# The Ordered Set of Rough Sets

Jouni Järvinen

Turku Centre for Computer Science (TUCS)
Lemminkäisenkatu 14 A, FIN-20520 Turku, Finland
jjarvine@cs.utu.fi

**Abstract.** We study the ordered set of rough sets determined by relations which are not necessarily reflexive, symmetric, or transitive. We show that for tolerances and transitive binary relations the set of rough sets is not necessarily even a semilattice. We also prove that the set of rough sets determined by a symmetric and transitive binary relation forms a complete Stone lattice. Furthermore, for the ordered sets of rough sets that are not necessarily lattices we present some possible canonical completions.

## 1 Different Types of Indiscernibility Relations

The rough set theory introduced by Pawlak (1982) deals with situations in which the objects of a certain universe of discourse $U$ can be identified only within the limits determined by the knowledge represented by a given indiscernibility relation. Based on such indiscernibility relation the lower and the upper approximation of subsets of $U$ may be defined. The lower and the upper approximation of a subset $X$ of $U$ can be viewed as the sets of elements which certainly and possibly belong to $X$, respectively.

Usually it is presumed that indiscernibility relations are equivalences. However, some authors, for example, Järvinen (2001), Pomykała (2002), and Skowron and Stepaniuk (1996) have studied approximation operators which are defined by tolerances. Slowinski and Vanderpooten (2000) have studied approximation operators defined by reflexive binary relations, and Greco, Matarazzo, and Slowinski (2000) considered approximations based on reflexive and transitive relations. Yao and Lin (1996) have studied approximations determined by arbitrary binary relations, and in a recent survey Düntsch and Gediga (2003) explored various types of approximation operators based on binary relations. Furthermore, Cattaneo (1998) and Järvinen (2002), for instance, have studied approximation operations in a more general lattice-theoretical setting.

The structure of the ordered set of rough sets defined by equivalences was examined by Gehrke and Walker (1992), Iwiński (1987), and J. Pomykała and J.A. Pomykała (1988). In this work we study the structure of the ordered sets of rough sets based on indiscernibility relations which are not necessarily reflexive, symmetric, or transitive.

## 2 Lattices and Orders

Here we recall some basic notions of lattice theory which can be found, for example, in the books by Davey and Priestly (2002) and Grätzer (1998). A binary relation $\leq$ on a

S. Tsumoto et al. (Eds.): RSCTC 2004, LNAI 3066, pp. 49–58, 2004.

set $P$ is called an *order*, if it is reflexive, antisymmetric, and transitive. An *ordered set* is a pair $\mathcal{P} = (P, \leq)$, with $P$ being a set and $\leq$ an order on $P$.

Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be two ordered sets. A map $\varphi\colon P \to Q$ is an *order-embedding*, if $a \leq b$ in $\mathcal{P}$ if and only if $\varphi(a) \leq \varphi(b)$ in $\mathcal{Q}$. An order-embedding $\varphi$ onto $Q$ is called an *order-isomorphism* between $\mathcal{P}$ and $\mathcal{Q}$. When there exists an order-isomorphism between $\mathcal{P}$ and $\mathcal{Q}$, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *order-isomorphic* and write $\mathcal{P} \cong \mathcal{Q}$.

An ordered set $\mathcal{P} = (P, \leq)$ is a *lattice*, if for any two elements $x$ and $y$ in $P$, the *join* $x \vee y$ and the *meet* $x \wedge y$ always exist. The ordered set $\mathcal{P}$ is called a *complete lattice* if the *join* $\bigvee S$ and the *meet* $\bigwedge S$ exist for any subset $S$ of $P$. The greatest element of $\mathcal{P}$, if it exists, is called the *unit* element and it is denoted by 1. Dually, the smallest element 0 is called the *zero* element. An ordered set is *bounded* if it has a zero and a unit.

A lattice $\mathcal{P} = (P, \leq)$ is *distributive* if it satisfies the conditions

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all $x, y, z \in P$. Let $\mathcal{P} = (P, \leq)$ be a bounded lattice. An element $x' \in P$ is a *complement* of $x \in P$, if $x' \vee x = 1$ and $x' \wedge x = 0$. A bounded lattice is a *Boolean lattice* if it is complemented and distributive.

*Example 1.* If $X$ is any set and $\mathcal{P} = (P, \leq)$ is an ordered set, we may order the set $P^X$ of all maps from $X$ to $P$ by the *pointwise order*:

$$f \leq g \text{ in } P^X \overset{\text{def}}{\iff} (\forall x \in P)\ f(x) \leq g(x) \text{ in } P.$$

We denote by $\mathbf{2}$ and $\mathbf{3}$ the chains obtained by ordering the sets $\{0, 1\}$ and $\{0, u, 1\}$ so that $0 < 1$ and $0 < u < 1$, respectively.

Let us denote by $\wp(U)$ the set of all subsets of $U$. It is well-known that the ordered set $(\wp(U), \subseteq)$ is a complete Boolean lattice such that for all $\mathcal{H} \subseteq \wp(U)$,

$$\bigvee \mathcal{H} = \bigcup \mathcal{H} \text{ and } \bigwedge \mathcal{H} = \bigcap \mathcal{H}.$$

Each set $X \subseteq U$ has a complement $U - X$. Furthermore, $(\wp(U), \subseteq) \cong (\mathbf{2}^U, \leq)$.

Let $\mathcal{P} = (P, \leq)$ be a lattice with 0. An element $x^*$ is a *pseudocomplement* of $x$ if $x \wedge x^* = 0$ and $x \wedge a = 0$ implies $a \leq x^*$. A lattice is *pseudocomplemented* if every element has a pseudocomplement. If a lattice $\mathcal{P}$ with 0 is distributive, pseudocomplemented, and it satisfies the Stone identity $x^* \vee x^{**} = 1$ for any element $x \in P$, then $\mathcal{P}$ is a *Stone lattice*. It is obvious that every Boolean lattice is a Stone lattice and that every finite distributive lattice is pseudocomplemented.

## 3 Rough Sets Defined by Equivalences

This section is devoted to the structure of the ordered set of rough sets determined by equivalence relations. Let $U$ be a set and let $E$ be an equivalence relation on $U$. For any $x$, we denote by $[x]_E$ the *equivalence class* of $x$, that is,

$$[x]_E = \{y \in U \mid x\, E\, y\}.$$

For any set $X \subseteq U$, let

$$X^{\blacktriangledown} = \{x \in U \mid [x]_E \subseteq X\};$$
$$X^{\blacktriangle} = \{x \in U \mid [x]_E \cap X \neq \emptyset\}.$$

The sets $X^{\blacktriangledown}$ and $X^{\blacktriangle}$ are called the *lower* and the *upper approximation* of $X$, respectively. Two sets are said to be *roughly equivalent*, denoted by $X \equiv Y$, if $X^{\blacktriangledown} = Y^{\blacktriangledown}$ and $X^{\blacktriangle} = Y^{\blacktriangle}$. The equivalence classes of the relation $\equiv$ are called *rough sets*. The family of all rough sets is denoted by $\mathcal{R}$, that is,

$$\mathcal{R} = \{\, [X]_{\equiv} \mid X \subseteq U\}.$$

*Example 2.* Let $U = \{a, b, c\}$ and let $E$ be an equivalence on $U$ such that

$$[a]_E = \{a, c\}, \quad [b]_E = \{b\}, \quad [c]_E = \{a, c\}.$$

The approximations are presented in Table 1. The rough sets are $\{\emptyset\}$, $\{\{a\}, \{c\}\}$, $\{\{b\}\}$, $\{\{a, b\}, \{b, c\}\}$, $\{\{a, c\}\}$, and $\{U\}$.

**Table 1.** Approximations of subsets

| $X$ | $X^{\blacktriangledown}$ | $X^{\blacktriangle}$ |
|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\{a\}$ | $\emptyset$ | $\{a, c\}$ |
| $\{b\}$ | $\{b\}$ | $\{b\}$ |
| $\{c\}$ | $\emptyset$ | $\{a, c\}$ |
| $\{a, b\}$ | $\{b\}$ | $U$ |
| $\{a, c\}$ | $\{a, c\}$ | $\{a, c\}$ |
| $\{b, c\}$ | $\{b\}$ | $U$ |
| $U$ | $U$ | $U$ |

Next we will briefly consider the structure of $\mathcal{R}$. The results presented here can be found in the works of Gehrke and Walker (1992), Iwiński (1987), and J. Pomykała and J.A. Pomykała (1988).

It is clear that rough sets can also be viewed as pairs of approximations $(X^{\blacktriangledown}, X^{\blacktriangle})$, since each approximation uniquely determines a rough set. The set of rough approximations can be ordered by

(3.1) $$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \overset{\text{def}}{\iff} X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

It is known that $(\mathcal{R}, \leq)$ is a complete Stone lattice such that for any $\mathcal{H} \subseteq \wp(U)$,

$$\bigvee\{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \in \mathcal{H}\} = (\bigcup\{X^{\blacktriangledown} \mid X \in \mathcal{H}\}, \bigcup\{X^{\blacktriangle} \mid X \in \mathcal{H}\});$$
$$\bigwedge\{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \in \mathcal{H}\} = (\bigcap\{X^{\blacktriangledown} \mid X \in \mathcal{H}\}, \bigcap\{X^{\blacktriangle} \mid X \in \mathcal{H}\}).$$

Each element $(X^{\blacktriangledown}, X^{\blacktriangle})$ has a pseudocomplement $(U - X^{\blacktriangle}, U - X^{\blacktriangle})$. Furthermore,

$$(\mathcal{R}, \leq) \cong (\mathbf{2}^I \times \mathbf{3}^J, \leq),$$

where $I$ is the set of the equivalence classes of $E$ which have exactly one element, and $J$ consists of $E$-classes having at least two members. Note that if all elements are pairwise discernible, that is, $E$ is the identity relation $\{(x, x) \mid x \in U\}$, then $(\mathcal{R}, \leq) \cong (\mathbf{2}^U, \leq)$.

*Example 3.* The ordered set of rough sets of Example 2 is presented in Fig. 1.

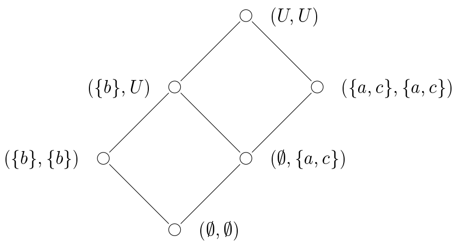


**Fig. 1.** Ordered set of rough sets

## 4 Structure of Generalized Rough Sets

Here we study ordered sets of rough sets defined by arbitrary binary relations. The motivation for this is that it is noted (see Järvinen (2002), for example) that neither reflexivity, symmetry, nor transitivity are necessary properties of indiscernibility relations, and we may present examples of indiscernibility relations that do not have these properties.

Let $R$ be a binary relation on $U$. Let us denote

$$R(x) = \{y \in U \mid x\, R\, y\}.$$

We may now generalize the approximation operators by setting

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\};$$
$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\}$$

for all $X \subseteq U$. The relation $\equiv$ and the set $\mathcal{R}$ of rough sets may be defined as in Section 3. Furthermore, the order $\leq$ on $\mathcal{R}$ is now defined as in (3.1).

### 4.1 Tolerance Relations

First we consider the ordered set $(\mathcal{R}, \leq)$ in case of tolerance relations. As noted in the previous section, the ordered set of rough sets defined by equivalences is a complete Stone lattice. Surprisingly, if we omit the transitivity, the structure of rough sets changes quite dramatically. Let us consider a tolerance $R$ on a set $U = \{a, b, c, d, e\}$ defined in Fig. 2 – the figure can be interpreted so that if $x\,R\,y$ holds, then there is an arrow from the point corresponding the element $x$ to the point that corresponds $y$. Järvinen (2001) has shown that the ordered set of rough sets determined by the tolerance $R$ is not even a $\vee$-semilattice nor a $\wedge$-semilattice. In that article one may also find the Hasse diagram of this ordered set.

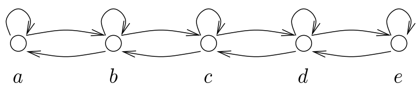


**Fig. 2.** Tolerance relation $R$

### 4.2 Transitive Relations

The removal of transitivity affects quite unexpectedly the structure of rough sets. Here we study rough sets determined by relations which are always at least transitive. We start by an example showing that the ordered sets of rough sets defined by merely transitive relations are not necessarily semilattices.

*Example 4.* Let $U = \{a, b, c, d, e, f, g, h, i, k\}$ and let $R$ be the transitive relation on $U$ depicted in Fig. 3. Note that since $R$ is not reflexive, $X^{\blacktriangledown} \subseteq X^{\blacktriangle}$ does not hold.

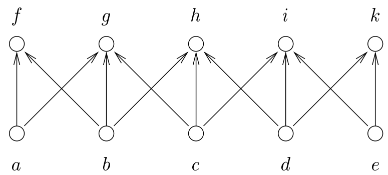


**Fig. 3.** Transitive relation $R$

For simplicity, let us denote the subsets of $U$ which differ from $\emptyset$ and $U$ by sequences of letters. For instance, $\{a, b, c\}$ is written as $abc$. The set of approximations determined by $R$ is the 22-element set

$$\{(fghik, \emptyset), (fghik, ab), (fghik, abc), (fghik, bcd), (fghik, cde), (fghik, de), (afghik, abc), (fghik, abcd), (fghik, abcde), (fghik, abde), (fghik, bcde), (efghik, cde), (abfghik, abcd), (afghik, abcde), (cfghik, abcde), (defghik, bcde), (abcfghik, abcde), (abfghik, abcde), (aefghik, abcde), (defghik, abcde), (cdefghik, abcde), (abcdefghik, abcde)\}.$$

Now, for example, $(abfghik, abcd) \wedge (afghik, abcde)$ does not exist; the set of lower bounds of this pair is $\{(afghik, abc), (fghik, abcd), (fghik, abc), (fghik, ab), (fghik, bcd), (fghik, \emptyset)\}$, which does not have a greatest element. Similarly,

$(afghik, abc) \vee (fghik, abcd)$ does not exist because this pair of elements has two minimal upper bounds.

Hence, $(\mathcal{R}, \leq)$ is neither $\vee$-semilattice nor a $\wedge$-semilattice.

Our next proposition shows that the rough sets defined by a symmetric and transitive binary relation form a complete Stone lattice.

**Proposition 5.** *For a symmetric and transitive binary relation, the ordered set of rough sets $(\mathcal{R}, \leq)$ is a complete Stone lattice.*

*Proof.* Let $R$ be a symmetric and transitive binary relation on a set $U$. Let us denote $U^* = \{x \in U \mid R(x) \neq \emptyset\}$. It is now obvious that $R \subseteq U^* \times U^*$. We start by showing that $R$ is an equivalence on $U^*$. The relation $R$ is symmetric and transitive by the definition. Suppose that $x \in U^*$. Then there exists a $y \in U^*$ such that $x\,R\,y$. Because $R$ is symmetric, also $y\,R\,x$ holds. But this implies $x\,R\,x$ by the transitivity. Thus, $R$ is an equivalence on $U^*$, and the resulting ordered set of rough sets on $U^*$ is a complete Stone lattice.

Let us denote by $\mathcal{R}$ the set of rough sets on $U$, and by $\mathcal{R}^*$ the set of rough set on $U^*$. We show that $(\mathcal{R}^*, \leq) \cong (\mathcal{R}, \leq)$. Let $\Sigma = U - U^*$ and let us define a map

$$\varphi\colon \mathcal{R}^* \to \mathcal{R}, (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangledown} \cup \Sigma, X^{\blacktriangle}).$$

Assume that $x \in \Sigma$. Because $R(x) = \emptyset$, $R(x) \subseteq X$ and $R(x) \cap X = \emptyset$ hold for all $X \subseteq U$. By applying this it is easy to see that the map $\varphi$ is an order-isomorphism, and hence $(\mathcal{R}, \leq)$ is a complete Stone lattice. □

Note that if $R$ is symmetric and transitive, but not reflexive, the elements that are not related even to themselves behave quite absurdly: they belong to every lower approximation, but not in any upper approximation as shown in the previous proof.

## 5 Completions

We have shown that for tolerances and transitive binary relations, the set of rough sets is not necessarily even a semilattice. Further, it is not known whether $(\mathcal{R}, \leq)$ is always a lattice, when the underlying relation $R$ is reflexive and transitive. We end this work by presenting some possible completions of $(\mathcal{R}, \leq)$. We will need the following definition. Let $\mathcal{P} = (P, \leq)$ be an ordered set and let $\mathcal{L} = (L, \leq)$ be a complete lattice. If there exists an order-embedding $\varphi\colon P \to L$, we say that $\mathcal{L}$ is a *completion* of $\mathcal{P}$.

### 5.1 Arbitrary Relations

Let us denote by $B^{\blacktriangledown}$ and by $B^{\blacktriangle}$ the sets of all lower and upper approximations of the subsets of $U$, respectively, that is, $B^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\}$ and $B^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}$. It is shown by Järvinen (2002) that $(B^{\blacktriangledown}, \subseteq)$ and $(B^{\blacktriangle}, \subseteq)$ are complete lattices for an arbitrary relation $R$. This means that also $(B^{\blacktriangledown} \times B^{\blacktriangle}, \leq)$ is a complete lattice; the order $\leq$ is defined as in (3.1). Thus, $(B^{\blacktriangledown} \times B^{\blacktriangle}, \leq)$ is always a completion of $(\mathcal{R}, \leq)$ for any $R$.

### 5.2 Reflexive Relations

Let us now assume that $R$ is reflexive. As we have noted, now $X^{\blacktriangledown} \subseteq X^{\blacktriangle}$ for any $X \subseteq U$. Let us denote

$$[B^{\blacktriangledown} \times B^{\blacktriangle}] = \{(X,Y) \in B^{\blacktriangledown} \times B^{\blacktriangle} \mid X \subseteq Y\}.$$

Obviously, $\mathcal{R} \subseteq [B^{\blacktriangledown} \times B^{\blacktriangle}]$. Because $[B^{\blacktriangledown} \times B^{\blacktriangle}]$ is a subset of $B^{\blacktriangledown} \times B^{\blacktriangle}$, we may order $[B^{\blacktriangledown} \times B^{\blacktriangle}]$ with the order inherited from $B^{\blacktriangledown} \times B^{\blacktriangle}$. It is also obvious that $([B^{\blacktriangledown} \times B^{\blacktriangle}], \leq)$ is a complete sublattice of $(B^{\blacktriangledown} \times B^{\blacktriangle}, \leq)$. Hence, we can write the following proposition.

**Proposition 6.** *If $R$ is reflexive, then $([B^{\blacktriangledown} \times B^{\blacktriangle}], \leq)$ is a completion of $(\mathcal{R}, \leq)$.*

Next, we present another completion for $(\mathcal{R}, \leq)$ in case $R$ is at least reflexive. As mentioned in Section 3, $(\mathcal{R}, \leq)$ is isomorphic to $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$, where $I$ is the set of the equivalence classes of $E$ which have exactly one element, and $J$ consists of $E$-classes having at least two members. Here we show that for reflexive relations this same ordered set can act as a completion. Note also for the proof of the next proposition that if $R$ is reflexive, then $X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$ and $R(x) \in I$ implies $R(x) = \{x\}$.

**Proposition 7.** *If $R$ is a reflexive relation, then $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$ is a completion of $(\mathcal{R}, \leq)$, where $I = \{R(x) \mid |R(x)| = 1\}$ and $J = \{R(x) \mid |R(x)| > 1\}$.*

*Proof.* Let us define a map $\varphi\colon \mathcal{R} \to \mathbf{2}^I \times \mathbf{3}^J$ by setting $\varphi(X^{\blacktriangledown}, X^{\blacktriangle}) = (f, g)$, where the maps $f\colon I \to \mathbf{2}$ and $g\colon J \to \mathbf{3}$ are defined by

$$f(R(x)) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases} \qquad \text{and} \qquad g(R(x)) = \begin{cases} 1 & \text{if } x \in X^{\blacktriangledown} \\ u & \text{if } x \in X^{\blacktriangle} - X^{\blacktriangledown} \\ 0 & \text{if } x \notin X^{\blacktriangle}. \end{cases}$$

Let us denote $\varphi(X^{\blacktriangledown}, X^{\blacktriangle}) = (f_1, g_1)$ and $\varphi(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (f_2, g_2)$.

Assume that $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$. We will show that $(f_1, g_1) \leq (f_2, g_2)$. If $f_1(R(x)) = 1$ for some $R(x) \in I$, then $x \in X$, and $R(x) = \{x\}$ implies $x \in X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \subseteq Y$. Thus, $f_2(R(x)) = 1$ and $f_1 \leq f_2$. If $g_1(R(x)) = 1$, then $x \in X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ and $g_2(R(x)) = 1$. If $g_1(R(x)) = u$, then $x \in X^{\blacktriangle} \subseteq Y^{\blacktriangle}$, which implies $g_2(R(x)) \geq u$. Hence, also $g_1 \leq g_2$.

Conversely, assume that $(f_1, g_1) \leq (f_2, g_2)$. We will show that $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$. Suppose that $x \in X^{\blacktriangledown}$. Then $1 = g_1(R(x)) \leq g_2(R(x))$ implies $x \in Y^{\blacktriangledown}$. If $x \in X^{\blacktriangle}$, then $u \leq g_1(R(x)) \leq g_2(R(x))$. This implies $x \in Y^{\blacktriangledown}$ or $x \in Y^{\blacktriangle} - Y^{\blacktriangledown}$, which obviously means that $x \in Y^{\blacktriangle}$ since $Y^{\blacktriangledown} \subseteq Y^{\blacktriangle}$. We have now proved that $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ and $X^{\blacktriangle} \subseteq Y^{\blacktriangledown}$. □

We end this section by presenting an example of the above-mentioned completions.

*Example 8.* Let us consider the relation $R$ defined in Fig. 4. Obviously, $R$ is reflexive, but not symmetric nor transitive.

Now the set of rough sets determined by the relation $R$ is

$$\mathcal{R} = \{(\emptyset,\emptyset), (\emptyset,\{a,b\}), (\emptyset,\{a,c\}), (\emptyset,\{b,c\}), (\{a\},U), (\{b\},U), (\{c\},U), (U,U)\}.$$

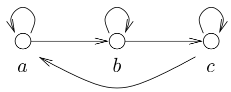


**Fig. 4.** Reflexive relation $R$

It is easy to observe that $(\mathcal{R}, \leq)$ is not a $\vee$-semilattice, because, for example, the elements $(\emptyset, \{a,b\})$ and $(\emptyset, \{a,c\})$ have the upper bounds $(\{a\}, U)$, $(\{b\}, U)$, $(\{c\}, U)$, and $(U,U)$ – but they do not have a smallest lower bound. Similarly, $(\mathcal{R}, \leq)$ is not a $\wedge$-semilattice, because the elements $(\{a\}, U)$ and $(\{b\}, U)$ have the lower bounds $(\emptyset, \emptyset)$, $(\emptyset, \{a\})$, $(\emptyset, \{b\})$, and $(\emptyset, \{c\})$, but not a greatest lower bound. The Hasse diagram of $(\mathcal{R}, \leq)$ presented in Fig. 5.

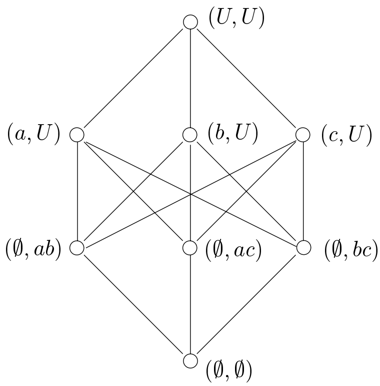


**Fig. 5.** Ordered set $(\mathcal{R}, \leq)$

The completions for $(\mathcal{R}, \leq)$ considered above are $(B^\blacktriangledown \times B^\blacktriangle, \leq)$, $([B^\blacktriangledown \times B^\blacktriangle], \leq)$, and $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$, where $I = \emptyset$, $J = \{R(a), R(b), R(c)\}$, and $R(a) = \{a,b\}$, $R(b) = \{b,c\}$, $R(c) = \{a,c\}$. It is easy to notice that $B^\blacktriangledown \times B^\blacktriangle$ contains 25 elements, $[B^\blacktriangledown \times B^\blacktriangle]$ has 15 elements, and $\mathbf{2}^I \times \mathbf{3}^J$ consists of 27 elements.

## Conclusions

In this paper we have considered rough sets determined by indiscernibility relations which are not necessarily reflexive, symmetric, or transitive. We have proved that if an indiscernibility relation is at least symmetric and closed, the the ordered set of rough sets is a complete Stone lattice. We have also shown that for tolerances and transitive binary relations, $(\mathcal{R}, \leq)$ is not necessarily even a semilattice. Additionally, it is not known whether the ordered set of rough sets $(\mathcal{R}, \leq)$ is a lattice, when the indiscernibility $R$ is reflexive and transitive, but not symmetric. These observations are depicted in Fig. 6.

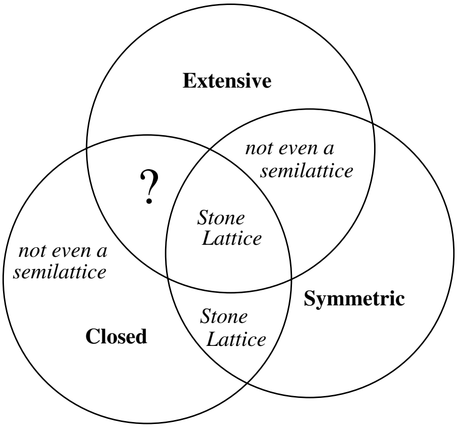


**Fig. 6.** Properties of ordered sets of rough sets

We also presented several possible and intuitive completions of $(\mathcal{R}, \leq)$. But as we saw in Example 8, the sizes of the completions are "too big". For example, we could made a completion of $(\mathcal{R}, \leq)$ of Example 8 just by adding the element $(\emptyset, U)$, and this completion has the size of only 9 elements, which much less than in the other completions presented. Therefore, we conclude this work by introducing the problem of determining the smallest completion of $(\mathcal{R}, \leq)$.

It would also interesting to study approximation operations which are defined as follows for any set $X \subseteq U$:

$$X^{\blacktriangledown} = X \cap \{x \in U \mid R(x) \subseteq X\};$$
$$X^{\blacktriangle} = X \cup \{x \in U \mid R(x) \cap X \neq \emptyset\}.$$

If the operations are defined as above, then

$$X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$$

for any relation $R$ and for any set $X \subseteq U$. As we noticed in Example 4 and Proposition 5, for example, this does not generally hold.

## Acknowledgements

Many thanks are due to Jari Kortelainen and Magnus Steinby for the careful reading of the manuscript and for their valuable comments and suggestions.

## References

G. Cattaneo, Abstract Approximation Spaces for Rough Theories, in: L. Polkowski, A. Skowron (eds.), *Rough Sets in Knowledge Discovery I* (Physica, Heidelberg, 1998) 59–98.

B.A. Davey, H.A. Priestley, *Introduction to Lattices and Order. Second Edition* (Cambridge University Press, Cambridge, 2002).

I. Düntsch, G. Gediga, Approximation Operators in Qualitative Data Analysis, in: H. de Swart, E. Orlowska, G. Schmidt, M. Roubens (eds.), *Theory and Applications of Relational Structures as Knowledge Instruments: COST Action 274, TARSKI. Revised Papers*, Lecture Notes in Artificial Intelligence **2929** (Springer, Heidelberg, 2001) 214–230.

M. Gehrke, E. Walker, On the Structure of Rough Sets, *Bulletin of the Polish Academy of Sciences, Mathematics* **40** (1992) 235–245.

G. Grätzer, *General Lattice Theory. Second Edition* (Birkhäuser, Basel, 1998).

S. Greco, B. Matarazzo, R. Slowinski, Rough Set Approach to Decisions Under Risk, in: W. Ziarko, Y. Yao (eds.), *Proceedings of The Second International Conference on Rough Sets and Current Trends in Computing (RSCTC 2000)*, Lecture Notes in Artificial Intelligence **2005** (Springer, Heidelberg, 2001) 160–169.

T.B. Iwiński, Algebraic Approach to Rough Sets, *Bulletin of the Polish Academy of Sciences, Mathematics* **35** (1987) 673–683.

J. Järvinen, Approximations and Rough Sets Based on Tolerances, in: W. Ziarko, Y. Yao (eds.), *Proceedings of The Second International Conference on Rough Sets and Current Trends in Computing (RSCTC 2000)*, Lecture Notes in Artificial Intelligence **2005** (Springer, Heidelberg, 2001) 182–189.

J. Järvinen, On the Structure of Rough Approximations, *Fundamenta Informaticae* **50** (2002) 135–153.

Z. Pawlak, Rough Sets, *International Journal of Computer and Information Sciences* **5** (1982) 341–356.

J. Pomykała, J.A. Pomykała, The Stone Algebra of Rough Sets, *Bulletin of the Polish Academy of Sciences, Mathematics* **36** (1988) 495–512.

J.A Pomykała, About Tolerance and Similarity Relations in Information Systems, in: J.J. Alpigini, J.F. Peters, A. Skowron, N. Zhong (eds.), *Proceedings of The Third International Conference on Rough Sets and Current Trends in Computing (RSCTC 2002)*, Lecture Notes in Artificial Intelligence **2475** (Springer, Heidelberg, 2002) 175 – 182.

A. Skowron, J. Stepaniuk, Tolerance Approximation Spaces, *Fundamenta Informaticae* **27** (1996) 245–253.

R. Slowinski, D. Vanderpooten, A Generalized Definition of Rough Approximations Based on Similarity, *IEEE Transactions on Knowledge and Data Engineering* **12** (2000) 331–336.

Y.Y. Yao, T.Y. Lin, Generalization of Rough Sets using Modal Logics, *Intelligent Automation and Soft Computing. An International Journal* **2** (1996) 103–120.