J. of Mult.-Valued Logic & Soft Computing., Vol. 14, pp. 1–32
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$(\delta, \star)$-Equality of Fuzzy Sets

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Received: July 28, 2004. Accepted: June 26, 2006.

The Cai δ-equality of fuzzy sets corresponds to the Lukasiewicz t-norm. In this paper we study the notion of $(*, \delta)$-equality, a concept which generalizes the δ-equality to the case of the fuzzy set theory based on an arbitrary continuous t-norm $*$. We investigate the robustness of some fuzzy implication operators in terms of $(*, \delta)$-equality.

Keywords: $(*, \delta)$-equality, fuzzy implication operators, fuzzy relations

1 INTRODUCTION

If A, B are two fuzzy sets of a universe X, then $d(A, B) = sup_{x \in X} |A(x) - B(x)|$ is *the distance* between A and B. In Pappis's paper [16], A and B are said to be approximately equal (denoted by $A \approx B$) if $d(A, B) \leq \epsilon$ where ϵ is a small non negative real number. ϵ is called *a proximity measure* of A and B. This definition was reformulated in [11] by using the similarity measure [12]: A and B are *α-similar* ($A \approx_\alpha B$) if $S(A, B) \geq \alpha$, where $S(A, B) = 1 - d(A, B)$. An axiomatic definition of distance measure and similarity measure was done in [12]. Three similarity measures have been considered in [17] and others in [25].

To each of these similarity measures a notion of "approximate equality of fuzzy sets" corresponds.

[5] and [26] remarked that this definition of approximative equality of two fuzzy sets causes some inconveniences. Therefore Cai [5] introduced the δ-equality of two fuzzy sets: A and B are δ-equal if $sup_{x \in X} |A(x) - B(x)| \leq 1 - \delta$ ($0 \leq \delta \leq 1$). Using the similarity measure associated with an implication

operator in the sense of [1], Wang et al. defined in [26] a more general concept of δ-equality.

Most of these papers analyze the way some implication operators and some operations of fuzzy sets and fuzzy relations behave with respect to δ-equalities. Such operators appear in fuzzy logic and are usually applied in fuzzy control. The results obtained in the above-mentioned papers reflect how the errors in premises influence the conclusions in fuzzy reasoning. Particularly, [5] and [6] contain plenty of results on δ-equality with respect to operations of fuzzy sets, fuzzy relations, extension principle, t-norms and s-norms as well as some robustness results on fuzzy implication operators and fuzzy inference rules. [5] and [6] distinguish themselves by the fact that in the study of different operations with respect to δ-equality, the real number δ is not fixed, but varies with the terms of the operations. It is easy to see that Cai δ-equality can be expressed in terms of the biresiduum corresponding to Lukasiewicz t-norm. All the results in [5] and [6] are obtained in the fuzzy set theory based on Lukasiewicz t-norm.

Changing the t-norm leads to another analysis of the fuzzy reasoning and to another way of "identifying" the fuzzy sets.

Thus a natural problem is if the Cai theory can be developed in a more general setting offered by an arbitrary continuous t-norm $*$. This paper is an answer to this problem.

We shall study the $(*, \delta)$-equality of fuzzy sets, a concept that generalizes the one of δ-equality.

The first objective of this paper is to extend some of Cai's results to a framework offered by a continuous t-norm. Besides these generalizations, results that do not arise from [5], [6] are obtained.

Our second objective is to prove the theorem in an uniform way based on the residuated structure of the interval $[0, 1]$ corresponding to a continuous t-norm. Our proofs are more natural and bring more clarity even for the particular case of [5] and [6].

The third objective is to show how the $(*, \delta)$-equality can be put to work in fuzzy revealed preference theory [8, 9].

Section 2 contains some basic results on a continuous t-norm $*$ and its residuum $\rightarrow$. In Section 3 we put in relation the Cai δ-equality and the Lukasiewicz t-norm. This suggests to us the $(*, \delta)$-equality, a concept obtained by using the biresiduum of the t-norm $*$.

Section 4 investigates how the basic operations on fuzzy sets preserve the $(*, \delta)$-equality. The effect of some fuzzy implication operators on the $(*, \delta)$-equality is studied in Section 5. Section 6 is concerned with the manner in which the composition of fuzzy relations and the transitive closure operator preserves the $(*, \delta)$-equality.

In Section 7 we relate the $(*, \delta)$-equality to some fuzzy operators defined by an s-norm. The operator P studied in Section 8 is analogous to the fuzzy operator PC defined in [27] p. 627. P has the same form with PC but it is defined using the Sugeno integral instead of the classical integral. The results of Section 8 point out the behaviour of some operators including P with respect to $(*, \delta)$-equality.

In Section 9 the notion of $(*, \delta)$-equality of two fuzzy choice functions is defined [2, 8, 9]. According to these papers, to each fuzzy choice function C a fuzzy revealed preference R_C is associated; conversely, to each fuzzy preference relation Q on the set of alternatives a fuzzy choice function is associated. The two theorems of this section establish how these two functions determine the translation from $(*, \delta)$-equality of the fuzzy choice functions to the $(*, \delta)$-equality of the fuzzy preference relations and conversely.

2 PRELIMINARIES

In this section we present some basic facts on continuous t-norms and residua. The background for these results can be found in [10, 13, 15, 21].

A mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm iff it is symmetric, associative, non-decreasing in each argument and $a * 1 = a$ for all $a \in [0, 1]$.

A t-norm is said to be continuous if it is continuous as a function on the unit interval. With any continuous t-norm $*$ we associate its *residuum*:

$a \to b = \bigvee\{c \in [0, 1] | a * c \leq b\}$.

The most well-known continuous t-norms are:

Lukasiewicz t-norm: $a *_L b =$ max $(0, a + b - 1)$; $a \to_L b =$ min $(1, 1 - a + b)$

Gödel t-norm: $a *_G b =$ min (a, b); $a \to_G b = \begin{cases} 1 \text{ if } a \leq b \\ b \text{ if } a > b \end{cases}$

Product t-norm: $a *_P b = ab$; $a \to_P b = \begin{cases} 1 \text{ if } a \leq b \\ b/a \text{ if } a > b \end{cases}$

Lemma 2.1. *([21]) For any $a, b, c \in [0, 1]$ the following properties hold:*

(1) $a * b \leq c \Leftrightarrow a \leq b \to c$; (2) $a * (a \to b) = a \wedge b$; (3) $a * b \leq a, a * b \leq b$; (4) $b \leq a \to b$; (5) $a \leq b \Leftrightarrow a \to b = 1$; (6) $a = 1 \to a$; (7) $1 = a \to a$; (8) $1 = a \to 1$; (9) $a * (b \vee c) = (a * b) \vee (a * c)$; (10) $a \leq b$ implies $b \to c \leq a \to c$ and $c \to a \leq c \to b$.

The negation operation $\neg$ associated with $*$ is defined by

$$\neg a = a \to 0 = \bigvee\{c \in [0, 1] | a * c = 0\}.$$

Lemma 2.2. *([21]) For any $a, b, c \in [0, 1]$ the following properties hold:*

(1) $a \leq \neg b \Leftrightarrow a * b = 0$; (2) $a * \neg a = 0$; (3) $a \leq \neg\neg a$; (4) $\neg 0 = 1, \neg 1 = 0$; (5) $\neg a = \neg\neg\neg a$; (6) $a \to b \leq \neg b \to \neg a$.

This lemma shows that $([0, 1], \vee, \wedge, *, 0, 1)$ is a residuated lattice [21].

The biresiduum associated with the continuous t-norm $*$ is defined by $\rho(a,b) = a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$.

Lemma 2.3. *([21]) For any $a, b, c, d \in [0,1]$ the following properties hold:*
(1) $\rho(a,1) = a$; (2) $a = b \Leftrightarrow \rho(a,b) = 1$; (3) $\rho(a,b) = \rho(b,a)$; (4) $\rho(a,b) \leq \rho(\neg a, \neg b)$; (5) $\rho(a,b) * \rho(b,c) \leq \rho(a,c)$; (6) $\rho(a,b) \wedge \rho(c,d) \leq \rho(a \wedge c, b \wedge d)$; (7) $\rho(a,b) \wedge \rho(c,d) \leq \rho(a \vee c, b \vee d)$; (8) $\rho(a,b) * \rho(c,d) \leq \rho(a * c, b * d)$; (9) $\rho(a,b) * \rho(c,d) \leq \rho(a \rightarrow c, b \rightarrow d)$; (10) $\rho(a,b) * a \leq b$; (11) $a \wedge b \leq \rho(a,b)$; (12) $\rho(a,b) * \rho(c,d) \leq \rho(\rho(a,c), \rho(b,d))$.

Proof: The proof of (1)-(3) and (5)-(9) can be found in [21], p. 14. (4) By Lemma 2.2 (6)

$$\rho(a,b) = (a \rightarrow b) \wedge (b \rightarrow a) \leq (\neg b \rightarrow \neg a) \wedge (\neg a \rightarrow \neg b) = \rho(\neg a, \neg b).$$

(10) By Lemma 2.1 (2), $\rho(a,b) * a \leq a * (a \rightarrow b) = a \wedge b \leq b$.

(11) By Lemma 2.1 (4), $a \leq b \rightarrow a$ and $b \leq a \rightarrow b$, hence $a \wedge b \leq (a \rightarrow b) \wedge (b \rightarrow a) = \rho(a,b)$.

(12) $\rho(\rho(a,c), \rho(b,d)) = \rho((a \rightarrow c) \wedge (c \rightarrow a), (b \rightarrow d) \wedge (d \rightarrow b)) \leq \rho(a \rightarrow c, b \rightarrow d) \wedge \rho(c \rightarrow a, d \rightarrow b) \leq [\rho(a,b) * \rho(c,d)] \wedge [\rho(c,d) * \rho(a,b)] = \rho(a,b) * \rho(c,d)$. □

Lemma 2.4. *([21]) For any $\{a_i\}_{i \in I} \subseteq [0,1]$, $\{b_i\}_{i \in I} \subseteq [0,1]$ and $a \in [0,1]$ the following properties hold:*
(1) $a \rightarrow (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I} (a \rightarrow a_i)$; (2) $(\bigvee_{i \in I} a_i) \rightarrow a = \bigwedge_{i \in I} (a_i \rightarrow a)$; (3) $\bigvee_{i \in I} (a_i \rightarrow a) \leq (\bigwedge_{i \in I} a_i) \rightarrow a$; (4) $\bigvee_{i \in I} (a \rightarrow a_i) \leq a \rightarrow (\bigvee_{i \in I} a_i)$; (5) $(\bigvee_{i \in I} a_i) * (\bigvee_{j \in I} b_j) = \bigvee_{i,j \in I} (a_i * b_j)$; (6) $(\bigwedge_{i \in I} a_i) * (\bigwedge_{j \in I} b_j) \leq \bigwedge_{i,j \in I} (a_i * b_j)$.

Lemma 2.5. *Let X be a non-empty set and $f : X \rightarrow [0,1]$, $g : X \rightarrow [0,1]$ two arbitrary functions. Then*

(1) $\rho(\bigwedge_{x \in X} f(x), \bigwedge_{x \in X} g(x)) \geq \bigwedge_{x \in X} \rho(f(x), g(x))$;
(2) $\rho(\bigvee_{x \in X} f(x), \bigvee_{x \in X} g(x)) \geq \bigwedge_{x \in X} \rho(f(x), g(x))$.

Proof: (1) By Lemma 2.3 (10), we have for each $z \in X$:

$$[\bigwedge_x \rho(f(x), g(x))] * (\bigwedge_y f(y)) \leq \rho(f(z), g(z)) * f(z) \leq g(z).$$

Then, by Lemma 2.1 (1), $\bigwedge_x \rho(f(x), g(x)) \leq (\bigwedge_y f(y)) \rightarrow g(z)$.

This inequality holds for any $z \in X$, hence by Lemma 2.4 (1):

$$\bigwedge_x \rho(f(x), g(x)) \le \bigwedge_z ((\bigwedge_y f(y)) \to g(z)) =$$

$$= (\bigwedge_y f(y)) \to (\bigwedge_z g(z))$$

Similarly, $\bigwedge_x \rho(f(x), g(x)) \le (\bigwedge_z (g(z)) \to (\bigwedge_y f(y))$, therefore $\bigwedge_x \rho(f(x), g(x)) \le [(\bigwedge_y f(y)) \to (\bigwedge_z g(z))] \wedge [(\bigwedge_z g(z)) \to (\bigwedge_y f(y))] = \rho(\bigwedge_x f(x), \bigwedge_x g(x))$.

(2) For any $y \in X$ we have

$[\bigwedge_x \rho(f(x), g(x))] * f(y) \le \rho(f(y), g(y)) * f(y) \le g(y) \le \bigvee_z g(z)$.

In accordance with Lemma 2.1 (1), $\bigwedge_x \rho(f(x), g(x)) \le f(y) \to (\bigvee_z g(z))$.

This inequality holds for any $y \in X$, therefore, by Lemma 2.4 (2) $\bigwedge_x \rho(f(x), g(x)) \le \bigwedge_y (f(y) \to (\bigvee_z g(z))) = (\bigvee_y f(y)) \to (\bigvee_z g(z))$

Similarly, $\bigwedge_x \rho(f(x), g(x)) \le (\bigvee_z g(z)) \to (\bigvee_y f(y))$ hence

$\bigwedge_x \rho(f(x), g(x)) \le [(\bigvee_y f(y)) \to (\bigvee_z g(z))] \wedge [(\bigvee_z g(z)) \to (\bigvee_y f(y))] = \rho(\bigvee_x f(x), \bigvee_x g(x))$. □

Let X be a non-empty set. A *fuzzy subset* of X is a function $A : X \to [0, 1]$. If $x \in X$ then $A(x)$ is called the degree of membership of x in A. Let us denote by $\mathcal{F}(X)$ the set of fuzzy subsets of X.

If $A, B \in \mathcal{F}(X)$ we denote $A \subseteq B$ if $A(x) \le B(x)$ for each $x \in X$. For any $A, B \in \mathcal{F}(X)$ we define the fuzzy subsets $A \cup B$, $A \cap B$ by

$(A \cup B)(x) = A(x) \vee B(x)$; $(A \cap B)(x) = A(x) \wedge B(x)$.

3 LUKASIEWICZ T-NORM AND CAI δ-EQUALITY

In this section we shall prove that the Cai δ-equality ([5], [6]) can be expressed in terms of the biresiduum of Lukasiewicz t-norm. This result is not new (see example [26], Proposition 3.1) but we shall briefly prove it.

Let us consider the Lukasiewicz t-norm $a *_L b = 0 \vee (a + b - 1)$ and its residuum $a \rightarrow_L b = 1 \wedge (1 - a + b)$. The biresiduum of $*_L$ will be given by

$$\rho_L(a, b) = (a \rightarrow_L b) \wedge (b \rightarrow_L a) = \begin{cases} b \rightarrow_L a & \text{if} \quad a \leq b \\ a \rightarrow_L b & \text{if} \quad a \geq b \end{cases}.$$

Lemma 3.1. *For any* $a, b \in [0, 1]$, $\rho_L(a, b) = 1 - |a - b|$.

Proof: Assume $a \leq b$, then
$\rho_L(a, b) = b \rightarrow_L a = 1 - b + a = 1 - |a - b|$.
The case $b \leq a$ follows similarly. □

Now we recall the Cai definition of δ-equality.

Definition 3.2. *([5], [6]) Let* X *be a non-empty set,* A, B *two fuzzy subsets of* X *and* $0 \leq \delta \leq 1$. *Then* A, B *are* δ*-equal* ($A = (\delta)B$ *in symbols) if the following condition holds:*

$$\bigvee_{x \in X} |A(x) - B(x)| \leq 1 - \delta.$$

Lemma 3.3. *If* $0 \leq \delta \leq 1$ *and* A, B *are two fuzzy subsets of* X *then the following are equivalent:*

(i) $A = (\delta)B$;
(ii) $\bigwedge_{x \in X} \rho_L(A(x), B(x)) \geq \delta$.

Proof: By Lemma 3.1 we remark that

$$1 - \bigvee_{x \in X} |A(x) - B(x)| = \bigwedge_{x \in X} (1 - |A(x) - B(x)|) = \bigwedge_{x \in X} \rho_L(A(x), B(x))$$

Then the equivalence of (i) and (ii) follows immediately. □

4 $(*, \delta)$-EQUALITY OF FUZZY SETS

In this section we shall introduce the $(*, \delta)$-equality and we shall discuss this notion with respect to algebraic operations of fuzzy sets and fuzzy relations. We shall relate the $(*, \delta)$-equality with Zadeh's extension principle.

In accordance with Lemma 3.3, the Cai δ-equality is a notion which corresponds to the Lukasiewicz t-norm. This lemma suggests to us the notion of $(*, \delta)$-equality, a concept corresponding to an arbitrary continuous t-norm.

Definition 4.1. *Let* $*$ *be a continuous t-norm and* X *a non-empty set. If* A, B *are two fuzzy subsets of* X *and* $0 \leq \delta \leq 1$ *then we shall say that* A, B *are* $(*, \delta)$*-equal* ($A = (*, \delta)B$ *in symbols) if the following condition holds*

$$\bigwedge_{x\in X} \rho(A(x), B(x)) \geq \delta,$$

where ρ is the biresiduum of $*$.

For the case when $*$ is the Lukasiewicz t-norm $*_L$ we obtain the Cai notion of δ-equality.

$\bigwedge_{x\in X} \rho(A(x), B(x))$ can represent the degree of similarity of the fuzzy sets A and B. Then $A = (*, \delta)B$ means that A and B are "equal to a degree greater than δ".

Example 4.2. Suppose two approximative pieces of information "about 2" lead to triangular fuzzy numbers $A = (2, 2)$ and $B = (2, 1)$:

$$A(x) = \begin{cases} x/2 & \text{if} \quad 0 \leq x \leq 2 \\ (4-x)/2 & \text{if} \quad 2 \leq x \leq 4 \\ 0 & \textit{otherwise}, \end{cases}$$

$$B(x) = \begin{cases} x-1 & \text{if} \quad 1 \leq x \leq 2 \\ 3-x & \text{if} \quad 2 \leq x \leq 3 \\ 0 & \textit{otherwise}. \end{cases}$$

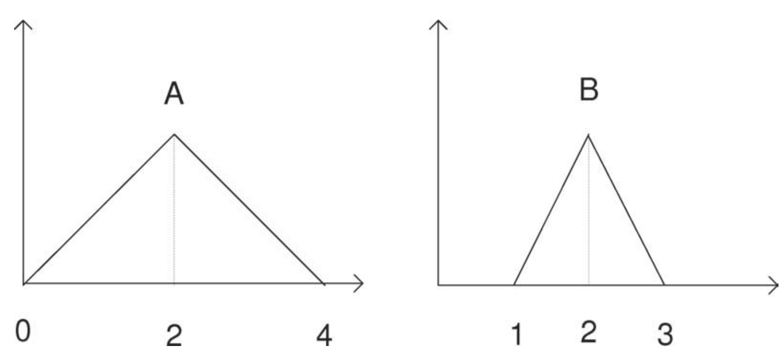

FIGURE 1
Fuzzy numbers A and. B

We want to see to what extent the fuzzy numbers A and B are $(*, \delta)$-equal.

We want to calculate $\rho(A(x), B(x)), x \in \Re$ for an arbitrary continuous t-norm. An easy computation leads to

$$\rho(A(x), B(x)) = \begin{cases} 1 & \text{if } x \leq 0 \\ \neg A(x) & \text{if } 0 < x < 1 \\ A(x) \to B(x) & \text{if } 1 \leq x \leq 3 \\ \neg A(x) & \text{if } 3 < x < 4 \\ 1 & \text{if } x \geq 4. \end{cases}$$

We will explicate $\rho(A(x), B(x))$ for Lukasiewicz, Gödel and product t-norms.

a) Lukasiewicz t-norm

$$\rho_L(A(x), B(x)) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - A(x) & \text{if } 0 < x < 1 \\ 1 - A(x) + B(x) & \text{if } 1 \leq x \leq 3 \\ 1 - A(x) & \text{if } 3 < x < 4 \\ 1 & \text{if } x \geq 4. \end{cases}$$

By computation we get

$$\rho_L(A(x), B(x)) = \begin{cases} 1 & \text{if } x \leq 0 \\ (2-x)/2 & \text{if } 0 < x \leq 1 \\ x/2 & \text{if } 1 \leq x \leq 2 \\ 2 - x/2 & \text{if } 2 \leq x \leq 3 \\ (x-2)/2 & \text{if } 3 \leq x \leq 4 \\ 1 & \text{if } x \geq 4. \end{cases}$$

We conclude that $\bigwedge_{x \in \Re} \rho_L(A(x), B(x)) = 1/2$ (see Fig. 2) hence $A = (*_L, 1/2)B$.

b) Gödel t-norm

$$\rho_G(A(x), B(x)) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } 0 < x < 1 \\ B(x) & \text{if } 1 \leq x \leq 3 \\ 0 & \text{if } 3 < x < 4 \\ 1 & \text{if } x \geq 4. \end{cases}$$

We notice that $\bigwedge_{x \in \Re} \rho_G(A(x), B(x)) = 0$, hence $A = (*_G, 0)B$.

FIGURE 2
$\rho_L(A(x), B(x))$.

c) product t-norm

$$\rho_P(A(x), B(x)) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } 0 < x < 1 \\ B(x)/A(x) & \text{if } 1 \leq x \leq 3 \\ 0 & \text{if } 3 < x < 4 \\ 1 & \text{if } x \geq 4. \end{cases}$$

We notice that $\bigwedge_{x \in \Re} \rho_P(A(x), B(x)) = 0$, hence $A = (*_P, 0)B$.

For this example, the only interesting case is the Lukasiewicz t-norm.

For the rest of the paper we fix a continuous t-norm $*$, its residuum $\to$ and its biresiduum ρ.

Let A, B be two fuzzy subsets of X. Let us define the *relational intersection* $A \sqcap B$ and the *relational union* $A \sqcup B$ as the fuzzy relations on X defined by $(A \sqcap B)(x, y) = A(x) \wedge B(y)$, $(A \sqcup B)(x, y) = A(x) \vee B(y)$ for all $x, y \in X$.

Proposition 4.3 *Let A, A', B, B' be fuzzy subsets of X. If $A = (*, \delta_1)A'$ and $B = (*, \delta_2)B'$ then $A \sqcap B = (*, \delta_1 \wedge \delta_2)A' \sqcap B'$ and $A \cap B = (*, \delta_1 \wedge \delta_2)A' \cap B'$.*

Proof: By hypothesis, $\bigwedge_{x \in X} \rho(A(x), A'(x)) \geq \delta_1$, $\bigwedge_{x \in X} \rho(B(x), B'(x)) \geq \delta_2$.

Then using Lemma 2.3 (6), one gets for all $x, y \in X$:

$$\begin{aligned} \rho((A \sqcap B)(x, y), (A' \sqcap B')(x, y)) &= \rho(A(x) \wedge B(y), A'(x) \wedge B'(y)) \\ &\geq \rho(A(x), A'(x)) \wedge \rho(B(y), B'(y)). \end{aligned}$$

Hence,

$$\bigwedge_{x,y\in X} \rho((A \sqcap B)(x, y), (A' \sqcap B')(x, y)) \geq$$

$$\bigwedge_{x,y\in X} (\rho(A(x), A'(x)) \wedge \rho(B(y), B'(y))) =$$

$$= [\bigwedge_{x} \rho(A(x), A'(x))] \wedge [\bigwedge_{y} \rho(B(y), B'(y))] \geq \delta_1 \wedge \delta_2.$$

Then $A \sqcap B = (*, \delta_1 \wedge \delta_2)A' \sqcap B'$. The second relation follows by

$$\bigwedge_{x\in X} \rho(A(x) \wedge B(x), A'(x) \wedge B'(x)) \geq \bigwedge_{x,y\in X} \rho(A(x) \wedge B(y), A'(x) \wedge B'(y)) \geq \delta_1 \wedge \delta_2.$$

□

Proposition 4.4 *If $A = (*, \delta_1)A'$ and $B = (*, \delta_2)B'$ then $A \sqcup B = (*, \delta_1 \wedge \delta_2)A' \sqcup B'$ and $A \cup B = (*, \delta_1 \wedge \delta_2)A' \cup B'$.*

Proof: Similarly, using Lemma 2.3 (7). □

Let $A_1, \ldots, A_n$ be fuzzy subsets of X. Let us define

$$\prod_{i=1}^{n} A_i : X^n \to [0, 1], \coprod_{i=1}^{n} A_i : X^n \to [0, 1].$$

by putting

$$(\prod_{i=1}^{n} A_i)(x_1, \ldots, x_n) = A_1(x_1) \wedge A_2(x_2) \wedge \ldots \wedge A_n(x_n)$$

$$(\coprod_{i=1}^{n} A_i)(x_1, \ldots, x_n) = A_1(x_1) \vee A_2(x_2) \vee \ldots \vee A_n(x_n)$$

for all $(x_1, \ldots, x_n) \in X^n$.

The following result generalizes Propositions 4.3 and 4.4.

Proposition 4.5 *Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be fuzzy subsets of X. If $A_i = (*, \delta_i)B_i$ for $i = 1, \ldots, n$ then $\prod_{i=1}^{n} A_i = (*, \bigwedge_{i=1}^{n} \delta_i) \prod_{i=1}^{n} B_i$, $\coprod_{i=1}^{n} A_i = (*, \bigwedge_{i=1}^{n} \delta_i) \coprod_{i=1}^{n} B_i$, $\bigcup_{i=1}^{n} A_i = (*, \bigwedge_{i=1}^{n} \delta_i) \bigcup_{i=1}^{n} B_i$, $\bigcap_{i=1}^{n} A_i = (*, \bigwedge_{i=1}^{n} \delta_i) \bigcap_{i=1}^{n} B_i$.*

If A is a fuzzy subset of X then $\neg A$ is the fuzzy subset of X defined by $(\neg A)(x) = \neg A(x)$ for each $x \in X$.

Proposition 4.6 *If $A = (*, \delta)B$ then $\neg A = (*, \delta)\neg B$.*

Proof: By Lemma 2.3 (4), $\bigwedge_x \rho(\neg A(x), \neg B(x)) \geq \bigwedge_x \rho(A(x), B(x)) \geq \delta$. □

If A, B are two fuzzy subsets of X then $A * B$ will be the fuzzy relation on X defined by $(A * B)(x, y) = A(x) * B(y)$ for all $x, y \in X$.

Proposition 4.7 *If $A = (*, \delta_1)A'$ and $B = (*, \delta_2)B'$ then $A * B = (*, \delta_1 * \delta_2)A' * B'$.*

Proof: By hypothesis we have

(a) $\bigwedge_x \rho(A(x), A'(x)) \geq \delta_1$, $\bigwedge_y \rho(B(y), B'(y)) \geq \delta_2$.

Now we shall prove the inequality

(b) $$\bigwedge_{x,y} \rho(A(x) * B(y), A'(x) * B'(y)) \geq [\bigwedge_x \rho(A(x), A'(x))] * [\bigwedge_y \rho(B(y), B'(y))].$$

Let $x, y \in X$. By Lemma 2.3 (8)

$$\begin{aligned}[\bigwedge_x \rho(A(x), A'(x))] * [\bigwedge_y \rho(B(y), B'(y))] &\leq \rho(A(x), A'(x)) * \rho(B(y), B'(y)) \\ &\leq \rho(A(x) * B(y), A'(x) * B'(y)).\end{aligned}$$

This inequality holds for any $x, y \in X$ therefore we obtain (b).

By (a) and (b) one can infer that

$$\begin{aligned}&\bigwedge_{x,y} \rho((A * B)(x, y), (A' * B')(x, y)) \\ &\quad = \bigwedge_{x,y} \rho(A(x) * B(y), A'(x) * B'(y)) \geq \delta_1 * \delta_2.\end{aligned}$$

□

Let A, B be two fuzzy subsets of X. Denote by $A \to B$ the fuzzy relation on X defined by $(A \to B)(x, y) = A(x) \to B(y)$ for all $x, y \in X$.

Proposition 4.8 *If $A = (*, \delta_1)A'$ and $B = (*, \delta_2)B'$ then $(A \to B) = (*, \delta_1 * \delta_2)(A' \to B')$.*

Proof: Similarly, using Lemma 2.3 (9). □

If A, B are two fuzzy subsets of X then $A\nabla B$ will be the fuzzy subset on X defined by $(A\nabla B)(x) = \neg\rho(A, B)(x)$ for all $x \in X$.

Proposition 4.9 *If $A = (*, \delta_1)A'$ and $B = (*, \delta_2)B'$ then $\nabla(A, B) = (*, \delta_1 * \delta_2)\nabla(A', B')$.*

Proof: By hypothesis we know $\bigwedge_x \rho(A(x), A'(x)) \geq \delta_1$, $\bigwedge_x \rho(B(x), B'(x)) \geq \delta_2$.

Using Lemma 2.3 (12) and the inequality (b) in the proof of Proposition 4.8 we have

$$\bigwedge_x \rho(\nabla(A(x), B(x)), \nabla(A'(x), B'(x))) =$$
$$\bigwedge_x \rho(\neg\rho(A(x), B(x)), \neg\rho(A'(x), B'(x)))$$
$$\geq \bigwedge_x \rho(\rho(A(x), B(x)), \rho(A'(x), B'(x))) \geq \bigwedge_x (\rho(A(x), A'(x)) *$$
$$\rho(B(x), B'(x))) \geq \delta_1 * \delta_2,$$

hence $\nabla(A, B) = (*, \delta_1 * \delta_2)\nabla(A', B')$. □

Proposition 4.10 *Let X and Y be two non-empty sets and f a mapping from X to Y, i.e. $f : X \to Y$. Let A and A' be fuzzy sets defined on X and B and B' fuzzy sets defined on Y by the extension principle with respect to f:*

$$B(y) = \begin{cases} \bigvee_{y=f(x)} A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$B'(y) = \begin{cases} \bigvee_{y=f(x)} A'(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If $A = (*, \delta)A'$ then $B = (*, \delta)B'$.

Proof: By hypothesis, $\bigwedge_{x\in X} \rho(A(x), A'(x)) \geq \delta$. According to Lemma 2.5 (2) we have

$$\bigwedge_{y\in Y} \rho(B(y), B'(y)) = \bigwedge_{y\in Y} \rho\Big(\bigvee_{y=f(x)} A(x), \bigvee_{y=f(x)} A'(x)\Big) \geq$$
$$\bigwedge_{y\in Y} \bigwedge_{y=f(x)} \rho(A(x), A'(x)) \geq \bigwedge_{x\in X} \rho(A(x), A'(x)) \geq \delta.$$

Thus $B = (*, \delta)B'$. □

The following result is a generalization of Proposition 4.7.

Proposition 4.11 *Let $X_1, \ldots, X_n$ be non-empty sets and A_i, B_i fuzzy subsets of X_i. Let us consider $A = A_1 * \ldots * A_n$, $B = B_1 * \ldots * B_n$ the fuzzy subsets of the cartesian product $X = X_1 \times \ldots \times X_n$ defined by*

$A(x_1, \ldots, x_n) = A_1(x_1) * \ldots * A_n(x_n)$,

$B(x_1, \ldots, x_n) = B_1(x_1) * \ldots * B_n(x_n)$

for any $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$.

If $A_i = (*, \delta_i)B_i$, $i = 1, \ldots, n$ then $A = (*, \delta_1 * \ldots * \delta_n)B$.

Proposition 4.12 *Let $X_1, \ldots, X_n, Y$ be non-empty sets and $f : X_1 \times \ldots \times X_n \to Y$. Let $A_i, A'_i \in \mathcal{F}(X_i)$, $i = 1, \ldots, n$ and $B, B' \in \mathcal{F}(Y)$ defined by*

$$B(y) = \begin{cases} \bigvee\{A_1(x_1) * \ldots * A_n(x_n) | f(x_1, \ldots, x_n) = y\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$B'(y) = \begin{cases} \bigvee\{A'_1(x_1) * \ldots * A'_n(x_n) | f(x_1, \ldots, x_n) = y\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If $A_i = (*, \delta_i)A'_i$, $i = 1, \ldots, n$ then $B = (*, \delta)B'$ where $\delta = \delta_1 * \ldots * \delta_n$.

Proof: If $X = X_1 \times \ldots \times X_n$ then f is a mapping from X to Y, so we can apply Proposition 4.11 to f and to the fuzzy subsets A, A' of X defined by $A(x_1, \ldots, x_n) = A_1(x_1) * \ldots * A_n(x_n)$, $A'(x_1, \ldots, x_n) = A'_1(x_1) * \ldots * A'_n(x_n)$.

By Proposition 4.7, $A = (*, \delta_1 * \ldots * \delta_n)A'$, hence, by Proposition 4.11, $B = (*, \delta_1 * \ldots * \delta_n)B'$. □

Proposition 4.13 *Let $X_1, \ldots, X_n$ be non-empty sets and $f : X_1 \times \ldots \times X_n \to Y$. Let $A_i, A'_i \in \mathcal{F}(X_i)$, $i = 1, \ldots, n$ and $B, B' \in \mathcal{F}(Y)$ defined by*

$$B(y) = \begin{cases} \bigvee\{A_1(x_1) \wedge \ldots \wedge A_n(x_n) | f(x_1, \ldots, x_n) = y\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$B'(y) = \begin{cases} \bigvee\{A'_1(x_1) \wedge \ldots \wedge A'_n(x_n) | f(x_1, \ldots, x_n) = y\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If $A_i = (*, \delta_i)A'_i$, $i = 1, \ldots, n$ then $B = (*, \delta)B'$ where $\delta = \delta_1 \wedge \ldots \wedge \delta_n$.

Proof: Similar to the proof of Proposition 4.12, using Propositions 4.5 and 4.11. □

5 SOME FUZZY OPERATORS

Let X be a non-empty set and $\mathcal{F}(X)$ the set of fuzzy subsets of X.

A *fuzzy operator* will be a function $I : (\mathcal{F}(X))^n \to \mathcal{F}(X^k)$ where n, k are non-zero natural numbers.

In this section we will investigate how some fuzzy operators preserve the $(*, \delta)$-equality.

Any function $\tau : [0,1]^n \to [0,1]$ provides a fuzzy operator $I : (\mathcal{F}(X))^n \to \mathcal{F}(X)$ defined by

$I(A_1, \ldots, A_n)(x_1, \ldots, x_n) = \tau(A_1(x_1), \ldots, A_n(x_n))$ for all $A_1, \ldots, A_n \in \mathcal{F}(X)$ and $x_1, \ldots, x_n \in X$.

Particularly, τ can be a fuzzy implicator, i.e. a function $\tau : [0,1]^2 \to [0,1]$ for which $\tau(0,0) = \tau(0,1) = \tau(1,1) = 1$, $\tau(1,0) = 0$ and whose first (partial) functions are decreasing (increasing). A list with the main fuzzy implicators can be found in [18], p. 24. Then the fuzzy operator $I : (\mathcal{F}(X))^2 \to \mathcal{F}(X^2)$ associated with a fuzzy implicator is given by $I(A_1, A_2)(x, y) = \tau(A_1(x), A_2(y))$ for all $A_1, A_2 \in \mathcal{F}(X)$ and $x, y \in X$.

The following result extends Proposition 4.1 of [6] to an arbitrary continuous t-norm $*$.

Proposition 5.1 *Let us consider the fuzzy operator $I : (\mathcal{F}(X))^2 \to \mathcal{F}(X^2)$ associated with the Gödel implicator $\to_G$:*

$$I(A, B)(x, y) = A(x) \to_G B(y) = \begin{cases} 1 & \text{if} \quad A(x) \leq B(y) \\ B(y) & \text{if} \quad A(x) > B(y) \end{cases}.$$

for any $A, B \in \mathcal{F}(X)$ and $x, y \in X$. If $A = (*, \delta)A'$ and $B = (*, \delta)B'$ then $I(A, B) = (*, \delta)I(A', B')$ where

$$\delta = [(\bigwedge_y B(y)) \vee (\bigwedge_y B'(y))] * (\bigwedge_y B(y) \wedge B'(y))$$

Proof: By Lemma 2.3 (5) we have for any $x, y \in X$:

(a)$\rho(I(A, B)(x, y), I(A', B')(x, y)) \geq$
$\geq \rho(I(A, B)(x, y), I(A', B)(x, y)) * \rho(I(A', B)(x, y), I(A', B')(x, y))$.

First we will prove the inequality

(b) $\rho(I(A, B)(x, y), I(A', B)(x, y)) \geq B(y)$.

We must consider the following cases:

(I) $A(x) = A'(x)$

Then $I(A, B)(x, y) = I(A', B)(x, y)$, hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = 1$.

(II) $A(x) < A'(x)$ We have three subcases:

- $A(x) < A'(x) \leq B(y)$

Then $I(A, B)(x, y) = I(A', B)(x, y) = 1$ hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = 1$.

- $A(x) \leq B(y) \leq A'(x)$

Then $I(A, B)(x, y) = 1$, $I(A', B)(x, y) = B(y)$ hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = \rho(1, B(y)) = B(y)$.

- $B(y) < A(x) < A'(x)$

Then $I(A, B)(x, y) = I(A', B)(x, y) = B(y)$ hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = \rho(B(y), B(y)) = 1$

(III) $A'(x) < A(x)$ We also have three subcases:

- $A'(x) < A(x) \leq B(y)$

Then $I(A, B)(x, y) = I(A', B)(x, y) = 1$ hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = 1$.

- $A'(x) \leq B(y) < A(x)$

Then $I(A, B)(x, y) = B(y)$, $I(A', B)(x, y) = 1$ hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = \rho(B(y), 1) = B(y)$.

- $B(y) < A'(x) < A(x)$

Then $I(A, B)(x, y) = I(A', B)(x, y) = 1$ hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = 1$.

Therefore the inequality (b) is verified in all the cases.

Secondly, we will establish the following inequality:

(c) $\rho(I(A', B)(x, y), I(A', B')(x, y)) \geq B(y) \wedge B'(y)$.

We must consider the following cases:

(I) $B(y) = B'(y)$

Then $I(A', B)(x, y) = I(A', B')(x, y)$ hence $\rho(I(A', B)(x, y), I(A', B')(x, y)) = 1$.

(II) $B'(y) < B(y)$ We have three subcases:

- $B'(y) < B(y) < A'(x)$

Then $I(A', B)(x, y) = B(y)$, $I(A', B')(x, y) = B'(y)$, hence, by Lemma 2.3 (11):

$\rho(I(A', B)(x, y), I(A', B')(x, y)) = \rho(B(y), B'(y)) \geq B(y) \wedge B'(y)$.

• $B'(y) < A'(x) \leq B(y)$
Then $I(A', B)(x, y) = 1$, $I(A', B')(x, y) = B'(y)$, hence
$\rho(I(A', B)(x, y), I(A', B')(x, y)) = \rho(1, B'(y)) = B'(y)$.

• $A'(x) \leq B'(y) < B(y)$
Then $I(A', B)(x, y) = I(A', B')(x, y) = 1$ hence $\rho(I(A', B)(x, y), I(A', B')(x, y)) = 1$.

(III) $B(y) < B'(y)$ We have three subcases:

• $A'(x) \leq B(y) < B'(y)$
Then $I(A', B)(x, y) = I(A', B')(x, y) = 1$ hence $\rho(I(A', B)(x, y), I(A', B')(x, y)) = 1$.

• $B(y) < A'(x) \leq B'(y)$
Then $I(A', B)(x, y) = B(y)$, $I(A', B')(x, y) = 1$ hence
$\rho(I(A', B)(x, y), I(A', B')(x, y)) = \rho(B(y), 1) = B(y)$.

• $B(y) < B'(y) < A'(x)$
Then $I(A', B)(x, y) = B(y)$, $I(A', B')(x, y) = B'(y)$ hence
$\rho(I(A', B)(x, y), I(A', B')(x, y)) \geq B(y) \wedge B'(y)$.

Thus the inequality (c) is verified in all cases. By the inequalities (a), (b), (c) and Lemma 2.4 (6) we obtain

$$\begin{aligned}
&\bigwedge_{x,y} \rho(I(A, B)(x, y), I(A', B')(x, y)) \\
&\geq \bigwedge_{x,y} \rho(I(A, B)(x, y), I(A', B)(x, y)) * \rho(I(A', B)(x, y), I(A', B')(x, y)) \\
&\geq [\bigwedge_{x,y} \rho(I(A, B)(x, y), I(A', B)(x, y))] * [\bigwedge_{x,y} \rho(I(A', B)(x, y), I(A', B')(x, y)] \\
&\geq [\bigwedge_{y} B(y)] * [\bigwedge_{y} (B(y) \wedge B'(y))].
\end{aligned}$$

By symmetry we get:

$$\bigwedge_{x,y} \rho(I(A, B)(x, y), I(A', B')(x, y)) \geq [\bigwedge_{y} B'(y)] * [\bigwedge_{y} (B(y) \wedge B'(y))]$$

Therefore, by Lemma 2.1 (9)

$$\bigwedge_{x,y} \rho(I(A, B)(x, y), I(A', B')(x, y))$$
$$\geq \{[\bigwedge_y B(y)] * [\bigwedge_y B(y) \wedge B'(y))]\} \vee \{[\bigwedge_y B'(y)] * [\bigwedge_y B(y) \wedge B'(y))]\}$$
$$= [(\bigwedge_y B(y)) \vee (\bigwedge_y B'(y))] * [\bigwedge_y (B(y) \wedge B'(y))].$$

□

In the proof of the above proposition the properties of $\rightarrow_G$ are used. An open problem is whether a similar result holds true for the fuzzy operators associated with other implicators.

A second class of fuzzy operators is obtained by using infinitary operators $\bigvee$ and $\bigwedge$ on $[0, 1]$. Let us consider a function $\tau : [0, 1]^n \rightarrow [0, 1]$ and $1 \leq k < n$. Then a fuzzy operator $I : (\mathcal{F}(X))^n \rightarrow \mathcal{F}(X^{n-k})$ is defined by

$$I(A_1, \ldots, A_n)(x_{k+1}, \ldots, x_n) = \bigvee_{x \in X} \tau(A_1(x), \ldots, A_k(x), A_{k+1}(x_{k+1}), \ldots, A_n(x_n))$$

for all $A_1, \ldots, A_n \in \mathcal{F}(X)$ and $x_1, \ldots, x_n \in X$. A similar fuzzy operator can be defined using $\bigwedge$ instead of $\bigvee$. In particular τ can be a term, i.e the composition of some of the operations of the residuated lattice $([0, 1], \vee, \wedge, *, \rightarrow, 0, 1)$.

Instead of formulating a general result about the way $(*, \delta)$-equality is preserved by the fuzzy operators induced by such terms, we will treat this problem in some particular cases.

Proposition 5.2 *Let us consider the fuzzy operators* $I_1, I_2, I_3, I_4 : (\mathcal{F}(X))^3 \rightarrow \mathcal{F}(X)$ *defined by*

$$I_1(A, B, C)(y) = \bigvee_{x \in X} [C(x) \wedge (\neg A(x) \vee B(y))];$$
$$I_2(A, B, C)(y) = \bigvee_{x \in X} [C(x) * (A(x) \rightarrow B(y))];$$
$$I_3(A, B, C)(y) = \bigvee_{x \in X} [C(x) * (\neg A(x) \vee B(y))];$$
$$I_4(A, B, C)(y) = \bigvee_{x \in X} [C(x) \wedge (\neg A(x) \rightarrow B(y))].$$

for any $A, B, C \in \mathcal{F}(\mathcal{X})$ *and* $y \in X$. *If* $A = (*, \delta_1)A'$, $B = (*, \delta_2)B'$ *and* $C = (*, \delta_3)C'$ *then*

$$I_1(A, B, C) = (*, \delta_1 \wedge \delta_2 \wedge \delta_3)I_1(A', B', C'),$$
$$I_2(A, B, C) = (*, \delta_1 * \delta_2 * \delta_3)I_2(A', B', C'),$$
$$I_3(A, B, C) = (*, \delta_3 * (\delta_1 \wedge \delta_2))I_3(A', B', C'),$$
$$I_4(A, B, C) = (*, \delta_3 \wedge (\delta_1 * \delta_2))I_4(A', B', C').$$

Proof: In accordance with Lemma 2.5 (2)

$$\bigwedge_y \rho(I_1(A, B, C)(y), I_1(A', B', C')(y)) = \bigwedge_y \rho(\bigvee_x [C(x) \wedge (\neg A(x) \vee B(y))],$$
$$\bigvee_x [C'(x) \wedge (\neg A'(x) \vee B'(y))])$$
$$\geq \bigwedge_y \bigwedge_x \rho([C(x) \wedge (\neg A(x) \vee B(y))], [C'(x) \wedge (\neg A'(x) \vee B'(y))]).$$

Let $x, y \in X$. By Lemma 2.3 (6),(7) and (4) we have:

$$\rho([C(x) \wedge (\neg A(x) \vee B(y))], [C'(x) \wedge (\neg A'(x) \vee B'(y))])$$
$$\geq \rho(C(x), C'(x)) \wedge \rho(\neg A(x) \vee B(y), \neg A'(x) \vee B'(y))$$
$$\geq \rho(C(x), C'(x)) \wedge \rho(\neg A(x), \neg A'(x)) \wedge \rho(B(y), B'(y))$$
$$\geq \rho(C(x), C'(x)) \wedge \rho(A(x), A'(x)) \wedge \rho(B(y), B'(y)).$$

We conclude that

$$\bigwedge_y \rho(I_1(A, B, C)(y), I_1(A', B', C')(y))$$
$$\geq \bigwedge_{x,y} [\rho(A(x), A'(x)) \wedge \rho(B(y), B'(y)) \wedge \rho(C(x), C'(x))]$$
$$= [\bigwedge_x \rho(A(x), A'(x))] \wedge [\bigwedge_y \rho(B(y), B'(y))] \wedge [\bigwedge_x \rho(C(x), C'(x))].$$

By hypothesis

$$\bigwedge_x \rho(A(x), A'(x)) \geq \delta_1, \bigwedge_y \rho(B(y), B'(y)) \geq \delta_2, \bigwedge_x \rho(C(x), C'(x)) \geq \delta_3,$$

therefore

$$\bigwedge_y \rho(I_1(A, B, C)(y), I_1(A', B', C')(y)) \geq \delta_1 \wedge \delta_2 \wedge \delta_3.$$

For the operators I_2, I_3 and I_4 the results are obtained similarly. □

6 $(*, \delta)$-EQUALITY AND FUZZY RELATIONS

In this section we shall investigate how the composition of fuzzy relations and the transitive closure operator preserve the $(*, \delta)$-equality.

Let R, S be two fuzzy relations on X. Recall that $R \circ S$ is the fuzzy relation defined by

$$(R \circ S)(x, z) = \bigvee_{y \in X} R(x, y) * S(y, z) \text{ for all } x, z \in X.$$

The following result generalizes Proposition 4.1 [5] (see also [11]).

Proposition 6.1 *Let R, R', S, S' be fuzzy relations on X. If $R = (*, \delta_1)R'$ and $S = (*, \delta_2)S'$ then $R \circ S = (*, \delta_1 * \delta_2)R' \circ S'$.*

Proof: By hypothesis

$$\text{(a)} \bigwedge_{x,z} \rho(R(x, z), R'(x, z)) \geq \delta_1, \bigwedge_{x,z} \rho(S(x, z), S'(x, z)) \geq \delta_2.$$

By Lemma 2.5 (2) we have

$$\begin{aligned}\text{(b)} &\bigwedge_{x,z} \rho((R \circ S)(x, z), (R' \circ S')(x, z)) \\ &= \bigwedge_{x,z} \rho(\bigvee_y R(x, y) * S(y, z), \bigvee_y R'(x, y) * S'(y, z)) \\ &\geq \bigwedge_{x,z} \bigwedge_y \rho(R(x, y) * S(y, z), R'(x, y) * S'(y, z)).\end{aligned}$$

Let $x, y, z \in X$. By Lemma 2.3 (8) and Lemma 2.4 (6)

$$\begin{aligned}&\rho(R(x, y) * S(y, z), R'(x, y) * S'(y, z)) \\ &\geq \rho(R(x, y), R'(x, y)) * \rho(S(y, z), S'(y, z)) \\ &\geq \bigwedge_{s,t,u,v} \rho(R(s, t), R'(s, t)) * \rho(S(u, v), S'(u, v)) \\ &\geq [\bigwedge_{s,t} \rho(R(s, t), R'(s, t))] * [\bigwedge_{u,v} \rho(S(u, v), S'(u, v))] \geq \delta_1 * \delta_2.\end{aligned}$$

These inequalities hold for all $x, y, z \in X$, hence

(c) $\bigwedge_{x,z} \bigwedge_{y} \rho(R(x,y) * S(y,z), R'(x,y) * S'(y,z)) \geq \delta_1 * \delta_2$.

From (b) and (c) it follows that $\bigwedge_{x,z} \rho((R \circ S)(x,z), (R' \circ S')(x,z)) \geq \delta_1 * \delta_2$, i.e. $R \circ S = (*, \delta_1 * \delta_2) R' \circ S'$. □

Lemma 6.2. *Let $(R_i)_{i \in I}$, $(S_i)_{i \in I}$ be two families of fuzzy relations on X and $R = \bigcup_{i \in I} R_i$, $S = \bigcup_{i \in I} S_i$. If $R_i = (*, \delta_i) S_i$ for any $i \in I$ then $R = (*, \bigwedge_{i \in I} \delta_i) S$.*

Proof: By hypothesis, $\bigwedge_{x,y} \rho(R_i(x,y), S_i(x,y)) \geq \delta_i$ for any $i \in I$. In accordance with Lemma 2.5 (2)

$$\bigwedge_{x,y} \rho(R(x,y), S(x,y)) = \bigwedge_{x,y} \rho(\bigvee_{i \in I} R_i(x,y), \bigvee_{i \in I} S_i(x,y))$$

$$\geq \bigwedge_{x,y} \bigwedge_{i \in I} \rho(R_i(x,y), S_i(x,y)) = \bigwedge_{i \in I} \bigwedge_{x,y} \rho(R_i(x,y), S_i(x,y)) \geq \bigwedge_{i \in I} \delta_i$$

□

A fuzzy relation R on X is $*$-*transitive* if $R(x,y) * R(y,z) \leq R(x,z)$ for any $x, y, z \in X$. If R is an arbitrary fuzzy relation on X then the $*$-*transitive closure* of R is the intersection $T(R)$ of all $*$-transitive fuzzy relations containing R.

The following result is well-known.

Lemma 6.3. *If R is a fuzzy relation then $T(R) = \bigcup_{n=1}^{\infty} R^n$ where $R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n-\text{times}}$ for each n.*

Theorem 6.4. *Let R, S be two fuzzy relations on X. If $R = (*, \delta) S$ then $T(R) = (*, \epsilon) T(S)$ where $\epsilon = \bigwedge_{n=1}^{\infty} \delta^{(n)}$ and $\delta^{(n)} = \underbrace{\delta * \delta * \ldots * \delta}_{n-\text{times}}$ for each* $n \geq 1$.

Proof: By Proposition 6.1, $R^n = (*, \delta^{(n)}) S^n$ for each $n \geq 1$. Then we apply Lemmas 6.2 and 6.3. □

7 $(*, \delta)$-EQUALITY AND S-NORMS

An s-norm is a binary operation on $[0, 1]$ by which one can define a generalized union of two fuzzy sets. [6], p. 744 studies how an s-norm behaves with respect to δ-equality. In this section we shall generalize this result of Cai investigating how the fuzzy operator introduced by an s-norm preserves the $(*, \delta)$-equality.

Applying the s-norm one defines a class of fuzzy operators that generalize the implication operator Dienes-Rescher (or Kleene-Dienes, by [18], p. 24). For this class of fuzzy operators one proves a preservation theorem of $(*, \delta)$-equality that extends Proposition 4.7, [6].

Let X be a non-empty set.

Proposition 7.1 *Let $A_1, A_2, B_1, B_2, C_1, C_2$ fuzzy subsets of X such that $A_1 \subseteq B_1 \subseteq C_1$, $A_2 \subseteq B_2 \subseteq C_2$. If $A_1 = (*, \delta_1)A_2$, $A_1 = (*, \delta_2)C_1$ and $A_2 = (*, \delta_3)C_2$ then $B_1 = (*, \delta_1 * (\delta_2 \wedge \delta_3))B_2$.*

Proof: By hypothesis

(a) $\bigwedge_x \rho(A_1(x), A_2(x)) \geq \delta_1, \bigwedge_x \rho(A_1(x), C_1(x)) \geq \delta_2, \bigwedge_x \rho(A_2(x), C_2(x)) \geq \delta_3$.

We shall prove that for each $y \in X$

(b) $[\bigwedge_x \rho(A_1(x), A_2(x))] * [\bigwedge_x \rho(A_1(x), C_1(x))] \leq B_1(y) \to B_2(y)$.

Let $y \in X$. We have $B_1(y) \leq C_1(y)$ hence, by Lemma 2.1 (10)

$$C_1(y) \to A_1(y) \leq B_1(y) \to A_1(y).$$

Thus, by Lemma 2.1 (2)

$$\begin{aligned}
&B_1(y) * [C_1(y) \to A_1(y)] * [A_1(y) \to A_2(y)] \leq B_1(y) * [B_1(y) \\
&\quad \to A_1(y)] * [A_1(y) \to A_2(y)] = [A_1(y) \wedge B_1(y)] * [A_1(y) \to A_2(y)] \\
&\quad \leq A_1(y) * [A_1(y) \to A_2(y)] = A_1(y) \wedge A_2(y) \leq A_2(y) \leq B_2(y).
\end{aligned}$$

In accordance with Lemma 2.1 (1)

$$[C_1(y) \to A_1(y)] * [A_1(y) \to A_2(y)] \leq B_1(y) \to B_2(y).$$

Thus

$$[\bigwedge_x \rho(A_1(x), A_2(x))] * [\bigwedge_x \rho(A_1(x), C_1(x))] \leq \rho(A_1(y), A_2(y)) * \rho(A_1(y), C(y))$$
$$\leq [C(y) \to A_1(y)] * [A_1(y) \to A_2(y)] \leq B_1(y) \to B_2(y).$$

Similarly

$$\text{(c)}[\bigwedge_x \rho(A_1(x), A_2(x))] * [\bigwedge_x \rho(A_2(x), C_2(x))] \leq B_2(y) \to B_1(y).$$

By (a), (b) and (c) we get for each $y \in X$:

$$\delta_1 * (\delta_2 \wedge \delta_3) \leq (\delta_1 * \delta_2) \wedge (\delta_1 * \delta_3) \leq [B_1(y) \to B_2(y)] \wedge [B_2(y) \to B_1(y)]$$
$$= \rho(B_1(y), B_2(y)).$$

It follows that

$$\delta_1 * (\delta_2 \wedge \delta_3) \leq \bigwedge_y \rho(B_1(y), B_2(y)).$$

□

Now let us recall the definition of s-norm.

Definition 7.2. *An s-norm is a function* $s : [0, 1] \times [0, 1] \to [0, 1]$ *such that the following axioms hold for any* $a, b, c \in [0, 1]$*:*

(A_1) $s(1, 1) = 1, s(0, a) = s(a, 0) = a$;
(A_2) $s(a, b) = s(b, a)$ *(commutativity axiom);*
(A_3) *If* $a \leq b$ *then* $s(a, c) \leq s(b, c)$*;*
(A_4) $s(s(a, b), c) = s(a, s(b, c))$ *(associativity axiom).*

The join operation $\vee$ is the most usual s-norm.
Let us consider the s-norm s_w defined by

$$s_w(a, b) = \begin{cases} a & \text{if} \quad b = 0 \\ b & \text{if} \quad a = 0 \\ 1 & \text{otherwise} \end{cases}.$$

The following result is Lemma 5.1, [6]:

Lemma 7.3. *For any s-norm* s *and for any* $a, b \in [0, 1]$ *we have* $a \vee b \leq s(a, b) \leq s_w(a, b)$*.*

If A, B are two fuzzy subsets of X and s is an s-norm, then $s(A, B)$ will be the fuzzy operator defined by $s(A, B)(x, y) = s(A(x), B(y))$ for all $x, y \in X$.

Proposition 7.4 *If* $A = (*, \delta_1)A'$ *and* $B = (*, \delta_2)B'$ *then* $s(A, B) = (*, \delta)s(A', B')$ *where* $\delta = (\delta_1 \wedge \delta_2) * \bigwedge_{x,y} ((A(x) \vee B(y)) \wedge (A'(x) \vee B'(y)))$.

Proof: By Lemma 7.3 we have

$A(x) \vee B(y) \leq s(A(x), B(y)) \leq s_w(A(x), B(y))$; $A'(x) \vee B'(y) \leq s(A'(x), B'(y)) \leq s_w(A'(x), B'(y))$ hence

$A \sqcup B \subseteq s(A, B) \subseteq s_w(A, B)$ and $A' \sqcup B' \subseteq s(A', B') \subseteq s_w(A', B')$.

By Proposition 4.5, $A \sqcup B = (*, \delta_1 \wedge \delta_2)A' \sqcup B'$.

Let $x, y \in X$. Then

$$
\begin{aligned}
& s_w(A(x), B(y)) \to (A(x) \vee B(y)) \\
&= \begin{cases} A(x) \to A(x) & \text{if} \quad B(y) = 1 \\ B(y) \to B(y) & \text{if} \quad A(x) = 1 \\ 1 \to (A(x) \vee B(y)) & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad B(y) = 1 \\ 1 & \text{if} \quad A(x) = 1 \\ A(x) \vee B(y) & \text{otherwise} \end{cases} .
\end{aligned}
$$

$A(x) \vee B(y) \leq s_w(A(x), B(y)) \to A(x) \vee B(y) = \rho(s_w(A(x), B(y)), A(x) \vee B(y))$ hence

$$\bigwedge_{x,y} (A(x) \vee B(y)) \leq \bigwedge_{x,y} \rho(s_w(A(x), B(y)), A(x) \vee B(y)).$$

Therefore $A \sqcup B = (*, \epsilon_1)s_w(A, B)$ where $\epsilon_1 = \bigwedge_{x,y} (A(x) \vee B(y))$. Similarly, $A' \sqcup B' = (*, \epsilon_2)s_w(A', B')$ where $\epsilon_2 = \bigwedge_{x,y} (A'(x) \vee B'(y))$.

Now we apply Proposition 7.1 to that situation, hence $s(A, B) = (*, \delta)s(A', B')$ where $\delta = (\delta_1 \wedge \delta_2) * (\epsilon_1 \wedge \epsilon_2)$. It is easy to see that δ has the desired form. □

Let us consider the fuzzy operator I defined by

$$I(A, B)(x, y) = \begin{cases} 1 \text{ if } A(x) \leq B(y) \\ s(\neg A(x), B(y)) \text{ if } A(x) > B(y) \end{cases}$$

for any $A, B \in \mathcal{F}(X)$ and $x, y \in X$.

If $*$ is the Lukasiewicz t-norm and s the join operation $\vee$ we obtain operator I from [6], Proposition 4.7:

$$I(A, B)(x, y) = \begin{cases} 1 \text{ if } A(x) \le B(y) \\ (1 - A(x)) \vee B(y) \text{ if } A(x) > B(y). \end{cases}$$

The following result extends Proposition 4.7 [6] to a very general setting.

Proposition 7.5 *If* $A = (*, \delta_1)A'$ *and* $B = (*, \delta_2)B'$ *then* $I(A, B) = (*, \delta)$ $I(A', B')$ *where* $\delta = [\bigwedge_{x,y} s(\neg A(x) \wedge \neg A'(x), B(y))] * [\bigwedge_{x,y} s(\neg A'(x), B(y) \wedge B'(y))]$.

Proof: By Lemma 2.3 (5) we have for any $x, y \in X$:

(a) $\rho(I(A, B)(x, y), I(A', B')(x, y)) \ge$
$\ge \rho(I(A, B)(x, y), I(A', B)(x, y)) * \rho(I(A', B)(x, y), I(A', B')(x, y))$.

Firstly we shall prove the inequality:

(b) $\rho(I(A, B)(x, y), I(A', B)(x, y)) \ge s(\neg A(x) \wedge \neg A'(x), B(y))$.

We must consider the cases:

(I) $A(x) = A'(x)$ (b) is obviously verified.

(II) $A(x) < A'(x)$ We have three subcases:

- $A(x) < A'(x) \le B(y)$ Then $I(A, B)(x, y) = I(A', B)(x, y) = 1$, hence $\rho(I(A, B)(x, y), I(A', B)(x, y)) = 1$.

- $B(y) < A(x) < A'(x)$ Then $I(A, B)(x, y) = s(\neg A(x), B(y))$; $I(A', B)(x, y) = s(\neg A'(x), B(y))$

We remark that $\neg A'(x) \le \neg A(x)$ hence $s(\neg A'(x), B(y)) \le s(\neg A(x), B(y))$, i.e. $s(\neg A'(x), B(y)) \to s(\neg A(x), B(y)) = 1$. Thus

$$\rho(I(A, B)(x, y), I(A', B)(x, y)) = \rho(s(\neg A(x), B(y)), s(\neg A'(x), B(y))) =$$

$= s(\neg A(x), B(y)) \to s(\neg A'(x), B(y)) \ge s(\neg A'(x), B(y)) \ge s(\neg A(x) \wedge \neg A'(x), B(y))$.

- $A(x) \le B(y) < A'(x)$ Then $I(A, B)(x, y) = 1$, $I(A', B)(x, y) = s(\neg A'(x), B(y))$ hence

$\rho(I(A, B)(x, y), I(A', B)(x, y)) = s(\neg A'(x), B(y)) \ge s(\neg A(x) \wedge \neg A'(x), B(y))$.

Therefore (b) is verified in all subcases.

(III) Similar to (II).

Now we shall establish the inequality

(c)$\rho(I(A', B)(x, y), I(A', B')(x, y)) \geq s(\neg A'(x), B(y) \wedge B'(y))$.
We also consider three cases:

(I) $B(y) = B'(y)$ (c) is obviously verified.

(II) $B'(y) < B(y)$ We shall analyze three subcases

• $B'(y) < B(y) < A'(x)$ Then $I(A', B)(x, y) = s(\neg A'(x), B(y))$, $I(A', B')(x, y) = s(\neg A'(x), B'(y))$.

But $B'(y) < B(y)$ implies $s(\neg A'(x), B'(y)) \leq s(\neg A'(x), B(y))$, therefore

$$\rho(I(A', B)(x, y), I(A', B')(x, y)) = \rho(s(\neg A'(x), B(y)), s(\neg A'(x), B'(y))) =$$

$$= s(\neg A'(x), B(y)) \to s(\neg A'(x), B'(y)) \geq s(\neg A'(x), B'(y)) \geq s(\neg A'(x), B(y) \wedge B'(y)).$$

• $B'(y) < A'(x) \leq B(y)$ Then $I(A', B)(x, y) = 1$, $I(A', B')(x, y) = s(\neg A'(x), B'(y))$ hence $\rho(I(A', B)(x, y), I(A', B')(x, y)) = s(\neg A'(x), B'(y)) \geq s(\neg A'(x), B(y) \wedge B'(y))$.

• $A'(x) \leq B'(y) < B(y)$ Then $I(A', B)(x, y) = I(A', B')(x, y) = 1$ hence $\rho(I(A', B)(x, y), I(A', B')(x, y)) = 1$.

Then (c) is verified in all subcases.

(III) $B(y) < B'(y)$ Similar to (II).
In accordance with (a), (b) and (c) we conclude

$$\bigwedge_{x,y} \rho(I(A, B)(x, y), I(A', B')(x, y)) \geq$$
$$\geq [\bigwedge_{x,y} s(\neg A(x) \wedge \neg A'(x), B(y))] * [\bigwedge_{x,y} s(\neg A'(x), B(y) \wedge B'(y))]. \qquad \square$$

8 Sugeno integral and $(*, \delta)$-equality

The fuzzy operator $PC : (\mathcal{F}(X))^3 \to \mathcal{F}(X)$ was introduced in [27], p. 627, as a probabilistic version of Zadeh's compositional rule of fuzzy inference [23]. The universe of discourse X has a structure of probability space (X, σ, P) and the definition of PC uses the integral corresponding to the probability P.

In this section we shall introduce a new probabilistic version $P : (\mathcal{F}(X))^3 \to \mathcal{F}(X)$ of compositional rule of inference using Sugeno integral [20] instead of the classical one.

The main theorem in this section establishes how the fuzzy operator P preserves the $(*, \delta)$-equality.

A *(discrete) fuzzy measure* on a finite set X is a function $\mu : \mathcal{P}(X) \to [0, 1]$ verifying the following properties: (M_1) $\mu(\phi) = 0$; (M_2) If $K \subseteq L \subseteq X$ then $\mu(K) \leq \mu(L)$; (M_3) $\mu(X) = 1$.

Definition 8.1. *Let X be a finite non-empty set, μ be a fuzzy measure on X and A a fuzzy subset of X. The discrete Sugeno integral* of A with respect to μ is defined by

$$\int A(x) d\mu(x) = \bigvee_{K \subseteq X} \bigwedge_{u \in K} (A(u) \wedge \mu(K)).$$

Let us consider the fuzzy operator $P : (\mathcal{F}(X))^3 \to \mathcal{F}(X)$ defined by

$P(A, B, A')(y) = \int A'(x) * (A(x) \to B(y)) d\mu(x)$ for any $A, A', B \in \mathcal{F}(X)$ and $y \in X$.

Remark 8.2 The fuzzy operator P is similar to the fuzzy operator PC defined in [27] p. 627. P has the same form with PC but it is defined using the Sugeno integral instead of the classical integral.

Proposition 8.3 *Let $A_1, A_2, A'_1, A'_2, B_1, B_2$ be fuzzy subsets of X. If $A_1 = (*, \delta_1) A_2$, $A'_1 = (*, \delta_2) A'_2$, $B_1 = (*, \delta_3) B_2$, then $P(A_1, B_1, A'_1) = (*, \delta_1 * \delta_2 * \delta_3) P(A_2, B_2, A'_2)$.*

Proof: We have:

(a) $\bigwedge_x \rho(A_1(x), A_2(x)) \geq \delta_1$, $\bigwedge_x \rho(A'_1(x), A'_2(x)) \geq \delta_2$, $\bigwedge_y \rho(B_1(y), B_2(y)) \geq \delta_3$.

Let $x, y \in X$. Then by Lemma 2.3 (8) and (9)

$$\begin{aligned} &\rho(A'_1(x) * (A_1(x) \to B_1(y)), A'_2(x) * (A_2(x) \to B_2(y))) \geq \\ &\geq \rho(A'_1(x), A'_2(x)) * \rho(A_1(x) \to B_1(y), A_2(x) \to B_2(y)) \geq \\ &\geq \rho(A'_1(x), A'_2(x)) * \rho(A_1(x), A_2(x)) * \rho(B_1(y), B_2(y)). \end{aligned}$$

Using these inequalities and Lemma 2.5 we obtain:

$$\begin{aligned} &\rho(P(A_1, B_1, A'_1)(y), P(A_2, B_2, A'_2)(y)) = \\ &= \rho(\bigvee_{K \subseteq X} \bigwedge_{x \in K} [A'_1(x) * (A_1(x) \to B_1(y))) \wedge \mu(K)], \\ &\bigvee_{K \subseteq X} \bigwedge_{x \in K} [(A'_2(x) * (A_2(x) \to B_2(y))) \wedge \mu(K)]) \geq \end{aligned}$$

$$\geq \bigwedge_{K\subseteq X} \bigwedge_{x\subseteq K} \rho([(A'_1(x) * (A_1(x)$$

$$\to B_1(y))) \wedge \mu(K)], [(A'_2(x) * (A_2(x) \to B_2(y))) \wedge \mu(K)]).$$

In accordance with Lemma 2.3 (6) we get from any $K \subseteq X$ and $y \in K$:

$$\rho([(A'_1(x) * (A_1(x) \to B_1(y))) \wedge \mu(K)], [(A'_2(x) * (A_2(x) \to B_2(y))) \wedge \mu(K)]) \geq$$
$$\geq \rho(A'_1(x) * (A_1(x) \to B_1(y)), A'_2(x) * (A_2(x) \to B_2(y))) \wedge \rho(\mu(K), \mu(K)) =$$
$$= \rho(A'_1(x) * (A_1(x) \to B_1(y)), A'_2(x) * (A_2(x) \to B_2(y))) \geq$$
$$\geq \rho(A_1(x), A_2(x)) * \rho(A'_1(x), A'_2(x)) * \rho(B_1(y), B_2(y)).$$

Thus

$$\rho(P(A_1, B_1, A'_1)(y), P(A_2, B_2, A'_2(y))) \geq$$
$$\geq \bigwedge_{K\subseteq X} \bigwedge_{x\in K} \rho(A_1(x), A_2(x)) * \rho(A'_1(x), A'_2(x)) * \rho(B_1(y), B_2(y)) \geq$$
$$\geq [\bigwedge_{K\subseteq X} \bigwedge_{x\in K} \rho(A_1(x), A_2(x))] * [\bigwedge_{K\subseteq X} \bigwedge_{x\in K} \rho(A'_1(x), A'_2(x))] * \rho(B_1(y), B_2(y)).$$

We remark that $\bigwedge_{K\subseteq X} \bigwedge_{x\in K} \rho(A_1(x), A_2(x)) = \bigwedge_{x\in X} \rho(A_1(x), A_2(x)) \geq \delta_1$ hence $\rho(P(A_1, B_1, A'_1)(y), P(A_2, B_2, A'_2)(y)) \geq \delta_1 * \delta_2 * \rho(B_1(y), B_2(y))$.

Therefore

$$\bigwedge_{y} P(A_1, B_1, A'_1)(y), P(A_2, B_2, A'_2)(y) \geq$$
$$\geq \bigwedge_{y} (\delta_1 * \delta_2 * \rho(B_1(y), B_2(y))) \geq$$
$$\geq \delta_1 * \delta_2 * \bigwedge_{y} \rho(B_1(y), B_2(y)) \geq$$
$$\geq \delta_1 * \delta_2 * \delta_3.$$

□

Consider the fuzzy operators $P_1, P_2, P_3, P_4, P_5 : (\mathcal{F}(X))^3 \to \mathcal{F}(X)$ defined by

$$P_1(A, B, A')(y) = \int A'(x) * (\neg A(x) \vee B(y)) d\mu(x),$$
$$P_2(A, B, A')(y) = \int A'(x) \wedge (\neg A(x) \vee B(y)) d\mu(x),$$

$P_3(A, B, A')(y) = \int A'(x) \wedge (A(x) \to B(y))d\mu(x),$
$P_4(A, B, A')(y) = \int A'(x) * \rho(A(x), B(y))d\mu(x),$
$P_5(A, B, A')(y) = \int A'(x) \wedge \rho(A(x), B(y))d\mu(x)$

for any $A, A', B \in \mathcal{F}(X)$ and $y \in X$.

Proposition 8.4. *Let* $A_1, A_2, A'_1, A'_2, B_1, B_2$ *be fuzzy subsets of* X. *If* $A_1 = (*, \delta_1)A_2, A'_1 = (*, \delta_2)A'_2, B_1 = (*, \delta_3)B_2$, *then*

$P_1(A_1, B_1, A'_1) = (*, \delta_3 * (\delta_1 \wedge \delta_2))P_1(A_2, B_2, A'_2),$
$P_2(A_1, B_1, A'_1) = (*, \delta_1 \wedge \delta_2 \wedge \delta_3)P_2(A_2, B_2, A'_2),$
$P_3(A_1, B_1, A'_1) = (*, \delta_3 \wedge (\delta_1 * \delta_2))P_3(A_2, B_2, A'_2),$
$P_4(A_1, B_1, A'_1) = (*, \delta_1 * \delta_2 * \delta_3)P_4(A_2, B_2, A'_2),$
$P_5(A_1, B_1, A'_1) = (*, \delta_3 \wedge (\delta_1 * \delta_2))P_5(A_2, B_2, A'_2).$

Proof: Similarly as Proposition 8.3. □

9 $(*, \delta)$-equality of fuzzy choice functions

In this section we shall introduce the notion of $(*, \delta)$-equality for fuzzy choice functions and we shall prove that $(*, \delta)$-equality is preserved by some fundamental constructions of fuzzy revealed preference.

A *fuzzy choice space* is a pair $\langle X, \mathcal{B}\rangle$ where X is a universe of alternatives and $\mathcal{B}$ is a non-empty family of non-zero fuzzy subsets of X. A *fuzzy choice function* on $\langle X, \mathcal{B}\rangle$ is a function $C : \mathcal{B} \to \mathcal{F}(X)$ such that for each $S \in \mathcal{B}$, $C(S)$ is non-zero and $C(S) \subseteq S$. Starting from Banerjee's paper [2] we have developed a revealed preference theory for fuzzy choice functions [8, 9].

We fix a continuous t-norm $*$. Let C be a fuzzy choice function on $\langle X, \mathcal{B}\rangle$. With C we associate the fuzzy revealed preference relation R_C defined by [8] $R_C(x, y) = \bigvee_{S\in\mathcal{B}} (C(S)(x) * S(y))$ for all $x, y \in X$. R_C is a fuzzy form of the revealed preference relation R introduced by Samuelson in 1938 [19].

The assignment $C \mapsto R_C$ defines a function from fuzzy choice functions on $\langle X, \mathcal{B}\rangle$ to fuzzy relations on X. Conversely, let us start with a fuzzy preference relation Q on X and we define a function $C_Q : \mathcal{B} \to \mathcal{F}(X)$ by

$$C_Q(S)(x) = S(x) * \bigwedge_{y\in X} [S(y) \to Q(x, y)]$$

for all $S \in \mathcal{B}$ and $x \in X$. In general C_Q is not a fuzzy choice function. If C is a fuzzy choice function and $Q = R_C$ then C_Q is also a fuzzy choice function. For a fuzzy choice function C denote $\hat{C} = C_{R_C}$; for $S \in \mathcal{B}$ and $x \in X$ we have

$$\hat{C}(S)(x) = S(x) * \bigwedge_{y \in X} [S(y) \to R_C(x, y)].$$

Let Q be a fuzzy preference relation on X; denote $\hat{Q} = R_{C_Q}$.

If C_1, C_2 are two fuzzy choice functions on $\langle X, \mathcal{B} \rangle$ then we define the *degree of similarity* of C_1 and C_2 by

$$E(C_1, C_2) = \bigwedge_{S \in \mathcal{B}} \bigwedge_{x \in X} \rho(C_1(S)(x), C_2(S)(x)).$$

Definition 9.1. *Let C_1 and C_2 be two fuzzy choice functions on $\langle X, \mathcal{B} \rangle$. For $0 \leq \rho \leq 1$ we say that C_1 and C_2 are $(\delta, *)$-equal ($C_1 = (*, \delta)C_2$ in symbols) if $E(C_1, C_2) \geq \delta$.*

Lemma 9.2. *For all $a, b, c \in [0, 1]$ we have $\rho(a * c, b * c) \geq \rho(a, b)$, $\rho(a \to c, b \to c) \geq \rho(a, b)$.*

Proof: By Lemma 2.3 (2), (9). □

The following result shows that the $(*, \delta)$-equality of fuzzy preference relations is preserved by the assignment $Q \mapsto \hat{Q}$.

Proposition 9.3. *Let Q_1, Q_2 be two fuzzy preference relations on X and $0 \leq \delta \leq 1$. If $Q_1 = (*, \delta)Q_2$ then $\hat{Q}_1 = (*, \delta)\hat{Q}_2$.*

Proof: Denoting $C_1 = C_{Q_1}$, $C_2 = C_{Q_2}$ we have by Lemma 2.5 (2) and Lemma 9.2:

$$\bigwedge_{x,y \in X} \rho(\hat{Q}_1(x, y), \hat{Q}_2(x, y)) =$$
$$= \bigwedge_{x,y \in X} \rho(\bigvee_{S \in \mathcal{B}} (C_1(S)(x) * S(y)), \bigvee_{S \in \mathcal{B}} (C_2(S)(x) * S(y))) \geq$$
$$\geq \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} \rho(C_1(S)(x) * S(y), C_2(S)(x) * S(y)) \geq$$
$$\geq \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} \rho(C_1(S)(x), C_2(S)(x)) = \bigwedge_{x \in X} \bigwedge_{S \in \mathcal{B}} \rho(C_1(S)(x), C_2(S)(x)).$$

For any $x \in X$ and $S \in \mathcal{B}$ we have by Lemma 2.5 (2) and Lemma 9.2:

$$\rho(C_1(S)(x), C_2(S)(x)) =$$
$$= \rho(S(x) * \bigwedge_{y \in X} [S(y) \to Q_1(x, y)], S(x) * \bigwedge_{y \in X} [S(y) \to Q_2(x, y)]) \geq$$
$$\geq \rho(\bigwedge_{y \in X} [S(y) \to Q_1(x, y)], \bigwedge_{y \in X} [S(y) \to Q_2(x, y)]) \geq$$
$$\geq \bigwedge_{y \in X} \rho(S(y) \to Q_1(x, y), S(y) \to Q_2(x, y)) \geq$$
$$\bigwedge_{y \in X} \rho(Q_1(x, y), Q_2(x, y)).$$

In accordance with hypothesis $Q_1 = (*, \delta)Q_2$ we obtain:

$$\bigwedge_{x,y\in X} \rho(\hat{Q}_1(x,y), \hat{Q}_2(x,y)) \geq \bigwedge_{x\in X}\bigwedge_{S\in\mathcal{B}}\bigwedge_{y\in X} \rho(Q_1(x,y), Q_2(x,y)) =$$

$$= \bigwedge_{x,y\in X} \rho(Q_1(x,y), Q_2(x,y)) \geq \delta.$$

We proved $\hat{Q}_1 = (*, \delta)\hat{Q}_2$. □

Now we shall prove that $(*, \delta)$-equality of fuzzy choice functions is preserved by the assignment $C \mapsto \hat{C}$.

Proposition 9.4. *Let C_1, C_2 be two fuzzy choice functions on $\langle X, \mathcal{B}\rangle$ and $0 \leq \delta \leq 1$. If $C_1 = (*, \delta)C_2$ then $\hat{C}_1 = (*, \delta)\hat{C}_2$.*

Proof: Denote $R_1 = R_{C_1}$, $R_2 = R_{C_2}$. According to Lemma 9.2 and Lemma 2.5 (1)

$$\bigwedge_{S\in\mathcal{B}}\bigwedge_{x\in X} \rho(\hat{C}_1(S)(x), \hat{C}_2(S)(x)) =$$

$$= \bigwedge_{S\in\mathcal{B}}\bigwedge_{x\in X} \rho(S(x) * \bigwedge_{y\in X}[S(y) \to R_1(x,y)], S(x) * \bigwedge_{y\in X}[S(y) \to R_2(x,y)]) \geq$$

$$\geq \bigwedge_{S\in\mathcal{B}}\bigwedge_{x\in X} \rho(\bigwedge_{y\in X}[S(y) \to R_1(x,y)], \bigwedge_{y\in X}[S(y) \to R_2(x,y)]) \geq$$

$$\geq \bigwedge_{S\in\mathcal{B}}\bigwedge_{x,y\in X} \rho(S(y) \to R_1(x,y), S(y) \to R_2(x,y)) \geq$$

$$\geq \bigwedge_{S\in\mathcal{B}}\bigwedge_{x,y\in X} \rho(R_1(x,y), R_2(x,y)) = \bigwedge_{x,y\in X} \rho(R_1(x,y), R_2(x,y))$$

Let $x, y \in X$. Then according to Lemma 2.5 (2) and Lemma 9.2

$$\rho(R_1(x,y), R_2(x,y)) = \rho(\bigvee_{S\in\mathcal{B}}(C_1(S)(x) * S(y)), \bigvee_{S\in\mathcal{B}}(C_2(S)(x) * S(y))) \geq$$

$$\bigwedge_{S\in\mathcal{B}} \rho(C_1(S)(x) * S(y), C_2(S)(x) * S(y)) \geq \bigwedge_{S\in\mathcal{B}} \rho(C_1(S)(x), C_2(S)(x)).$$

Knowing that $C_1 = (*, \delta)C_2$ we obtain:

$$\bigwedge_{S\in\mathcal{B}}\bigwedge_{x\in X} \rho(\hat{C}_1(S)(x), \hat{C}_2(S)(x)) \geq \bigwedge_{S\in\mathcal{B}}\bigwedge_{x\in X} \rho(C_1(S)(x), C_2(S)(x)) \geq \delta.$$

Therefore $\hat{C}_1 = (*, \delta)\hat{C}_2$. □

10 Some final remarks

In this paper we introduced the $(\star, \delta)$-equality, a concept that indicates the degree of nearness of two fuzzy sets or two fuzzy relations. Our concept generalizes the δ-equality of fuzzy sets studied by Cai in [5], [6].

The starting point of this paper was the observation that the δ-equality can be defined in terms of the biresiduum associated with the Lukasiewicz t-norm. Our main contribution is the extension of Cai theory to the more general context of fuzzy set theory corresponding to an arbitrary continuous t-norm $*$. Most results of the paper lay emphasis on the behaviour of some fuzzy operators with respect to $(\star, \delta)$-equality.

Such fuzzy operators appear in fuzzy reasoning and their investigation using other types of t-norms may bring new information.

As further research we will study how the concept of $(*, \delta)$-equality can be applied to fuzzy reasoning for fuzzy optimization.

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