Conservation Laws in Rectangular CA

TIM BOYKETT[1 2], JARKKO KARI[3]*, SIAMAK TAATI[4 3]

[1] *Time's Up Research Department, Austria*
[2] *Department of Algebra, Johannes-Kepler University, Austria*
[3] *Department of Mathematics, University of Turku, Finland*
[4] *Turku Centre for Computer Science, Finland*

A 1D Reversible Cellular Automata (RCA) with forward and backward radius-$\frac{1}{2}$ neighborhoods is called Rectangular. It was previously conjectured that the conservation laws in 1D Rectangular RCA can be described as linear combinations of independent constant-speed flows to the right or to the left. This is indeed the case; so is a similar statement about a more general class of Rectangular RCA in any dimension.

Key words: Conservation Laws, Noether's Theorem, Cellular Automata, Reversibility

1 INTRODUCTION

In classical formulations of physical systems, an understanding of a phenomenon in terms of causes and effects is used to obtain a functional of the evolution of the state variables of the system through time (i.e., the *action*), whose stationary point singles out a particular trajectory for the system as the one which, according to the model, actually happens in reality (cf. [6]). Noether's Theorem establishes a one-to-one correspondence between the conservation laws of the system and the symmetries of this functional, in virtually every formalism of this type. In cellular automata, in contrast, the dynamical rules of the system are given explicitly, in a matter-of-fact

* Partially supported by the Academy of Finland grant 54102.

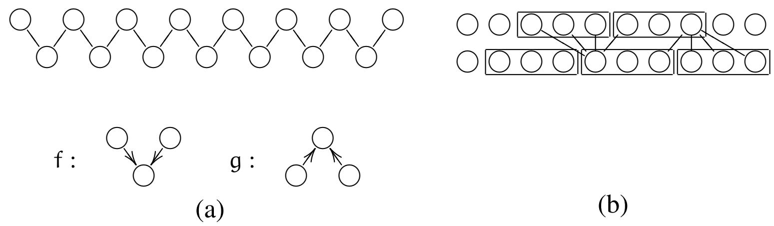

FIGURE 1
(a) A Rectangular RCA with forward rule f and backward rule g. (b) Transforming an arbitrary RCA into a rectangular one.

manner. Therefore, it is not clear how one can find a similar structural interpretation of the conservation laws in cellular automata.

We suggest a structural interpretation for the range-1 additive conservation laws in one-dimensional reversible cellular automata. The significance of the Noether's Theorem in classical physics is two-fold: On the one hand it identifies all the additive conservation laws of a system described in terms of partial differential equations. On the other hand it provides a geometric insight into the nature of conservation laws in such a system. Our interpretation addresses only this latter. Note that for any k, all range-k additive conservation laws of a CA are already identified as the solution space of a set of linear equations [4].

2 THE SETUP

Let $\mathcal{A} = (S, f, g)$ be a one-dimensional reversible CA (RCA) with state set S, and forward and backward local rules $f, g : S \times S \to S$, respectively, where f is applied on the neighborhood $(0, 1)$ and g on the neighborhood $(-1, 0)$ (Figure 1a). Any RCA can be transformed into such form, possibly by grouping blocks of consecutive cells into *super-cells* of a new CA and composing with a suitable translation (Figure 1b). However, our discussion below then gives information only about the cumulative value of a conserved quantity over these super-cells. The details of the dynamics of the conserved quantity within such super-cells remains to be investigated.

An RCA of the above form has the property that the pre-image of each state under the (forward or backward) local rule is the product of the left and right Welch sets (see e.g. [1, 3]). That is, whenever $f(a, b) = f(c, d) = x$, we also have $f(a, d) = f(c, b) = x$. Hence, it is appropriate to call such a CA, *rectangular*, echoing McLean's use of the term in the associative case for rectangular semigroups [5].

Combining with a permutation of the states, every RCA $\mathcal{A}$ can be turned into an idempotent one; i.e., one with $f(x, x) = x$ for all states x. Namely, for each $x \in S$, let $\pi(x) \triangleq f(x, x)$. The RCA $\hat{\mathcal{A}}$ defined by the local rule $\hat{f} \triangleq \pi^{-1} \circ f$ is idempotent. We call $\hat{\mathcal{A}}$, the idempotent *lifting* of $\mathcal{A}$.

A mapping $\mu : S \to \mathbb{R}$ is seen as the local distribution of a quantity such as energy, mass, etc. For a set $K \subseteq \mathbb{Z}$ of cells, and a configuration $c \in S^{\mathbb{Z}}$, $M_K(c) \triangleq \sum_{i \in K} \mu(c(i))$ is the μ-*content* of K under c, whenever it converges. To avoid irrelevant technicalities we always assume that there is a state $e \in S$ with $\mu(e) = 0$. If not, we can simply consider the mapping $\hat{\mu}(x) \triangleq \mu(x) - \mu(e)$ instead.

If $\mu(e) = 0$, the function $M_{\mathbb{Z}}$ converges on any configuration c that contains e almost everywhere. Then, we can say μ is *conserved* by a CA $\mathcal{A}$, if $M_{\mathbb{Z}}(f(c)) = M_{\mathbb{Z}}(c)$ for all such configurations. Equivalently, conservation of μ by $\mathcal{A}$ can be expressed by the local condition

$$\mu(x) - \mu(f(e, x)) + \mu(f(e, y)) - \mu(f(x, y)) = 0$$

for all $x, y \in S$ [4]. This gives a system of linear equations whose solutions are the conserved quantities of $\mathcal{A}$.

Let $\mathcal{A} = (S, f, g)$ be a rectangular RCA as above. If a quantity μ is conserved by $\mathcal{A}$, it is also conserved by its idempotent lifting $\hat{\mathcal{A}}$. Conversely, any quantity μ which is conserved by the idempotent lifting $\hat{\mathcal{A}}$ is also conserved by $\mathcal{A}$, provided that μ is constant over each of the cycles of the permutation π (see [2]). Therefore, hereafter we assume that $\mathcal{A}$ is idempotent. Otherwise we can study its idempotent lifting $\hat{\mathcal{A}}$. The conservation laws of $\mathcal{A}$, then, are obtained simply by ruling out the quantities that are not constant over the cycles of the lifting permutation.

In summary, we only need to study the CA that are normalized in the sense that, are rectangular and idempotent. The conserved quantities are normalized so that they map at least one state to zero.

3 THE FLOW INTERPRETATION

Let $\mathcal{A}$ and $\mathcal{A}'$ be 1D idempotent rectangular RCA, and $h : \mathcal{A} \to \mathcal{A}'$ a state-to-state morphism. Then any conserved quantity μ for $\mathcal{A}'$ gives a conserved

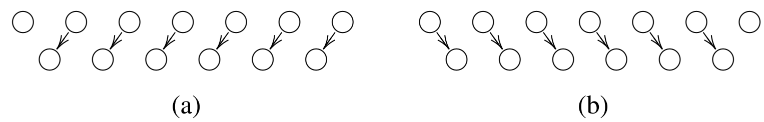

FIGURE 2
(a) Shift-to-the-left CA. (b) Shift-to-the-right CA.

quantity $\mu \circ h$ for $\mathcal{A}$. It was conjectured that all the conservation laws of a rectangular RCA $\mathcal{A}$ are linearly generated by those that are obtained this way from the binary factors of $\mathcal{A}$, i.e., the factors $\mathcal{A}'$ with $S' = \{0, 1\}$, and had been argued that this would somehow resemble the Noether's theorem [2].

Note that *shift-to-the-left* (Figure 2a) and *shift-to-the-right* (Figure 2b) are the only 1D binary idempotent rectangular RCA. The conserved quantities in a binary shift are exactly the multiples of the identity mapping $\mu(x) = k.x$ ($k \in \mathbb{R}$) (modulo an additive constant).

Let $\mathcal{A} = (S, f, g)$ be an idempotent rectangular RCA, and μ a conserved quantity with $\mu(e) = 0$. For any $x \in S$, define the *left flow* $\mu_L(x) \triangleq \mu(f(e, x))$ and the *right flow* $\mu_R(x) \triangleq \mu(f(x, e))$. We have $\mu = \mu_L + \mu_R$. Note also that $\mu_L(e) = \mu_R(e) = 0$. For the inverse rule, we define the left and right flows $\eta_L(x) \triangleq \mu(g(e, x))$ and $\eta_R(x) \triangleq \mu(g(x, e))$ analogously.

Clearly for all $x, y, z \in S$, if $z = f(x, y)$, then $\mu(z) = \mu_R(x) + \mu_L(y)$. (Just consider the configuration $\cdots\ e\ x\ y\ e\ \cdots$.) Similarly, $\mu(x) = \eta_R(u) + \eta_L(v)$, whenever $x = g(u, v)$.

Lemma 1. *If* $z = f(x, y)$, *then* $\mu_R(x) = \eta_L(z)$.

Proof. Look at the consecutive configurations

$$\begin{array}{ll} (t = 0) & \cdots\ e\ e\ x\ y\ \cdots \\ (t = 1) & \cdots\ e\ u\ z\ \cdots \end{array}$$

where $u = f(e, x)$. We have

$$\mu(x) = \mu_L(x) + \mu_R(x) = \eta_R(u) + \eta_L(z)$$

and

$$\mu(u) = \mu_L(x) = \eta_R(u)$$

from which the claim follows immediately. $\square$

Since $z = f(z,z)$ (we assumed that the CA is idempotent), it follows:

Corollary 1. $\mu_R(z) = \mu_R(x)$, *whenever* $z = f(x,y)$.

Hence the right flow μ_R is itself a conserved quantity. Analogous results for the left flow μ_L. Now, observe that for a fixed number k the mapping

$$h_k : S \to \{0,1\}$$
$$h_k(x) \triangleq \begin{cases} 1 & \text{if } \mu_R(x) = k \\ 0 & \text{otherwise,} \end{cases}$$

is a morphism to the binary shift-to-the-right CA (besides being a conserved quantity). It can be seen that μ_R is a linear combination of such mappings:

$$\mu_R(x) = \sum_k k.h_k(x)$$

Similarly, $\mu_L(x)$ can be written as a linear combination of morphisms to the binary shift-to-the-left CA. Therefore we see the validity of the conjecture. In other words:

Theorem 1. *In every 1D idempotent rectangular RCA, every conserved quantity is a sum of non-interacting constant-speed flows to the left and to the right.*

4 RECTANGULAR RCA

The above argument does not readily work in higher dimensions. But it can be generalized to a class of RCA over arbitrary lattices which have a similar rectangularity property.

Let us now define such a rectangularity property, in a more general setting, for the CA over an arbitrary lattice $\mathbb{L}^\star$ and with arbitrary neighborhoods. Let $F : S^{\mathbb{L}} \to S^{\mathbb{L}}$ be the global mapping of a CA with state set S and local rule f. We say the CA (or simply F) is *rectangular* if for every state $x \in S$ and every cell $i \in \mathbb{L}$, there is a set $S_i(x) \subseteq S$ of states, such that the pre-image of the set

$$A(x) \triangleq \{\, c \mid c \text{ is a configuration with } c(0) = x \,\}$$

is the Cartesian product of the sets $S_i(x)$; i.e.,

$$F^{-1}(A(x)) = \prod_{i \in \mathbb{L}} S_i(x)\,.$$

$^\star\mathbb{L}$ can be $\mathbb{Z}^d$, or more generally, any finitely generated group.

Note that $S_i(x) \subsetneq S$ may happen only for a finite number of cells i. In fact, i is in the neighborhood of the cell 0, if and only if $S_i(x) \subsetneq S$ for some x.

It turns out that for the reversible CA the rectangularity is a symmetric property:

Lemma 2. *If an RCA is rectangular, so is its inverse. If the minimal forward neighborhood is* N*, the minimal inverse neighborhood is* $-N$.

Proof. Let $\mathcal{A}$ be an RCA over a lattice $\mathbb{L}$, S its state set, and $F, G : S^{\mathbb{L}} \to S^{\mathbb{L}}$ its forward and backward global mappings. Suppose that F is rectangular, and define the sets $S_i(x)$ as before.

Let $F(a_1) = a_2$ and $F(b_1) = b_2$, where $a_1(0) = b_1(0) = x$. For a fixed $k \in \mathbb{L}$, let c_2 be a configuration obtained from a_2 by switching only the state of the cell k to its value in b_2:

$$c_2(i) \triangleq \begin{cases} b_2(k) & \text{if } i = k, \\ a_2(i) & \text{otherwise,} \end{cases}$$

and define $c_1 = G(c_2)$. For any $i \in \mathbb{L}$, we know that x is in $S_{-i}(c_2(i))$. Therefore, we must have $c_1(0) = x$, since F is rectangular. (Otherwise we could switch the $c_1(0)$ to x, and the state of all cells remained unchanged in $F(c_1)$—a contradiction!) This implies that if for each $k \in \mathbb{L}$ we define

$$\begin{aligned} T_k(x) \;\triangleq\; \{y \mid\ & \text{there exist } c_1 \text{ and } c_2 \text{ with } F(c_1) = c_2, \\ & \text{and } c_1(0) = x \text{ and } c_2(k) = y\}, \end{aligned}$$

then the image $F(A(x))$ is the Cartesian product of the sets $T_k(x)$, and hence G is rectangular, too.

Furthermore, for each $i \in \mathbb{L}$ and $x, y \in S$, we have

$$x \in S_i(y) \qquad \text{if and only if} \qquad y \in T_{-i}(x)\,,$$

which means that the minimal neighborhoods of F and G are the reflections of each other with respect to $i = 0$. $\square$

Consider a rectangular reversible CA with the above notations, and let $\mu : S \to \mathbb{R}$ be a conserved quantity. As before assume that $\mu(e) = 0$ for some state e. Observe that we can actually assume μ to be also non-negative: if not, simply take e to be the state that minimizes μ and study $\hat{\mu}(.) \triangleq \mu(.) - \mu(e)$.

The good point about rectangular RCA is that, similar to the 1D case, we can decompose any conserved quantity into non-interacting flows, each moving with constant speed on a straight line.

For any state x, let us define a configuration $\delta_x : \mathbb{L} \to S$ with

$$\delta_x(k) \triangleq \begin{cases} x & \text{if } k = 0, \\ e & \text{otherwise.} \end{cases}$$

The flows in direction $i \in \mathbb{L}$, for the forward and backward CA, are defined as

$$\mu_i(x) \triangleq \mu\big(F(\delta_x)(i) \big),$$

$$\eta_i(x) \triangleq \mu\big(G(\delta_x)(i) \big).$$

Our claim is that these indeed have a flow-like behavior.

Lemma 3. *Whenever y is in $T_i(x)$, we have $\mu_i(x) = \eta_{-i}(y)$.*

Proof. Let $y \in T_i(x)$. Choose configurations a_1 and a_2 such that the following conditions hold:

i) $F(a_1) = a_2$,

ii) $a_1(0) = x$ and $a_2(i) = y$,

iii) $M_{\mathbb{L}}$ converges over a_1, and $M_{\mathbb{L}}(a_1)$ has the minimum possible value.

Define $m \triangleq M_{\mathbb{L}}(a_1) = M_{\mathbb{L}}(a_2)$.

It is easy to see that, for any $j \neq i$, we have $a_2(j) \in T_j(x)$, and

$$\mu\,(a_2(j)) = \min\{\, \mu(z) \mid z \in T_j(x) \,\}. \tag{1}$$

Similarly, for any $j \neq i$, we have $a_1(j) \in S_{j-i}(y)$, and

$$\mu\,(a_1(j)) = \min\{\, \mu(z) \mid z \in S_{j-i}(y) \,\}. \tag{2}$$

Now let $b_2 \triangleq F(\delta_x)$. From equations (1) and (2) it is clear that $\mu(b_2(j)) \geq \mu(a_2(j))$, for any $j \neq i$. If $\mu(b_2(j)) > \mu(a_2(j))$ for some j, define a new configuration c_2 with

$$c_2(k) \triangleq \begin{cases} a_2(j) & \text{if } k = j, \\ b_2(k) & \text{otherwise,} \end{cases}$$

and let $c_1 = G(c_2)$. Clearly $c_1(0) = x$ (the CA is rectangular), but

$$M_{\mathbb{L}}(c_1) = M_{\mathbb{L}}(c_2) < M_{\mathbb{L}}(b_2) = M_{\mathbb{L}}(\delta_x) = \mu(x),$$

which is a contradiction. Therefore we must have $\mu(b_2(j)) = \mu(a_2(j))$ for any $j \neq i$.

Now, writing the conservation law for δ_x we have

$$\mu(x) = m - \mu(y) + \mu(b_2(i))$$

which implies

$$\mu_i(x) = \mu(b_2(i)) = \mu(x) + \mu(y) - m$$

Due to symmetry, we also have

$$\eta_{-i}(y) = \mu(x) + \mu(y) - m$$

which gives the required result. □

This, of course, implies that in general, the value of μ on each cell is equal to the sum of flows coming from its neighbors one step before. If we further require idempotency (i.e., every homogeneous configuration to be a fixed point of F), similar to the 1D case we can argue that for each i, the flow μ_i in direction i is conserved by its own.

Theorem 2. *In every idempotent rectangular RCA, every conserved quantity is a sum of non-interacting constant-speed flows each streaming to a cell from one of its neighbors.*

5 CONCLUSION

We have seen that the form of conserved quantities in idempotent one-dimensional rectangular RCA reduce to the sum of non-interacting flows, confirming a conjecture based upon experiments. We have then extended this result to a more general case without the dimensionality constraint.

The non-idempotent case (in 1D) allows flows to interact. This is similar to the case in Margolus neighborhood scheme [7]. Their two-step model has such an idempotent flow step followed by a cell-internal permutation step, similar to the general situation described above in section 2.

In general we conjecture that a form of general "idempotence" can be defined and that "idempotent" rectangular RCA will have non-interacting flows. Then general rectangular RCA have "context-independent" (a generalization of idempotent) flows over single time steps and that a form of density summation is possible. However this remains work for the future.

REFERENCES

[1] Tim Boykett. (2004). Efficient exhaustive listings of reversible one dimensional cellular automata. *Theoretical Computer Science*, 325:215–247.

[2] Tim Boykett. (2005). Towards a Noether-like conservation law theorem for one dimensional reversible cellular automata. `arXiv:nlin.CG/0312003`.

[3] Eugen Czeizler and Jarkko Kari. (2005). A tight linear bound on the neighborhood of inverse cellular automata. In *Proceedings of ICALP 2005*, number 3580 in Lecture Notes in computer Science, pages 410–420.

[4] Tetsuya Hattori and Shinji Takesue. (1991). Additive conserved quantities in discrete-time lattice dynamical systems. *Physica D*, 49:295–322.

[5] David McLean. (1954). Idempotent semigroups. *American Mathematical Monthly*, 61:110–113.

[6] Gerald J. Sussman, Jack Wisdom, and Meinhard E. Mayer. (2001). *Structure and Interpretation of Classical Mechanics*. MIT Press.

[7] T. Toffoli and N. Margolus. (1987). *Cellular Automata Machines*. MIT Press.