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Congruence indicators for fuzzy choice functions

Irina Georgescu

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Abstract We introduce the congruence indicators $WFCA(\cdot)$ and $SFCA(\cdot)$ corresponding to fuzzy congruence axioms $WFCA$ and $SFCA$. These indicators measure the degree to which a fuzzy choice function verifies the axioms $WFCA$ and $SFCA$, respectively. The main result of the paper establishes for a given choice function the relationship between its congruence indicators and some rationality conditions. One obtains a fuzzy counterpart of the well-known Arrow–Sen theorem in crisp choice functions theory.

1 Introduction

In economic and social life, the behaviour of individuals and groups is subject to vague preferences.

There exist several factors that can lead to vague preferences. A large number of attributes can determine the vagueness of preferences; some of the attributes have a more important role than others in evaluating an option. The criteria for evaluating the alternatives can be to some extent incomparable to one another. We add to this the partial information on the object of preferences and the human subjectivity as well.

Similar with preferences, the choices can be exact or vague.

According to the relationship exact–vague, there have been studied the following situations:

(a) exact preferences and exact choices;
(b) vague preferences and exact choices;
(c) vague preferences and vague choices.

I. Georgescu (✉)
Department of Economic Cybernetics, Academy of Economic Studies,
Calea Dorobanţilor 15–17, Sector 1, Bucharest, Romania
e-mail: irina.georgescu@csie.ase.ro

The first situation represents the subject of the classic theory of choice functions (Arrow 1959; Sen 1971; Suzumura 1976).

In the second situation, even if the preferences are vague, the act of choice is characterized by the precise specification of the chosen alternatives (Barrett et al. 1990; De Baets and Fodor 1997, 2003; Kulshreshtha and Shekar 2000, etc.)

There are cases when both the preferences and the choices are vague (e.g. in different moments of a decision when the option is not definitive Banerjee 1995). This is related to the third situation.

One of the dominant paradigms of social choice is the revealed preference theory. The classical economic theory has as a basic assumption the rationality of the consumer behaviour, as an optimizing behavior subject to some budgetary constraints. Revealed preference is a concept introduced by Samuelson in 1938, in the attempt to postulate the rationality of a consumer's behaviour in terms of a preference relation associated to a demand function.

Revealed preference theory has been developed in an axiomatic framework by Uzawa 1956; Arrow 1959; Richter 1966; Sen 1971; Suzumura 1976 and many others. Banerjee (1995) studied the rationality of choice functions whose domain consists of crisp sets of alternatives and the range consists of fuzzy subsets of the set of alternatives.

In Georgescu (2004a,b, 2005) we have tried to develop a theory of revealed preference for a large class of fuzzy choice functions: both the domain and the range of the choice function consist of fuzzy subsets. By identifying a crisp set with its characteristic function, our choice function includes that of Banerjee.

In the above mentioned papers there have been studied the axioms of revealed preference $WAFRP$, $SAFRP$ and the axioms of congruence $WFCA$, $SFCA$ in relationship with the rationality of fuzzy choice functions. $WAFRP$ (resp. $SAFRP$) is the fuzzy form of the Weak Axiom of Revealed Preference $WARP$ (resp. The Strong Axiom of Revealed Preference $SARP$), and $WFCA$ (resp. $SFCA$) is the fuzzy form of the Weak Congruence Axiom WCA (resp. the Strong Congruence Axiom SCA). The main result of Georgescu (2004a) establishes the relationship between the axioms $WAFRP$, $SAFRP$, $WFCA$ and $SFCA$, by obtaining a fuzzy extension of the Arrow–Sen theorem (Arrow 1959; Sen 1971).

The analysis of the fuzzy phenomena requires reasonings that operate with values of truth of the statements expressed by numbers in the real interval [0, 1]. Based on this fact, instead of checking whether a fuzzy choice function C has a property P or not, it is more adequate to have a "measure" of the degree to which C verifies P.

The indicators $WFCA(C)$ and $SFCA(C)$ defined in this paper evaluate the degree to which the choice function C verifies the axioms $WFCA$ and $SFCA$, respectively.

Section 2 contains definitions and basic facts on t-norms, fuzzy sets and fuzzy preference relations.

In Sect. 3 there are presented the fuzzy choice functions, the notions of fuzzy rationality and fuzzy normality, the revealed preference relations R_C, $\bar{R}_C$ and $\tilde{P}_C$ associated with a fuzzy choice function, as well as the axioms $WFCA$ and $SFCA$.

Section 4 deals with the similarity degree of two fuzzy choice functions, notion that the study of the indicators of Sect. 6 relies on.

The indicators $Ref(Q)$, $Trans(Q)$ and $SC(Q)$ studied in Sect. 5 express the degree to which a fuzzy preference relation Q verifies the properties of reflexivity, transitivity, and strong completeness, respectively.

Section 6 contains the main contributions of this paper. In this section there are defined the indicators $WFCA(C)$ and $SFCA(C)$ which express the degree to which the fuzzy choice function C verifies $WFCA$ and $SFCA$, respectively. The main result of the paper (Theorem 6.6) shows that, in the presence of two natural hypotheses H_1 and H_2, $WFCA(C)$ and $SFCA(C)$ are identical and can be expressed in terms of a normality degree and of the transitivity degree of R_C or $\bar{R}_C$. A corollary of this result is the fuzzy form of Arrow–Sen theorem (Georgescu 2004a).

2 Basic definitions and results

In this section we present some basic facts on continuous t-norms and residua. The background for these results can be found in Bělohlávek (2002), Fodor and Roubens (1994), and Klement et al. (2000).

A mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm iff it is commutative, associative, non-decreasing in each argument and $a * 1 = a$ for all $a \in [0, 1]$.

With any continuous left–continuous t-norm $*$ we associate its *residuum*: $a \to b = \bigvee\{c \in [0, 1] | a * c \leq b\}$.

The most well-known continuous t-norms are:

Lukasiewicz t-norm: $a *_L b = max\ (0, a + b - 1)$*;* $a \to_L b = min\ (1, 1 - a + b)$

minimum operator: $a *_G b = min\ (a, b)$*;* $a \to_G b = \begin{cases} 1 & if\ a \leq b \\ b & if\ a > b \end{cases}$

Product t-norm: $a *_P b = ab$*;* $a \to_P b = \begin{cases} 1 & if\ a \leq b \\ b/a & if\ a > b \end{cases}$

The negation operation $\neg$ *associated with* $*$ *is defined by* $\neg a = a \to 0 = \bigvee\{c \in [0, 1] | a * c = 0\}$.

Let us consider the nilpotent minimum:

$$x *_{nM} y = \begin{cases} 0 & if \quad x + y \leq 1, \\ min(x, y) & otherwise. \end{cases}$$

The nilpotent minimum is left-continuous, but not continuous (see Fodor and Roubens 1994; Klement et al. 2000).

Let $*$ *be a left-continuous t-norm.*

The properties mentioned in the following three lemmas reflect the main connections between the t-norm $*$ *and its residuum* $\to$*.*

Lemma 2.1 (Bělohlávek 2002; Fodor and Roubens 1994) *For any* $a, b, c \in [0, 1]$ *the following properties hold:*

(1) $a * b \leq c \Leftrightarrow a \leq b \to c$;
(2) $a * (a \to b) \leq a \wedge b$; *if* $*$ *is continuous, then* $a * (a \to b) = a \wedge b$;
(3) $a \leq b \Leftrightarrow a \to b = 1$;

(4) $a = 1 \to a$;
(5) $(a \to b) * (b \to c) \leq a \to c$;
(6) $a \leq b$ *implies* $b \to c \leq a \to c$ *and* $c \to a \leq c \to b$.

Lemma 2.2 (Bělohlávek 2002; Fodor and Roubens 1994) *For any* $\{a_i\}_{i \in I}, \{b_i\}_{i \in I} \subseteq [0, 1]$ *and* $a \in [0, 1]$ *the following properties hold:*

(1) $a \to \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} (a \to a_i)$;

(2) $\left(\bigvee_{i \in I} a_i\right) \to a = \bigwedge_{i \in I} (a_i \to a)$;

(3) $\bigvee_{i \in I} (a_i \to a) \leq \left(\bigwedge_{i \in I} a_i\right) \to a$;

(4) $\left(\bigvee_{i \in I} a_i\right) * a = \bigvee_{i \in I} (a_i * a)$;

(5) $\left(\bigwedge_{i \in I} a_i\right) * \left(\bigwedge_{j \in I} b_j\right) \leq \bigwedge_{i, j \in I} (a_i * b_j)$.

The biresiduum associated with the left-continuous t-norm $*$ *is defined by* $\rho(a, b) = a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Let X *be a non-empty set. A fuzzy subset* of X is a function $A : X \to [0, 1]$. We denote by $\mathcal{P}(X)$ the family of crisp subsets of X and by $\mathcal{F}(X)$ the family of fuzzy subsets of X. As a crisp subset of X is defined by its characteristic function, then we have $\mathcal{P}(X) \subseteq \mathcal{F}(X)$. For any $A, B \in \mathcal{F}(X)$, by $A \subseteq B$ we mean that $A(x) \leq B(x)$ for each $x \in X$. A fuzzy subset A of X is *non-zero* if $A(x) \neq 0$ for some $x \in X$; A is *normal* if $A(x) = 1$ for some $x \in X$.

If $x_1, \ldots, x_n \in X$ then $[x_1, \ldots, x_n]$ will denote the characteristic function of $\{x_1, \ldots, x_n\}$:

$$[x_1, \ldots, x_n](y) = \begin{cases} 1 & \text{if } y \in \{x_1, \ldots, x_n\}, \\ 0 & \text{if } y \notin \{x_1, \ldots, x_n\}. \end{cases}$$

For $A, B \in \mathcal{F}(X)$ let us denote

$$I(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)) \quad \text{and} \quad E(A, B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)).$$

It is clear that $A \subseteq B$ iff $I(A, B) = 1$ and $A = B$ iff $E(A, B) = 1$. For any $x \in X$ we have:

$$I(A, B) \leq A(x) \to B(x) \quad \text{and} \quad E(A, B) \leq A(x) \leftrightarrow B(x).$$

$I(A, B)$ is called the *subsethood degree* of A and B and $E(A, B)$ the *degree of equality* (degree of similarity) of A and B. Intuitively $I(A, B)$ expresses the truth value of the statement "A is included in B" and $E(A, B)$ the truth value of the statement "A and B contain the same elements" (see Bělohlávek 2002, p. 82).

Let $*$ be a left–continuous t-norm.

A *fuzzy preference relation* Q on X is a function $Q : X^2 \to [0, 1]$.

Let X be a non–empty set and Q a fuzzy relation on X. Q is said to be:

reflexive if $Q(x, x) = 1$ for all $x \in X$;
symmetric if $Q(x, y) = Q(y, x)$ for all $x, y \in X$;
$*$-*transitive* if $Q(x, y) * Q(y, z) \leq Q(x, z)$ for all $x, y, z \in X$;
total if $Q(x, y) > 0$ or $Q(y, x) > 0$ for all distinct $x, y \in X$;
strongly total if $Q(x, y) = 1$ or $Q(y, x) = 1$ for all distinct $x, y \in X$;
complete if $Q(x, y) > 0$ or $Q(y, x) > 0$ for all $x, y \in X$;
strongly complete if $Q(x, y) = 1$ or $Q(y, x) = 1$ for all $x, y \in X$.

We remark that Q is strongly complete iff it is reflexive and strongly total.

A fuzzy preference relation Q is said to be *regular* if it is $*$-transitive and strongly complete.

Let Q be a fuzzy relation on X. Denote by $T(Q)$ the intersection of all $*$-transitive fuzzy relations that contain Q:

$T(Q) = \bigcap\{Q' | Q \subseteq Q'$ and Q' is $*$-transitive$\}$.

$T(Q)$ is called the $*$-*transitive closure* of Q. Remark that $T(Q) = Q$ iff Q is $*$-transitive.

Proposition 2.3 *For any* $x, y \in X$,

$$T(Q)(x, y) = Q(x, y) \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1,\ldots,t_n \in X} Q(x, t_1) * \ldots * Q(t_n, y).$$

3 Fuzzy choice functions

In real life there are cases when both preferences and choices are vague [e.g. in different moments of a negotiation when the option is not definitive (Klaue et al. 2001; Kurbel and Loutschko 2003)]. At the mathematical level the choices are modeled by fuzzy choice functions.

Dasgupta and Deb (1991) study properties of rationality for a class of fuzzy choice functions. The authors consider that:

"Even when one is analyzing *precise* choice when preferences are fuzzy, this approach is useful. Fuzzy choice sets provide an important intermediate step analogous to the "substitution effect" in the theory of consumer demand. Thus, precise choice with fuzzy preference may be viewed as taking place in two steps: (a) fuzzy choice; (b) fuzzy choice being "made" precise in some "natural" way".

A similar concept of fuzzy choice functions has been studied by Banerjee (1995). His motivation is the following:

"For instance, a decision maker, faced with the problem of deciding whether not to choose an alternative x from a set of alternatives A, may feel that he/she is inclined to the extent 0.8 (on, say, a scale from 0 to 1) toward choosing it. Moreover, this fuzziness of choice is, at least potentially, *observable*. For instance, the decision maker in the example will be able to tell an interviewer the degree of his/her inclinations, or demonstrate these inclinations to an observer by the degree of eagerness or enthusiasm which he/she displays. Hence, while there may be problems of estimation, fuzzy choice functions are, in theory, observable."

In this section we shall work with a notion of fuzzy choice function larger than Banerjee's. It has been studied by Georgescu (2004a,b, 2005) and Ovchinnikov (2004).

A *fuzzy choice space* is a pair $(X, \mathcal{B})$ where X is a non-empty set of alternatives and $\mathcal{B}$ is a non-empty family of non-zero fuzzy subsets of X. A *fuzzy choice function* on $(X, \mathcal{B})$ is a function $C : \mathcal{B} \to \mathcal{F}(X)$ such that for any $S \in \mathcal{B}$, $C(S)$ is a non-zero fuzzy subset of X and $C(S) \subseteq S$.

The members of $\mathcal{B}$ can be interpreted as available fuzzy sets. If $x \in X$ is an alternative and $S \in \mathcal{B}$ is an available fuzzy set then the real number $S(x)$ can be viewed as the availability degree of x with respect to S. The degree $C(S)(x)$ to which x is chosen subject to S naturally belongs to the interval $[0, S(x)]$.

In the end of a decision-making process, an agent has to select an alternative from a feasible set of alternatives. In some cases, the decision process is complex and assumes intermediary evaluations, when the information is partial (e.g. negotiations on electronic marketplaces (Klaue et al. 2001; Kurbel and Loutschko 2003)). The fuzzy choice functions defined above may offer a mathematical modeling appropriate for such situations.

Since a crisp set is defined by its characteristic function, our definition of a fuzzy choice function generalizes Banerjee (1995). In Banerjee (1995) the domain of a choice function is made of all non-empty finite subsets and the range is made of fuzzy subsets of X. In our approach, both the domain and the range of a choice function contain fuzzy subsets of X.

The results of Georgescu (2004a,b) are proved provided the choice function C verifies the following hypotheses:

H_1 Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X;
H_2 $\mathcal{B}$ includes the fuzzy sets $[x_1, \ldots, x_n]$ for any $n \geq 1$ and $x_1, \ldots, x_n \in X$.

Since $C(S) \subseteq S$, in H_1 it suffices to assume that $C(S)$ is normal for each $S \in \mathcal{B}$. For the crisp choice functions the hypothesis H_1 is automatically fulfilled by the definition of such choice function: any S and $C(S)$ are non-empty. In the same case H_2 asserts that $\mathcal{B}$ includes all non-empty finite subsets of X, hypothesis assumed in (Arrow 1959; Sen 1971; Uzawa 1956).

Let $(X, \mathcal{B})$ be a fuzzy choice space and Q a fuzzy preference relation on X. For any $S \in \mathcal{B}$ let us define the fuzzy subsets $M(S, Q)$ and $G(S, Q)$ of X.

$$M(S, Q)(x) = S(x) * \bigwedge_{y \in X} [(S(y) * Q(y, x)) \to Q(x, y)]$$

$$G(S, Q)(x) = S(x) * \bigwedge_{y \in X} [S(y) \to Q(x, y)].$$

In the crisp case, $M(S, Q)$ represents the set of Q-maximal elements of S and $G(S, Q)$ represents the set of Q-greatest elements of S (see Suzumura 1976):

$$G(S, Q) = \{x \in S | (x, y) \in Q \text{for all} y \in S\};$$
$$M(S, Q) = \{x \in S | (y, x) \notin P_Q \text{for all} y \in S\},$$

where P_Q is the strict preference relation associated with Q ($P_Q = \{(x, y) \in X^2 | (x, y) \in Q$ and $(y, x) \notin Q\}$).

In general the functions $M(\cdot, Q) : \mathcal{B} \to \mathcal{F}(X)$ and $G(\cdot, Q) : \mathcal{B} \to \mathcal{F}(X)$ are not fuzzy choice functions.

The functions $M(\cdot, Q)$ and $G(\cdot, Q)$ allow for introducing the notion of rationality of fuzzy choice functions. A fuzzy choice function C on $(X, \mathcal{B})$ is called M-rational (resp. G-rational) if $C = M(\cdot, Q)$ (resp. $C = G(\cdot, Q)$) for some preference relation Q on X. In case of the minimum operator $\wedge$, if Q is reflexive and strongly total, then $M(\cdot, Q) = G(\cdot, Q)$ Georgescu (2005), therefore M-rationality and G-rationality coincide.

Consider now a fuzzy choice function C on $(X, \mathcal{B})$. To C one assigns the fuzzy revealed preferences R_C, $\bar{R}_C$ and $\tilde{P}_C$ on X defined by

$$R_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y));$$
$$\bar{R}_C(x, y) = C([x, y])(x);$$
$$\tilde{P}_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y))$$

for any $x, y \in X$. R_C, $\bar{R}_C$ and $\tilde{P}_C$ are fuzzy versions of some preference relations studied in classical revealed preference theory (Arrow 1959; Richter 1966; Sen 1971).

Let x, y be two alternatives. By interpreting the t–norm as a conjunction, the real number $R_C(x, y)$ is the degree of truth of the statement "there is an S such that the alternative x is chosen with respect to S and alternative y verifies S". $\bar{R}_C(x, y)$ represents the degree of truth of the statement "from the set $\{x, y\}$ is chosen at least the alternative x", and $\tilde{P}_C(x, y)$ is the degree of truth of the statement "there is an S with respect to which x is chosen and y is rejected".

By Georgescu (2004a), under the hypotheses H_1 and H_2, in case of the minimum operator, $\bar{R}_C \subseteq R_C$ and R_C, $\bar{R}_C$ are strongly complete.

A fuzzy choice function C is said to be G-normal (resp. M-normal) if $C = G(\cdot, R_C)$ (resp. $C = M(\cdot, R_C)$) Georgescu (2004a). G-normality (resp. M-normality) is a special case of G-rationality (resp. M-rationality). For simplicity we shall write $\hat{C} = G(\cdot, R_C)$. In case of the minimum operator, $C(S) \subseteq \hat{C}(S)$ for any $S \in \mathcal{B}$ (Georgescu 2004a).

Now we shall state two axioms of congruence for fuzzy choice functions.

Let $C : \mathcal{B} \to \mathcal{F}(X)$ be a fuzzy choice function.

$WFCA$ (Weak Fuzzy Congruence Axiom)

For any $S \in \mathcal{B}$ and $x, y \in X$, the following inequality holds:

$$R_C(x, y) * C(S)(y) * S(x) \leq C(S)(x).$$

$SFCA$ (Strong Fuzzy Congruence Axiom)
For any $S \in \mathcal{B}$ and $x, y \in X$, the following inequality holds:

$$W_C(x, y) * C(S)(y) * S(x) \leq C(S)(x).$$

W_C represents the transitive closure of fuzzy preference relation R_C.

Axiom $WFCA$ expresses the fact that the degree to which alternative x is chosen with respect to S, y verifies S and x is revealed preferred to y is less or equal than the degree to which y is chosen with respect to S.

Similar interpretations can be given to the axiom $SFCA$.

Remark 3.1 Axioms $WFCA$, $SFCA$ are fuzzy versions of axioms WCA, SCA in crisp choice function theory.

4 Similarity of fuzzy choice functions

Approximate reasoning deals with variables that are not identical, but have close behaviour becoming identifiable. Therefore we need some concepts to express situations when fuzzy sets and/or fuzzy relations are identifiable. The notion of similarity is very satisfiable for describing such situations.

The notion of similarity relation was introduced by Zadeh (1971) as a generalization of the concept of (crisp) equivalence relation.

We fix a continuous t-norm.

Let X be a non-empty set. A fuzzy relation Q on X is said to be a *similarity relation* if it is reflexive, symmetric and $*$-transitive. If $x, y \in X$ then $Q(x, y)$ will be called the similarity degree of x and y.

Lemma 4.1 (Bělohlávek 2002) *The function* $E(\cdot, \cdot) : \mathcal{F}(X)^2 \to [0, 1]$ *defined by the assignment* $(A, B) \mapsto E(A, B)$ *is a similarity relation on* $\mathcal{F}(X)$.

Lemma 4.2 (Bělohlávek 2002) *If* $A, B \in \mathcal{F}(X)$ *and* $x \in X$ *then* $E(A, B) * A(x) \leq B(x)$.

If Q_1, Q_2 are two fuzzy relations on X then the degree of similarity of Q_1 and Q_2 has the form: $E(Q_1, Q_2) = \bigwedge_{x,y \in X}(Q_1(x, y) \leftrightarrow Q_2(x, y))$.

According to Lemma 4.2, for any $x, y \in X$ we have $E(Q_1, Q_2) * Q_1(x, y) \leq Q_2(x, y)$.

If one interprets the fuzzy relations Q_1 and Q_2 as preferences of two agents then the real number $E(Q_1, Q_2)$ expresses how "similar" these preferences are.

The next definition introduces the degree of similarity of two fuzzy choice functions.

Definition 4.3 Let C_1, C_2 be two fuzzy choice functions on $(X, \mathcal{B})$. The *degree of similarity* $E(C_1, C_2)$ of C_1 and C_2 is a real defined by

$$E(C_1, C_2) = \bigwedge_{x \in X} \bigwedge_{S \in \mathcal{B}} (C_1(S)(x) \leftrightarrow C_2(S)(x)).$$

Lemma 4.4 (Georgescu forthcoming) *The assignment* $(C_1, C_2) \mapsto E(C_1, C_2)$ *defines a similarity relation on the set of fuzzy choice functions on* $(X, \mathcal{B})$.

The following three lemmas are true for the case when $*$ is the minimum operator $\wedge$.

Lemma 4.5 *Let* C, C' *be two fuzzy choice functions. Then for any* $S \in \mathcal{B}$ *and* $x \in X$, *we have*

(i) $E(C, C') \wedge C(S)(x) \leq C'(S)(x)$;
(ii) $E(C, C') \wedge \neg C(S)(x) \leq \neg C'(S)(x)$.

Proof By Lemma 2.1(2), we have

(i)

$$\begin{aligned} E(C, C') \wedge C(S)(x) &= C(S)(x) \wedge \bigwedge_{y \in X} \bigwedge_{T \in \mathcal{B}} [C(T)(y) \leftrightarrow C'(T)(y)] \\ &\leq C(S)(x) \wedge [C(S)(x) \rightarrow C'(S)(x)] \\ &= C(S)(x) \wedge C'(S)(x) \leq C'(S)(x). \end{aligned}$$

(ii) According to (i) we have

$$E(C, C') \wedge \neg C(S)(x) \wedge C'(S)(x) \leq C(S)(x) \wedge \neg C(S)(x) = 0.$$

It follows that $E(C, C') \wedge \neg C(S)(x) \wedge C(S')(x) = 0$, therefore $E(C, C') \wedge \neg C(S)(x) \leq \neg C'(S)(x)$. □

The previous lemma shows how the similarity preserves the vague choices.

Lemma 4.6 *Let* C *and* C' *be two fuzzy choice functions and* $x, y \in X$. *Then*

(i) $E(C, C') \wedge R_C(x, y) \leq R_{C'}(x, y)$;
(ii) $E(C, C') \wedge \neg R_C(x, y) \leq \neg R_{C'}(x, y)$.

Proof (i) By applying Lemma 4.5(i), we have:

$$\begin{aligned} E(C, C') \wedge R_C(x, y) &= E(C, C') \wedge \bigvee_{S \in \mathcal{B}} [C(S)(x) \wedge S(y)] \\ &= \bigvee_{S \in \mathcal{B}} [E(C, C') \wedge C(S)(x) \wedge S(y)] \\ &\leq \bigvee_{S \in \mathcal{B}} [C'(S)(x) \wedge S(y)] = R_{C'}(x, y). \end{aligned}$$

(ii) By (i) we have $E(C, C') \wedge \neg R_C(x, y) \wedge R_{C'}(x, y) \leq R_C(x, y) \wedge \neg R_C(x, y) = 0$, from which it follows that $E(C, C') \wedge \neg R_C(x, y) \leq \neg R_{C'}(x, y)$. □

Lemma 4.7 *If C and C' are two fuzzy choice functions and $x, y \in X$ then $E(C, C') \wedge \tilde{P}_C(x, y) \leq \tilde{P}_{C'}(x, y)$.*

Proof According to Lemma 4.5(i), we have

$$
\begin{aligned}
& E(C, C') \wedge \tilde{P}_C(x, y) \\
& \quad = E(C, C') \wedge \bigvee_{S \in \mathcal{B}} [C(S)(x) \wedge S(y) \wedge \neg C(S)(y)] \\
& \quad = \bigvee_{S \in \mathcal{B}} [(E(C, C') \wedge C(S)(x)) \wedge S(y) \wedge (E(C, C') \wedge \neg C(S)(y))] \\
& \quad \leq \bigvee_{S \in \mathcal{B}} [C'(S)(x) \wedge S(y) \wedge \neg C'(S)(y)] = \tilde{P}_{C'}(x, y).
\end{aligned}
$$

□

The two previous lemmas show the way the revealed preferences (represented by the fuzzy preference relations R_C and $\tilde{P}_C$) are preserved by the similarity of fuzzy choice functions.

The degree of similarity $E(C_1, C_2)$ allows us to define the indicators of congruence in Sect. 6, and the three lemmas from above will be intensely used for proving the main theorem of the paper (Theorem 6.6).

5 Indicators of fuzzy preference relations

We have seen that fuzzy relations model vague preferences. The properties of the fuzzy relations studied in this section (reflexivity, transitivity, etc.) throw a better light on the way preference relations are connected with various alternatives.

The indicators of fuzzy preference relations introduced in this section have the following significance: instead of checking whether a fuzzy relation R has the property P, we evaluate the degree to which R verifies P.

If Q is a crisp relation on X then the sentence "Q is reflexive" is a statement in the bivalent logic: it is true or false. For a fuzzy relation Q on X it is more appropriate to place the sentence "Q is reflexive" in the setting of a fuzzy logic (Klement et al. 2000; Bělohlávek 2002, etc.). Then instead of saying that Q is reflexive or not we shall consider the degree of truth of that statement. This will be a real number in the interval [0, 1] and it will express "how reflexive" the fuzzy relation Q is. The indicators introduced by Definition 5.1 represent degrees of truth which correspond to properties of fuzzy relations such as reflexivity, $*$-transitivity etc.

Definition 5.1 (Bělohlávek 2002) Let Q be a fuzzy relation on X. We define the following indicators:

(a) the degree of reflexivity of Q:

$$Ref(Q) = \bigwedge_{x \in X} Q(x, x);$$

(b) the degree of $*$-transitivity of Q:

$$Trans(Q) = \bigwedge_{x,y,z \in X} (Q(x, y) * Q(y, z) \to Q(x, z));$$

(c) the degree of strong completeness of Q:

$$SC(Q) = \bigwedge_{x,y \in X} (Q(x, y) \vee Q(y, x)).$$

Lemma 5.2 *For any fuzzy relation Q on X the following assertions hold:*

(a) $Ref(Q) = 1$ *iff Q is reflexive;*
(b) $Trans(Q) = 1$ *iff Q is $*$-transitive;*
(c) $SC(Q) = 1$ *iff Q is strongly complete.*

Proof We shall prove (b) for example. By Lemma 2.1(3),

$$\begin{aligned} Trans(Q) = 1 &\text{ iff } Q(x, y) * Q(y, z) \to Q(x, z) = 1 \text{ for all} x, y, z \in X; \\ &\text{iff } Q(x, y) * Q(y, z) \le Q(x, z) \text{ for all } x, y, z \in X; \\ &\text{iff } Q \text{is} * \text{-transitive.} \end{aligned}$$

□

Example 5.3 Let $X = \{x, y\}$ and the fuzzy preference relation given by the matrix

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $0 \le a, b, c, d \le 1$. Then $Ref(Q) = a \wedge d$; $Trans(Q) = (b \wedge c) \leftrightarrow (a \wedge d)$; $SC(Q) = a \vee b \vee c \vee d$.

Proposition 5.4 *Let Q be a fuzzy relation on X and $x, y, z \in X$. Therefore*

(a) $Ref(Q) \le Q(x, x)$*;*
(b) $Trans(Q) * Q(x, y) * Q(y, z) \le Q(x, z)$*;*
(c) $SC(Q) \le Q(x, y)$ *or* $SC(Q) \le Q(y, x)$.

Proof To exemplify, we prove (b). By the definition of $Trans(Q)$ we have $Trans(Q) \le Q(x, y) * Q(y, z) \to Q(x, z)$ from where, by applying Lemma 2.1(2), one obtains $Trans(Q) * Q(x, y) * Q(y, z) \le Q(x, z)$. □

The following two propositions show how the three indicators defined above are preserved by the fuzzy equality $E(\cdot, \cdot)$.

Proposition 5.5 *Let Q_1, Q_2 be two fuzzy relations on X. Then*

(1) $Ref(Q_1) * E(Q_1, Q_2) \leq Ref(Q_2)$;
(2) $SC(Q_1) * E(Q_1, Q_2) \leq SC(Q_2)$.

Proof (1) For any $x \in X$, by Lemma 2.1(2), we have

$$\begin{aligned} Ref(Q_1) * E(Q_1, Q_2) &\leq Q_1(x, x) * (Q_1(x, x) \leftrightarrow Q_2(x, x)) \\ &\leq Q_1(x, x) * (Q_1(x, x) \rightarrow Q_2(x, x)) \\ &\leq Q_1(x, x) \wedge Q_2(x, x) \leq Q_2(x, x). \end{aligned}$$

It follows

$$Ref(Q_1) * E(Q_1, Q_2) \leq \bigwedge_{x \in X} Q_2(x, x) = Ref(Q_2).$$

(2) With Lemma 2.2(4) and Lemma 4.2, the proof is similar as (1).

We assume throughout that $*$ is the minimum operator.

Proposition 5.6 *For any two fuzzy relations Q_1, Q_2 on X we have:*

$$Trans(Q_1) \wedge E(Q_1, Q_2) \leq Trans(Q_2).$$

Proof Let $x_1, x_2 \in X$. Then by Lemma 4.2 and Proposition 5.4 (b) and

since $\wedge$ is idempotent, $Trans(Q_1) \wedge E(Q_1, Q_2) \wedge Q_2(x, y) \wedge Q_2(y, z)$

$$\begin{aligned} &= Trans(Q_1) \wedge E(Q_1, Q_2) \wedge E(Q_1, Q_2) \wedge Q_2(x, y) \wedge E(Q_1, Q_2) \wedge Q_2(y, z) \\ &\leq E(Q_1, Q_2) \wedge Trans(Q_1) \wedge Q_1(x, y) \wedge Q_1(y, z) \\ &\leq E(Q_1, Q_2) \wedge Q_1(x, z) \leq Q_2(x, z). \end{aligned}$$

By applying Lemma 2.1(1), for any $x, y, z \in X$ we have

$$Trans(Q_1) \wedge E(Q_1, Q_2) \leq (Q_2(x, y) \wedge Q_2(y, z)) \rightarrow Q_2(x, z)$$

from where

$$\begin{aligned} Trans(Q_1) \wedge E(Q_1, Q_2) &\leq \bigwedge_{x, y, z \in X} [(Q_2(x, y) \wedge Q_2(y, z)) \rightarrow Q_2(x, z)] \\ &= Trans(Q_2). \end{aligned}$$

□

Proposition 5.7 *Let Q be a fuzzy relation on X. For any $x, y \in X$ we have*

$$Trans(Q) \wedge T(Q)(x, y) \leq Q(x, y).$$

Proof Let $n \geq 1$ and $t_1, \ldots, t_n \in X$. Applying the idempotence of $\wedge$ and several times Proposition 5.4 (b) we get

$$Trans(Q) \wedge Q(x, t_1) \wedge \cdots \wedge Q(t_n, y) \leq Q(x, y).$$

Therefore by Lemma 2.2(4) we have

$$\begin{aligned} &Trans(Q) \wedge T(Q)(x, y) \\ &= Trans(Q) \wedge \left[Q(x, y) \vee \bigvee_{n=1}^{\infty} \bigvee_{t_1, \ldots, t_n \in X} Q(x, t_1) \wedge \cdots \wedge Q(t_n, y) \right] \\ &= Trans(Q) \wedge Q(x, y) \vee \left[\bigvee_{n=1}^{\infty} \bigvee_{t_1, \ldots, t_n \in X} Trans(Q) \wedge Q(x, t_1) \wedge \cdots \wedge Q(t_n, y) \right] \\ &\leq Q(x, y). \end{aligned}$$

□

Remark 5.8 By Proposition 5.7 and Lemma 2.1(1), we have $Trans(Q) \leq T(Q)(x, y) \rightarrow Q(x, y) = T(Q)(x, y) \leftrightarrow Q(x, y)$.

Hence $Trans(Q) \leq \bigwedge_{x, y \in X} (T(Q)(x, y) \leftrightarrow Q(x, y)) = E(Q, T(Q))$.

6 Congruence indicators for fuzzy choice functions

The goal of this section is to define and study the indicators of the axioms of congruence $WFCA$, $SFCA$. These indicators express the degree to which the axioms $WFCA$ and $SFCA$ are verified by a fuzzy choice function. The main result proved in this section is an Arrow–Sen theorem for fuzzy choice functions formulated in terms of indicators (Theorem 6.6).

Definition 6.1 For a fuzzy choice function C on $(X, \mathcal{B})$ we define the following indicators of the axioms $WFCA$ and $SFCA$:

(i) $WFCA(C) = \bigwedge_{x, y \in X} \bigwedge_{S \in \mathcal{B}} [S(x) * C(S)(y) * R_C(x, y) \rightarrow C(S)(x)]$;

(ii) $SFCA(C) = \bigwedge_{x, y \in X} \bigwedge_{S \in \mathcal{B}} [S(x) * C(S)(y) * W_C(x, y) \rightarrow C(S)(x)]$.

The previous definition of the two indicators comes from the formulation of the axioms $WFCA$ and $SFCA$ in the natural language and it is based on the fact that $*$ models the fuzzy conjunction and the residuum $\rightarrow$ models the fuzzy implication.

Remark 6.2 For a choice function C the following equivalences hold:

$WFCA(C) = 1$ iff C verifies $WFCA$;
$SFCA(C) = 1$ iff C verifies $SFCA$.

The indicator $WFCA(C)$ (resp. $SFCA(C)$) expresses the degree to which the fuzzy choice function C verifies $WFCA$ (resp. $SFCA$). These indicators allow an analysis of the behaviour of a fuzzy choice function with respect to the axioms $WFCA$ (resp. $SFCA$). With these indicators we can compare the fuzzy choice functions: if C_1 and C_2 are two fuzzy choice functions and $WFCA(C_1) \leq WFCA(C_2)$ then we can say that C_2 is "more congruous" than C_1 in the sense of the axiom $WFCA$. It follows that each of these indicators produces a ranking of a family of fuzzy choice functions.

One can define similarly the revealed preference indicators $WAFRP(C)$ and $SAFRP(C)$ corresponding to the fuzzy revealed preference axioms $WAFRP$ and $SAFRP$.

In this section we suppose that hypotheses H_1 and H_2 are verified.

To prove the main result of this section (Theorem 6.6), we need some preliminary technical results, which have an intrinsic interest as well.

According to Sen (1971), if the crisp choice function C verifies the Weak Congruence Axiom WCA then the crisp revealed preference relation R_C associated with C is transitive. The following proposition generalizes this result to fuzzy choice functions.

Proposition 6.3 *If C is a fuzzy choice function on $(X, \mathcal{B})$ then $WFCA(C) \leq Trans(R_C)$.*

Proof Let $x, y, z \in X$. We shall prove the inequality

(a) $WFCA(C) \wedge R_C(x, y) \wedge R_C(y, z) \leq R_C(x, z)$

Let us denote $T = [x, y, z]$. By H_2 $C(T)$ is normal hence $C(T)(x) = 1$ or $C(T)(y) = 1$ or $C(T)(z) = 1$. We analyze these three cases.

If $C(T)(x) = 1$, then $1 = C(T)(x) \wedge T(z) \leq R_C(x, z)$, hence $R_C(x, z) = 1$.

If $C(T)(y) = 1$, then by Lemma 2.1 (2), we have

$$\begin{aligned}
&WFCA(C) \wedge R_C(x, y) \wedge R_C(y, z)\\
&\quad \leq R_C(x, y) \wedge R_C(y, z) \wedge [(T(x) \wedge C(T)(y) \wedge R_C(x, y)) \to C(T)(x)]\\
&\quad = R_C(x, y) \wedge R_C(y, z) \wedge [R_C(x, y) \to C(T)(x)]\\
&\quad \leq R_C(x, y) \wedge [R_C(x, y) \to C(T)(x)] = R_C(x, y) \wedge C(T)(x)\\
&\quad \leq C(T)(x) = C(T)(x) \wedge T(z) \leq R_C(x, z).
\end{aligned}$$

If $C(T)(z) = 1$, then

$$\begin{aligned}
&WFCA(C) \wedge R_C(x, y) \wedge R_C(y, z)\\
&\quad \leq R_C(x, y) \wedge R_C(y, z) \wedge [(T(y) \wedge C(T)(z) \wedge R_C(y, z)) \to C(T)(y)] \wedge WFCA(C)\\
&\quad = R_C(x, y) \wedge R_C(y, z) \wedge [R_C(y, z) \to C(T)(y)] \wedge WFCA(C)\\
&\quad = R_C(x, y) \wedge R_C(y, z) \wedge C(T)(y) \wedge WFCA(C)\\
&\quad \leq R_C(x, y) \wedge C(T)(y) \wedge [(T(x) \wedge C(T)(y) \wedge R_C(x, y)) \to C(T)(x)]\\
&\quad = R_C(x, y) \wedge C(T)(y) \wedge [(R_C(x, y) \wedge C(T)(y)) \to C(T)(x)]\\
&\quad = R_C(x, y) \wedge C(T)(y) \wedge C(T)(x) \leq C(T)(x) = C(T)(x) \wedge T(z) \leq R_C(x, z).
\end{aligned}$$

Therefore the inequality (a) is verified for all the three cases. From (a) it follows immediately that

$WFCA(C) \leq (R_C(x, y) \wedge R_C(y, z)) \to R_C(x, z)$ for any $x, y, z \in X$, therefore

$WFCA(C) \leq \bigwedge_{x,y,z \in X} [(R_C(x, y) \wedge R_C(y, z)) \to R_C(x, z)] = Trans(R_C)$. □

The following proposition expresses in terms of indicators the fact that the G-normality of C and the transitivity of R_C assure the fulfillment of the fuzzy congruence axiom $WFCA$.

Proposition 6.4 *If C is a fuzzy choice function then $E(C, \hat{C}) \wedge Trans(R_C) \leq WFCA(C)$.*

Proof Let $S \in \mathcal{B}$ and $x, y \in X$. We shall prove that

(a) $E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$.

Let $z \in X$. Knowing that $C(S)(y) \wedge S(z) \leq R_C(y, z)$ and applying Proposition 5.4 (b):

$$
\begin{aligned}
&E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge C(S)(y) \wedge S(x) \wedge S(z) \\
&\quad \leq E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge R_C(y, z) \wedge S(x) \leq R_C(x, z).
\end{aligned}
$$

From this, Lemma 2.1(1) gives:

$$
E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge C(S)(y) \wedge S(x) \leq S(z) \to R_C(x, z)
$$

for each $z \in X$, therefore

$$
E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge C(S)(y) \wedge S(x) \leq \bigwedge_{z \in X} [S(z) \to R_C(x, z)].
$$

We deduce that

$$
\begin{aligned}
&E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge C(S)(y) \wedge S(x) \\
&\quad \leq S(x) \wedge \bigwedge_{z \in X} [S(z) \to R_C(x, z)] = \hat{C}(S)(x).
\end{aligned}
$$

By applying Lemma 4.5 (i) we obtain

$$
\begin{aligned}
&E(C, \hat{C}) \wedge Trans(R_C) \wedge R_C(x, y) \wedge C(S)(y) \wedge S(x) \\
&\quad \leq E(C, \hat{C}) \wedge \hat{C}(S)(x) \leq C(S)(x)
\end{aligned}
$$

and (a) is proved. According to (a), by Lemma 2.1 (1), we have:

$$
E(C, \hat{C}) \wedge Trans(R_C) \leq (S(x) \wedge C(S)(x) \wedge R_C(x, y)) \to C(S)(x)
$$

from where

$$E(C,\hat{C}) \wedge Trans(R_C) \leq \bigwedge_{x,y\in X} \bigwedge_{S\in\mathcal{B}} [(S(x) \wedge C(S)(y) \wedge R_C(x,y)) \rightarrow C(S)(x)]$$
$$= WFCA(C).$$

□

The following theorem shows that the similarity of the fuzzy preference relations R_C and $\bar{R}_C$ is assured by the degree of G-normality of C.

Theorem 6.5 *For any fuzzy choice function C on $(X,\mathcal{B})$ we have $E(C,\hat{C}) \leq E(R_C,\bar{R}_C)$.*

Proof According to the definition of $\hat{C}$ we have

$$E(C,\hat{C}) = \bigwedge_{x\in X} \bigwedge_{S\in\mathcal{B}} [\hat{C}(S)(x) \leftrightarrow C(S)(x)] = \bigwedge_{x\in X} \bigwedge_{S\in\mathcal{B}} [\hat{C}(S)(x) \rightarrow C(S)(x)]$$

since $C \subseteq \hat{C}$.

Using the definition of $\bar{R}_C$ we get

$$E(R_C,\bar{R}_C) = \bigwedge_{x,y\in X} [R_C(x,y) \leftrightarrow \bar{R}_C(x,y)]$$
$$= \bigwedge_{x,y\in X} [R_C(x,y) \rightarrow \bar{R}_C(x,y)]$$

since $\bar{R}_C \subseteq R_C$.

Since R_C is reflexive and $[x,y](x) = [x,y](y) = 1$, $[x,y](z) = 0$ for $z \notin \{x,y\}$, one obtains by Lemma 2.1 (4)

$$\hat{C}([x,y])(x) = [x,y](x) \wedge \bigwedge_{z\in X} ([x,y](z) \rightarrow R_C(x,z))$$
$$= [[x,y](x) \rightarrow R_C(x,x)] \wedge [[x,y](y) \rightarrow R_C(x,y)]$$
$$= R_C(x,x) \wedge R_C(x,y)$$
$$= R_C(x,y).$$

Then

$$R_C(x,y) \rightarrow \bar{R}_C(x,y) = \hat{C}([x,y])(x) \rightarrow \bar{R}_C(x,y) = \hat{C}([x,y])(x) \rightarrow C([x,y])(x).$$

Therefore,

$$E(C,\hat{C}) = \bigwedge_{x\in X} \bigwedge_{S\in\mathcal{B}} [\hat{C}(S)(x) \rightarrow C(S)(x)]$$

$$\leq \bigwedge_{x \in X} \bigwedge_{y \in X} (\hat{C}([x, y])(x) \to C([x, y])(x))$$
$$= \bigwedge_{x, y \in X} [R_C(x, y) \to \bar{R}_C(x, y)] = E(R_C, \bar{R}_C).$$

□

A well-known Arrow–Sen theorem (Arrow 1959; Sen 1971) establishes for crisp choice functions the equivalence between the congruence axioms WCA, SCA and other rationality conditions. The following result provides the relationship between the congruence indicators $WFCA(C)$, $SFCA(C)$, the degree of similarity of C and $\hat{C}$ and the transitivity indicators of the fuzzy revealed preference relations R_C, $\bar{R}_C$. In this way, one obtains, in terms of indicators and similarity, a fuzzy counterpart of the Arrow–Sen theorem.

Theorem 6.6 *If C is a fuzzy choice function on $(X, \mathcal{B})$ then*

$$WFCA(C) = SFCA(C) = E(C, \hat{C}) \wedge Trans(R_C)$$
$$= E(C, \hat{C}) \wedge Trans(\bar{R}_C).$$

Proof Let us show that $E(C, \hat{C}) \wedge Trans(R_C) = E(C, \hat{C}) \wedge Trans(\bar{R}_C)$.

According to Theorem 6.5, $E(C, \hat{C}) \leq E(R_C, \bar{R}_C)$, hence, by Proposition 5.6:

$$E(C, \hat{C}) \wedge Trans(R_C) \leq E(R_C, \bar{R}_C) \wedge Trans(R_C) \leq Trans(\bar{R}_C);$$
$$E(C, \hat{C}) \wedge Trans(\bar{R}_C) \leq E(R_C, \bar{R}_C) \wedge Trans(\bar{R}_C) \leq Trans(R_C).$$

From these two inequalities it follows immediately

$$E(C, \hat{C}) \wedge Trans(R_C) = E(C, \hat{C}) \wedge Trans(\bar{R}_C).$$

Let us show that $WFCA(C) = SFCA(C)$.

Let $S \in \mathcal{B}$ and $x, y \in X$. Since $R_C(x, y) \leq W_C(x, y)$ by Lemma 2.1 (6) we have

$$(S(x) \wedge C(S)(y) \wedge W_C(x, y)) \to C(S)(x)$$
$$\leq (S(x) \wedge C(S)(y) \wedge R_C(x, y)) \to C(S)(x).$$

From this, $SFCA(C) \leq WFCA(C)$. For the converse inequality $WFCA(C) \leq SFCA(C)$, we have to show that for any $S \in \mathcal{B}$ and $x, y \in X$,

(a) $WFCA(C) \leq (S(x) \wedge C(S)(y) \wedge W_C(x, y)) \to C(S)(x)$.

By Lemma 2.1 (1) inequality (a) is equivalent with

(b) $WFCA(C) \wedge S(x) \wedge C(S)(y) \wedge W_C(x, y) \leq C(S)(x)$.

According to Proposition 6.3, $WFCA(C) \leq Trans(R_C)$ and according to Proposition 5.7, $Trans(R_C) \wedge W_C(x, y) \leq R_C(x, y)$.

Then

$$\begin{aligned} &WFCA(C) \wedge S(x) \wedge C(S)(y) \wedge W_C(x, y) \\ &\quad \leq Trans(R_C) \wedge S(x) \wedge C(S)(y) \wedge W_C(x, y) \\ &\quad \leq S(x) \wedge C(S)(y) \wedge R_C(x, y) \end{aligned}$$

from where one obtains by means of Lemma 2.1(2)

$$\begin{aligned} &WFCA(C) \wedge S(x) \wedge C(S)(y) \wedge W_C(x, y) \\ &\quad \leq S(x) \wedge C(S)(y) \wedge R_C(x, y) \wedge WFCA(C) \\ &\quad \leq S(x) \wedge C(S)(y) \wedge R_C(x, y) \wedge [(S(x) \wedge C(S)(y) \wedge R_C(x, y)) \rightarrow C(S)(x)] \\ &\quad = S(x) \wedge C(S)(y) \wedge R_C(x, y) \wedge C(S)(x) \leq C(S)(x). \end{aligned}$$

With this (b) is proved, therefore $WFCA(C) \leq SFCA(C)$. It follows $WFCA(C) = SFCA(C)$.

Let us show that $WFCA(C) = E(C, \hat{C}) \wedge Trans(R_C)$. Since Proposition 6.4 implies that $E(C, \hat{C}) \wedge Trans(R_C) \leq WFCA(C)$, it rests to show the converse.

Let $S \in \mathcal{B}$ and $x \in X$. Since $C(S)$ is a normal fuzzy subset of X there exists $z \in X$ such that $C(S)(z) = 1$, therefore $S(z) = 1$. Then, by Lemma 2.1 (4)

$$\begin{aligned} \hat{C}(S)(x) &= S(x) \wedge \bigwedge_{u \in X} [S(u) \rightarrow R_C(x, u)] \leq S(x) \wedge [S(z) \rightarrow R_C(x, z)] \\ &= S(x) \wedge [1 \rightarrow R_C(x, z)] = S(x) \wedge R_C(x, z). \end{aligned}$$

Then by applying Lemma 2.1 (2) it follows that

$$\begin{aligned} &WFCA(C) \wedge \hat{C}(S)(x) \leq WFCA(C) \wedge S(x) \wedge R_C(x, z) \\ &\quad = S(x) \wedge C(S)(z) \wedge R_C(x, z) \wedge \bigwedge_{u,v \in X} \bigwedge_{T \in \mathcal{B}} [(T(u) \wedge C(T)(v) \wedge R_C(u, v)) \\ &\qquad \rightarrow C(T)(u)] \\ &\quad \leq S(x) \wedge C(S)(z) \wedge R_C(x, z) \wedge [(S(x) \wedge C(S)(z) \wedge R_C(x, z)) \rightarrow C(S)(x)] \\ &\quad = S(x) \wedge C(S)(z) \wedge R_C(x, z) \wedge C(S)(x) \leq C(S)(x). \end{aligned}$$

By Lemma 2.1(1) one obtains $WFCA(C) \leq \hat{C}(S)(x) \rightarrow C(S)(x)$. Since $C(S)(x) \leq \hat{C}(S)(x)$, then by Lemma 2.1(3), $C(S)(x) \rightarrow \hat{C}(S)(x) = 1$ therefore

$$WFCA(C) \leq \bigwedge_{S \in \mathcal{B}} \bigwedge_{x,y \in X} [C(S)(x) \leftrightarrow \hat{C}(S)(x)] = E(C, \hat{C}).$$

By Proposition 6.3, it follows that $WFCA(C) \leq E(C, \hat{C}) \wedge Trans(R_C)$. □

Corollary 6.7 (Georgescu 2004a) *For a fuzzy choice function C on $(X, \mathcal{B})$ the following assertions are equivalent:*

(1) *C verifies $WFCA$;*
(2) *C verifies $SFCA$;*
(3) *C is G-normal and R_C is regular;*
(4) *C is G-normal and $\bar{R}_C$ is regular.*

Proof The equivalence (1) $\Leftrightarrow$ (2) follows by Theorem 6.6 and Remark 6.2. We prove now (1) $\Leftrightarrow$ (3). It is known that under hypotheses H_1 and H_2, R_C and $\bar{R}_C$ are reflexive and strongly total, therefore the regularity of R_C and $\bar{R}_C$ means their $*$-transitivity. According to Theorem 6.6, Remark 6.2 and Lemma 5.2 (b), the following assertions are equivalent:

- C verifies $SFCA$;
- $WFCA(C) = 1$;
- $E(C, \hat{C}) = 1$ and $Trans(R_C) = 1$;
- $C = \hat{C}$ and R_C is $\wedge$-transitive;
- C is G-normal and R_C is regular.

The other equivalences are established similarly. □

Remark 6.8 The degree of similarity $E(C, \hat{C})$ of C and $\hat{C}$ can be regarded as a measure of the G-normality of C. Then Theorem 6.6 shows that the indicators $WFCA(C)$ and $SFCA(C)$ are equal and that they can be expressed in terms of the G-normality of C and of the degree of transitivity of R_C or $\bar{R}_C$.

Theorem 6.6 has been stated and proved for the minimum operator $\wedge$. In the proof of this theorem and the preliminary propositions we have used the idempotency and other specific properties of minimum operator. The following example shows that in case of the nilpotent minimum, the Lukasiewicz t-norm and the product t-norm, Theorem 6.6 cannot be stated.

Example 6.9 Let $X = \{x, y\}$ and $\mathcal{B} = \{[x], [y], [x, y], A\}$ where $A \in \mathcal{F}(X)$ is defined by $A = 0.3\chi_{\{x\}} + \chi_{\{y\}}$.

Consider function $C : \mathcal{B} \to \mathcal{F}(X)$ defined by

$$C([x]) = \chi_{\{x\}}; \quad C([y]) = \chi_{\{y\}}; \quad C([x, y]) = 0.25\chi_{\{x\}} + \chi_{\{y\}}; \quad C(A) = 0.25\chi_{\{x\}} + \chi_{\{y\}}.$$

C is a fuzzy choice function fulfilling H_1 and H_2. We determine first the fuzzy relation R_C:

$$R_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y)) = 0.25.$$

Analogously $R_C(x, x) = R_C(y, y) = R_C(y, x) = 1$, thus

$$R_C = \begin{pmatrix} 1 & 0.25 \\ 1 & 1 \end{pmatrix}.$$

It is clear that R_C is $*$-transitive, strongly complete, and $R_C = \bar{R}_C$. Thereby $W_C = R_C$ and $WFCA(C) = SFCA(C)$.

By computation, we obtain that $WFCA(C) = SFCA(C) = 1$.
We compute next $\hat{C}$.

$$\begin{aligned}
\hat{C}([x])(x) &= 1 = C([x])(x),\\
\hat{C}([x])(y) &= 0 = C([x])(y),\\
\hat{C}([y])(x) &= 0 = C([y])(x),\\
\hat{C}([y])(y) &= 1 = C([y])(y),\\
\hat{C}([x,y])(x) &= [x,y](x) * [([x,y](x) \to R_C(x,x)) \wedge ([x,y](y) \to R_C(x,y))],\\
&= R_C(x,y) = 0.25 = C([x,y])(x),\\
\hat{C}([x,y])(y) &= R_C(y,x) = 1 = C([x,y])(y),\\
\hat{C}(A)(y) &= A(y) * [((A(x) \to R_C(y,x)) \wedge (A(y) \to R_C(y,y))],\\
&= 0.3 \to R_C(y,x) = 1 = C(A)(y),\\
\hat{C}(A)(x) &= A(x) * [((A(x) \to R_C(x,x)) \wedge (A(y) \to R_C(x,y))],\\
&= 0.3 * R_C(x,y) = 0.3 * 0.25; C(A)(x) = 0.25.
\end{aligned}$$

For any left-continuous t-norm $*$, we compute the degree of similarity of C and $\hat{C}$ and we obtain that $E(C, \hat{C}) = (0.3 * 0.25) \leftrightarrow 0.25$. Particularizing the t-norm $*$ one obtains:

For the minimum operator, $E(C, \hat{C}) = 0.25 \leftrightarrow 0.25 = 1$;
For the Lukasiewicz t-norm $*_L$, $E(C, \hat{C}) = 0 \leftrightarrow 0.25 = 0.75$;
For the product t-norm $*_P$, $E(C, \hat{C}) = 0.075 \leftrightarrow 0.25 = 0.3$;
For the nilpotent minimum $*_{nM}$, $E(C, \hat{C}) = 0 \leftrightarrow 0.25 = 0.75$.

In conclusion, Theorem 6.6 is verified only for the minimum operator.

7 Concluding remarks

The Arrow–Sen theorem, in the form given in Sen (1971), establishes the equivalence of revealed preference axioms $WARP$, $SARP$, of congruence axioms WCA, SCA and of another four conditions of rationality of choice functions.

A fuzzy extension of this theorem (Georgescu 2004a) connects the axioms *WAFRP*, $SAFRP$, $WFCA$ and $SFCA$, fuzzy forms of $WARP$, $SARP$, WCA and SCA, respectively.

For fuzzy choice functions, some of these equivalences of the Arrow–Sen theorem hold true for an arbitrary continuous t-norm, others for the minimum operator and others for the Lukasiewicz t-norm.

The core of this paper is a stronger version of the part in Arrow–Sen theorem (Georgescu 2004a) that holds for the minimum operator. This result expresses the relationship between the congruence indicators $WFCA(C)$ and $SFCA(C)$ associated with a fuzzy choice function C , the similarity of C and its G-normal fuzzy choice function, and transitivity degree of the fuzzy revealed preference relations R_C and $\bar{R}_C$.

These indicators enable us to compare two fuzzy choice functions with respect to each axiom. For example, for two fuzzy choice functions C_1 and C_2, if $WFCA(C_1) \leq$

$WFCA(C_2)$ then we can say that C_2 verifies $WFCA$ to a greater extent than C_1. On this basis, any family of fuzzy choice functions can be ranked with respect to each axiom.

Let us consider a family of fuzzy choice functions, each of them representing a model resulting from an expertise. Then by using the congruence indicators we can select the expertise with the highest "degree of rationality".

The main result of this paper (Theorem 6.6) has two limitations: one is due to the presence of hypotheses H_1 and H_2 , other due to the fact that it holds only for the minimum operator. These limitations would be eventually overcome by the modification of definition of $WFCA(C)$ and $SFCA(C)$, or by using other indicators. An open problem is to use the revealed preference indicators $WAFRP(C)$ and $SAFPR(C)$ in obtaining a fuzzy generalization of that part of the Arrow–Sen theorem related to the revealed preference axioms $WARP$ and $SARP$.

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