Dense Full-Diversity Matrix Lattices for Four Transmit Antenna MISO Channel

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Abstract— **We construct some geometrically dense matrix lattices with good minimum determinants for 4 transmit antenna MISO applications. The construction is based on the theory of rings of algebraic integers and related subrings of the Hamiltonian quaternions. Simulations in a quasi-static Rayleigh fading channel show that our dense quaternionic constructions outperform the earlier rectangular lattices as well as the DAST-lattice.**

I. Background and basic definitions

We are interested in the coherent multiple input-single output (MISO) case where the receiver perfectly knows the channel coefficients. The received signal is

$$\mathbf{y_{1\times n}} = \mathbf{h_{1\times k}X_{k\times n}} + \mathbf{n_{1\times n}},$$

where $\mathbf{X}$ is the transmitted codeword taken from Space-Time Block Code (STBC), $\mathbf{h}$ is the Rayleigh fading channel response and the components of the noise vector $\mathbf{n}$ are i.i.d. complex Gaussian random variables.

A *lattice* is a discrete finitely generated free abelian subgroup $\mathbf{L}$ of a real (or complex) finite dimensional vector space $\mathbf{V}$, called the ambient space. In the space-time setting a natural ambient space is the space $\mathcal{M}_n(\mathbf{C})$ of complex $n\times n$-matrices. When a code is a subset of a lattice $\mathbf{L}$ in this ambient space, the *rank criterion* states that any non-zero matrix in $\mathbf{L}$ must be invertible. This follows from the fact that the difference of any two matrices from $\mathbf{L}$ is again in $\mathbf{L}$. As a main design criterion we recall the *minimum determinant* of the code $\mathcal{C}$. In the case of square matrix lattice this takes the form

$$\delta_{\mathcal{C}} = \ \min_{\mathbf{M}\in\mathcal{C},\mathbf{M}\neq 0}\{\det(\mathbf{M}\mathbf{M}^*)^{\frac{1}{k}}\},$$

where $\mathbf{M}^*$ is the adjoint of the matrix $\mathbf{M}$ and k is the number of transmit antennas. The receiver, however, (recall that we work in the MISO setting) sees vector lattices instead of matrix lattices. When the channel state is $\mathbf{h}$, the receiver expects to see the lattice $\mathbf{hL}$.

This work is a continuation of the reports [1] and [2]. The reader interested in more background is referred to [3]-[9].

II. Rings of algebraic numbers, quaternions and lattice constructions

It is widely known how the so called *Alamouti design* represents multiplication in the ring of quaternions. As the quaternions form a division algebra, such matrices must be invertible, i.e. the resulting STBC meets the rank criterion. Matrix representations of other division algebras have been proposed as STBC codes at least in [2],[10],[11],[12], and (though without explicitly saying so) [13]. The most recent work ([11],[12] and [13]) has concentrated on adding multiplexing gain (i.e. MIMO applications) and/or combining it with a good minimum determinant. We do not seek any multiplexing gains, but want to improve upon e.g. the DAST-lattices introduced in [10] by using non-commutative division algebras.

The set $\{a_1 + a_2 i + a_3 j + a_4 k |\ a_i \in \mathbf{R}\ \forall i\}$, where $i^2 = j^2 = k^2 = -1$, $ij = k$, is recalled as the ring of Hamiltonian quaternions. We shall use extension rings of the Gaussian integers $\mathcal{G} = \{a + bi | a, b \in \mathbf{Z}\}$ inside a given division algebra as they fit nicely with the popular 16-QAM and QPSK alphabets. Natural examples of such rings are the rings of algebraic integers inside an extension field of the quotient fields of $\mathcal{G}$, as well as their counterparts inside the quaternions. To that end we need division algebras A that are also 4-dimensional vectors spaces over the field $K = \mathbf{Q}(i)$. Let $\zeta = e^{\pi i/8}$ (resp. $\xi = e^{\pi i/4} = (1 + i)/\sqrt{2}$) be primitive 16^{th} (resp. 8^{th}) root of unity. Our main examples of such division algebras are the number field $L = \mathbf{Q}(\zeta)$ and the division algebra $\mathbf{H} = \mathbf{Q}(\xi) \oplus \mathbf{Q}(\xi)j$. As $zj = jz^*$ for all complex numbers z, and as the field $\mathbf{Q}(\xi)$ is stable under the usual complex conjugation $(^*)$, the set $\mathbf{H}$ is a subskewfield of the quaternions.

As always, multiplication (from the left) by a non-zero element of the division algebra A is an invertible $\mathbf{Q}(i)$-linear mapping (with $\mathbf{Q}(i)$ acting from the right). Therefore its matrix with respect to a chosen $\mathbf{Q}(i)$-basis $\mathcal{B}$ of A is also invertible. Our example division algebras L and $\mathbf{H}$ have as natural $\mathbf{Q}(i)$-bases the sets $\mathcal{B}_L = \{1, \zeta, \zeta^2, \zeta^3\}$ and $\mathcal{B}_H = \{1, \xi, j, j\xi\}$ respectively. Thus we immediately arrive at the following matrix representations of our division algebras.

Proposition 2.1: Let the variables c_1, c_2, c_3, c_4 range over all the elements of $\mathbf{Q}(i)$. The division algebras L and $\mathbf{H}$ can be identified via an isomorphism ϕ with the following rings of matrices

$$L = \left\{ \left(\begin{array}{cccc} c_1 & ic_4 & ic_3 & ic_2 \\ c_2 & c_1 & ic_4 & ic_3 \\ c_3 & c_2 & c_1 & ic_4 \\ c_4 & c_3 & c_2 & c_1 \end{array} \right) \right\},$$

and

$$\mathbf{H} = \left\{ M = M(c_1, c_2, c_3, c_4) = \left(\begin{array}{cccc} c_1 & ic_2 & -c_3^* & -c_4^* \\ c_2 & c_1 & ic_4^* & -c_3^* \\ c_3 & ic_4 & c_1^* & c_2^* \\ c_4 & c_3 & -ic_2^* & c_1^* \end{array} \right) \right\}.$$

The isomorphism ϕ from L into the matrix ring is determined by $\mathbf{Q}(i)$-linearity and the fact that ζ corresponds to the choice $c_2 = 1, c_1 = c_3 = c_4 = 0$. The isomorphism ϕ from $\mathbf{H}$ into the matrix ring is determined by $\mathbf{Q}(i)$-linearity and the facts that ξ corresponds to the choice $c_2 = 1$, $c_1 = c_3 = c_4 = 0$, and j corresponds to the choice $c_3 = 1$, $c_1 = c_2 = c_4 = 0$. In particular the determinants of these matrices are non-zero whenever at least one of the coefficients c_1, c_2, c_3, c_4 is non-zero. ■

Remark 2.1: The algebra $\mathbf{H}$ could also be viewed as cyclic division algebra in the sense of [11]. As it is a subring of the Hamiltonian quaternions, its center consists of the intersection $\mathbf{H} \cap \mathbf{R} = \mathbf{Q}(\sqrt{2})$. Also $\mathbf{Q}(\xi)$ is an example of a splitting field of $\mathbf{H}$. In the notation of section 7 of [11] we have an obvious isomorphism

$$\mathbf{H} \simeq (\mathbf{Q}(\xi)/\mathbf{Q}(\sqrt{2}), \sigma, -1),$$

where σ is the usual complex conjugation.

In order to get STBC-lattices and useful bounds for the minimum determinant we need to identify suitable subrings R of the algebra $\mathbf{H}$. We shall do this by placing certain restrictions for the elements c_1, c_2, c_3, c_4.

In the case of the field L we are only interested in its ring of integers $\mathcal{O}_L = \mathbf{Z}[\zeta]$ that is a free $\mathcal{G}$-module with basis $\mathcal{B}_L$. In this case the ring $\phi(\mathcal{O}_L)$ consists of those matrices of L that have all the coefficients $c_1, c_2, c_3, c_4 \in \mathcal{G}$. Similarly, the $\mathcal{G}$-module spanned by our earlier basis $\mathcal{B}_H$ is a ring $\mathcal{L}$ of the required type. We call this ring the ring of Lipschitz' integers of $\mathbf{H}$. Again $\phi(\mathcal{L})$ consists of those matrices of $\mathbf{H}$ that have all the coefficients $c_1, c_2, c_3, c_4 \in \mathcal{G}$. While $\mathcal{O}_L$ is known to be maximal among the rings satisfying our requirements, the same is not true about $\mathcal{L}$. The ring of Hurwitz' integral quaternions also has an extension of the prescribed type inside $\mathbf{H}$. This ring, denoted by $\mathcal{H}$, is the (right) $\mathcal{G}$-module generated by the basis $\mathcal{B}_{Hur} = \{\rho, \rho\xi, j, j\xi\}$, where $\rho = (1 + i + j + k)/2$. The fact that $\mathcal{H}$ is a subring can easily be verified by straightforward computations, e.g. $\xi\rho = \rho\xi - j\xi$. For future use we express the ring $\mathcal{H}$ in terms of the basis $\mathcal{B}_H$ of Proposition 2.1. We easily see that the quaternion $q = c_1 + \xi c_2 + jc_3 + j\xi c_4$ is an element of $\mathcal{H}$, if and only if the coefficients $c_t, t = 1, 2, 3, 4$ satisfy the requirements $(1 + i)c_t \in \mathcal{G}$ for all t and $c_1 + c_3, c_2 + c_4 \in \mathcal{G}$. As the ideal generated by $1 + i$ is of index two in $\mathcal{G}$, we see that $\mathcal{L}$ is an additive subgroup of index four in $\mathcal{H}$. We summarize these findings in the next proposition. The bound on the minimum determinant is a consequence of the fact that all the elements of $\mathcal{G}$ have norm at least 1.

Proposition 2.2: The following rings of matrices form STBC-lattices of minimum determinant 1.

$$L_1 = \left\{ \left(\begin{array}{cccc} c_1 & ic_4 & ic_3 & ic_2 \\ c_2 & c_1 & ic_4 & ic_3 \\ c_3 & c_2 & c_1 & ic_4 \\ c_4 & c_3 & c_2 & c_1 \end{array} \right) | c_1, c_2, c_3, c_4 \in \mathcal{G} \right\},$$

$$L_2 = \{M(c_1, c_2, c_3, c_4) | c_1, c_2, c_3, c_4 \in \mathcal{G}\},$$

$$L_3 = \{M(c_1, c_2, c_3, c_4) | c_1, c_2, c_3, c_4 \in \frac{1+i}{2}\mathcal{G},$$
$$c_1 + c_3 \in \mathcal{G}, c_2 + c_4 \in \mathcal{G}\}.$$

■

Remark 2.2: The lattice L_1 is quite similar to the DAST-lattice in the sense that all of its matrices can be diagonalized simultaneously. The lattice L_2, for its part, is a more developed case from the so-called *quasi-orthogonal* STBC suggested e.g. in [14]. The matrix of L_2 can be found as an example also in [11], but no optimization has been done there by using, for example, ideals as we do here.

A drawback shared by the lattices L_1 and L_2 is that in the ambient space of the transmitter they are isometric to the rectangular lattice $\mathbf{Z}^8$. The rectangular shape does carry the advantage that the sets of information carrying coefficients of the basic matrices are simple and all identical (this is useful in e.g. sphere decoding), but this shape is very wasteful in terms of transmission power. Geometrically denser sublattices of $\mathbf{Z}^8$, e.g. the checkerboard lattice D_8 and the diamond lattice E_8 are well known (cf. e.g. [15]). However, we must be careful when picking the copies of the sublattices, as it is the minimum determinant we want to keep an eye on.

As our earlier simulations ([1],[2]) showed that L_2 outperforms L_1, we concentrate on finding good sublattices of L_2. The units of the ring L_2 are exactly the non-zero matrices, whose determinants have the minimal absolute value of one. Thus an intuitive way to find a sublattice with a better minimum determinant is to take the lattice $\phi(\mathcal{I})$, where $\mathcal{I} \subset R$ is a proper ideal. This idea has appeared in [2] and [12]. Even earlier, ideals of rings of algebraic integers were used in [8] to produce dense lattices. Let us first record the following simple fact. For the proof, see [2].

Lemma 2.3: Let A and B be diagonalizable complex square matrices of the same size. Assume that they commute and that their eigenvalues are all real and non-negative. Then

$$\det(A + B) \geq \det A + \det B,$$

and we have a strict inequality, if both A and B are invertible. ■

Proposition 2.4: Let $\mathcal{I}$ be the prime ideal of the ring $\mathcal{G}$ generated by $1 + i$. Define

$$\mathcal{I}_{\mathcal{L}} = \{(c_1 + \xi c_2) + j(c_3 + \xi c_4) \in \mathcal{L} | c_1 + c_2 + c_3 + c_4 \in \mathcal{I}\}.$$

Then $\mathcal{I}_\mathcal{L}$ is an ideal of index two in $\mathcal{L}$. The corresponding lattice

$$L_4 = \{M \in L_2 | c_1 + c_2 + c_3 + c_4 \in \mathcal{I}\}$$

is a rank 2 sublattice in L_2. Furthermore, the absolute value of $\det(MM^*)$, $M \in L_4 \setminus \{0\}$, is then at least 4.

Proof: It is straightforward to check that $\mathcal{I}_\mathcal{L}$ is stable under (left or right) multiplication with the quaternions ξ and j, so $\mathcal{I}_\mathcal{L}$ is an ideal in $\mathcal{L}$.

Let us consider a matrix $M \in L_4$ and write it in the block form

$$M = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix}.$$

We see that

$$MM^* = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & AA^* + BB^* \end{pmatrix},$$

and

$$AA^* + BB^* = \begin{pmatrix} \alpha & k^* \\ k & \alpha \end{pmatrix},$$

where $\alpha = \sum_{j=1}^4 |c_j|^2$ is a non-negative integer and $k = -ic_1c_2^* + c_2c_1^* - ic_3c_4^* + c_4c_3^*$ is a Gaussian integer with the property $k^* = ik$. We are to prove that $\det MM^* = \left(\alpha^2 - |k|^2\right)^2 \geq 4$. Assume first that $c_3 = c_4 = 0$, i.e. the block $B = 0$. Then $\det(A)$ is the relative norm $\det(A) = N^{\mathbf{Q}(\xi)}_{\mathbf{Q}(i)}(c_1 + \xi c_2)$, which is a Gaussian integer. As $c_1 + \xi c_2$ is a non-zero element of the ideal $\mathcal{I}$, we conclude that $\det(A)$ is a non-zero non-unit. Therefore $\det(A)\det(A^*) \geq 2$, and the claim follows.

Let us then assume that both A and B are non-zero. Then $\det(A)$ and $\det(B)$ are non-zero Gaussian integers and have a norm of at least one. The matrices A, A^*, B, B^* all commute, so by Lemma 2.3 we get

$$\det(MM^*) > \det(AA^*)^2 + \det(BB^*)^2 \geq 2.$$

As $\det(MM^*) = \left(\alpha^2 - |k|^2\right)^2$ is the square of a rational integer, it must be at least 4. ∎

Remark 2.3: It is easy to see that in the previous proposition, $a+bi \in \mathcal{I}$, iff $a+b$ is an even integer. Thus geometrically the matrix lattice L_4 is, indeed, isometric to D_8.

We proceed to describe two more interesting sublattices of L_2 with even better minimum determinants. To that end we use the ring $\mathcal{H}$ (or the lattice L_3). The first sublattice is isometric to the direct sum $D_4 \perp D_4$ of two 4-dimensional checkerboard lattices.

Proposition 2.5: Let again $\mathcal{I}$ be the ideal $(1+i)\mathcal{G}$ and $M(c_1, c_2, c_3, c_4)$ be the matrices of Proposition 2.1. The lattice

$$L_5 = \{M(c_1, c_2, c_3, c_4) \in L_2 | c_1 + c_3, c_2 + c_4 \in \mathcal{I}\}$$

has minimum determinant equal to 2.

Proof: The matrices A in the lattice L_5 are of the form $A = (1+i)M$, where M is a matrix in the lattice L_3 of Proposition 2.2. Thus $\det(AA^*) = 16\det(MM^*)$, so the claim follows from Proposition 2.2. ∎

The root lattice E_8 can be described in terms of Gaussian integers as follows (cf. [16]):

$$E_8 = \frac{1}{1+i}\{(c_1, c_2, c_3, c_4) \in \mathcal{G}^4 | c_1 + \mathcal{I} = c_t + \mathcal{I},$$
$$t = 2, 3, 4, \sum_{t=1}^4 c_t \in 2\mathcal{G}\}.$$

By our identification of quadruples $(c_1, c_2, c_3, c_4) \in \mathcal{G}^4$ and elements of $\mathbf{H}$ it is readily verified that $\Lambda = (1+i)E_8$ has $\{2, (1+i) + (1+i)\xi, (1+i)\xi + (1+i)j, 1 + \xi + j + j\xi\} \subseteq \mathcal{L}$ as a $\mathcal{G}$-basis, whence the set $\{1+i, 1+\xi, \xi+j, \rho+\rho\xi\} \subseteq \mathcal{H}$ is a $\mathcal{G}$-basis for E_8. By another simple computation we see that $E_8 = \mathcal{H}(1+\xi)$, i.e. E_8 is the left ideal of the ring $\mathcal{H}$ generated by $1+\xi$.

Proposition 2.6: The lattice

$$L_6 = \{M(c_1, c_2, c_3, c_4) \in L_2 | \ c_1 + \mathcal{I} = c_t + \mathcal{I},$$
$$t = 2, 3, 4, \sum_{t=1}^4 c_t \in 2\mathcal{G}\}$$

is an index 16 sublattice of L_2. Furthermore, the minimum determinant of L_6 is $2\sqrt{2}$.

Proof: Let $M_I = M(1,1,0,0)$ be the matrix $\phi(1+\xi)$ under the isomorphism of Proposition 2.1. We see that $\det(M_I M_I^*) = 4$. By the preceding discussion any matrix A of the lattice L_6 is of the form $A = MM_I(1+i)$, where M is a matrix from $\mathcal{H}$. As in the proof of Proposition 2.5, we see that $\det AA^* = 16\det(M_I M_I^*)\det(MM^*)$. Therefore the claim on the minimum distance follows from Proposition 2.2. We see that the coefficient c_1 can be chosen arbitrarily within $\mathcal{G}$. The coefficients c_2 and c_3 then must belong to the coset $c_1 + \mathcal{I}$, and c_4 must be chosen such that $c_1 + c_2 + c_3 + c_4 \in 2\mathcal{G} = \mathcal{I}^2$. As $\mathcal{I}$ is of index two in $\mathcal{G}$, we see that the index of L_6 in L_2 is 16 as claimed. ∎

Remark 2.4: We have now produced a nested sequence of lattices

$$2\mathbf{Z}^8 = 2L_2 \subseteq L_6 \subseteq L_5 \subseteq L_4 \subseteq L_2 = \mathbf{Z}^8 (\subseteq L_3).$$

We concentrate on the lattices that are sandwiched between $2\mathbf{Z}^8$ and $\mathbf{Z}^8$. Such lattices are in a bijective correspondence with binary linear code of length 8 by "projection modulo 2". As it happens, within this sequence of lattices the minimum Hamming distance of the binary linear code and the minimum determinant of the lattice are somewhat related.

The 8-dimensional rectangular grid $\mathbf{Z}^8$ (no coding)
↓
The checkerboard lattice D_8 (↔ overall parity check code of length 8)
↓
The lattice $D_4 \perp D_4$ (↔ two blocks of the overall parity check code of length 4)
↓
The root lattice E_8 (↔ extended Hamming-code of length 8).

The obvious question now is what happens if we simply concatenate the use of L_2 with a good binary code (extended over several L_2-blocks, if need be), and then were done with it. While the binary linear codes appearing above are the first ones that come to mind, we want to caution the unwary end-user. Namely, the ring in question may contain high weight units. If such binary words are included, then the minimum determinant of the corresponding lattice is equal to 1, i.e. no coding gain will take place. E.g. the unit $(1-\xi^3)/(1-\xi) = 1+\xi+\xi^2 = (1+i)+\xi$ of the ring $\mathcal{L}$ corresponds to the matrix $M(1+i,1,0,0)$ of determinant 1, and thus we must not allow such words of weight 3 to be included in the code. If the lattice L_1 were used, the situation would be even worse, as then we would have units like $(1-\zeta^7)/(1-\zeta)$ in the ring $\mathcal{O}_L$ that would be mapped to a word of Hamming weight 7. A construction based on ideals provides a mechanism to avoid this problem caused by units.

III. Simulation results

We summarize the findings of Propositions 2.2–2.6 in the following.

Proposition 3.1: (1) The lattice L_2 is isometric to the rectangular lattice $\mathbf{Z}^8$ and has minimum determinant 1.

(2) The lattice L_4 is an index two sublattice of L_2 and has minimum determinant $\sqrt{2}$.

(3) The lattice L_5 is an index four sublattice of L_2 and has minimum determinant 2.

(4) The lattice L_6 is an index 16 sublattice of L_2 and has minimum determinant $2\sqrt{2}$. ■

In order to compare these lattices we scale them to the same minimum determinant. When a real scaling factor ρ is used the minimum determinant is multiplied by ρ^2. As all the lattices have rank 8, the fundamental volume is then multiplied by ρ^8. Let us choose the units so that the fundamental volume of L_2 is $m(L_2) = 1$. Then after scaling $m(L_4) = 1/2$, $m(L_5) = 1/4$ and $m(L_6) = 1/4$. As the density of a lattice is inversely proportional to the fundamental volume, we thus expect the codes constructed within e.g. the lattices L_4 and L_6 to outperform the codes of the same size within L_2.

Figure 1 shows the block error rates of the various competing lattice codes at the rate 2 bits/s/Hz, i.e. all the codes contain 256 matrices. For the lattices L_1, L_2, L_{DAST} and L_{ABBA} [17] this simply amounted to letting the coefficients c_1, c_2, c_3, c_4 take all the values in the QPSK-alphabet. Therefore, it would have been easy to obtain bit error rates as well. For the lattices L_4, L_5 and L_6 a more or less random set of 256 shortest vectors was chosen. As there is no natural way to assign bit patterns to vectors of D_8, $D_4 \perp D_4$ or E_8, we chose to show the block error rates instead of the bit error rates. Figure 1 shows that the lattice L_6 wins over all the other lattices.

The simulations were set up here so that the 95 per cent reliability range amounts to a relative error of about 3 per cent at the low SNR end, and to about 10 per cent at the high SNR end (or to about 4000 and 400 error events respectively). One receiver was used for all the lattices.

Figure 2 shows the block error rates of the code within L_6 and the Golden code [12] at the rate 4 bits/s/Hz with two receivers. At the rate 4 bits/s/Hz one block of our code consists of 16 bits, whereas one block of the Golden code carries 8 bits only. For that reason we decided to show the error rate of two consecutive blocks of the Golden code; i.e. if the usual error rate of the block of length two is p, the rate we show is $2p-p^2$. To maintain the quasi-static channel assumption, the channel matrix was changed only after every fourth time slot for both codes.

We can conclude that the lattice L_6 outperforms the Golden code when SNR reaches about 13 dB. However, this is an unfair comparison because our code uses four transmit antennas while the Golden code uses only two — this is just a manifestation of the diversity gain, but we were interested in finding the approximate crossing point. The fact that the Golden code triumphs over our lattice at the low SNR end is not such a severe drawback either, since our codes are designed mainly for MISO channels while the Golden code is intended wholly for MIMO channels.

IV. Conclusions and suggestions for further research

In this paper, we present new constructions of rate one, full diversity, and energy efficient 4×4 space-time codes by using the theory of rings of algebraic integers and their counterparts within the division rings of Lipschitz' and Hurwitz' integral quaternions. A comfortable, purely number theoretic way to improve space-time lattice constellations is introduced. The use of ideals provides us denser lattices and an easy way to present the exact proofs for the minimum determinants. The constructions can be extended to a larger number of transmit antennas and they fit nicely with the popular Q^2-QAM and QPSK modulation alphabets.

Comparisons with DAST-code show that our codes provide lower energy and block error rates due to their good minimum determinant and high density. Despite the fact that our codes are mainly designed to use only one receiver antenna, comparisons with the Golden code give hope that, with some slight changes, the ideas of this paper will also work with multiple receivers. For that reason, our next goal is to improve these ideas and codes so that they would perform well also in MIMO channels. At the moment we are searching for well-performing MIMO codes arising from the theory of crossed product algebras and maximal orders of cyclic division algebras. As a matter of fact, we have already proved that our densest lattice code L_6 corresponds to a maximal order within the cyclic division algebra $\mathbf{H}$.

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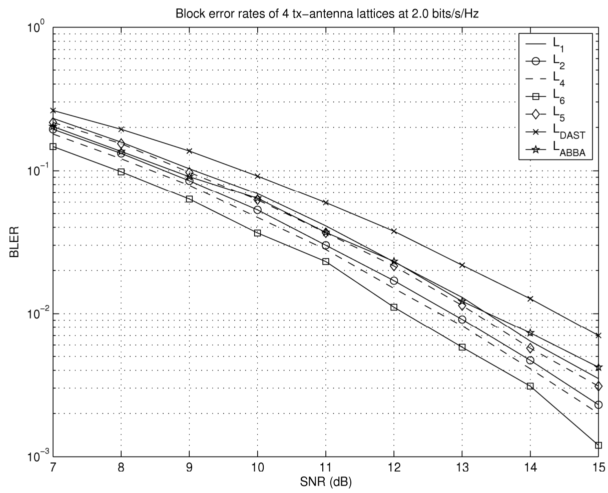

Fig. 1. Block error rates of 4 tx-antenna lattices at 2.0 bits/s/Hz

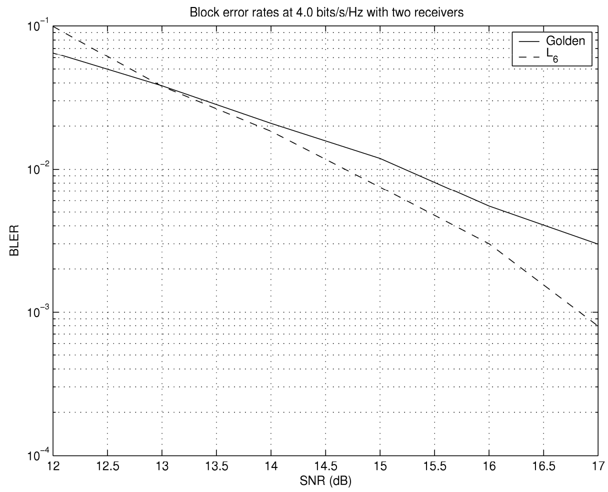

Fig. 2. Block error rates at 4.0 bits/s/Hz with two receivers

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