A PARTICLE DISPLACEMENT REPRESENTATION FOR CONSERVATION LAWS IN TWO-DIMENSIONAL CELLULAR AUTOMATA

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Abstract. The problem of describing the dynamics of a conserved energy in a cellular automaton in terms of local movements of "particles" (quanta of that energy) has attracted some people's attention. The one-dimensional case was already solved by Fukś (2000) and Pivato (2002). For the two-dimensional cellular automata, we show that every (context-free) conservation law can be expressed in terms of such particle displacements.

Introduction

Let $\mathbb{L} = \mathbb{Z}^d$ be the d-dimensional square *lattice*, and S a finite set of *states*. Every cellular automaton (CA for short) $F : S^{\mathbb{L}} \to S^{\mathbb{L}}$ maps the uniform configurations into the uniform configurations. Two configurations $x, y : \mathbb{L} \to S$ are *asymptotic* if they agree on all but finitely many cells of the lattice. The image of asymptotic configurations under every cellular automaton remain asymptotic.

A (context-free) *energy* assignment is a function $\mu : S \to \mathbb{R}$. The *μ-content* of a finite pattern $p : A \to S$ ($A \subseteq \mathbb{L}$ finite) is the sum $M(p) \triangleq \sum_{i \in A} \mu(p[i])$. For every two asymptotic configurations $x, y \in S^{\mathbb{L}}$, the corresponding *energy difference* is

$$\delta M(x, y) \triangleq \sum_{i \in \mathbb{L}} [\mu(y[i]) - \mu(x[i])] \tag{0.1}$$

which is clearly well-defined (only a finite number of terms are non-zero). The energy μ is *conserved* by a cellular automaton $F : S^{\mathbb{L}} \to S^{\mathbb{L}}$, if

$$\delta M(Fx, Fy) = \delta M(x, y) \tag{0.2}$$

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for every two asymptotic configurations x and y. This is equivalent to the formulations in terms of finite or periodic configurations [7, 3]. In particular, one can show that, if F maps an a-uniform configuration to a b-uniform configuration, we must have $\mu(a) = \mu(b)$.

For a conserved energy μ, it is desirable to find a local rule that explains the microscopic dynamics of μ under the iteration of F, in terms of "flows" of energy from one cell to another. More specifically, a *flow* for μ is a mapping $x, i, j \mapsto \Phi_{i\to j}(x) \in \mathbb{R}$ for $x \in S^{\mathbb{L}}$ and $i, j \in \mathbb{L}$ that satisfies the following conditions:

a) For every configuration x and every cell a,
$$\mu\left(x[a]\right) = \sum_{j\in\mathbb{L}} \Phi_{a\to j}(x)\,, \tag{0.3}$$

b) For every configuration x and every cell a,
$$\sum_{i\in\mathbb{L}} \Phi_{i\to a}(x) = \mu\left((Fx)[a]\right)\,, \tag{0.4}$$

c) There exist finite sets $K, I \subseteq \mathbb{L}$, and a rule $\varphi : S^K \times I \to \mathbb{R}$ such that,
$$\Phi_{i\to j}(x) = \begin{cases} \varphi\left(x[j+K], i-j\right) & \text{if } i-j \in I, \\ 0 & \text{otherwise,} \end{cases} \tag{0.5}$$
for every $x \in S^{\mathbb{L}}$ and $i, j \in \mathbb{L}$.

Here, $f[A]$ denotes the restriction of a function f to a subset A of its domain. Equations (0.4) and (0.3) are called the *continuity equations*. Equation (0.5) states that the amount of the flows toward each cell is decided locally, by looking at a finite *neighborhood* K of that cell. The set I is the set of *directions* from which energy flows into a cell. The local rule φ is called an *inflow*. An energy μ is *locally conserved* by F, if it has an inflow.

Proposition 0.1 (Hattori and Takesue [7])**.** *In cellular automata, conserved energies are locally conserved.*

We remark that the third condition in the above definition could equivalently be formulated in terms of an *outflow*.

Recently a number of people have shown interest in flows that can be interpreted as displacement of "particles" (see e.g. [6, 10, 9, 1]). In such a case, the μ-content of a pattern p is seen as the sum of the energies (or masses) of the particles in p.

Recall that every finitely generated subgroup of $\mathbb{R}$ is isomorphic to $\mathbb{Z}^m$ for some $m \geq 0$. If an energy $\mu : S \to \mathbb{Z}^m$ is conserved by a cellular automaton F, each of its m components must be conserved independently. If we are able to find "particle flows" for each component, we can extend our interpretation for μ by assuming m different types of particles which flow independently. So, without loss of generality, we can concentrate on the case $\mu : S \to \mathbb{Z}$ or $\mu : S \to \mathbb{Q}$. Now, if the particles are indistinguishable and each have the same energy $\varepsilon \in \mathbb{R}$, for every state $s \in S$ we must have $\mu(s) = \nu(s) \cdot \varepsilon$, where $\nu(s) \in \mathbb{Z}^{\geq 0}$ is the number of particles in s. Thus, it makes sense to assume μ is everywhere non-negative (or everywhere non-positive).

Formally, let $\mu : S \to \mathbb{Q}^{\geq 0}$ be a conserved energy for a cellular automaton F. A *particle flow* for μ is a flow Φ whose values are from non-negative rationals $Q^{\geq 0}$. Let Φ be a particle flow which is defined by an inflow $\varphi : S^K \times I \to \mathbb{Q}^{\geq 0}$. Let $\varepsilon > 0$ be such that $\varphi(p, i)/\varepsilon \in \mathbb{Z}^{\geq 0}$ for every p and i. Then ε is the *μ-content of a particle* and the function $\rho(\cdot,\cdot) \triangleq \varphi(\cdot,\cdot)/\varepsilon$

is called a *particle displacement rule*. For every state $s \in S$, $\nu(s) \triangleq \mu(s)/\varepsilon$ is the *number of particles* in s.

Proposition 0.2 (Fukś [6] and Pivato [10])**.** *Let $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a one-dimensional cellular automaton, and $\mu : S \to \mathbb{N}$ an energy conserved by F. Then, μ has a particle flow.*

We extend this result to the two-dimensional CA. In Section 1, we show how to construct a particle flow for a conserved energy in a two-dimensional radius-$\frac{1}{2}$ CA. In Section 2, we give a sketch of how this can be exploited in the case of arbitrary neighborhoods. Some open problems are proposed in Section 3.

1. Particle Flows in Radius One Half CA

Let $F : S^{\mathbb{Z}^2} \to S^{\mathbb{Z}^2}$ be a two-dimensional CA with neighborhood

$$N = \{(0,0), (0,1), (1,1), (1,0)\} \tag{1.1}$$

and local rule $f : S^N \to S$. The neighbors $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$ are interpreted, respectively, as the down-left (`dl`), up-left (`ul`), up-right (`ur`) and down-right (`dr`) neighbors. Such a neighborhood is often called *radius-*$\frac{1}{2}$.

To simplify our exposition, let us distinguish between *neighbors* of a cell i, and the cells *adjacent* to it. The former are the cells $i+\mathtt{dl}$, $i+\mathtt{ul}$, $i+\mathtt{ur}$ and $i+\mathtt{dr}$ one step before, while the latter are the cells $i+\mathtt{r}$, $i+\mathtt{u}$, $i+\mathtt{l}$ and $i+\mathtt{d}$ at the same time step, where $\mathtt{r} = (1,0)$, $\mathtt{u} = (0,1)$, $\mathtt{l} = (-1,0)$ and $\mathtt{d} = (0,-1)$.

Let $\mu : S \to \mathbb{Q}^{\geq 0}$ be a conserved energy. Without loss of generality, we can assume that $\mu(\diamond) = 0$ for a state $\diamond \in S$ which we call *blank*. For every state $x \in S$ we define the *free* flows going out of x by looking at the configurations

$$\begin{array}{ccc} \diamond & \diamond & \diamond \\ \diamond & x & \diamond \\ \diamond & \diamond & \diamond \end{array} \quad \xrightarrow{F} \quad \begin{array}{cc} x_2 & x_3 \\ x_1 & x_4 \end{array} \tag{1.2}$$

That is, $\varphi_{\swarrow}(x) \triangleq \mu(f(\begin{smallmatrix} \diamond & x \\ \diamond & \diamond \end{smallmatrix}))$, $\varphi_{\nwarrow}(x) \triangleq \mu(f(\begin{smallmatrix} \diamond & \diamond \\ \diamond & x \end{smallmatrix}))$, $\varphi_{\nearrow}(x) \triangleq \mu(f(\begin{smallmatrix} \diamond & \diamond \\ x & \diamond \end{smallmatrix}))$ and $\varphi_{\searrow}(x) \triangleq \mu(f(\begin{smallmatrix} x & \diamond \\ \diamond & \diamond \end{smallmatrix}))$. By the conservation of μ, we have

$$\varphi_{\swarrow}(x) + \varphi_{\nwarrow}(x) + \varphi_{\nearrow}(x) + \varphi_{\searrow}(x) = \mu(x)\,. \tag{1.3}$$

When two states are put next to each other, their touching out-going flows *interfere* and as a result we have a flow *deflection* from one cell toward another.

$$\begin{array}{cccc} \diamond & \diamond & \diamond & \diamond \\ \diamond & x & y & \diamond \\ \diamond & \diamond & \diamond & \diamond \end{array} \quad \xrightarrow{F} \quad \begin{array}{ccc} x_2 & a & y_3 \\ x_1 & b & y_4 \end{array} \tag{1.4}$$

Specifically, for every $x, y \in S$, define

$$\psi_{\uparrow}(x\ \ y) \triangleq \max\left\{0,\, \mu(f(\begin{smallmatrix} \diamond & \diamond \\ x & y \end{smallmatrix})) - \varphi_{\nearrow}(x) - \varphi_{\nwarrow}(y)\right\}, \tag{1.5}$$

and

$$\psi_{\downarrow}(x\ \ y) \triangleq \max\left\{0,\, \mu(f(\begin{smallmatrix} x & y \\ \diamond & \diamond \end{smallmatrix})) - \varphi_{\searrow}(x) - \varphi_{\swarrow}(y)\right\}. \tag{1.6}$$

By the conservation of μ we have

$$\mu(f(\begin{smallmatrix}\diamond & \diamond\\ x & y\end{smallmatrix})) = \varphi_{\nearrow}(x) + \varphi_{\nwarrow}(y) + \psi_{\uparrow}(x\ \ y) - \psi_{\downarrow}(x\ \ y)\,, \tag{1.7}$$

$$\mu(f(\begin{smallmatrix}x & y\\ \diamond & \diamond\end{smallmatrix})) = \varphi_{\searrow}(x) + \varphi_{\swarrow}(y) + \psi_{\downarrow}(x\ \ y) - \psi_{\uparrow}(x\ \ y)\,, \tag{1.8}$$

and either $\psi_{\downarrow}(x\ \ y)$ or $\psi_{\uparrow}(x\ \ y)$ is zero. The deflections $\psi_{\rightarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix})$ and $\psi_{\leftarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix})$ are defined similarly, and in the same way we have

$$\mu(f(\begin{smallmatrix}x & \diamond\\ y & \diamond\end{smallmatrix})) = \varphi_{\searrow}(x) + \varphi_{\nearrow}(y) + \psi_{\rightarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix}) - \psi_{\leftarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix})\,, \tag{1.9}$$

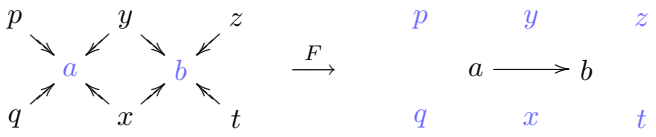
$$\mu(f(\begin{smallmatrix}\diamond & x\\ \diamond & y\end{smallmatrix})) = \varphi_{\swarrow}(x) + \varphi_{\nwarrow}(y) + \psi_{\leftarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix}) - \psi_{\rightarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix})\,, \tag{1.10}$$

and either $\psi_{\leftarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix})$ or $\psi_{\rightarrow}(\begin{smallmatrix}x\\ y\end{smallmatrix})$ is zero. The deflections summarize all the interactions between the free flows:

Lemma 1.1. *For every $x, y, z, t \in S$ we have*

$$\begin{aligned}\mu(f(\begin{smallmatrix}y & z\\ x & t\end{smallmatrix})) &= \varphi_{\nearrow}(x) + \varphi_{\searrow}(y) + \varphi_{\swarrow}(z) + \varphi_{\nwarrow}(t)\\ &+ \psi_{\rightarrow}(\begin{smallmatrix}y\\ x\end{smallmatrix}) + \psi_{\downarrow}(y\ \ z) + \psi_{\leftarrow}(\begin{smallmatrix}z\\ t\end{smallmatrix}) + \psi_{\uparrow}(x\ \ t)\\ &- \psi_{\leftarrow}(\begin{smallmatrix}y\\ x\end{smallmatrix}) - \psi_{\uparrow}(y\ \ z) - \psi_{\rightarrow}(\begin{smallmatrix}z\\ t\end{smallmatrix}) - \psi_{\downarrow}(x\ \ t)\,.\end{aligned} \tag{1.11}$$

We shall think of the free flows and the flow deflections as weighted arrows from one cell (in the space-time) to another. For example, in the consecutive configurations



(1.12)

the free flow $\varphi_{\searrow}(p)$ determines an arrow toward the cell with state a from its up-left neighbor in the previous time step, and so forth. Similarly, there is a deflection arrow from a to b with weight $\psi_{\rightarrow}(\begin{smallmatrix}y\\ x\end{smallmatrix}) \geq 0$ and one in the opposite direction with weight $\psi_{\leftarrow}(\begin{smallmatrix}y\\ x\end{smallmatrix}) \geq 0$, even though at least one of them is zero. For two consecutive configurations x and $y = F(x)$, let us write $\Phi_{\nearrow}[i]$ for the free flow arrow with value $\varphi_{\nearrow}(x[i + \mathtt{dl}])$ from the cell $i + \mathtt{dl}$ (on x), to the cell i (on y), and so forth. Similarly, let $\Psi_{\uparrow}[i]$, $\Psi_{\rightarrow}[i]$, $\Psi_{\downarrow}[i]$ and $\Psi_{\leftarrow}[i]$ be the deflection arrows going out from i (on y) to the cells $i + \mathtt{u}$, $i + \mathtt{r}$, $i + \mathtt{d}$ and $i + \mathtt{l}$ (on y), respectively.

Deflections represent the deviation of an actual flow from the free flows. If we split each deflection $\psi_{\uparrow}$ (resp. $\psi_{\rightarrow}$, $\psi_{\downarrow}$, $\psi_{\leftarrow}$) into two parts $\psi_{\upharpoonleft}$ and $\psi_{\uparrow}$ (resp. $\psi_{\rightharpoonup}$ and $\psi_{\rightarrow}$, $\psi_{\downharpoonleft}$ and $\psi_{\downarrow}$, $\psi_{\leftharpoondown}$ and $\psi_{\leftarrow}$) and use these parts to correct the free flows, we obtain an actual flow for μ. To be precise, at each cell i, the arrow $\Phi_{\nearrow}[i]$ is corrected to

$$\Phi'_{\nearrow}[i] \triangleq \Phi_{\nearrow}[i] - \Psi_{\uparrow}[i] + \Psi_{\downarrow}[i + \mathtt{u}] - \Psi_{\rightharpoonup}[i] + \Psi_{\leftharpoondown}[i + \mathtt{r}] \tag{1.13}$$

and so forth (Figure 1). We require the splits $\psi_{\upharpoonleft}, \psi_{\uparrow}, \ldots$ to be non-negative and rational. Otherwise, the splitting can be arbitrary and may depend on the local neighborhood pattern. The main challenge here is to do the splitting in such a way that the corrected flow has only non-negative rational values.

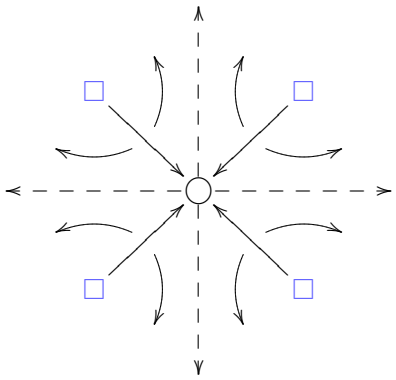


Figure 1: Correcting the flows.

Let us say that a cell i on y is *balanced*, if

a) $\Phi_{\searrow}[i] + \Phi_{\swarrow}[i] \geq \Psi_{\uparrow}[i]$ (and its rotations),
b) $\Phi_{\searrow}[i] + \Phi_{\swarrow}[i] + \Phi_{\nwarrow}[i] \geq \Psi_{\uparrow}[i] + \Psi_{\rightarrow}[i]$ (and its rotations), and
c) $\Phi_{\searrow}[i] + \Phi_{\swarrow}[i] + \Phi_{\nwarrow}[i] + \Phi_{\nearrow}[i] \geq \Psi_{\uparrow}[i] + \Psi_{\rightarrow}[i] + \Psi_{\downarrow}[i] + \Psi_{\leftarrow}[i]$.

Observation 1.2. For every $a, b, c \in S$ we have

a) $\varphi_{\searrow}(a) + \varphi_{\swarrow}(b) \geq \psi_{\uparrow}(a \quad b)$.
b) $\varphi_{\searrow}(a) + \varphi_{\swarrow}(b) + \varphi_{\nwarrow}(c) \geq \psi_{\uparrow}(a \quad b) + \psi_{\rightarrow}\binom{b}{c}$.

Lemma 1.3. *If a cell is balanced, we can split its out-going deflections properly, so that its corrected in-coming flows remain non-negative and rational.*

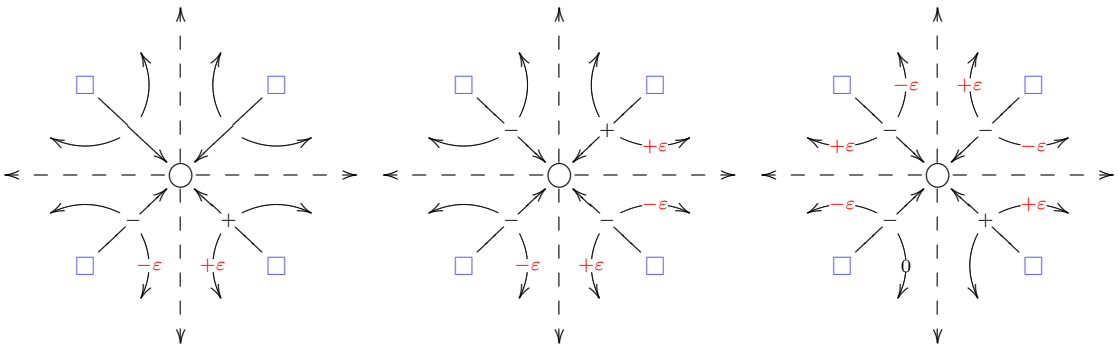
Proof. Let i be a balanced cell. Let us do the splitting in such a way that (the splits are non-negative and rational, and) when the flows are corrected, the total amount of negative flows coming into i (let us call it M) is minimal. We claim that the corrected in-coming flows of i are non-negative.

Suppose the contrary. Without loss of generality, assume that $\Phi'_{\nearrow} < 0$. Since the sum of the corrected in-coming flows of i is non-negative (requirement (c) of balancedness), at least one of the other in-coming flows is strictly positive.

First assume that all the splits are strictly positive. If $\Phi'_{\nwarrow} > 0$, we could get a splitting with smaller M, by choosing

$$\Psi'_{\downarrow} = \Psi_{\downarrow} - \varepsilon \tag{1.14}$$
$$\Psi'_{\downarrow} = \Psi_{\downarrow} + \varepsilon \tag{1.15}$$

(see Figure 2.a) for a sufficiently small $\varepsilon > 0$.



(a) (b) (c)

Figure 2: Proper splitting of the deflections.

By symmetry, $\Phi'_{\searrow} > 0$ cannot happen either. So let $\Phi'_{\nwarrow}, \Phi'_{\searrow} \leq 0$ and $\Phi'_{\swarrow} > 0$. Again, this is not possible, because if (for a sufficiently small $\varepsilon > 0$) we chose

$$\Psi'_{\downarrow} = \Psi_{\downarrow} - \varepsilon \quad (1.16)$$
$$\Psi'_{\downarrow} = \Psi_{\downarrow} + \varepsilon \quad (1.17)$$
$$\Psi'_{\rightharpoondown} = \Psi_{\rightharpoondown} - \varepsilon \quad (1.18)$$
$$\Psi'_{\rightharpoonup} = \Psi_{\rightharpoonup} + \varepsilon \quad (1.19)$$

(see Figure 2.b) we would get a smaller M.

If $\Psi_{\downarrow} = 0$, then $\Phi_{\nearrow} < \Psi_{\leftharpoondown}$ and by requirement (a) of balancedness $\Phi_{\searrow} > \Psi_{\leftharpoondown} \geq 0$. If $\Phi'_{\searrow} > 0$, by choosing a different splitting like before, we could get a smaller M. So $\Phi'_{\searrow} \leq 0$ and $\Psi_{\uparrow} > 0$. Now, by requirement (b) of balancedness $\Phi_{\swarrow} > \Psi_{\uparrow} \geq 0$. Again, if $\Phi'_{\swarrow} > 0$, by choosing a different splitting like before, leads us to a smaller M. So $\Phi'_{\swarrow} \leq 0$ and $\Psi_{\rightharpoonup} > 0$.

At this point, since $\Phi'_{\nearrow} < 0$ and $\Phi'_{\searrow}, \Phi'_{\swarrow} \leq 0$, we must have $\Phi'_{\nwarrow} > 0$. But again, by taking a sufficiently small $\varepsilon > 0$ and choosing the splitting

$$\Psi'_{\leftharpoondown} = \Psi_{\leftharpoondown} - \varepsilon \quad (1.20)$$
$$\Psi'_{\leftharpoonup} = \Psi_{\leftharpoonup} + \varepsilon \quad (1.21)$$
$$\Psi'_{\uparrow} = \Psi_{\uparrow} - \varepsilon \quad (1.22)$$
$$\Psi'_{\uparrow} = \Psi_{\uparrow} + \varepsilon \quad (1.23)$$
$$\Psi'_{\rightharpoonup} = \Psi_{\rightharpoonup} - \varepsilon \quad (1.24)$$
$$\Psi'_{\rightharpoondown} = \Psi_{\rightharpoondown} + \varepsilon \quad (1.25)$$

(see Figure 2.c) we would get a smaller M.

Finally, let $\Psi_{\downarrow}, \Psi_{\leftharpoondown} > 0$. If either $\Phi'_{\nwarrow} > 0$ or $\Phi'_{\searrow} > 0$ we could make M smaller like before. So $\Phi'_{\nwarrow}, \Phi'_{\searrow} \leq 0$ and $\Phi'_{\swarrow} > 0$. Now, by the requirement (b) of balancedness, either $\Psi_{\rightharpoondown} > 0$ or $\Psi_{\uparrow} > 0$. So, again we could get a smaller M as before.

Hence, in any case $\Phi'_{\nearrow} < 0$ leads to a contradiction, which by symmetry means the corrected in-coming flows of i must all be non-negative. ∎

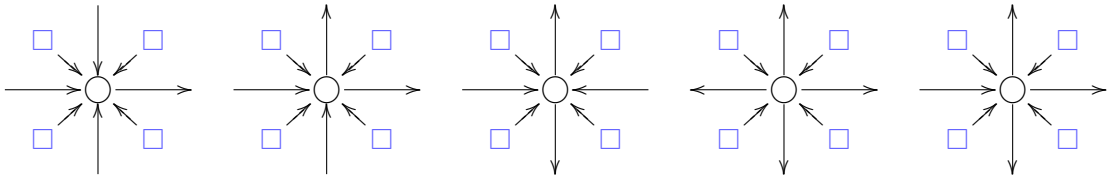


Figure 3: The situations that may happen around a cell.

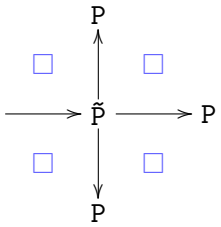


Figure 4: Doubly problematic cell.

Figure 3 shows a cell and the various situations that may happen, based on the direction of the deflection arrows around it. The other possibility are all symmetrically identical to these five case. According to the Lemma 1.3, Observation 1.2 guarantees that, unless there is exactly one deflection directing toward a cell (i.e., the cases (1-4)), one can correct the in-coming free flows of that cell to satisfy its out-going deflections, in such a way that the corrected flows remain non-negative. Let us call a cell *problematic* (P in symbol) if the situation around it is as in case (5) (or its symmetrically identical variants). We call a cell *doubly problematic* ($\tilde{\mathtt{P}}$ in symbol) if it is problematic, and furthermore, the endpoints of its out-going deflection arrows are also problematic (Figure 4). For two adjacent cells i and j, let us say j *follows* i, if there is a deflection arrow from i to j.

Observation 1.4. If j follows i, both of i and j cannot be doubly problematic at the same time.

Theorem 1.5. *Let $F : S^{\mathbb{Z}^2} \to S^{\mathbb{Z}^2}$ be a two-dimensional radius-$\frac{1}{2}$ cellular automaton. Then, every conserved energy $\mu : S \to \mathbb{Q}^{\geq 0}$ has a particle flow, with flows entering each cell only from its four neighbors.*

Proof. Let x and y be two consecutive configurations in $S^{\mathbb{Z}^2}$. Let us set the free flows as an initial approximation of the desired flow. That is, let $\Phi_d^{(0)} \triangleq \Phi_d$ ($d \in \{\nearrow, \searrow, \swarrow, \nwarrow\}$). We construct a non-negative rational flow for μ, by correcting this approximation in three steps. At each step a number of deflections are split and redistributed into the affected flows.

Step 1.
Splitting. For every cell i which is in either cases (1-4) of Figure 3 we split the out-going deflections as suggested in Lemma 1.3. If the cell is doubly problematic, we leave the splitting of its out-going deflections for the next step. If the cell is problematic, but not

doubly problematic, we leave one of its out-going deflections that leads to a non-problematic cell for the next step, and split the other two, as described in Lemma 1.3.

CORRECTING. We use the already split deflections to correct the flows. Let $\Phi_d^{(1)}$ ($d \in \{\nearrow, \searrow, \swarrow, \nwarrow\}$) be the corrected flow arrows of this step.

Step 2.

SPLITTING. Let i be a problematic cell. Notice that unless i follows a doubly problematic cell, its in-coming deflection is already resolved in the previous step. So, in this case i is no more problematic, and we can split its out-going deflections as explained in Lemma 1.3. In particular, all the out-going deflections of (formerly) doubly problematic cells are split in this step (Observation 1.4).

CORRECTING. We correct the flows using the newly split deflections. Let $\Phi_d^{(2)}$ ($d \in \{\nearrow, \searrow, \swarrow, \nwarrow\}$) be the corrected flow arrows of this step.

Step 3.

SPLITTING. The only unresolved deflections are those leaving a problematic cell (such as i) which follows an initially doubly problematic cell. But the out-going deflections of the doubly problematic cells are already resolved. So i is no further problematic. We split its unresolved out-going deflection using Lemma 1.3.

CORRECTING. We correct the flows using the newly split deflections. Let $\Phi_d^{(3)}$ ($d \in \{\nearrow, \searrow, \swarrow, \nwarrow\}$) be the corrected flow arrows of this step.

At this point, all the deflections are resolved. The corrected arrows $\Phi_d^{(3)}$ define a flow Φ by

$$\Phi_{i\to j} \triangleq \begin{cases} \Phi^{(3)}_{\nearrow}[j] & \text{if } i = j + \mathtt{dl}, \\ \Phi^{(3)}_{\searrow}[j] & \text{if } i = j + \mathtt{ul}, \\ \Phi^{(3)}_{\swarrow}[j] & \text{if } i = j + \mathtt{ur}, \\ \Phi^{(3)}_{\nwarrow}[j] & \text{if } i = j + \mathtt{dr}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.26}$$

for μ, which satisfies the continuity equations, and its values are locally determined. Also, by construction, the values of Φ are all non-negative and rational. Therefore, Φ is a particle flow. ∎

2. Particle Flows in CA with Arbitrary Neighborhood

Every CA can be transformed into a radius-$\frac{1}{2}$ one, using a combination of a translation and moving to a higher block representation. A conserved energy $\mu : S \to \mathbb{Q}^{\geq 0}$ gives a conserved energy $\hat{\mu} : \hat{S} \to \mathbb{Q}^{\geq 0}$ for the new CA, which simply measures the collective energy of the super-cells of this new CA. By the discussion of the previous chapter, we can find a particle flow for $\hat{\mu}$. This can be turned into a particle flow for μ in the following way.

Let x and $y = F(x)$ be two consecutive configurations in the original CA. For each super-cell $\hat{i}$ on x let us order its out-going flow arrows using, for example, the lexicographic ordering of their end points, and distribute them properly into its constituent cells, so that each cell with state s gives out exactly $\mu(s)$. Now for each cell i on x and each super-cell $\hat{j}$ on y, we have a flow arrow from i to $\hat{j}$. Then, for each super-cell on y order its in-coming flow arrows according to the lexicographic ordering of their starting points, and

distribute them properly into its constituent cells, so that each cell with state s receives exactly $\mu(s)$. Clearly, this can be done locally, but the obtained flow may not be translation-invariant. However, if we translate the partitioning of the cells and take the average of the flows obtained from each partitioning, we obtain a translation-invariant flow which is still non-negative and rational-valued.

3. Open Problems

With enough patience, one should be able to find a particle representation for the conservation laws of the three-dimensional CA, or the CA on the hexagonal or triangular lattices, using similar analysis. Is there a unified approach that works for any lattice, in any number of dimensions?

A drawback of our solution is the arbitrariness involved. There are infinite number of ways one can assign a flow to a given conservation law. Can we (possibly by putting some extra constraints, or by formalizing the concept of the flows in a different way) obtain a "natural" flow for each conservation law, which is *unique*? One criterion for naturalness is that for a reversible CA, the flows in the backward direction of time should be obtained from the flows in the forward direction, only by reversing the direction of the arrows. Such a concept of flow would definitely give a better understanding of the dynamics of the conserved energy.

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Appendix A

Proof of Proposition 0.1. Let $F : S^{\mathbb{L}} \to S^{\mathbb{L}}$ be a cellular automaton with neighborhood $N \subseteq \mathbb{L}$ and local rule $f : S^N \to S$, and let $\mu : S \to \mathbb{R}$ be an energy which is conserved by F. Fix a state $\diamond \in S$ as *blank*. We construct an inflow $\bar{\varphi}$ for the conserved energy $\bar{\mu} \triangleq \mu(\cdot) - \mu(\diamond)$. An inflow for μ follows by superposing $\bar{\varphi}$ with a constant inflow $\varphi_\diamond(\cdot, 0) \equiv \mu(\diamond)$. That is, $\varphi = \bar{\varphi} + \varphi_\diamond$.

Let $\preceq$ be the lexicographic order on $\mathbb{L}$. For every cell $a \in \mathbb{L}$, let $\gamma(a)$ be its successor according to the total ordering $\preceq$. For every pattern $p : A \to S$ ($A \subseteq \mathbb{L}$) and every cell a, define the pattern $\chi_a p : A \to S$ with

$$(\chi_a p)[i] = \begin{cases} \diamond & \text{if } a \preceq i, \\ p[i] & \text{otherwise.} \end{cases} \tag{A.1}$$

Notice that for every configuration x and every cell a, the configurations $\chi_a x$ and $\chi_{\gamma(a)} x$ are asymptotic, and we have

$$\delta\bar{M}\left(\chi_a x, \chi_{\gamma(a)} x\right) = \bar{\mu}(x[a]) - \bar{\mu}(\diamond) = \bar{\mu}(x[a])\,. \tag{A.2}$$

We define the amount of flow from the cell a to the cell b as the difference between the $\bar{\mu}$-content of the cell b in $F(\chi_a x)$ and $F(\chi_{\gamma(a)} x)$.

More precisely, set $K = I = N$. For every pattern $p : N \to S$ and a direction $i \in N$, let

$$\varphi(p, i) \triangleq \bar{\mu}\left(f(\chi_{\gamma(i)} p)\right) - \bar{\mu}\left(f(\chi_i p)\right)\,. \tag{A.3}$$

We prove that $\bar{\varphi}$ satisfies the continuity equations.

First, let $i_1 \prec i_2 \prec i_3 \prec \cdots \prec i_n$ ($n = |N|$) be the enumeration of the elements of N according to the order $\preceq$. Let x be an arbitrary configuration and a an arbitrary cell. Note that in $\chi_{a+i_1} x$, all the cells in the neighborhood of a are blank, so $F(\chi_{a+i_1} x)[a] = f(\diamond^N)$. Similarly, $F(\chi_{a+\gamma(i_n)} x)[a] = (Fx)[a]$, and $F(\chi_{a+\gamma(i_t)} x)[a] = F(\chi_{a+i_{t+1}} x)[a]$, for every $1 \leq t < n$. Therefore, we have

$$\begin{aligned}
\sum_{i \in N} \bar{\varphi}\left(x[a+N], i\right) &= \sum_{i \in I} \left[\bar{\mu}\left(F(\chi_{a+\gamma(i)} x)[a]\right) - \bar{\mu}\left(F(\chi_{a+i} x)[a]\right)\right] && \text{(A.4)} \\
&= \sum_{t=1}^{n} \left[\bar{\mu}\left(F(\chi_{a+i_{t+1}} x)[a]\right) - \bar{\mu}\left(F(\chi_{a+i_t} x)[a]\right)\right] && \text{(A.5)} \\
&= \bar{\mu}\left((Fx)[a]\right) - \bar{\mu}(\diamond) && \text{(A.6)} \\
&= \bar{\mu}\left((Fx)[a]\right) && \text{(A.7)}
\end{aligned}$$

Next, notice that the configurations $F(\chi_a x)$ and $F(\chi_{\gamma(a)} x)$ may differ only on the cells in $a - N$. Thus, conservation of $\bar{\mu}$ implies that

$$\begin{aligned}
\bar{\mu}\left(x[a]\right) &= \delta\bar{M}\left(\chi_a x, \chi_{\gamma(a)} x\right) && \text{(A.8)} \\
&= \delta\bar{M}\left(F(\chi_a x), F(\chi_{\gamma(a)} x)\right) && \text{(A.9)} \\
&= \sum_{i \in N} \left[\bar{\mu}\left(F(\chi_{\gamma(a)} x)[a-i]\right) - \bar{\mu}\left(F(\chi_a x)[a-i]\right)\right] && \text{(A.10)} \\
&= \sum_{i \in I} \bar{\varphi}\left(x[a-i+N], i\right)\,. && \text{(A.11)}
\end{aligned}$$

Hence, $\bar{\varphi}$ is an inflow for $\bar{\mu}$ and we are done! ∎