Risk Aversion through Fuzzy Numbers

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Abstract

This paper is concerned with an approach of risk aversion by possibility theory. We introduce and study new possibilistic risk indicators.The main notions are the possibilistic risk premium and the possibilistic relative risk premium associated with a fuzzy number and a utility function. We also give formulae for computing them.

1. Introduction

Risk theory is developed traditionally in the context of probability theory. Possibility theory initiated by Zadeh in [9] is an alternative to probability theory in the treatment of uncertainty. Possibility theory is applied in cases of uncertainty in which phenomena occur with a reduced frequency or in which we have few data.

This paper proposes a possibilistic treatment of risk aversion. We define the possibilistic risk premium as a measure of risk aversion in situations of possibilistic uncertainty. A formula is proved for the computation of the possibilistic risk premium in terms of possibilistic indicators. We shall consider well-known definitions of fuzzy numbers and their main properties ([3]).

2. Possibilistic indicators

Probability theory and possibility theory are two alternative ways of describing uncertain situations. The second one is preferred to the first one in cases when the frequency of occurrence of a phenomenon is scarce or when we do not have a sufficiently large database. Being a relatively new field, possibility theory has tried to get inspired from the notions and results of probability theory and to have a parallel evolution. Schematically speaking, the transition from probability theory to possibility theory has as main components:

(a) the concept of probability has been replaced by the concepts of possibility and necessity;

(b) possibilistic distributions have replaced probabilistic distributions;

(c) possibilistic indicators similar to probabilistic indicators have been defined.

Fuzzy numbers represent the most important class of possibilistic distributions. In various interpretations, they generalize in fuzzy context the real numbers. By using Zadeh's extension principle [10], the fundamental operations with real numbers are extended to fuzzy numbers [3].

The notion of possibilistic mean value of a fuzzy number A was defined by Carlsson and Fullér in [2] and has been generalized in [4] by the introduction of the notion of expected value of a fuzzy number A w.r.t. a weighting function f.

A function $f:[0,1]\to \mathbf{R}$ is a weighting function if it is non-negative, monotone increasing and verifies the normality condition

$$\int_0^1 f(x)dx = 1 .$$

Let us fix a weighting function f and a fuzzy number A. For any $\gamma\in [0, 1]$ let us consider the set $[A]^{\gamma}=\{x\in \mathbf{R} | A(\gamma)\geq\gamma\}$. Then $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$, where $a_1(\gamma)=\min [A]^{\gamma}$ and $a_2(\gamma)=\max [A]^{\gamma}$.

Definition 1. The possibilistic expected value of A w.r.t. f is defined by

$$(1)\ E_f(A)=\int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)d\gamma \ .$$

For a continuous function g: $\mathbf{R}\to \mathbf{R}$ and for a fuzzy number A we define the fuzzy subset g(A) of **R** by using Zadeh's extension principle [10]:

(2) $g(A)(y)= \sup_{g(x)=y} A(x)$ if there is $x\in \mathbf{R}$ such that $g(x)=y$ and $g(A)(y)=0$ otherwise.

g(A) is not always a fuzzy number, therefore we cannot define $E_f(g(A))$ using (1). In [4]

the following definition of $E_f(g(A))$ has been proposed.

$$(3)\ E_f(g(A))=\int_0^1 \left[\frac{1}{a_2(\gamma)-a_1(\gamma)}\int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx\right]f(\gamma)d\gamma \ .$$

If g is the identity function then we obtain formula (1).

Let A be a fuzzy number, g: **R**→ **R,** h: **R**→ **R** two continuous functions and a, b∈ **R.** We consider the continuous function u: **R**→ **R** defined by u(x)=ag(x)+bh(x) for any x∈ **R.**

Proposition 1. [5] $E_f(u(A))=a\ E_f(g(A))+b\ E_f(h(A))$.

The variance is one of the main indicators associated with a random variable. One poses the question of defining a concept of possibilistic variance.

In [4] the possibilistic variance $Var_f(A)$ of a fuzzy number A w.r.t. a weighting function f has been introduced by

(4) $$Var_f(A)=\int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))^2}{12} f(\gamma)d\gamma\ .$$

We present a second way of defining a notion of possibilistic variance (see [5]). We consider the continuous function g: **R**→ **R** given by $g(x)=(x-E_f(A))^2$ for any x∈ **R**.

Definition 2. [5] The strong possibilistic variance $Var^*_f(A)$ of A w.r.t. f is defined by

(5) $Var^*_f(A)=E_f(g(A))$.

According to formula (3), (5) can be written as

(6) $$Var^*_f(A)=\int_0^1 \left[\frac{1}{a_2(\gamma)-a_1(\gamma)}\int_{a_1(\gamma)}^{a_2(\gamma)} (x-E_f(A))^2 dx\right] f(\gamma)d\gamma\ .$$

The following result establishes the relationship between $Var_f(A)$ and $Var^*_f(A)$.

Proposition 2. $Var^*_f(A)=4Var_f(A)-E_f^2(A)+\int_0^1 a_1(\gamma)a_2(\gamma)f(\gamma)d\gamma$.

3. Towards possibilistic risk aversion

Risk aversion is one of the most important themes in risk theory. The mathematical framework in which risk theory is developed is assured by a utility function and a random variable.

Let u: **R**→ **R** be a utility function and X a random variable. We suppose that u is a continuous function. The *risk premium* $\rho_{X,u}$ associated with X and u is defined by the identity:

(1) $E(u(X))=u(E(X)-\rho_{X,u})$.

In interpretation, the utility function u represents the attitude of an agent related to an uncertain situation described by the random variable X. For simplicity, we can consider X as representing a lottery. The, the risk premium is interpreted as "the maximum amount by which the agent is willing to decrease the expected return from the lottery ticket to have assure return" ([6], p. 19).

Besides the risk premium, we have the notions of *relative risk premium* $\hat{\rho}_{X,u}$ defined by the equality

(2) $E(u(X))=u(E(X)(1-\hat{\rho}_{X,u}))$.

Proposition 3. ([6], p. 21) Assume that u is twice differentiable, strictly concave and increasing. Then

(i) $\rho_{X,u} = -\frac{1}{2}\sigma_X^2\, u(E(X))$ where σ_X^2 is the variance of X;

(ii) $\hat{\rho}_{X,u} = -\frac{1}{2} E(X)\, \sigma_X^2\, u'(E(X))$.

Since we have surveyed those elements of probabilistic theory of risk aversion, we intend now to make a possibilistic treatment of the topic.

The possibilistic framework for a possibilistic theory of risk aversion has the following three components:

(a) a fuzzy number A which describes an uncertain situation;

(b) a weighting function u which represents the attitude of an agent w.r.t. A;

(c) a weighting function f.

Now we define the notions of possibilistic risk premium and the possibilistic relative risk premium.

Definition 3. The possibilistic risk premium $\rho_A = \rho_{A,f,u}$ (associated with A, f, and u) is defined by the equality:

(3) $E_f(u(A)) = u(E_f(A) - \rho_A)$.

Definition 4. The possibilistic relative risk premium $\hat{\rho}_{A,f,,u}$ (associated with A, f, and u) is defined by the equality:

(4) $E_f(u(A)) = u(E_f(A)(1 - \hat{\rho}_A))$.

We notice that (3) and (4) are possibilistic versions of the equalities (1) and (2); the random variable X is replaced by a fuzzy number A and E(X) is replaced by $E_f(A)$.

The following results is a possibilistic version of Proposition 3.

Proposition 4. Assume that u is twice differentiable, strictly concave and increasing. Then the following equalities hold:

(i) $\rho_A = -\frac{1}{2} Var^*{}_f(A) \frac{u''(E_f(A))}{u'(E_f(A))}$

(ii) $\hat{\rho}_A = -\frac{1}{2} Var_f^*(A) E_f(A) \frac{u''(E_f(A))}{u'(E_f(A))}$.

10. References

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