# An Algebraic Tool for Obtaining Conditional Non-Vanishing Determinants

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*Abstract*— **An algebraic tool from the theory of central simple algebras is proposed to obtain families of complex matrices satisfying the conditional non-vanishing determinant (CNVD) property. Such property is useful in e.g. the design of multiuser space-time (ST) codes, in which context it is not always crucial for the transmission matrix to be invertible, but whenever it IS invertible, it is important that it has a non-vanishing determinant. Also any submatrix of any subset of users multiplied with its transpose conjugate should preferably have a non-vanishing determinant, provided it is non-zero. In a recent submission by Lu *et al.* it was shown that, albeit not alone, such property yields a construction of two-user space-time codes that achieve the optimal diversity-multiplexing tradeoff (DMT) and outperform the previously known two-user ST codes.**

## I. Introduction

During the past five years extensive research has been carried out on single-user (SU) multiple-input multiple-output (MIMO) space-time lattice codes based on cyclic division algebras (CDAs) [1], [2], [3]. At its best, this research has resulted in codes that get very close to the outage capacity, in particular the so-called maximal order codes [4]. Motivated by the promising outcome in the SU-MIMO scenario, the aim in [5] was to adapt the machinery provided by CDAs to the multiuser (MU) MIMO scenario as well, with the ultimate goal of producing diversity-multiplexing tradeoff (DMT) achieving codes in mind. One of the crucial properties the code should satisfy in order to achieve the optimal DMT turned out to the so-called conditional non-vanishing determinant property.

For other works on multiuser codes and the design criteria and DMT for such codes, see [6], [7], [8], [9], [10], [11].

The main goals in [5] were to construct explicit, sphere-decodable codes for the $(2 \times 2)$ situation, where both of the two users are equipped with two transmitting antennas, and two antennas are available at the receiving end, and further to design a general, DMT-achieving, sphere-decodable $(n_t \times n_r)$ MU-MIMO scheme for two users, that would yield good performance also at the low SNR end.

This paper will concentrate on the question as to how to obtain the CNVD property which is a necessary, albeit not alone sufficient, condition for achieving good performance and the optimal DMT. Hence, we shall present an algebraic tool coming from the theory of central simple algebras, telling us that certain types of matrices always yield the CNVD property. The main motivation for this paper came from the fact that, with the proposed algebraic Center argument tool, we are able to avoid lengthy calculations that were needed for proving the CNVD property for the codes proposed in [5]. The Center argument generalizes to maximal orders as well, thus being the first method (according to the best of the authors' knowledge) with which one can prove the CNVD property for maximal order multiuser ST codes. For the use of matrix representations of cyclic algebras and their orders as space-time codes in general, we refer the reader to [12], [13].

## II. Cyclic division algebras and orders

In this section we introduce some concepts and results from the theory of central simple algebras for later use. For the proofs of these results and for a proper introduction we refer the reader to [14].

In the rest of the paper we assume that all the fields are finite extensions of the field $\mathbf{Q}$.

**Definition II.1.** Let $K$ be an algebraic number field and assume that $E/K$ is a cyclic Galois extension of degree $n$ with the Galois group $\mathrm{Gal}(E/K) = \langle\sigma\rangle$. We can now define an associative $K$-algebra

$$\mathscr{A} = (E/K, \sigma, \gamma) = E \oplus uE \oplus u^2E \oplus \cdots \oplus u^{n-1}E,$$

where $u \in \mathscr{A}$ is an auxiliary generating element subject to the relations $xu = u\sigma(x)$ for all $x \in E$ and $u^n = \gamma \in K^*$.

This work was supported in part by the Emil Aaltonen Foundation, Finland, and the Finnish Academy of Science and Letters, Finland, through grants for R. Vehkalahti and C. Hollanti.

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We call this type of algebra a *cyclic algebra* and the field $K$ the *center* of the algebra. The center is the set of elements of $\mathscr{A}$ that commute with all the elements of $\mathscr{A}$. Throughout the paper, $K$ denotes the center, and $F$ denotes its subfield $F \subseteq K$. The inclusion may also be trivial, i.e., we allow $K = F$.

**Definition II.2.** An algebra $\mathscr{A}$ is called *simple* if it has no non-trivial ideals. A $K$-algebra $\mathscr{A}$ is *central* if its center $Z(\mathscr{A}) = \{a \in \mathscr{A} \mid aa' = a'a \ \forall a' \in \mathscr{A}\} = K$.

**Definition II.3.** A *central simple* $K$-algebra is a simple algebra which is finite dimensional over its center $K$.

**Proposition II.1.** *Every cyclic algebra is central simple.*

**Definition II.4.** A cyclic algebra is a division algebra if and only if all the non-zero elements of the algebra are invertible.

**Proposition II.2** (Norm Condition)**.** *The cyclic algebra $\mathscr{A} = (E/K, \sigma, \gamma)$ of degree $n$ is a division algebra if and only if the smallest factor $t \in \mathbf{Z}_+$ of $n$ such that $\gamma^t$ is the norm of some element of $E^*$ is $n$.*

Due to the above proposition, the element $\gamma$ is often referred to as the *non-norm element*.

**Definition II.5.** Let $\mathscr{A}$ be a $K$-central division algebra. We then call $\sqrt{[\mathscr{A} : K]}$ the index of the algebra.

**Example II.1.** Let $\mathscr{A} = (E/K, \sigma, \gamma)$ be a cyclic division algebra and let $\gamma \in \mathscr{O}_K^*$ be an algebraic integer of $K$. We immediately see that the $\mathscr{O}_K$-module

$$\Lambda = \mathscr{O}_E \oplus u\mathscr{O}_E \oplus \cdots \oplus u^{n-1}\mathscr{O}_E,$$

where $\mathscr{O}_E$ is the ring of integers of $E$, is a subring in the cyclic algebra $(E/K, \sigma, \gamma)$. We refer to this ring as the *natural order*. Note also that if $\gamma$ is not an algebraic integer, then $\Lambda$ fails to be closed under multiplication.

Let $K/F$ be a finite extension (could be also the trivial extension) of algebraic number fields and $\mathscr{A}$ a $K$-central division algebra of degree $n$.

**Definition II.6.** An $\mathscr{O}_F$-order $\Lambda$ in $\mathscr{A}$ is a subring of $\mathscr{A}$, having the same identity element as $\mathscr{A}$, and such that $\Lambda$ is a finitely generated module over $\mathscr{O}_F$ and generates $\mathscr{A}$ as a linear space over $F$.

**Definition II.7.** An $\mathscr{O}_F$-order $\Lambda$ is called *maximal*, if it is not properly contained in any other $\mathscr{O}_F$-order.

**Proposition II.3.** *Any $K$-central division algebra $\mathscr{A}$ has a maximal $\mathscr{O}_F$-order, and any order inside $\mathscr{A}$ is contained in at least one maximal order.*

**Example II.2.** Suppose that $E/K$ is a cyclic extension of algebraic number fields. Let $\mathscr{A} = (E/K, \sigma, \gamma)$ be a cyclic division algebra.

We can consider $\mathscr{A}$ as a right vector space over $E$, and every element $a = x_0 + ux_1 + \cdots + u^{n-1}x_{n-1} \in \mathscr{A}$ has the following representation as a matrix

$$A = \begin{pmatrix} x_0 & \gamma\sigma(x_{n-1}) & \gamma\sigma^2(x_{n-2}) & \cdots & \gamma\sigma^{n-1}(x_1) \\ x_1 & \sigma(x_0) & \gamma\sigma^2(x_{n-1}) & & \gamma\sigma^{n-1}(x_2) \\ \vdots & & & & \vdots \\ x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_{n-3}) & \cdots & \sigma^{n-1}(x_0) \end{pmatrix}.$$

We call this representation the *left regular representation* and denote $A = \psi(a)$. We often identify an element $a$ with its representation $\psi(a)$.

**Definition II.8.** The determinant (resp. trace) of the matrix $A$ above is called the *reduced norm* (resp. *reduced trace*) of the element $c \in \mathscr{A}$ and is denoted by $nr_{\mathscr{A}/K}(c)$ (resp. $tr_{\mathscr{A}/K}(c)$).

**Proposition II.4.** *Let $\mathscr{A}$ be a $K$-central division algebra and $a$ an element of $\mathscr{A}$. Then $nr(a)$ and $tr(a) \in K$.*

**Proposition II.5.** *The norm and trace maps do not depend on the maximal representation, i.e. the left regular representation is not the only representation we can use. However, we stick to $\psi$ for simplicity.*

**Definition II.9.** We then define the reduced trace and norm of $a$ to $F$ by

$$tr_{\mathscr{A}/F}(a) = tr_{K/F}(tr_{\mathscr{A}/K}(a))$$

$$nr_{\mathscr{A}/F}(a) = nr_{K/F}(nr_{\mathscr{A}/K}(a)).$$

where $nr_{K/F}$ and $tr_{K/F}$ are the usual relative norm and trace maps of a number field extension (sometimes also denoted by $\mathrm{N}_{K/F}$ and $\mathrm{T}_{K/F}$).

**Proposition II.6.** *Let $\Lambda$ be an $\mathscr{O}_F$-order in a $K$-central division algebra $\mathscr{A}$. Then for any element $a \in \Lambda$ its reduced norm $nr_{\mathscr{A}/F}(a)$ and reduced trace $tr_{\mathscr{A}/F}(a)$ are elements of the ring of integers $\mathscr{O}_F$ of the field $F$. If $a$ is non-zero, then so is $nr_{\mathscr{A}/F}(a)$.*

In [4], [15] it has been shown that the discriminant of an order is closely related to the density of the resulting code lattice, and should be minimized (see the remark below) in order to achieve the best coding gains. Let us now recall the notion of a discriminant of an order.

**Definition II.10.** Let $\mathscr{A}$ be a $K$-central division algebra and $m = dim_F\mathscr{A}$. The $\mathscr{O}_F$*-discriminant* of the $\mathscr{O}_F$-order $\Lambda$ is the ideal $d(\Lambda/\mathscr{O}_F)$ in $\mathscr{O}_F$ generated by the set

$$\{\det(tr_{\mathscr{A}/F}(x_i x_j))_{i,j=1}^m \mid (x_1, \ldots, x_m) \in \Lambda^m\}.$$

If $\Lambda$ is a free $\mathcal{O}_F$-module, then

$$d(\Lambda/\mathcal{O}_F) = \det(tr(x_i x_j))_{i,j=1}^{m}, \tag{1}$$

where $\{x_1,\ldots,x_m\}$ is any $\mathcal{O}_F$-basis of $\Lambda$.

**Remark II.1.** For the purposes of MIMO coding, we may assume here that $F$ is a principal ideal domain. Then the $\mathcal{O}_F$-discriminant is as given in (1), and by 'minimizing' we mean the minimization of $|d(\Lambda/\mathcal{O}_F)|$.

**Proposition II.7.** *All the maximal orders of a K-central division algebra share the same discriminant. The maximal orders have the smallest discriminant among all the orders within a given division algebra.*

Now we can define the following.

**Definition II.11.** Let $\mathscr{A}$ be a $K$-central division algebra and let $\Lambda$ be some maximal order in $\mathscr{A}$. Then we refer to $d(\Lambda/\mathcal{O}_K) = d_{\mathscr{A}}$ as the *discriminant of the algebra* $\mathscr{A}$.

The following lemma connects the discriminants $d(\Lambda/\mathcal{O}_K)$ and $d(\Lambda/\mathcal{O}_F)$.

**Lemma II.8.** *Let $\mathscr{A}$ be a K-central division algebra of index n and let $\Lambda$ be an $\mathcal{O}_K$-order. If $\Lambda$ is an $\mathcal{O}_F$-order in $\mathscr{A}$, then*

$$d(\Lambda/\mathcal{O}_F) = nr_{K/F}(d(\Lambda/\mathcal{O}_K))d(\mathcal{O}_K/\mathcal{O}_F)^{n^2}.$$

**Remark II.2.** In order to achieve the best coding gains that order codes can offer, one should use maximal orders instead of natural ones [13].

## III. The Center Argument: A tool for obtaining the conditional non-vanishing determinant property

Throughout this section, we assume that each of the $K$ users is equipped with $n_t$ transmit antennas. To avoid confusion, we will denote the center of the algebra by $F$ instead of $K$.

**Definition III.1.** If there exists a fixed positive constant $k$ independent of the size of the code $\mathscr{C}$ such that

$$\min_{S\in\mathscr{C},\det(S)\neq 0} |\det(S)| = k > 0,$$

we say that the space-time code $\mathscr{C}$ (and the matrix $S$) has the *conditional non-vanishing determinant (CNVD) property*.

The following proposition states basic but crucial properties of finite dimensional central simple algebras [16, p. 215], [14].

**Proposition III.1** (Center Argument)**.** *Let $\mathscr{A} = \mathscr{M}_n(\mathfrak{D})$ be a finite dimensional simple algebra, where $\mathfrak{D}$ is a finite dimensional division algebra. Now $\mathscr{A}$ is central simple over its center F, and the center is the same for $\mathscr{A}$ as for $\mathfrak{D}$. The norm of an element S of the matrix algebra $\mathscr{A}$ is the determinant of the matrix S. Hence,* $\det(S)\in F$*, and further* $\det(S)\in\mathcal{O}_F$*, when we are using an $\mathcal{O}_F$-order $\Lambda\subseteq\mathfrak{D}$. If $\mathcal{O}_F$ is either $\mathbf{Z}$ or $\mathbf{Z}[\sqrt{-m}]$, we get that* $\det(S)\geq 1$.

**Remark III.1.** There are a couple of things to be aware of.

1) To be able to say something about orders for a matrix algebra, we have to notice that the orders of $\mathscr{M}_n(\mathfrak{D})$ are of the form $\mathscr{M}_n(\Lambda)$, where $\Lambda$ is an order of $\mathfrak{D}$. Thus, if we choose the $\mathfrak{D}$-blocks from an order $\Lambda\subset\mathfrak{D}$, we get an order $\Pi = \mathscr{M}_n(\Lambda)\subset\mathscr{A}$. If $\Lambda$ is maximal, so is $\Pi$.

2) The center for $\mathscr{A}$ is the same as for $\mathfrak{D}$. This means that we want the center to be either $\mathbf{Q}$ or some quadratic imaginary number field $\mathbf{Q}(\sqrt{-m})$, $m$ positive and square-free, as otherwise we would end up with a vanishing determinant (or at least cannot prove the CNVD property with the proposed tool III.1).

Let us then consider the following form of transmission matrix involving $K$ users,

$$S = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1K} \\ \vdots & & & \vdots \\ A_{K1} & A_{K2} & \cdots & A_{KK} \end{pmatrix}_{Kn_t\times Kn_t} \in\mathscr{M}_K(\Lambda), \tag{2}$$

where $\Lambda$ is an $\mathcal{O}_F$-order of the index $n_t$ cyclic division algebra $\mathfrak{D} = (E/F,\sigma,\gamma\in\mathcal{O}_F)$, and the submatrices $A_{ij}\in\Lambda$ are like in Example II.2.

The next lemma from [17] is useful when searching for bounds for determinants of positive definite matrices.

**Lemma III.2** (Minkowski Determinant Inequality)**.** *Let $A_1,\ldots,A_k$ be complex $n\times n$ matrices. Consider the $n\times nk$ matrix $(A_1,A_2,\ldots,A_k) = A$. Now it holds that*

$$(\det(AA^\dagger)) \geq \sum_{i=1}^{k} |det(A_i)|^2.$$

**Proposition III.3.** *(i) A space-time code consisting of matrices of the form* (2) *has the conditional non-vanishing determinant property, provided that the center F of $\mathfrak{D}$ is either the field of rationals $\mathbf{Q}$ or an imaginary quadratic number field $\mathbf{Q}(\sqrt{-m})$, where m is a positive, square-free integer.*

*(ii) For a submatrix $X = (A_{i1}\cdots A_{iK})$ corresponding to a single user,* $\det(XX^\dagger)$ *also has the CNVD property.*

*(iii) In the case when E is closed under complex conjugation and $\gamma = -1$, for a submatrix X of S corresponding to any subset of h $(h = 1,...,K)$ users,* $\det(XX^\dagger)$ *satisfies the CNVD property.*

*Proof:* (i) Let us look at the algebra consisting of the matrices $S$ that contain $K^2$ $n_t \times n_t$ CDA blocks from the division algebra $\mathfrak{D}$. We can consider this matrix not only as a matrix consisting of 'random' blocks coming from the division algebra $\mathfrak{D}$, but also as a matrix algebra $\mathscr{M}_K(\mathfrak{D})$. Whereas on the one hand the matrix $S$ is just a mess of $\mathfrak{D}$-blocks (this being the reason why $\mathscr{M}_K(\mathfrak{D})$ is not a *division* algebra), on the other hand it is also an element in the algebra $\mathscr{M}_K(\mathfrak{D})$. The Center argument III.1 now gives us that $\det(S) \geq 1$.

(ii) For a single user, we can use the Minkowski determinant inequality from Lemma III.2. It tells us that for the $i$th user, $\det(XX^\dagger) = \det(\sum_j A_{ij}A_{ij}^\dagger) \geq \sum_j \det(A_{i1}A_{i1}^\dagger) = \sum_j |\det(A_{i1})|^2 \geq K$, provided that $\det(XX^\dagger) \neq 0$.

(iii) For $h = 2,...,K-1$, we easily see that we must have $A_{ij}^\dagger \in \mathfrak{D}$, which is satisfied only when both the transpose and complex conjugate of $A_{ij}$ are in $\mathfrak{D}$. By looking at the matrix representation, the transpose condition implies $\gamma^2 = 1$. □

**Remark III.2.** Having $\gamma = -1$ implies that $n_t = 2$ as otherwise $-1$ is always a norm element. However, this does not place any restrictions on the number of users $K$, hence the situation still remains entirely practical.

Let us now consider a two-user transmission matrix

$$S = \begin{pmatrix} A_1 & \tau(A_1) \\ A_2 & \tau(A_2) \end{pmatrix}_{2n_t \times 2n_t}, \tag{3}$$

where $A_i \in \Lambda \subseteq \mathfrak{D} = (E/F, \sigma, \gamma)$ and $\mathfrak{D}$ is a cyclic division algebra of index $n_t$, $\Lambda$ its order, $\gamma \in \mathscr{O}_F^*$, $[F : \mathbf{Q}] = 2$ and $\mathrm{Gal}(F/\mathbf{Q}) = \langle \tau \rangle$.

**Proposition III.4.** *Let $F$ be an imaginary quadratic number field $\mathbf{Q}(\sqrt{-m})$, where $m$ is a positive, square-free integer. Let us assume that the field $E$ is closed under $\tau$ (which is now just complex conjugation), and further that $\tau\sigma = \sigma\tau$. Then, a space-time code consisting of matrices of the form* (3) *has the conditional non-vanishing determinant property. Also the single user submatrix has the CNVD property.*

*Proof:* As $E$ is closed under $\tau$, we get that $\tau^i(A_j) \in \mathfrak{D}$ for all $i, j$. In particular, let now $A = A(x_0, x_1) \in \mathfrak{D}$. As $\tau\sigma = \sigma\tau$ and $\gamma \in F$, the matrix $\tau(A(x_0, x_1)) = A(\tau(x_0), \tau(x_1))$, which is again in $\mathfrak{D}$. From here, the proof goes as for Proposition III.3.

For a single user, the proof is exactly the same as in Proposition III.3, as $\det(\tau(A_i)) = \tau(\det(A_i))$, and $\tau$ is nothing but complex conjugation. Hence $|\det(\tau(A))|^2 = |\det(A)|^2$. □

**Remark III.3.** It can be shown with different methods that even when $K > 2$, i.e. the center $F$ has degree greater than two over $\mathbf{Q}$, a generalized form of (3)

$$S = \begin{pmatrix} A_1 & \cdots & \tau(A_K) \\ \vdots & & \vdots \\ A_2 & \cdots & \tau(A_K) \end{pmatrix}_{Kn_t \times Kn_t} \tag{4}$$

will result in codes satisfying the CNVD property for the whole matrix $S$ and for any of its $l$-user $(l = 1,...,K-1)$ submatrices. In [5], a DMT-optimal construction of this form was first proposed for the two-user case, and in another ISIT 2009 submission [18] for general $K$ users. In both cases, the proof for the DMT-optimality requires the CNVD property.

**Remark III.4.** Similar results are possible to achieve without the Center argument, but with brute force calculations one usually needs the fact that we use the ring of integers of the maximal subfield $E$, and hence the outcome is valid only for the natural order, not for arbitrary orders. As maximal orders are to be preferred over natural orders due to the higher density they provide, we feel that this Center argument tool (perhaps with some extensions or modifications) might be of good use in the design of multiuser space-time codes.

## IV. Example codes having the CNVD property for any subset of users

The first of the following two subsections features a code constructed from a maximal order or the Golden+ algebra [4]. There, $\gamma \neq -1$ so if we wish to achieve the CNVD property for any subset of users, we have to restrict to $K = 2$.

The latter subsection, for its part, describes a $K$-user code taking advantage of the Silver algebra [19]. The Silver code [20] has performance only slightly worse than that of the Golden or Golden+ code, but it is simpler to decode as it can be constructed with the aid of two Alamouti blocks. Similar methods can be used for the present multiuser code without losing the CNVD property.

Both of the described algebras have index two, hence $n_t = 2$ for both example codes.

### A. A two-user code from the Golden+ algebra

Let us consider the Golden+ division algebra $\mathfrak{D} = (\mathbf{Q}(s = \sqrt{2+i})/\mathbf{Q}(i), \sigma, \gamma = i)$, where $\sigma(s) = -s$. Let $\Lambda$ be a maximal order of $\mathfrak{D}$. For an explicit description of the basis of this order, see [4], [13]. Next, we form a code $\mathscr{C}$ as a finite subset of this order,

$$\mathscr{C} \subset \left\{ S = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{4\times 4} \in \mathscr{M}_2(\Lambda) \right\},$$

where

$$A_{ij}=\psi(x_{ij}=x_{0ij}+ux_{1ij})=\begin{pmatrix} x_{0ij} & \gamma\sigma(x_{1ij}) \\ x_{1ij} & \sigma(x_{0ij}) \end{pmatrix},\ x_{ij}\in\Lambda.$$

According to Proposition III.3, the code admits the CNVD property for both one user and two users.

### *B. A K-user code from the Silver algebra*

In [19] it was shown that the Silver code is a subset in the cyclic division algebra $\mathfrak{D}=(\mathbf{Q}(\sqrt{-7},i)/\mathbf{Q}(\sqrt{-7}),\sigma,\gamma=-1)$, where $\sigma(i)=-i$ and $\sigma(\sqrt{7})=-\sqrt{7}$. Let now $\Lambda\subseteq\mathfrak{D}$ be an order (a scaled version of the natural order if one wishes to have simple decoding, see [19] for more details), and $x_{ij}\in\Lambda,\ i,j\in\{1,...,K\}$. Now consider the code

$$\mathscr{C}\subset\left\{S=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1K} \\ \vdots & & & \vdots \\ A_{K1} & A_{K2} & \cdots & A_{KK} \end{pmatrix}_{2K\times 2K}\in\mathscr{M}_K(\Lambda)\right\},$$

where $A_{ij}=\psi(x_{ij})$ (cf. the previous example).

The code satisfies all the conditions of Proposition III.3, hence it has the CNVD property for any number of users $K$, and also for any subset of $l$ users, $l=1,...,K-1$.

## V. Conclusions

We have proposed an algebraic tool, called the Center argument, with the help of which one can obtain the conditional NVD property by using certain types of matrix algebras explicitly specified in this paper. Already earlier [5], the CNVD property has turned out to be significant in order to achieve good performance and the DMT-optimality for multiuser ST codes. The previous methods for proving the CNVD, however, have involved lengthy calculations. The new tool does not apply to all multiuser codes, but we have shown that it significantly shortens the CNVD proofs whenever it can be applied. Also, previous tools have required the use of a natural order, whereas the Center argument is valid for any order and thus generalizes the previous constructions. With the Center argument, also maximal orders can be taken advantage of.

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