P. Majlender, On weighted possibilistic mean and variance of fuzzy quantitives, *Fuzzy Sets and Systems*, 136(2003), 363–374. doi: 10.1016/S0165-0114(02)00216-6

[9] R. Fullér and P. Majlender, On interactive fuzzy quantitives, *Fuzzy Sets and Systems*, 143 (2004), 355–369. doi: 10.1016/S0165-0114(03)00180-5

[10] R. Fullér, J. Mezei and P. Várlaki, An improved index of interactivity for fuzzy numbers, *Fuzzy Sets and Systems*, 165(2011), 56–66. doi: 10.1016/j.fss.2010.06.001

[11] R. Goetschel and W. Voxman, Elementary Fuzzy Calculus, *Fuzzy Sets and Systems*, 18(1986) 31–43. doi: 10.1016/0165-0114(86)90026-6

[12] István Á. Harmati, A note on f-weighted possibilistic correlation for identical marginal possibility distributions, *Fuzzy Sets and Systems* (to appear). doi: 10.1016/j.fss.2010.11.005

[13] P. M. Larsen, Industrial applications of fuzzy logic control, International Journal of Man–Machine Studies, 12(1980), 3–10. doi: 10.1016/S0020-7373(80)80050-2

[14] E.H. Mamdani, Advances in the linguistic synthesis of fuzzy controllers, International Journal of Man–Machine Studies, 8(1976), issue 6, 669–678. doi: 10.1016/S0020-7373(76)80028-4

[15] R.R. Yager , A procedure for ordering fuzzy subsets of the unit interval, Information Sciences 24(1981) 143–161. doi: 10.1016/0020-0255(81)90017-7

[16] Y. Yoshida, M. Yasuda, J -i. Nakagami, M. Kurano, A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty, Fuzzy Sets and Systems 157 (2006) 2614–2626.

[17] L. A. Zadeh, Concept of a linguistic variable and its application to approximate reasoning, **I, II, III**, *Information Sciences*, 8(1975) 199–249, 301–357; 9(1975) 43–80.