On the Notion of Dominance of Fuzzy Choice Functions and Its Application in Multicriteria Decision Making

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Abstract. The aim of this paper is twofold: The first objective is to study the degree of dominance of fuzzy choice functions, a notion that generalizes Banerjee's concept of dominance. The second objective is to use the degree of dominance as a tool for solving multicriteria decision making problems. These types of problems describe concrete economic situations where partial information or human subjectivity appears. The mathematical modelling is done by formulating fuzzy choice problems where criteria are represented by fuzzy available sets of alternatives.

1 Introduction

The revealed preference theory was introduced by Samuelson in 1938 [14] in order to express the rational behaviour of a consumer by means of the optimization of an underlying preference relation. The elaboration of the theory in an axiomatic framework was the contribution of Arrow [1], Richter [12], Sen [15] and many others.

Fuzzy preference relations are a topic a vast literature has been dedicated to. Most authors admit that the preferences that appear in social choice are vague (hence modelled through fuzzy binary relations), but the act of choice is exact (hence choice functions are crisp) ([3], [4], [5]). They study crisp choice functions associated with a fuzzy preference relation.

In [2] Banerjee admits the vagueness of the act of choice and studies choice functions with a fuzzy behaviour. The domain of a Banerjee choice function C is made of all non-empty finite subsets of a set of alternatives X and its range is made of non-zero fuzzy subsets of X.

In [8], [9] we have considered choice functions C for which the domain and the range are made of fuzzy subsets of X. Banerjee fuzzifies only the range of a choice function; we use a fuzzification of both the domain and the range of a choice function. In our case, the available sets of alternatives are fuzzy subsets of X. In this way appears the notion of availability degree of an alternative x with respect to an available set S. The availability degree might be useful when

L. Godo (Ed.): ECSQARU 2005, LNAI 3571, pp. 257–268, 2005.

the decision-maker possesses partial information on the alternative x or when a criterion limits the possibility of choosing x. Therefore the available sets can be considered criteria in decision making.

Papers [2], [17] develop a theory of fuzzy revealed preference for a class of fuzzy choice functions. Papers [8], [9] study a larger class of fuzzy choice functions with respect to rationality and revealed preference.

The aim of this paper is to provide a procedure for ranking the alternatives according to fuzzy revealed preference. For this we introduce the degree of dominance of a fuzzy choice function, notion that refines the dominance from [2], [17]. This concept is derived from the fuzzy choice and not from the fuzzy preference. A problem of choice using the formulation of papers [8], [9] can be assimilated to a multicriteria decision problem. The criteria are mathematically modelled by the available sets of alternatives and the degree of dominance offers a hierarchy of alternatives for each criterion.

The paper is organized as follows. Section 2 is concerned with introductory aspects on fuzzy sets and fuzzy relations. Section 3 introduces some basic issues on fuzzy revealed preference. Section 4 recalls the Banerjee's concept of dominance. Section 5 introduces the degree of dominance and the main results around it. Three congruence axioms FC^*1, FC^*2 and FC^*3 are studied; they extend the congruence axioms $FC1$, $FC2$ and $FC3$ from [2], [17]. A new revealed preference axiom $WAFRP_D$ is formulated and the equivalence $WAFRP_D \Leftrightarrow FC^*1$ is proved. The last section presents a mathematical model for a concrete problem of multicriteria decision making.

2 Preliminaries

In this section we shall recall some properties of the Gödel t-norm and its residuum, as well as some basic definitions on fuzzy sets [6], [10].

Let $[0,1]$ be the unit interval. For any $a, b \in [0,1]$ we shall denote $a \vee b = \max\,(a,b)$; $a \wedge b = \min\,(a,b)$. More generally, for any $\{a_i\}_{i\in I} \subseteq [0,1]$ we denote $\bigvee\limits_{i\in I} a_i = sup\{a_i | i \in I\}$; $\bigwedge\limits_{i\in I} a_i = inf\{a_i | i \in I\}$.

Then $([0,1], \vee, \wedge, 0, 1)$ becomes a distributive complete lattice. The binary operation $\wedge$ is a continuous t-norm, called Gödel t-norm [6], [10].

The *residuum* of the Gödel t-norm $\wedge$ is defined by

$$a \to b = \bigvee\{c \in [0,1] | a \wedge c \leq b\} = \begin{cases} 1 \text{ if } a \leq b \\ b \text{ if } a > b \end{cases}$$

The corresponding biresiduum is defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Let X be a non-empty set. A fuzzy subset of X is a function $A : X \to [0,1]$. Denote by $\mathcal{F}(X)$ the family of fuzzy subsets of X. By identifying a (crisp) subset A of X with its characteristic function, the set $\mathcal{P}(X)$ of subsets of X can be considered a subset of $\mathcal{F}(X)$.

A fuzzy subset A of X is *non-zero* if $A(x) \neq 0$ for some $x \in X$; A is normal if $A(x) = 1$ for some $x \in X$. The *support* of $A \in \mathcal{F}(X)$ is *supp*

$A = \{x \in X | A(x) > 0\}$. For any $x_1, \ldots, x_n \in X$ denote by $[x_1, \ldots, x_n]$ the characteristic function of the set $\{x_1, \ldots, x_n\}$.

A *fuzzy preference relation* R is a fuzzy subset of X^2, i.e. a function $R : X^2 \rightarrow [0, 1]$; for $x, y \in X$ the real number $R(x, y)$ is the *degree of preference* of x with respect to y.

If R, Q are two fuzzy preference relations on X then the *composition* $R \circ Q$ is the fuzzy preference relation defined by $R \circ Q = \bigvee\{R(x, z) \wedge Q(z, y) | z \in X\}$ for any $x, y \in X$.

If $A, B \in \mathcal{F}(X)$ then we denote

$I(A, B) = \bigwedge\limits_{x \in X} (A(x) \rightarrow B(x))$; $E(A, B) = \bigwedge\limits_{x \in X} (A(x) \leftrightarrow B(x))$.

$I(A, B)$ is called the *subsethood degree* of A in B and $E(A, B)$ the *degree of equality* of A and B. Intuitively $I(A, B)$ expresses the truth value of the statement "A is included in B." and $E(A, B)$ expresses the truth value of the statement "A and B contain the same elements."(see [6]). We remark that $A \subseteq B$ if and only if $I(A, B) = 1$ and $A = B$ if and only if $E(A, B) = 1$.

3 Fuzzy Revealed Preference

Revealed preference is a concept introduced by Samuelson in 1938 [14] in the attempt to postulate the rationality of a consumer's behaviour in terms of a preference relation associated to a demand function. Revealed preferences are patterns that can be inferred indirectly by observing a consumer's behaviour. The consumer reveals by choices his preferences, hence the term *revealed preference* .

To study fuzzy revealed preferences and fuzzy choice functions associated to them is a natural problem. A vast literature has been dedicated to the case when preferences are fuzzy but the act of choice is exact [3], [4], [5]. In [2] Banerjee lifts this condition putting forth the idea of fuzzy choice functions (see also [16]). We give a short description of Banerjee's framework.

Let X be a non-empty set of alternatives, $\mathcal{H}$ the family of all non-empty finite subsets of X and $\mathcal{F}$ the family of non-zero fuzzy subsets of X with finite support. A Banerjee fuzzy choice function is a function $C : \mathcal{H} \rightarrow \mathcal{F}$ such that *supp* $C(S) \subseteq S$ for any $S \in \mathcal{H}$.

According to the previous definition the domain $\mathcal{H}$ of a Banerjee fuzzy choice function is the family of all non-empty finite subsets of X. In [8] and [9] we have developed a theory of fuzzy revealed preferences and fuzzy functions associated to them in an extended form, generalizing Banerjee's.

A fuzzy choice space is a pair $\langle X, \mathcal{B} \rangle$ where X is a non-empty set and $\mathcal{B}$ is a non-empty family of non-zero fuzzy subsets of X. A *fuzzy choice function* (=fuzzy consumer) on $\langle X, \mathcal{B} \rangle$ is a function $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ such that for each $S \in \mathcal{B}$, $C(S)$ is non-zero and $C(S) \subseteq S$.

Now we introduce the fuzzy revealed preference relation R associated to a fuzzy choice function $C : \mathcal{B} \rightarrow \mathcal{F}(X)$: $R(x, y) = \bigvee\limits_{S \in \mathcal{B}} (C(S)(x) \wedge S(y))$ for any $x, y \in X$.

R is the fuzzy form of the revealed preference relation originally introduced by Samuelson in [14] and studied in an axiomatic framework in [1], [15] etc.

Conversely, to a fuzzy preference relation Q one assigns a fuzzy choice function C defined by $C(S)(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \rightarrow Q(x,y)]$ for any $S \in \mathcal{B}$ and $x \in X$. $C(S)(x)$ is the degree of truth of the statement "x is one of the Q-greatest alternatives satisfying criterion S".

4 Banerjee's Concept of Dominance

Banerjee's paper [2] deals with the revealed preference theory for his fuzzy choice functions. He studies three congruence axioms $FC1$, $FC2$, $FC3$. In [17], Wang establishes the connection between $FC1$, $FC2$, $FC3$. These three axioms are formulated in terms of dominance of an alternative x in an available set S of alternatives.

In the literature of fuzzy preference relations there are several ways to define the dominance (see [11]). In general the dominance is related to a fuzzy preference relation [7]. The concept of dominance in [2] is related to the act of choice and is expressed in terms of the fuzzy choice function. For a fuzzy preference relation there exist a lot of ways to define the degree of dominance of an alternative [2], [3], [4], [5], [7], [11].

Let C be a fuzzy choice function, $S \in \mathcal{H}$ and $x \in S$. x is said to be *dominant* in S if $C(S)(y) \leq C(S)(x)$ for any $y \in S$. The dominance of x in S means that x has a higher potentiality of being chosen than the other elements of S. It is obvious that this definition of dominance is related to the act of choice, not to a preference relation.

Banerjee also considers a second type of dominance, associated to a fuzzy preference relation.

Let R be a fuzzy preference relation on X, $S \in \mathcal{H}$ and $x \in X$. x is said to be relation dominant in S in terms of R if $R(x,y) \geq R(y,x)$ for all $y \in S$.

Let $S \in \mathcal{H}$, $S = \{x_1, \ldots, x_n\}$. The *restriction* of R to S is $R|_S = (R(x_i, x_j))_{n \times n}$. Then we have the *composition* $R|_S \circ C(S) = \bigvee_{j=1}^{n} (R(x_i, x_j) \wedge C(S)(x_j))$.

In [2] Banerjee introduced the following *congruence axioms* for a fuzzy choice function C:

$FC1$ For any $S \in \mathcal{H}$ and $x, y \in S$, if y is dominant in S then $C(S)(x) = R(x,y)$.

$FC2$ For any $S \in \mathcal{H}$ and $x, y \in S$, if y is dominant in S and $R(y,x) \leq R(x,y)$ then x is dominant in S.

$FC3$ For any $S \in \mathcal{H}$, $\alpha \in (0,1]$ and $x, y \in S$, $\alpha \leq C(S)(y)$ and $\alpha \leq R(x,y)$ imply $\alpha \leq C(S)(x)$.

In [17], Wang proved that $FC3$ holds iff for any $S \in \mathcal{H}$, $R|_S \circ C(S) \subseteq C(S)$. Then $FC3$ is equivalent with any of the following statements:

∘ For any $S \in \mathcal{H}$ and $x \in S$, $\bigvee_{y \in S} (R(x,y) \wedge C(S)(y)) \leq C(S)(x)$;

∘ For any $S \in \mathcal{H}$ and $x, y \in S$, $R(x,y) \wedge C(S)(y) \leq C(S)(x)$.

In [17] it is proved that $FC1$ implies $FC2$, $FC3$ implies $FC2$ and $FC1$, $FC3$ are independent.

Some results from Sect. 5 are based on the following hypotheses:

$(H1)$ Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X;

(H_2) $\mathcal{B}$ includes all fuzzy sets $[x_1, \ldots, x_n]$, $n \geq 1$ and $x_1, \ldots, x_n \in X$.

5 Degree of Dominance and Congruence Axioms

In this section we shall define a notion of degree of dominance in the framework of the fuzzy choice functions introduced above. This kind of dominance is attached to a fuzzy choice function and not to a fuzzy preference relation. It shows to what extent, as the result of the act of choice, an alternative has a dominant position among others.

As seen in the previous section, the concept of dominance appears essentially in the expression of congruence axioms $FC1$-$FC3$. We define now the *degree of dominance* of an alternative x with respect to a fuzzy subset S. This will be a real number that shows the position of x among the other alternatives.

We fix a fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$.

Definition 1. *Let $S \in \mathcal{B}$ and $x \in X$. The degree of dominance* of x in S is given by

$$D_S(x) = S(x) \wedge \bigwedge_{y \in X} [C(S)(y) \to C(S)(x)]$$
$$= S(x) \wedge [(\bigvee_{y \in X} C(S)(y)) \to C(S)(x)].$$

If $D_S(x) = 1$ then we say that x is *dominant* in S.

Remark 1. Let S be a crisp subset of X. Identifying S with its characteristic function we have the equivalences:

$D_S(x) = 1$ iff $S(x) = 1$ and $C(S)(y) \leq C(S)(x)$ for any $y \in X$
iff $x \in S$ and $C(S)(y) \leq C(S)(x)$ for any $y \in S$.

This shows that in this case we obtain exactly the notion of dominance of Banerjee.

Remark 2. In accordance with Definition 1, x is dominant in S iff $S(x) = 1$ and $\bigvee_{y \in X} C(S)(y) = C(S)(x)$.

Remark 3. Assume that C satisfies $(H1)$, i.e. $C(S)(y_0) = 1$ for some $y_0 \in X$. In this case $\bigvee_{y \in X} C(S)(y) = 1$ therefore $D_S(x) = C(S)(x)$.

Lemma 1. *If $[x, y] \in \mathcal{B}$ then $D_{[x,y]}(x) = C([x,y])(y) \to C([x,y])(x)$.*

Proposition 1. *For any $S \in \mathcal{B}$ and $x, y \in X$ we have*
(i) $C(S)(x) \leq D_S(x) \leq S(x)$;
(ii) $S(x) \wedge D_S(y) \wedge [C(S)(y) \rightarrow C(S)(x)] \leq D_S(x)$.

Remark 4. By Proposition 6, $D_S(x) > 0$ for some $x \in X$. Then the assignment $S \mapsto D_S$ is a fuzzy choice function $D : \mathcal{B} \rightarrow \mathcal{F}(X)$. According to Remark 4, if C satisfies $(H1)$ then $C = D$. It implies that the study of the degree of dominance is interesting for the case when hypothesis $(H1)$ does not hold.

Remark 5. For $S \in \mathcal{B}$ and $x \in X$ we define the sequence $(D_S^n(x))_{n \geq 1}$ by induction:
$D_S^1(x) = D_S(x)$; $D_S^{n+1}(x) = S(x) \wedge \bigwedge_{y \in X} [D_S^n(y) \rightarrow D_S^n(x)]$.

By Proposition 6 (i) we have $C(S)(x) \leq D_S^1(x) \leq \ldots \leq D_S^n(x) \leq \ldots \leq D_S^{\infty}(x) \leq S(x)$, where $D_S^{\infty}(x) = \bigvee_{n=1}^{\infty} D_S^n(x)$. The assignments $S \mapsto D_S^n$, $n \geq 1$ and $S \mapsto D_S^{\infty}$ provide new fuzzy choice functions.

The following definition generalizes Banerjee's notion of dominant relation in S in terms of R.

Definition 2. *Let Q be a fuzzy preference relation on X, $S \in \mathcal{B}$ and $x \in X$. The degree of dominance* of x in S in terms of Q is defined by
$D_S^Q(x) = S(x) \wedge \bigwedge_{y \in X} [(S(y) \wedge Q(y,x)) \rightarrow Q(x,y)]$

If $D_S^Q(x) = 1$ then we say that x is *dominant in S in terms of Q* .
The congruence axioms $FC1$, $FC2$, $FC3$ play an important role in Banerjee's theory of revealed preference. The formulation of $FC1$, $FC2$ uses the notion of dominance and $FC3$ is a generalization of Weak Congruence Axiom (WCA).

Now we introduce the congruence axioms FC^*1, FC^*2, FC^*3 which are refinements of axioms $FC1$, $FC2$, $FC3$. Axioms FC^*1 and FC^*2 are formulated in terms of degree of dominance. FC^*3 is Weak Fuzzy Congruence Axiom ($WFCA$) defined in [8], [9].

FC^*1 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$S(x) \wedge D_S(y) \leq R(x,y) \rightarrow C(S)(x)$.
FC^*2 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$S(x) \wedge D_S(y) \wedge (R(y,x) \rightarrow R(x,y)) \leq D_S(x)$.
FC^*3 For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$S(x) \wedge C(S)(y) \wedge R(x,y) \leq C(S)(x)$.

The form FC^*1 is derived from FC^*3 by replacing $D_S(y)$ by $C(S)(y)$. By Remarks 4 and 7, $D_S(x)$ (resp. $D_S(y)$) can be viewed as a substitute of $C(S)(x)$ (resp. $C(S)(y)$).

If hypothesis $(H1)$ holds, then by Remark 4, $D_S(y) = C(S)(y)$ axioms FC^*1 and FC^*3 are equivalent.

Remark 6. Notice that FC^*3 appears under the name $WFCA$ (Weak Fuzzy Congruence Axiom).

Proposition 2. $FC^*1 \Rightarrow FC^*3$.

Proposition 3. $FC^*3 \Rightarrow FC^*2$.

Proposition 4. *If FC^*1 holds then $D_S(x) \leq D_S^R(x)$ for any $S \in \mathcal{B}$ and $x \in X$.*

Theorem 1. *Assume that the fuzzy choice function C fulfills $(H2)$. Then axiom FC^*1 implies that for any $S \in \mathcal{B}$ and $x \in X$ we have*

$$D_S(x) = S(x) \wedge \bigwedge_{y \in X} [S(y) \rightarrow D_{[x,y]}(x)].$$

The formulation of axiom FC^*3 has Lemma 2.1 in [17] as starting point. The following result establishes the equivalence of FC^*3 with a direct generalization of $FC3$.

Proposition 5. *The following assertions are equivalent:*
(1) The axiom FC^*3 holds;
(2) For any $S \in \mathcal{B}$, $x, y \in X$ and $\alpha \in (0, 1]$,

$$S(x) \wedge S(y) \wedge [\alpha \rightarrow C(S)(y)] \wedge [\alpha \rightarrow R(x, y)] \leq \alpha \rightarrow C(S)(x).$$

Definition 3. *Let C be a fuzzy choice function on $\langle X, \mathcal{B} \rangle$. We define the fuzzy relation R_2 on X by*

$$R_2(x, y) = \bigwedge_{S \in \mathcal{B}} [(S(x) \wedge D_S(y)) \rightarrow C(S)(x)].$$

Remark 7. Let C be a fuzzy choice function, $S \in \mathcal{B}$ and $x, y \in X$. By the definition of fuzzy revealed preference R

$$R(x, y) \wedge S(x) \wedge D_S(y) = [\bigvee_{T \in \mathcal{B}} (C(T)(x) \wedge T(y))] \wedge S(x) \wedge D_S(y)$$
$$= \bigvee_{T \in \mathcal{B}} [S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y)].$$

Then FC^*1 is equivalent to the following statement

- For any $S, T \in \mathcal{B}$ and $x, y \in X$
$S(x) \wedge T(y) \wedge C(T)(x) \wedge D_S(y) \leq C(S)(x)$.

In [9] the following revealed preference axiom was considered:
$WAFRP^\circ$ For any $S, T \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:
$[S(x) \wedge C(T)(x)] \wedge [T(x) \wedge C(S)(x)] \leq E(S \cap C(T), T \cap C(S))$.
In [9] it was proved that $WAFRP^\circ$ and $FC^*3 = WFCA$ are equivalent.

A problem is if we can find a similar result for condition FC^*1. In order to obtain an answer to this problem we introduce the following axiom:
$WAFRP_D$ For any $x, y \in X$ and $S, T \in \mathcal{B}$,
$[S(x) \wedge C(T)(x)] \wedge [T(y) \wedge D_S(y)] \leq I(S \cap C(T), T \cap C(S))$.

Theorem 2. *For a fuzzy choice function $C : \mathcal{B} \rightarrow \mathcal{F}(X)$ the following are equivalent:*

(i) C satisfies FC^*1;
(ii) $R \subseteq R_2$;
(iii) C satisfies $WAFRP_D$.

6 An Application to Multicriteria Decision Making

In making a choice, a set of alternatives and a set of criteria are usually needed.

According to [18], the alternatives and the criteria are defined as follows:

"Alternatives are usually *mutually exclusive* activities, objects, projects, or models of behaviour among which a choice is possible".

"Criteria are measures, rules and standards that guide decision making. Since decision making is conducted by selecting or formulating different attributes, objectives or goals, all three categories can be referred as criteria. That is, criteria are all those attributes, objectives or goals which have been judged relevant in a given decision situation by a particular decision maker (individual or group)".

In this section we shall present one possible application of fuzzy revealed preference theory. It represents a model of decision making based on the ranking of alternatives according to fuzzy choices. An agent's decision is based on the ranking of alternatives according to different criteria. This ranking is obtained by using fuzzy choice problems and the instrument by which it is established is the degree of dominance associated to a fuzzy choice function. In defining this fuzzy choice function the revealed preference theory is applied.

A producer manufactures m types of products $P_1, \ldots, P_m$. n companies $x_1, \ldots, x_n$ are interested in selling his products. The sales obtained in year T are given in the following table:

	P_1	P_2	$\ldots$	P_m
x_1	a_{11}	a_{12}	$\ldots$	a_{1m}
x_2	a_{21}	a_{22}	$\ldots$	a_{2m}
$\ldots$				
x_n	a_{n1}	a_{n2}	$\ldots$	a_{nm}

where a_{ij} denotes the number of units of product P_j sold by company x_i in year T. For the year $T+1$ the producer would like to increase the number of sales with the n companies. The companies give an estimation of the sales for year $T+1$ contained in a matrix (c_{ij}) with n rows and m columns; c_{ij} denotes the number of units of product P_j that the company x_i estimates to sell in year $T+1$.

Each product has to be sold by those companies that have an efficient sales market. In choosing these companies, an analysis will require two aspects:

(a) the sales a_{ij} for year T;

(b) the estimated sales c_{ij} for year $T+1$.

The sales for year T can be considered results of the act of choice, or more clearly, values of a choice function, and the preferences will be given by the revealed preference relation associated to these choice functions. With the resulting preference relation and the estimated sale for the year $T+1$, a fuzzy choice function can be defined. This choice function will be used to rank the companies with respect to each type of product. Dividing the values a_{ij} and c_{ij} respectively by a power of 10 conveniently chosen we may assume that $0 \leq a_{ij}, c_{ij} \leq 1$ for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

In establishing the mathematical model the following steps are needed:

(A) To build a fuzzy choice function from the sales of year T.

The set of alternatives is $X = \{x_1, \ldots, x_n\}$.

For each $j = 1, \ldots, m$ denote by S_j the subset of X whose elements are those companies that have had "good" sales for product P_j in year T. Only the companies whose sales are greater than a threshold e_j are considered.

If $\mathcal{H} = \{S_1, \ldots, S_m\}$ then $\langle X, \mathcal{H}\rangle$ is a fuzzy choice space (we will identify S_j with its characteristic function). The sales (a_{ij}) of year T lead to a choice function $C' : \mathcal{H} \to \mathcal{F}(X)$ defined by:

(1) $C'(S_j)(x_i) = a_{ij}$

for each $j = 1, \ldots, m$ and $x_i \in S_j$.

This context is similar to Banerjee [2]. There $\mathcal{H}$ contains all non-empty finite subsets of X.

(B) The choice function C' gives a fuzzy revealed preference relation R on X:

(2) $R(x_i, x_j) = \bigvee\{C'(S_k)(x_i) | x_i, x_j \in S_k\} = \bigvee\{a_{ik} | x_i, x_j \in S_k\}$

for any $x_i, x_j \in X$.

$R(x_i, x_j)$ represents the degree to which alternative x_i is preferred to alternative x_j as a consequence of current sales.

Since in most cases R is not reflexive, we replace it by its reflexive closure R'.

(C) From the fuzzy revealed preference matrix R' and the matrix c_{ij} of estimated sales one can define a fuzzy choice function C, whose values will estimate the potential sales for the year $T+1$. Starting from C one will rank the alternatives for each type of product.

The set of alternatives is $X = \{x_1, \ldots, x_n\}$. For each $j = 1, \ldots, m$ A_j will denote the fuzzy subset of X given by

(3) $A_j(x_i) = c_{ij}$ for any $i = 1, \ldots, n$.

Take $\mathcal{A} = \{A_1, \ldots, A_m\}$. One obtains the fuzzy choice space $\langle X, \mathcal{A}\rangle$. The choice function $C : \mathcal{A} \to \mathcal{F}(X)$ is defined by

$$(4)\ C(A_j)(x_i) = A_j(x_i) \wedge \bigwedge_{k=1}^{n} [A_j(x_k) \to R'(x_i, x_k)]$$

$$= c_{ij} \wedge \bigwedge_{k=1}^{n} [c_{ij} \to R'(x_i, x_k)]$$

for any $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Applying the degree of dominance for the fuzzy choice function C one will obtain a ranking of the companies with respect to each product. This ranking gives the information that the mathematical model described above offers to the producer with respect to the sales activity for the following year.

We present next the algorithm of this problem.

The input data are:

m= the number of types of products

n=the number of companies

a_{ij}=the matrix of sales for year T

c_{ij}=the matrix of estimated sales for year $T+1$

$(e_1, \ldots, e_m)$=the threshold vector

Assume $0 \leq a_{ij} \leq 1$, $0 \leq c_{ij} \leq 1$ for any $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

From the mathematical model we can derive the following steps:

Step 1. Determine the subsets $S_1, \ldots, S_m$ of $X = \{x_1, \ldots, x_n\}$ by $S_k = \{x_i \in X | a_{ik} \geq e_k\}$, $k = 1, \ldots, m$.

Step 2. Compute the matrix of revealed preferences $R = (R(x_i, x_j))$ by $R(x_i, x_j) = \bigvee_{x_i, x_j \in S_k} a_{ik}$.

Replace R with its reflexive closure R'.

Step 3. Determine the fuzzy sets $A_1, \ldots, A_m$

$A_j = \frac{c_{ij}}{x_1} + \ldots + \frac{c_{nj}}{x_n}$ for $j = 1, \ldots, m$

Step 4. Obtain the choice function C applying (3)

Step 5. Determine the degrees of dominance $D_{A_j}(x_i)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Step 6. Rank the set of alternatives with respect to each product P_j by ranking the set $\{D_{A_j}(x_1), \ldots, D_{A_j}(x_n)\}$.

For a better understanding of this model we present a numerical illustration. Consider the initial data $m = 3$ products and $n = 5$ companies willing to sell these products.

The sales for year T are given in the following table:

	P_1	P_2	P_3
x_1	0.3	0.6	0.7
x_2	0.8	0.1	0.5
x_3	0.7	0.6	0.1
x_4	0.1	0.8	0.7
x_5	0.8	0.1	0.7

The estimated sales for year $T+1$ are given in the following table:

	P_1	P_2	P_3
x_1	0.5	0.7	0.7
x_2	0.8	0.3	0.6
x_3	0.8	0.7	0.2
x_4	0.2	0.8	0.8
x_5	0.8	0.2	0.8

The thresholds are $e_1 = e_2 = e_3 = 0.2$.

We follow now the steps described above.

Step 1. The subsets S_1, S_2, S_3 of X are:
$S_1 = \{x_1, x_2, x_3, x_5\}$, $S_2 = \{x_1, x_3, x_4\}$, $S_3 = \{x_1, x_2, x_4, x_5\}$.

Step 2. We compute the matrix of revealed preferences R. Then we replace it by its reflexive closure R'.

$$R = \begin{pmatrix} 0.7 & 0.7 & 0.6 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.5 & 0.8 \\ 0.7 & 0.7 & 0.7 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.7 & 0.8 \end{pmatrix}; \; R' = \begin{pmatrix} 1 & 0.7 & 0.6 & 0.7 & 0.7 \\ 0.8 & 1 & 0.8 & 0.5 & 0.8 \\ 0.7 & 0.7 & 1 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.8 & 1 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.7 & 1 \end{pmatrix}.$$

For example, $R(x_1, x_2) = \bigvee_{x_1, x_2 \in S_k} a_{1k} = a_{11} \vee a_{13} = 0.3 \vee 0.7 = 0.7$.

Step 3. The fuzzy sets A_1, A_2, A_3 are:

$A_1 = \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5}$;
$A_2 = \frac{0.7}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5}$;
$A_3 = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.2}{x_3} + \frac{0.8}{x_4} + \frac{0.8}{x_5}$.

Step 4. The corresponding fuzzy choice functions are:

$C(A_1)(x) = \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5}$
$C(A_2)(x) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5}$
$C(A_3)(x) = \frac{0.7}{x_1} + \frac{0.5}{x_2} + \frac{0.2}{x_3} + \frac{0.7}{x_4} + \frac{0.7}{x_5}$.

Step 5. The corresponding degrees of dominance are represented in the table:

$D_{A_j}(x_i)$	x_1	x_2	x_3	x_4	x_5
A_1	0.5	0.8	0.7	0.2	0.8
A_2	0.6	0.3	0.6	0.8	0.2
A_3	0.7	0.5	0.2	0.8	0.7

The table of degrees of dominance establishes the ranking of alternatives according to each criterion.

According to criterion A_1,
$D_{A_1}(x_4) < D_{A_1}(x_1) < D_{A_1}(x_3) < D_{A_1}(x_2) = D_{A_1}(x_5)$.

According to criterion A_2,
$D_{A_2}(x_5) < D_{A_2}(x_2) < D_{A_2}(x_1) = D_{A_2}(x_3) < D_{A_2}(x_4)$.

According to criterion A_3,
$D_{A_3}(x_3) < D_{A_3}(x_2) < D_{A_3}(x_1) = D_{A_3}(x_5) < D_{A_3}(x_4)$.

7 Concluding Remarks

This paper completes the results of [8], [9]. Our main contribution is to introduce the concept of degree of dominance of an alternative, as a method of ranking the alternatives according to different criteria. These criteria can be taken as the available sets of alternatives.

The degree of dominance of an alternative x in an available set S of alternatives reflects x's position towards the other alternatives (with respect to S). This notion expresses the dominance of an alternative with regard to the act of choice, not to a preference relation. With the degree of dominance one can build a hierarchy of alternatives for each available set S. If one defines a concept of aggregated degree of dominance (that unifies the degrees of dominance with regard to various available sets) one obtains an overall hierarchy of alternatives.

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