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ON CONJUGACY OF LANGUAGES *

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Abstract. We say that two languages X and Y are conjugates if they satisfy the *conjugacy equation* $XZ = ZY$ for some language Z. We study several problems associated with this equation. For example, we characterize all sets which are conjugated via a two-element biprefix set Z, as well as all two-element sets which are conjugates.

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1. Introduction

Since a seminal paper of Makanin in 1976, *cf.* [12], word equations have been studied quite intensively. Despite the fact that many fundamental problems, such as the exact complexity of the satisfiability problem, *cf.* [14], or the maximal size of independent systems of equations in n variables, *cf.* [5], are not solved, one can say that there exists a deep and rich theory on word equations. A pioneering paper on modern combinatorics on words is [11], which actually deals with some problems related to our considerations.

If language equations, as extensions of word equations, are considered the situation changes drastically: almost nothing is known about those. The goal of this paper is to initiate a research on a particular language equation, namely on the conjugacy equation $XZ = ZY$. We point out differences from the word case, solve it in certain simple cases, as well as formulate some interesting open problems.

Keywords and phrases: conjugacy equation, languages, Conway's Problem

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Recently, a special case of the conjugacy equation, namely the commutation equation $XZ = ZX$ for languages has been studied in a number of papers. In certain cases, for example when X is a two-element set or X is a prefix set, *cf.* [2] and [15], it is completely solved: Z must be of the form $Z = \cup_{i \in I} \varrho(X)^i$ with $I \subseteq \mathbb{N}$, and $\varrho(X)$ being the root of X, *i.e.*, the minimal set having the set X as its power. On the other hand, the answer to an old problem of Conway, *cf.* [3], asking whether the (unique) maximal set Z commuting with a given rational X is also rational, is known only in some very special cases, like in the cases when X is a three-element set or X is an ω-code, *cf.* [8], [6] or [7] for a survey. Consequently, the conjugacy equation for languages cannot be easy.

Let us recall that the conjugacy equation $xz = zy$ for non-empty words has a well known general solution:

$$\exists p, q \in \Sigma^* \text{ such that } x = pq,\ y = qp \text{ and } z \in (pq)^* p.$$

As is immediate to check the words can be replaced by languages (or finite languages) to obtain solutions of the conjugacy equation $XZ = ZY$ for languages: triples

$$X = PQ,\ Y = QP \text{ and } Z = \bigcup_{i \in I} (PQ)^i P$$

for $P, Q \subseteq \Sigma^*$ and $I \subseteq \mathbb{N}$, are solutions. They are referred to as *word type solutions*. However, not all solutions are of word type even in the case when Z is a two-element prefix set, *cf.* Examples 3.4 and 3.5.

We associate with the conjugacy equation $XZ = ZY$ the following four different basic problems:

Problem 1.1. Given a set $Z \subseteq \Sigma^+$, describe all pairs (X, Y) such that $XZ = ZY$, *i.e.*, X and Y are conjugated via Z.

Problem 1.2. Given sets $X, Y \subseteq \Sigma^+$, describe all sets Z such that $XZ = ZY$, *i.e.*, X and Y are conjugated via Z.

Problem 1.3. Describe all sets $X, Y \subseteq \Sigma^+$ which are conjugates, *i.e.*, there exists a non-empty set Z such that $XZ = ZY$.

Problem 1.3 can be stated as a decision problem as well:

Problem 1.4. Decide whether two finite sets are conjugates, *i.e.*, there exists a non-empty set Z such that $XZ = ZY$.

What we are able to say about these problems is as follows. In Section 3 we solve Problem 1.1 for two-element biprefix codes Z, as well as show, via examples, that for arbitrary two-element sets the problem looks essentially more complicated. In Section 4 we characterize when two-element sets X and Y are conjugates, thus giving an implicit solution to this special case of Problem 1.3.

We consider Problem 1.4 particularly interesting. It formulates a very special case of satisfiability problem over the monoid of languages with finite constants.

Even this very simple case does not seem easy indicating that the general satisfiability problem might be very hard, if not even undecidable. Actually, Problem 1.4 (as well as Problem 1.3) has two variants: either general or finite Z is asked.

Finally, we emphasize that we consider language equations as generalizations of word equations, that is to say as equations with only one associative operation, the product. For certain special equations with two operations — union and product — a nice theory can be built based on a general fixed point approach. Computing a rational expression for a given finite automaton is an example of that, *cf.* [4] or [9] for a general treatment.

2. Preliminaries

Let Σ be a finite alphabet, and Σ^* (resp. Σ^+) the free monoid (resp. semigroup) generated by Σ. We use lowercase letters to denote words, *i.e.*, elements of Σ^*, and capital letters to denote languages, *i.e.*, subsets of Σ^*. Mostly we consider finite languages. The empty word is denoted by 1, and the length of a word w by $|w|$. For a set X by $|X|$ we mean its cardinality. Occasionally we consider infinite words, *i.e.*, elements of Σ^ω or infinite powers of languages. The set of all suffixes (resp. prefixes) of a set X is denoted by $\mathrm{Suff}(X)$ (resp. $\mathrm{Pref}(X)$).

We say that a word $u \in \Sigma^+$ is *primitive* if for any word $z \in \Sigma^+$ and any integer n, the equation $u = z^n$ implies $n = 1$. The *primitive root* of a word $u \in \Sigma^+$, denoted $\varrho(u)$, is the unique primitive word of which u is a power. Furthermore, we say that words x and y *commute* if they satisfy the equation $xy = yx$. The following conditions are equivalent:

- words x and y commute;
- words x and y have a common power;
- there exists a word t such that $x, y \in t^*$;
- $\varrho(x) = \varrho(y)$.

In our later considerations we will also need the following result, known as Fine and Wilf Theorem:

Lemma 2.1. *Let $x, y \in \Sigma^+$. If words x^ω and y^ω have a common prefix of length $|x| + |y| - \gcd(|x|, |y|)$, then x and y commute.*

By the *conjugacy equation* we mean the equation

$$XZ = ZY\,. \tag{1}$$

As is well known, in the case of words it characterizes when two words x and y are conjugates, *i.e.*, of the forms $x = pq$ and $y = qp$ for some words p and q. More precisely:

Lemma 2.2. *If two words $x, y \in \Sigma^+$ satisfy the conjugacy equation $xz = zy$ for some $z \in \Sigma^*$, then there exist words p and q such that pq is primitive and*

$$x = (pq)^i, \quad y = (qp)^i, \quad \textit{and } z \in p(qp)^*,$$

for some integer $i \geq 1$.

For the proofs and more details we refer the reader to [10] or [1].

For languages we take (1) as a definition of conjugacy. We say that languages X and Y are *conjugates*, in symbols $X \sim Y$, if there exists a non-empty set Z such that (X, Y, Z) is a solution of (1). If this is the case we also say that X and Y are *conjugated via* Z, and we write $X \sim_Z Y$. Note that a stricter definition of conjugacy of codes, corresponding to what we call word type solutions, was studied in [13].

Obviously, if $X_1 \sim_Z Y_1$ and $X_2 \sim_Z Y_2$ then also $X_1 \cup X_2 \sim_Z Y_1 \cup Y_2$. Consequently, for a given Z there exists the unique solution of (1) which is maximal with respect to both components. We call this solution the *maximal* solution of (1). We say that a solution (X_1, Y_1) is *contained* in a solution (X, Y), if $X_1 \subseteq X$, $Y_1 \subseteq Y$ and $(X_1, Y_1) \neq (X, Y)$. Dually to the notion of the maximal solution, we call a solution (X, Y) of (1), for a given Z, *minimal*, if there are no solutions (X_1, Y_1) and (X_2, Y_2) of (1) contained in the solution (X, Y) such that

$$X = X_1 \cup X_2 \quad \text{and} \quad Y = Y_1 \cup Y_2\,. \tag{2}$$

Of course, there might be several minimal solutions. Clearly, all finite solutions can be expressed as component-wise unions of minimal solutions. We have to use Zorn's Lemma to derive the same result in the general case:

Proposition 2.3. *All solutions* (X, Y) *of the conjugacy equation* (1)*, including the maximal one, can be obtained as component-wise unions of minimal solutions.*

Proof. Let x_0 be any element of X, and consider the set E of all solutions (X', Y') contained in (X, Y) such that x_0 belongs to X'. This is a partially-ordered set for the relation "contains", and every chain (X_i, Y_i) has a lower bound (X', Y') where X' is the intersection of the X_i's, and Y' the intersection of the Y_i's. Indeed, it is clear that $X'Z$ and ZY' are subsets of $X_iZ = ZY_i$ for all i. Conversely, assume that w is in the intersection of all X_iZ's, then for each i, $w = x_iz_i$ for some $x_i \in X_i$ and $z_i \in Z$. There are only finitely many different x_i's, forming a set X''. One of them, say x_i, must be in X' (otherwise, for each $x \in X''$ there is a j such that x is not in X_j, but then consider m the maximum of such j's, no element of X'' can be in X_m). Then for this x_i, we have that $w = x_iz_i$ is in $X'Z$. Similarly w is in ZY', so $X'Z = ZY'$. So (X', Y') is in E, and it is obviously contained in all (X_i, Y_i). Now, by Zorn's Lemma, E has a minimal element, which is precisely a minimal solution (X', Y') with X' containing x_0. So (X, Y) is the union of all the minimal solutions contained in it. □

Note that if one wants to avoid using Zorn's Lemma, the fact that every element x (resp. y) contained in the maximal solution belongs also to a minimal solution has to be shown. This can be straightforward in particular cases, *cf.* Theorem 3.13 and Corollary 3.14.

We mainly concentrate to study (1) in the cases when

(i) Z or X and Y are two-element sets; or
(ii) Z is a biprefix code, *i.e.*, no word of Z is a prefix or a suffix of another word of Z.

One can easily prove the following simple facts which we will use many times later on.

Proposition 2.4. *If sets X, Y and Z satisfy the conjugacy equation* (1)*, then*
(i) *for every positive integer n, $X^nZ = ZY^n$;*
(ii) $Z \subseteq \mathrm{Pref}(X^+) \cap \mathrm{Suff}(Y^+)$.

3. The conjugacy via a given Z

In this section we study Problem 1.1, *i.e.*, given a set Z we would like to describe all pairs of sets (X, Y) such that $XZ = ZY$.

As we already noted the conjugacy equation (1) has always word type solutions

$$X = PQ, \quad Y = QP, \quad \text{and} \quad Z = \bigcup_{i \in I} (PQ)^i P \tag{3}$$

for $P, Q \subseteq \Sigma^*$ and $I \subseteq \mathbb{N}$. In some cases these are the only possible solutions. For example, if the sets X, Y and Z are prefix codes, or if the sets X and Y are uniform, *i.e.*, consist of words of a fixed length, then the equation (1) has only word type solutions. This follows from the fact that the monoids of prefix codes, *cf.* [13], and of uniform non-empty languages are free. Consequently, we formulate:

Proposition 3.1. *If prefix codes X, Y and Z satisfy the conjugacy equation* (1)*, with $X, Y \neq \{1\}$, then there exist prefix codes $P, Q \subseteq \Sigma^*$ and an integer $i \in \mathbb{N}$ such that $X = PQ$, $Y = QP$ and $Z = (PQ)^i P$.*

If we assume that the sets X and Y are uniform, we can decompose the set Z into uniform subsets, and clearly, (X, Y) is a solution of (1) for each such subset of the set Z as well. Therefore, we have the following proposition:

Proposition 3.2. *If sets X, Y and Z satisfy the conjugacy equation* (1) *and X and Y are uniform, with $X, Y \neq \{1\}$, then there exist uniform sets $P, Q \subseteq \Sigma^*$ and $I \subseteq \mathbb{N}$ such that $X = PQ$, $Y = QP$ and $Z = \cup_{i \in I} (PQ)^i P$.*

However, as we shall see in the sequel, not all solutions are of the word type, even for unary Z.

If Z has only one element z, it is easy to prove the following statement:

Proposition 3.3. *If $Z = \{z\}$ is a singleton, then the maximal solution of the conjugacy equation* (1) *is $(X_{\max}, Y_{\max})$, where $X_{\max} = \{pq : \exists m \geq 0, z = (pq)^m p\}$ and $Y_{\max} = \{qp : \exists m \geq 0, z = (pq)^m p\}$.*

Proof. Since Z is a singleton, each element of X is conjugated via z to an element of Y. Consequently, minimal pairs are pairs of singletons, $(\{x\},\{y\})$, such that $xz = zy$: this is the word case, so that $x = pq$, $y = qp$ and $z = (pq)^m p$ for some integer $m \geq 0$ and words p and q. Taking the union of all minimal solutions we get the desired maximal one. □

Now consider a solution (X, Y) of (1) where Z is a singleton, which is a union of singleton pairs $(\{x_i\},\{y_i\})_{i\in J}$ such that for all of them the corresponding $m_i = 0$. Then for all $i \in J$ the corresponding p_i satisfies $p_i = z$. If we take $Q = \{q_i : i \in J\}$, then $X = zQ = ZQ$ and $Y = Qz = QZ$. Hence, the considered solution is of the form (ZQ, QZ), which is a special case of the word type solution (3) with $I = \{0\}$. On the other hand, the union of minimal solutions for which $m \neq 0$ cannot usually be described in such a compact way:

Example 3.4. $Z = \{aa\}$, $X = Y = \{a, aaa\}$ is a solution of the conjugacy equation (1), which is not of word type. However, this is not a minimal solution either.

An example of a binary prefix code Z, which allows a minimal solution not being of word type, is as follows:

Example 3.5. $Z = \{a, ba\}$, $X = \{a, ab, abb, ba, babb\}$, $Y = \{a, ba, bba, bbba\}$ is a solution of (1). This is a minimal solution, but not of word type. Indeed, the only solutions contained in (X, Y) are: $X_1 = Y_1 = \{a, ba\}$, $X_2 = \{abb, babb\}$, $Y_2 = \{bba, bbba\}$, and their union, which does not form the whole (X, Y).

Note that here X and Y are of different cardinality and satisfy $ZQ' \subseteq X \subseteq ZQ$ and $Y = QZ = Q'Z$ with $Q = \{1, b, bb\}$ and $Q' = \{1, bb\}$.

This is an example of a method to search for conjugates of different cardinality. Indeed if there exist three sets P, Q', Q such that $Z = \cup_{i\in I}(PQ)^i P$ for some $I \subseteq \mathbb{N}$ and $PQ' \subseteq X \subseteq PQ$, $QP \subseteq Y \subseteq Q'P$, then the sets X and Y are conjugates. This follows since the conditions imply that $PQ'P = PQP$.

Example 3.5 shows one interesting property of the relation $\sim$ for languages.

Lemma 3.6. *The relation $\sim$ for language is reflexive, transitive, but not symmetric.*

Proof. It is easy to check the reflexivity and the transitivity. To prove that $\sim$ is not symmetric, consider the sets X, Y, Z from Example 3.5. Assume for a contradiction that there exists a non-empty set W such that $WX = YW$. Take a $w \in W$. It can be written in the form

$$w = b^{n_0} ab^{n_1} a \ldots ab^{n_k},$$

where $k, n_0, \ldots, n_k \geq 0$. Now, $wab = b^{n_0} ab^{n_1} a \ldots ab^{n_k} ab \in WX = YW$, which is possible only if $n_0 \in \{0, 1, 2, 3\}$ and $b^{n_1} a \ldots ab^{n_k} ab \in W$. Repeating the same argument several times we get that $b(ab)^k \in W$. Now again, $bbbab(ab)^k \in YW = WX$, which implies that $bbb(ab)^k \in W$. Since $bbb \notin \mathrm{Suff}(X^+)$, we have also $bbb(ab)^k \notin \mathrm{Suff}(X^+)$. But this is a contradiction, since by Proposition 2.4, $W \subseteq \mathrm{Suff}(X^+)$. □

On the other hand the relation $\sim$ for words is clearly an equivalence relation, *i.e.*, it is reflexive, transitive and symmetric.

3.1. The case when Z is a biprefix code.

Let X, Y, Z, with Z a biprefix code, be a solution of the conjugacy equation. Then both products XZ and ZY are unambiguous, *i.e.*, for all $(x, z) \in X \times Z$, there exists exactly one pair $(z', y) \in Z \times Y$ such that $xz = z'y$, and conversely. This allows a more detailed analysis.

The conjugacy equation yields instances of word equations of type:

$$xz = z'y \quad \text{with } x \in X,\, y \in Y \text{ and } z, z' \in Z. \tag{4}$$

We call such an instance *overlapping* or *non-overlapping* depending on whether $|x| < |z'|$ or $|x| \geq |z'|$, respectively. The reason why we separate these two cases is the following result:

Proposition 3.7. *Let Z be a biprefix code. The minimal solutions of the conjugacy equation* (1) *consist entirely of x- and y- values of either overlapping or non-overlapping instances of* (4).

Proof. Let $Z = \{z_i\}_{i \in I}$ be a biprefix code and let us fix an $x \in X$. For all $k \in I$ we can write $xz_k = z'_k y_k$. We claim that if one of these instances, say $xz_i = z'_i y_i$, is non-overlapping, so are all, and moreover all z'_k's are equal. Indeed, if $xz_j = z'_j y_j$, then z'_j is a prefix of xz_j and since z'_i is a prefix of x, necessarily $z'_i = z'_j$; otherwise Z would not be biprefix.

The above allows us to write $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, where X_1 and Y_1 consist of exactly those x- and y- values of (4) which come from overlapping instances. Then, clearly, $X_1 Z \subseteq ZY_1$, and symmetrically $ZY_1 \subseteq X_1 Z$. Therefore $X_1 Z = ZY_1$, and similarly $X_2 Z = ZY_2$. □

We proceed by analysing the two disjoint cases of Proposition 3.7. In the non-overlapping case the situation is easier:

Proposition 3.8. *Let Z be a biprefix code. The solutions of the conjugacy equation* (1) *corresponding to non-overlapping instances of* (4) *are of the form: $Y_1 = QZ$ and $X_1 = ZQ$, for some $Q \subseteq \Sigma^*$. Conversely, every pair (X_1, Y_1) of this form is conjugated via Z.*

Proof. Since in any non-overlapping instance of word equation (4) $|z'| \leq |x|$ and $|z| \leq |y|$, z' is uniquely determined by x, and symmetrically z is uniquely determined by y. Moreover, for a given $x \in X_1$, there exists a unique q such that $x = z'q$. For a fixed $x \in X_1$ and arbitrary $z \in Z$ we have $qz = y \in Y_1$, and therefore $qZ \subseteq Y_1$. Consider the set

$$Q = \{q : \; x = z'q, z' \in Z, x \in X_1\}.$$

It is easy to see that $QZ \subseteq Y_1$. But also the reverse inclusion $Y_1 \subseteq QZ$ holds, since any $y \in Y_1$ can be written in the form $y = qz$ with $q \in Q$ and $z \in Z$.

Hence, $Y_1 = QZ$ and also $X_1 = ZQ$, since $X_1Z = ZY_1 = ZQZ$ and Z is a biprefix code. □

The message of the above proposition is that solutions obtained from non-overlapping instances of (4) are of word type. For the other solutions this need not be true in general, even in the case when Z is a biprefix code as shown in the following examples.

Example 3.9. $Z = \{aaa, aabaa\}$, $X = \{a, aab\}$, $Y = \{a, baa\}$ is a solution of (1) corresponding to overlapping instances. It is not of word type.

Example 3.10. $Z = \{aabaabaa, aabaaabaa\}$, $X = \{aab, aaba\}$, $Y = \{baa, abaa\}$ is another similar solution, as in Example 3.9, not of word type.

We are not able to characterize all solutions corresponding to overlapping instances of (4), except in the case when $|Z| = 2$. This will be done in Section 3.2, as an application of the following general considerations.

We assume that neither X nor Y contains the empty word. This can be assumed since $(\{1\}, \{1\})$ is always a minimal solution and if one of those sets contains 1 so does the other, due to the fact that Z is a biprefix code. By Proposition 2.4, $X^nZ = ZY^n$ for all n. Now, when n tends to infinity, we have $ZY^\omega \subseteq X^\omega$ and ${}^\omega XZ \subseteq {}^\omega Y$. Hence, any element of Z is a prefix of a word in X^ω and a suffix of a word in ${}^\omega Y$. Conversely, any word in X^ω has an element of Z as a prefix, and this element is unique since Z is a prefix code. Indeed, take an infinite word $\tilde{x} = x_1x_2 \cdots \in X^\omega$ and an arbitrary $z \in Z$. Since $X^nZ = ZY^n$ for all n, there are words $z' \in Z$ and $y_1, \ldots, y_n \in Y$ such that $x_1 \ldots x_n z = z'y_1 \ldots y_n$. If we take n such that $n \min_{y \in Y} |y| \geq |z|$, then z' is a prefix of $x_1 \ldots x_n$ which is a prefix of $\tilde{x}$. Since Z is a prefix code, such z' is unique. Similarly, any word in ${}^\omega Y$ has a unique element of Z as a suffix.

Now, fix an x in X, and let z be the element of Z which is a prefix of x^ω. Then $xz = z'y$ for some $z' \in Z$ and $y \in Y$. But z' is a prefix of xz which is a prefix of x^ω, and so by the uniqueness $z' = z$. Thus $x^nz = zy^n$ for all n and z is also the unique suffix of ${}^\omega y$.

We have defined two maps, $f : X \to Y$, $x \mapsto y$, and $g : X \to Z$, $x \mapsto z$. Moreover, f is a bijection, because y is uniquely defined from x, and symmetrically starting from y we find an x' such that $x'z = zy = xz$, implying that $x' = x$.

Lemma 3.11. *If Z is a biprefix code and $X \sim_Z Y$, then there exist two unique maps: $f : X \to Y$ and $g : X \to Z$ such that for every $x \in X$, the elements $y = f(x)$ and $z = g(x)$ satisfy $xz = zy$. Moreover, the map f is a bijection.*

Proof. Above we have shown that such maps exist. Now we prove that they are unique. Hence, assume for contradiction that for some $x \in X$ there are $z_1, z_2 \in Z$ and $y_1, y_2 \in Y$ such that $xz_1 = z_1y_1$ and $xz_2 = z_2y_2$. Since for any integer n, $x^nz_1 = z_1y_1^n$ and $x^nz_2 = z_2y_2^n$, this would imply that both z_1 and z_2 are prefixes of x^ω and we know that x^ω has the unique prefix from Z. Hence, the map g is unique.

For $x \in X$ and $z \in Z$, we have a unique y such that $xz = zy$, as already shown above. □

Lemma 3.11 has a nice consequence, *cf.* also Example 3.5.

Corollary 3.12. *If Z is a biprefix code and $X \sim_Z Y$, then necessarily $|X| = |Y|$.*

3.2. The case when Z is a binary biprefix set.

We now assume that $Z = \{z_1, z_2\}$ is a **binary** biprefix code and f and g are the mappings of Lemma 3.11. Let $X_1 = g^{-1}(z_1)$, $Y_1 = f(X_1)$, $X_2 = g^{-1}(z_2)$ and $Y_2 = f(X_2)$, where, however, X_i's and Y_i's are not as in the previous section. Then $X_1 z_1 = z_1 Y_1$ and $X_2 z_2 = z_2 Y_2$. Since the products XZ and ZY are unambiguous, we have $X_1 z_2 \cup X_2 z_1 = z_2 Y_1 \cup z_1 Y_2$. But it is impossible to have $x_1 z_2 = z_2 y_1$ with $x_1 \in X_1$ and $y_1 \in Y_1$, since the map g is unique. Therefore $X_1 z_2 = z_1 Y_2$ and $X_2 z_1 = z_2 Y_1$.

Now, given $x_1 \in X_1$, let $y_1 = f(x_1)$, so that $x_1 z_1 = z_1 y_1$, $y_2 \in Y_2$ such that $x_1 z_2 = z_1 y_2$, and $x_2 = f^{-1}(y_2)$, so that $x_2 z_2 = z_2 y_2$. Finally, let $y_3 \in Y_1$ such that $x_2 z_1 = z_2 y_3$. Then we have $|x_1| = |y_1|$, $|x_2| = |y_2|$, $|z_1| - |z_2| = |x_1| - |y_2| = |y_3| - |x_2|$, hence $|y_3| = |y_1|$. If $|z_1| \geq |y_1|$, then y_1 is a suffix of z_1, which is a suffix of $z_2 y_3$, hence $y_3 = y_1$. If $|z_1| < |y_1|$, then y_1 and y_3 have a common suffix z_1. Let then t be a word such that $y_1 = t z_1$ and $x_1 = z_1 t$. This implies $y_2 = t z_2$ and $x_2 = z_2 t$, and consequently $y_3 = t z_1 = y_1$. In both cases, we concluded that $y_3 = y_1$, therefore $(\{x_1, x_2\}, \{y_1, y_2\})$ is a minimal solution, and all minimal solutions have this form. As a conclusion, there exist words p_1, q_1, p_2, q_2 and integers m_1, m_2 such that

$$\begin{aligned} x_1 &= p_1 q_1, & y_1 &= q_1 p_1, & z_1 &= (p_1 q_1)^{m_1} p_1, \\ x_2 &= p_2 q_2, & y_2 &= q_2 p_2, & z_2 &= (p_2 q_2)^{m_2} p_2, \end{aligned} \tag{5}$$

since $x_1 z_1 = z_1 y_1$ and $x_2 z_2 = z_2 y_2$. The other relations $x_1 z_2 = z_1 y_2$ and $x_2 z_1 = z_2 y_1$ provide the equations

$$q_1 (p_2 q_2)^{m_2} = (q_1 p_1)^{m_1} q_2, \qquad q_2 (p_1 q_1)^{m_1} = (q_2 p_2)^{m_2} q_1, \tag{6}$$

which are sufficient and necessary conditions for $(\{x_1, x_2\}, \{y_1, y_2\})$ to be a solution. So we have proved:

Theorem 3.13. *Let Z be a binary biprefix code. Then all the minimal solutions of the conjugacy equation* (1) *are of the form $(\{x_1, x_2\}, \{y_1, y_2\})$ such that there exist words p_1, q_1, p_2, q_2 and integers m_1, m_2 satisfying* (5) *and* (6).

From the definition of a minimal solution we immediately obtain:

Corollary 3.14. *Let Z be a binary biprefix code. Then all sets X and Y which are conjugated via Z are obtained as component-wise unions of sets satisfying* (5) *and* (6).

Next, we will refine our considerations by analysing when and how (6) can be satisfied. Without loss of generality we can assume that $m_1 \geq m_2$.

We have the following cases:

Case $m_1 = m_2 = 0$. Then the equations reduce to $q_1 = q_2 = q$, $p_1 = z_1$ and $p_2 = z_2$, and we have the minimal solution (Zq, qZ), already found in Proposition 3.8.

Case $m_2 = 0$ and $m_1 > 0$. Then the equations become $q_1 = (q_1p_1)^{m_1}q_2 = q_2(p_1q_1)^{m_1}$, which is only possible if $m_1 = 1$ and $p_1 = q_2 = 1$. Then $x_1 = y_1 = z_1 = q_1$ and $x_2 = y_2 = z_2 = p_2$: this is a particular case of the previous solution (Zq, qZ), with $q = 1$.

Further on, we can suppose that $m_1 > 0$ and $m_2 > 0$. Now, both equations in (6) reduce to

$$(p_1q_1)^{m_1-1}p_1 = (p_2q_2)^{m_2-1}p_2\,, \tag{7}$$

and then $Xt = tY$ with $t = (p_1q_1)^{m_1-1}p_1$.

Case $m_1 = m_2 = 1$. Then $p_1 = p_2 = p$. Let $Q = \{q_1, q_2\}$. Then $Z = pQp$ and the minimal solution is (pQ, Qp), *i.e.*, a solution of word type.

Case $m_2 = 1$ and $m_1 > 1$. Then $p_2 = (p_1q_1)^{m_1-1}p_1$. Example 3.9 is of this type with $m_1 = 2$, $p_1 = a$, $q_1 = 1$, $q_2 = b$.

Case $m_1 = m_2 = 2$. Then $p_1q_1p_1 = p_2q_2p_2$. Assuming that $|p_1| \le |p_2|$, there are words r and s and an integer l such that $p_1 = (rs)^lr$, $p_2 = (rs)^{l+1}r$, $q_1 = srq_2rs$. Then

$$z_1 = (rs)^{l+1}rq_2(rs)^{l+2}rq_2(rs)^{l+1}r \quad \text{and} \quad z_2 = (rs)^{l+1}rq_2(rs)^{l+1}rq_2(rs)^{l+1}r\,.$$

Example 3.10 is of this type with $l = 0$, $r = a$, $s = 1$, $q_2 = b$.

Case $m_1 > 2$ and $m_2 > 2$. Then the word $w = (p_1q_1)^{m_1-1}p_1 = (p_2q_2)^{m_2-1}p_2$ has two periods $|p_1q_1|$ and $|p_2q_2|$, and its length is more than $|p_1q_1|+|p_2q_2|$. Hence, according to Lemma 2.1, there is a common period t for p_1q_1 and p_2q_2. But then t is a common period for z_1 and z_2, hence one is a prefix of the other, which contradicts the fact that Z is a biprefix. So there are no solutions of this type.

Case $m_2 = 2$ and $m_1 > 2$. Then $p_2q_2p_2 = (p_1q_1)^{m_1-1}p_1$. If $|p_2| \ge |p_1q_1|$, the above argument also applies. Let us therefore assume that $|p_2| < |p_1q_1|$, hence $|q_2| > |(p_1q_1)^{m_1-3}p_1|$. If $|p_2| \le |p_1|$, there are words r and s and an integer l such that $p_2 = (rs)^lr$, $p_1 = (rs)^{l+1}r$, $q_2 = srq_1(p_1q_1)^{m_1-2}rs$. Then

$$\begin{aligned} z_1 &= [(rs)^{l+1}rq_1]^{m_1}(rs)^{l+1}r \quad \text{and} \\ z_2 &= [(rs)^{l+1}rq_1]^{m_1-1}(rs)^{l+2}r[q_1(rs)^{l+1}r]^{m_1-1}\,. \end{aligned}$$

The following example shows the simplest solution of this case:

Example 3.15. With $l = 0$, $m_1 = 3$, $r = a$, $s = 1$, $q_1 = b$ we get $Z = \{aabaabaabaa, aabaabaaabaabaa\}$, $X = \{aab, aabaaba\}$, $Y = \{baa, abaabaa\}$.

For the case $|p_2| > |p_1|$ we do not have an exhaustive analysis. In this case more complicated solutions are possible, for instance:

Example 3.16. For $p_1 = a$, $q_1 = aba$, $p_2 = aa$, $q_2 = baaab$ and $m_1 = 3$ we have $Z = \{aabaaabaaabaa, aabaaabaabaaabaa\}$, $X = \{aaba, aabaaab\}$, $Y = \{abaa, baaabaa\}$. Note that here $|p_1q_1| > |p_2| > |p_1|$.

4. When binary sets X and Y are conjugates

In this section we study when two sets X and Y are conjugates. That is to say, we want to characterize all pairs (X, Y) which are conjugates, as well as characterize all sets Z such that $X \sim_Z Y$. We start with a few general observations.

Whenever $X \sim Y$, then necessarily

$$\min_{x\in X} |x| + \min_{z\in Z} |z| = \min_{z\in Z} |z| + \min_{y\in Y} |y|,$$

and therefore also $\min_{x\in X} |x| = \min_{y\in Y} |y|$. Moreover, the sets

$$X_1 = \{x_1 \in X : \quad |x_1| = \min_{x\in X} |x|\}$$

and

$$Y_1 = \{y_1 \in Y : \quad |y_1| = \min_{y\in Y} |y|\}$$

are conjugated via $Z_1 = \{z_1 \in Z : \quad |z_1| = \min_{z\in Z} |z|\}$. If $1 \in X$ then $X_1 = Y_1 = \{1\}$, so Y contains also 1. Obviously, all languages containing the empty word are conjugates (via Σ^*). In the sequel, we assume that $1 \notin X$ and $1 \notin Y$, i.e., $X_1, Y_1 \neq \{1\}$. Since all the sets X_1, Y_1 and Z_1 are uniform, by Proposition 3.2 necessarily

$$X_1 = PQ, \quad Y_1 = QP, \quad \text{and} \quad Z_1 = (PQ)^i P$$

for some non-negative integer i and uniform sets P and Q. Hence, we have the following proposition:

Proposition 4.1. *Let $X \sim_Z Y$ with $X, Y \subseteq \Sigma^+$ and Z non-empty. Let X_1 (resp. Y_1, Z_1) be the set of the elements of X (resp. Y, Z) of the minimal length. There exist uniform sets P and Q and an integer $i \geq 0$ such that $X_1 = PQ$, $Y_1 = QP$, and $Z_1 = (PQ)^i P$. In particular, if $|X_1| = 1$ or $|Y_1| = 1$, then P and Q must be singletons and $X_1 = \{(uv)^m\}$, $Y_1 = \{(vu)^m\}$, $Z_1 = \{(uv)^{k_1} u\}$ for some words u and v, where uv is primitive, and some integers $m \geq 1$ and $k_1 \geq 0$.*

In what follows we characterize the relation $X \sim Y$ for two-element sets X and Y. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ with $|x_1| \leq |x_2|$ and $|y_1| \leq |y_2|$. We assume that Z is a non-empty set such that $XZ = ZY$. By Proposition 4.1, clearly, $|x_1| = |y_1|$.

First, we state a simple result which will be used several times later. Then, we will distinguish two main cases depending on whether x_1 and x_2 commute or not. In the second case we will further consider cases $|x_1| = |x_2|$ and $|x_1| < |x_2|$, which, by Proposition 4.1, coincide with the cases $|y_1| = |y_2|$ and $|y_1| < |y_2|$, respectively. As a consequence, of several lemmas, each solving one of previously stated cases, we finally get a theorem characterizing all binary conjugated sets X and Y.

We will need the following lemma which belongs to the folklore of the theory of combinatorics on words.

Lemma 4.2. *If a word z satisfies the equation $(uv)^k z = z(vu)^k$ with uv primitive, $v \neq 1$ and $k \geq 1$, then $z \in (uv)^* u$.*

Proof. Let $x = (uv)^k$ and $y = zv$. Then $xy = (uv)^k zv = z(vu)^k v = zv(uv)^k = yx$. The words x and y commute, and therefore have the same primitive root, uv. Let $y = (uv)^l$ for some $l \geq 1$ (note that $|y| \geq |v| > 0$), then $z = (uv)^{l-1}u \in (uv)^*u$. □

4.1. The commutative case

Now, let us consider the case when x_1 and x_2 commute.

Lemma 4.3. *Let sets $X = \{x_1, x_2\} \subseteq \Sigma^+$ and $Y = \{y_1, y_2\} \subseteq \Sigma^+$, with $|x_1| \leq |x_2|$ and $|y_1| \leq |y_2|$, be conjugates via a non-empty set Z. If $x_1, x_2 \in t^+$, where t is primitive, then there is a word s such that $y_1, y_2 \in s^+$ and words t and s are conjugates,* i.e., *$t = uv$ and $s = vu$ for some $u, v \in \Sigma^*$. Moreover, the set Z satisfies that $Z \subseteq (uv)^*u$.*

Proof. Take an arbitrary word $z \in Z$. By Proposition 2.4, for any positive integer n and $i = 1, 2$ there are integers $i_1, \ldots, i_n \in \{1, 2\}$ and $z' \in Z$ such that

$$zy_i^n = x_{i_1} \ldots x_{i_n} z' \in t^+ z' .$$

If we take $n \geq 2$ such that $|z| + 2|y_i| \leq n|x_1|$, then zy_i^2 is a prefix of t^ω. This implies that $z = t^{m_z} u_z$, for some integer $m_z \geq 0$ and some word u_z, a proper prefix of t. Let $t = u_z v_z$ and $s_z = v_z u_z$. Then $z \in (u_z v_z)^* u_z$ and y_i^2 is a prefix of s_z^ω. Note that since t is primitive, so is s_z. Since $|y_2| \geq |y_1| = |x_1| \geq |t| = |s_z|$, by Lemma 2.1 we have that y_1 (resp. y_2) commutes with s_z. But due to the primitiveness of s_z we conclude that $y_1, y_2 \in s_z^+$.

Now, it suffices to prove that for all $z, \bar{z} \in Z$, $u_z = u_{\bar{z}}$ and $v_z = v_{\bar{z}}$. Since s_z and $s_{\bar{z}}$ have a common power (y_1), they commute and since they are both primitive, $s_z = s_{\bar{z}}$. We have $u_z v_z u_{\bar{z}} = u_{\bar{z}} v_{\bar{z}} u_{\bar{z}} = u_{\bar{z}} v_z u_z$. By Lemma 4.2, $u_{\bar{z}} \in (u_z v_z)^* u_z$, which implies $u_{\bar{z}} = u_z$. □

4.2. The non-commutative case

In what follows we will assume that x_1 and x_2, and similarly, y_1 and y_2, do not commute. As an immediate consequence of Proposition 4.1 we have that the lengths of x_1 and x_2 are equal if and only if the lengths of y_1 and y_2 are so:

Corollary 4.4. *Let sets $X = \{x_1, x_2\} \subseteq \Sigma^+$ and $Y = \{y_1, y_2\} \subseteq \Sigma^+$, with $|x_1| \leq |x_2|$ and $|y_1| \leq |y_2|$, be conjugates via a non-empty set Z. Words x_1 and x_2 have the same length, if and only if words y_1 and y_2 have so.*

Now, we will consider the simplest case when the sizes of words in X and Y are equal.

Lemma 4.5. *Let sets $X = \{x_1, x_2\} \subseteq \Sigma^+$ and $Y = \{y_1, y_2\} \subseteq \Sigma^+$ be conjugates via a non-empty set Z. If $|x_1| = |x_2| = |y_1| = |y_2|$, then there are words u, v and p such that $|u| = |v|$ and a set $I \subseteq \mathbb{N}$ such that one of the following conditions is satisfied:*

(i) $X = \{pu, pv\}$, $Y = \{up, vp\}$ and $Z = \bigcup_{i \in I} \{pu, pv\}^i p$,

(ii) $X = \{up, vp\}$, $Y = \{pu, pv\}$ *and* $Z = \bigcup_{i \in I} \{up, vp\}^i \{u, v\}$.

Proof. Notice that the sets X and Y are uniform, so as a consequence of Proposition 3.2 there are sets $P, Q \subseteq \Sigma^*$ and $I \subseteq \mathbb{N}$ such that $X = PQ$, $Y = QP$ and $Z = \cup_{i \in I} (PQ)^i P$. Now, if $|X| = 2$, then either $|P| = 1$ and $|Q| = 2$ (case *(i)* with $P = \{p\}$ and $Q = \{u, v\}$), or $|P| = 2$ and $|Q| = 1$ (case *(ii)* with $P = \{u, v\}$ and $Q = \{p\}$). □

In the case when lengths of words in X and Y are not all the same we need the following 2 lemmas:

Lemma 4.6. *Let sets* $X = \{x_1, x_2\} \subseteq \Sigma^+$ *and* $Y = \{y_1, y_2\} \subseteq \Sigma^+$, *with* $|x_1| \leq |x_2|$ *and* $|y_1| \leq |y_2|$, *be conjugates via a non-empty set* Z. *If* $|x_2| \neq |y_2|$, *then* x_1 *and* x_2 *commute.*

Proof. We will prove the claim only in the case $|x_2| < |y_2|$. If $|x_2| > |y_2|$ the claim follows symmetrically. Hence, assume that $|x_2| < |y_2|$, and let z_1 be an element of minimal length of Z. By Proposition 2.4, for any positive integer n, the word $w = x_1^n x_2 z_1$ belongs to the set ZY^{n+1}. Hence, we have $w = z' y_{i_1} \ldots y_{i_{n+1}}$ for some $i_1, \ldots, i_{n+1} \in \{1, 2\}$ and $z' \in Z$. As z_1 was chosen of minimal length, $|z'| \geq |z_1|$; recall also that $|y_{i_j}| \geq |y_1| = |x_1|$ and $|y_2| > |x_2|$. If for any $j \in \{1, \ldots, n+1\}$ we have $i_j = 2$, then

$$|w| = |z'| + |y_{i_1}| + \cdots + |y_{i_{n+1}}| \geq |z'| + n|y_1| + |y_2| > |z_1| + n|x_1| + |x_2|,$$

a contradiction since $|w| = n|x_1| + |x_2| + |z_1|$. Therefore $i_1 = \ldots = i_{n+1} = 1$, *i.e.*, $w = x_1^{n-1} x_1 x_2 z_1 = z' y_1^{n+1}$. By a similar argument, we obtain $x_1^{n-1} x_2 x_1 z_1 = z'' y_1^{n+1}$ for some $z'' \in Z$. If we take an integer n such that $(n+1)|y_1| \geq |x_1 x_2 z_1|$, we find that the words $x_1 x_2 z_1$ and $x_2 x_1 z_1$ have the same length and are both suffixes of y_1^{n+1}, therefore are equal. Hence, x_1 and x_2 commute. □

Lemma 4.7. *Let sets* $X = \{x_1, x_2\} \subseteq \Sigma^+$ *and* $Y = \{y_1, y_2\} \subseteq \Sigma^+$ *be conjugates via a non-empty set* Z. *If* $|x_1| = |y_1| < |x_2| = |y_2|$, *then either* x_1 *and* x_2 *commute, or* x_2 *and* y_2 *are conjugates. Moreover, in the last case there exists a word* t *such that either* $x_1 \sim_t y_1$ *and* $x_2 \sim_t y_2$, *or* $y_1 \sim_t x_1$ *and* $y_2 \sim_t x_2$.

Proof. By Proposition 4.1, we know that there exist words u and v and integers k_1 and m such that uv is primitive, $x_1 = (uv)^m$, $y_1 = (vu)^m$ and $Z_1 = \{z_1\}$, where $z_1 = (uv)^{k_1} u$. Note that $x_1 \sim_{(uv)^i u} y_1$ for any $i \geq 0$. We have either $x_2 z_1 = z_1 y_2$, or $x_2 z_1 = z' y_1$, for some $z' \in Z$. In the first case, we have immediately that x_2 and y_2 are conjugates via z_1, and we are done. In the second case, let Z' be the set of words in Z having the same length as z'.

We construct a sequence $\{z^{(i)}\}_{i \geq 1}$ in Z'. Let $z^{(1)} = z'$. For any $i \geq 1$ we have, either $x_1 z^{(i)} = z^{(i+1)} y_1$, or $x_1 z^{(i)} = z_1 y_2$. First, assume that the second case never happens. We have $x_1^i z^{(j)} = z^{(i+j)} y_1^i$ for all $i \geq 1$ and $j \geq 1$. Hence all $z^{(j)}$ are suffixes of y_1^i for some big enough integer i, and therefore they are equal. Then $x_1 z' = z' y_1$, and by Lemma 4.2, we have $z' \in (uv)^* u$. Using

$$x_2 z_1 = z' y_1, \tag{8}$$

we obtain $x_2 \in (uv)^+$, hence x_1 and x_2 commute.

Now, assume that there is a non-negative integer n such that for all $i = 1, \ldots, n$, $x_1 z^{(i)} = z^{(i+1)} y_1$ and $x_1 z^{(n+1)} = z_1 y_2$. These equalities imply that

$$x_1^{n+1} z' = x_1 x_1^n z^{(1)} = x_1 z^{(n+1)} y_1^n = z_1 y_2 y_1^n \,. \tag{9}$$

Equations (8) and (9) imply that

$$(uv)^{m(n+1)} x_2 (uv)^{k_1} u = x_1^{n+1} x_2 z_1 = z_1 y_2 y_1^{n+1} = (uv)^{k_1} u y_2 (vu)^{m(n+1)} \,.$$

Now, if $m(n+1) \leq k_1$, then we have that $x_1 \sim_t y_1$ and $x_2 \sim_t y_2$ for $t = (uv)^{k_1 - m(n+1)} u$. Otherwise, $y_1 \sim_t x_1$ and $y_2 \sim_t x_2$ for $t = (vu)^{m(n+1)-k_1-1} v$. In both cases we have that x_2 and y_2 are conjugates. □

Note that the notation $x \sim_z y$ means $xz = zy$, and therefore not necessarily implies that $y \sim_z x$. In fact, if words x, y, z satisfy both $x \sim_z y$ and $y \sim_z x$, then they all commute.

Combining all lemmas proved above we get the following characterization of all binary conjugated sets X and Y:

Theorem 4.8. *Let $X = \{x_1, x_2\} \subseteq \Sigma^+$ and $Y = \{y_1, y_2\} \subseteq \Sigma^+$, with $|x_1| \leq |x_2|$ and $|y_1| \leq |y_2|$, be conjugates,* i.e., *$X \sim Y$. Then at least one of the following conditions holds true:*

(i) $x_1 x_2 = x_2 x_1$, $y_1 y_2 = y_2 y_1$, i.e., *words x_1 and x_2 (resp. y_1 and y_2) commute, $|x_1| = |y_1|$, and moreover, the words x_1 and y_1 are conjugates;*

(ii) *there exists a word t such that, either $Xt = tY$, or $tX = Yt$.*

Conversely, if $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ with $|x_1| \leq |x_2|$ and $|y_1| \leq |y_2|$ satisfy either (i) or (ii), then $X \sim Y$.

Proof. The first part of theorem is a consequence of Lemmas 4.3, 4.5, 4.6 and 4.7: If $|x_1| = |x_2|$ or $|y_1| = |y_2|$, by Corollary 4.4 and Lemma 4.5 we are in case (ii) with $t = p$. If $|x_1| < |x_2| = |y_2|$, by Lemma 4.7 we are also in case (ii). Otherwise, by Lemma 4.6, x_1 and x_2 commute, and by Lemma 4.3 we are in case (i).

Conversely, assume that X and Y satisfy condition (i): there exist words p, q such that $x_1 = (pq)^{m_1}$, $x_2 = (pq)^{m_2}$, $y_1 = (qp)^{m_1}$ and $y_2 = (qp)^{m_3}$ with $m_1 < m_2$ and $m_1 < m_3$. Then the sets X and Y are conjugated via the set $Z = p(qp)^*$.

In case (ii), we have either $Xt = tY$ or $tX = Yt$. In the first case we have $XZ = ZY$ for $Z = \{t\}$. In the second case we take $Z = t^{-1} Y^n$, where n is large enough so that all elements of Y^n are longer than t. In case that Y contains the empty word, then we consider sets $X - \{1\}$ and $Y - \{1\}$ instead. Since $tX^n = Y^n t$, all elements of Y^n have t as a prefix, so we have $tZ = Y^n$. Then $tXZ = YtZ = Y^{n+1} = tZY$, *i.e.*, $XZ = ZY$. Hence X and Y are conjugated. □

Theorem 4.8 shows that the conjugacy of binary sets reduces to that of words. It provides a complete characterization, which, however, is not easy to state in a closed form. However, it does not give a characterization of the sets Z via which the sets X and Y are conjugated. This is due to the fact that, unlike Lemmas 4.3

and 4.5, Lemma 4.7 does not provide such characterization. Nevertheless, such result will be contained in the forthcoming paper.

The following two corollaries are approaches to merge conditions (i) and (ii) of Theorem 4.8 into one to obtain a more compact form. In the first one we restrict the lengths of elements of X and Y.

Corollary 4.9. *Let $X = \{x_1, x_2\} \subseteq \Sigma^+$ and $Y = \{y_1, y_2\} \subseteq \Sigma^+$ with $|x_1| = |y_1|$ and $|x_2| = |y_2|$. Then X and Y are conjugates if and only if there exists a single word t such that $Xt = tY$ or $tX = Yt$.*

In the second one we consider the conjugacy via finite sets Z. In such case, similarly as we show that the lengths of the shortest elements of X and Y are equal, one can show that the same is true for the longest elements. Therefore, the following corollary is an immediate consequence of the previous one:

Corollary 4.10. *Let $X = \{x_1, x_2\} \subseteq \Sigma^+$ and $Y = \{y_1, y_2\} \subseteq \Sigma^+$. Then X and Y are conjugated via a finite non-empty set Z if and only if there exists a single word t such that $Xt = tY$ or $tX = Yt$.*

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